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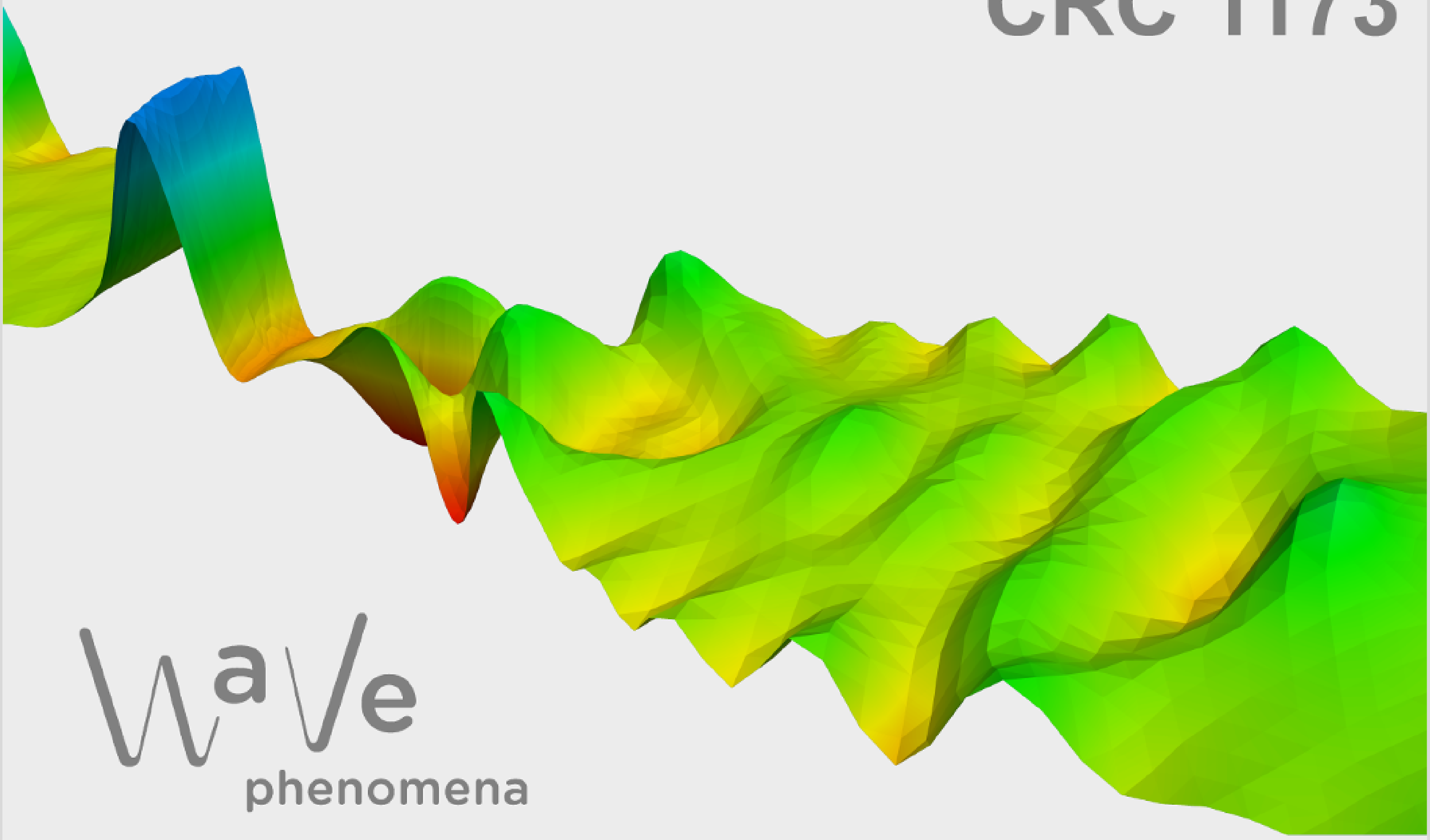

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THE LIMITING ABSORPTION PRINCIPLE AND A RADIATION CONDITION FOR THE SCATTERING BY A PERIODIC LAYER

ANDREAS KIRSCH AND ARMIN LECHLEITER

Abstract. Scattering of time-harmonic waves from periodic structures at some fixed real-valued wave number becomes analytically difficult whenever there arise surface waves: These non-zero solutions to the homogeneous scattering problem physically correspond to modes propagating along the periodic structure and clearly imply non-uniqueness of any solution to the scattering problem. In this paper, we consider a medium that is defined in the upper two-dimensional half-space by a penetrable and periodic contrast. We prove that there is a so-called limiting absorption solution to the associated scattering problem. By definition, such a solution is the limit of a sequence of unique solutions for artificial complex-valued wave numbers tending to the above-mentioned real-valued wave number. Our method of proof seems to be new: By the Floquet-Bloch transform we first reduce the scattering problem to a finite-dimensional one that is set in the linear space spanned by all surface waves. In this space, we then compute explicitly which modes propagate along the periodic structure to the left or to the right. This finally yields a representation for our limiting absorption solution which leads to a proper extension of the well known upward propagating radiation condition. Finally, we prove uniqueness of a solution under this radiation condition.

1. Introduction

Periodic non-absorbing surface structures feature surface waves that propagate along the structure without decaying. These waves do physically arise at certain exceptional values of the Bloch parameter, and mathematically they are eigenfunctions of a certain (quasi-)periodic eigenvalue problem involving the surface structure. The corresponding eigenvalue determines the surface wave's frequency, and the surface wave itself is the quasi-periodically extended eigenfunction.

Since the eigenfunction is a non-zero solution to a corresponding quasi-periodic scattering problem from the periodic structure, the latter scattering problem cannot be uniquely solvable at any of these eigenfrequencies. For this reason, such frequencies are usually excluded from the analysis of surface scattering. Consequently, scattering theory from periodic structures is a field that artificially always excludes surface waves, despite in applications both phenomena of course can hardly be separated.

In this paper we show for a model scattering problem involving a periodic contrast function that there always exists a unique solution that satisfies a so-called limiting absorption principle, independent of whether surface waves do exist or not. By construction, this solution is, in a certain topology, limit of the unique solutions to a family of coercive problems with artificial complex-valued wave numbers.

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This limiting absorption solution consists of two parts that we determine via the Floquet-Bloch transform: The first part belongs to H^1 on any strip of finite height and the second part is made up of surface waves. This second part vanishes if no surface waves exist and arises via a finite-dimensional eigenvalue problem by holomorphic perturbation theory in a neighborhood of any quasi-periodicity that features non-uniqueness. This paper seems to be a first instance of such a limiting absorption principle for a surface scattering problem (apart from flat surfaces and everywhere constant coefficients, where things are much simpler).

Precisely, we consider a problem in the upper half plane $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ that is formulated variationally in the waveguide $W := \mathbb{R} \times (0, h) \subset \mathbb{R}^2_+$. We assume that the index of refraction $q \in L^\infty(\mathbb{R}^2_+)$ is $2\pi-$periodic with respect to x_1 and equals to one for $x_2 > h$ to construct a (weak) limiting absorption solution $u \in H^2_{loc}(\mathbb{R}^2_+)$ solving

$$\Delta u + k^2 q\, u \;=\; 0 \quad \text{in } \mathbb{R}^2_+, \tag{1}$$

subject to Dirichlet boundary conditions $u = 0$ on $\Gamma_0 = \mathbb{R} \times \{0\}$ and a suitable radiating condition stated below.

The limiting absorption principle has been established for frequency scattering problems in free space, in closed waveguides, and in stratified media several times in the literature. We refer to [13, 10, 14, 1, 16, 15]. Further, [4] (see also [8]) recently showed a limiting absorption principle for scattering in a closed waveguide that relies fundamentally on the Floquet-Bloch transform and has substantially motivated our present paper. In [4], the authors decompose fields via the eigenfunctions of the generalized quasi-periodic Laplacian in the unit cell. This technique cannot be applied in our case, as such decompositions cannot be directly transferred to surface structures that form open instead of closed waveguides. Our analysis is indeed rather different compared to the one in [4], and also compared to the independent study in [7].

The limiting absorption principle leads to a special decomposition of the solution into a field $u^{(1)}$ which decays and a finite combination of surface waves. This allows us to formulate a radiation condition which includes the standard upward propagating radiation condition and provides uniqueness of the solution. Also radiation conditions for scattering problems by stratified media have a long history, see, e.g. [17, **?**, **?**] but, again, our work seems to be the first for scattering problems by periodic layers.

Our analysis carries over without difficulty to the case of a completely open wave guide; that is, formulated in all of $\mathbb{R}^2$ without the Dirichlet boundary, and to the case of closed waveguides as considered in [4]. For this case, our approach provides an independent proof of the limiting absorption principle and uniqueness.

The methods we apply are all well-known and in principle simple enough to extend our analysis to more involved scattering problems in linear elasticity or electromagnetics. To reduce technical difficulties, we are however merely considering the simple Helmholtz equation in (1).

To briefly comment on this paper's structure, the following Section 2 discusses the scattering problem in more detail and Section 3 introduces the Floquet-Bloch transform as well as the variational formulation of the scattering problem at complex-valued frequency. Section 4 shows properties of this formulation at wave numbers and quasi-periodicities where non-uniqueness holds; Section 5 then reduces the mathematical problem to a finite-dimensional one. Finally, Section 6 shows existence of a limiting absorption solution und uniqueness of the solution.

2. Surface Scattering and Non-Uniqueness

Let $k \in \mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k \geq 0$ be the wave number. We consider in the following the case that a point source at some point $y \in \mathbb{R}^2$ with $y_2 > 0$ is scattered by an inhomogeneous layer on top of a perfect conductor. The layer is assumed to be $2\pi-$periodic with respect to the variable x_1. Therefore, the incident field is given by the Dirichlet Green's function of the half plane $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$; that is, $u^{inc}(x) = \Phi_k(x, y) - \Phi_k(x, y^*)$ for $x_2 > 0$ and $x \neq y$. Here, $y^* = (y_1, -y_2)^\top$ denotes the reflected point at $\mathbb{R} \times \{0\}$, and

$$\Phi_k(x, y) \;=\; \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y,$$

is the radiating fundamental solution of the Helmholtz equation in $\mathbb{R}^2$.

We note that for real values $k = \hat{k}$ we have the asymptotics $|u^{inc}(x)| \leq c\,[1 + |x|^{3/2}]^{-1}$ for $x_2 \in [0, h]$ (for any $h > 0$ and $c = c(h)$) while in the case that $\operatorname{Im} k > 0$ the function u^{inc} decays exponentially as $|x|$ tends to infinity. Thus, in both cases $u^{inc} \in L^2\big(\mathbb{R} \times (0, h)\big)$ for every $h > 0$. By enlarging h we can assume that the source point y is inside the layer; that is, $0 < y_2 < h$.

The scattering problem is to determine the total field $u^t \in H^1_{loc}\big(\mathbb{R}^2_+ \setminus \{y\}\big)$ with

$$\Delta u^t + k^2 q\, u^t \;=\; 0 \text{ in } \mathbb{R}^2_+ \setminus \{y\}\,, \qquad u^t = 0 \text{ on } \Gamma_0\,, \tag{2}$$

and such that the scattered field $u^s = u^t - u^{inc}$ is more regular than the incident field, in particular $u^s \in H^1_{loc}(\mathbb{R}^2_+)$. So far, the solution is not uniquely determined, because some kind of radiation condition for the scattered field is required. We will comment on this in a moment. We transform this problem into an inhomogeneous equation in $H^1_{loc}(\mathbb{R}^2_+)$ with a source term of bounded support. Indeed, choose $\epsilon < \min\{y_2, h - y_2\}$ and a function $\chi \in C^\infty(\mathbb{R}^2)$ with $\chi(x) = 0$ for $|x - y| \leq \epsilon/2$ and $\chi(x) = 1$ for $|x - y| \geq \epsilon$ and set $u = u^s + \chi\, u^{inc}$. Then u vanishes for $x_2 = 0$ and solves

$$\Delta u + k^2 q\, u \;=\; f \quad \text{in } \mathbb{R}^2_+\,, \tag{3}$$

where $f := \;[k^2(1 - q)(\chi - 1) + \Delta\chi]\, u^{inc} + 2\nabla\chi \cdot \nabla u^{inc}$. We note that $f \in L^2(\mathbb{R}^2_+)$ has support in the disc $B(y, \epsilon) \subset W$ and depends analytically on k. From now on we treat $f = f_k \in L^2(\mathbb{R}^2_+)$ as an arbitrary function with compact support in the disc $B(y, \epsilon) \subset W$ such that $k \mapsto f_k \in L^2\big(B(y, \varepsilon)\big)$ is holomorphic in some (complex) neighborhood of some $\hat{k} \in \mathbb{R}_{>0}$. In the case of the scattering problem the scattered field is then given by $u^s = u - \chi\, u^{inc}$ and the total field by $u^t = u + (1 - \chi)u^{inc}$. The solution of (3) is understood in the variational sense:

Definition 2.1. *A function $u \in H^1_{loc}(\mathbb{R}^2_+)$ with $u = 0$ on $\mathbb{R} \times \{0\}$ is called variational solution of (3) if*

$$\int_{\mathbb{R}^2_+} \big[\nabla u \cdot \nabla \psi - k^2 q\, u\, \psi\big]\, dx \;=\; -\int_W f\, \psi\, dx \tag{4}$$

for all $\psi \in H^1(\mathbb{R}^2_+)$ with compact support.

By choosing $\psi \in H^1(\mathbb{R}^2_+)$ in (4) with compact support in $\mathbb{R} \times (h, \infty)$ we note that u is a classical solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $x_2 > h$ and thus analytic.

As mentioned above, a further condition is needed to assure uniqueness. For wave numbers with positive imaginary part we require that $u \in H^1(\mathbb{R}^2_+)$.

Theorem 2.2. *For $k \in \mathbb{C}$ with $\mathrm{Re}\, k > 0$ and $\mathrm{Im}\, k > 0$ and $q \geq q_0$ in $\mathbb{R}^2_+$ for some $q_0 > 0$ there exists a unique variational solution $u \in H^1(\mathbb{R}^2_+)$ of problem (4).*

Proof: We define the closed subspace $H^1_0(\mathbb{R}^2_+)$ of $H^1(\mathbb{R}^2_+)$ by $H^1_0(\mathbb{R}^2_+) = \{u \in H^1(\mathbb{R}^2_+) : u = 0 \text{ on } \mathbb{R} \times \{0\}\}$ and the bounded sesquilinear form $a : H^1_0(\mathbb{R}^2_+) \times H^1_0(\mathbb{R}^2_+) \to \mathbb{C}$ by

$$a(u, \psi) \;=\; \int_{\mathbb{R}^2_+} \left[\nabla u \cdot \nabla \overline{\psi} - k^2 q\, u\, \overline{\psi}\,\right] dx\,, \quad u, \psi \in H^1_0(\mathbb{R}^2_+)\,,$$

and the bounded anti-linear form $\ell : H^1_0(\mathbb{R}^2_+) \to \mathbb{C}$ by

$$\ell(\psi) \;=\; -\int_W f\, \overline{\psi}\, dx\,, \quad \psi \in H^1_0(\mathbb{R}^2_+)\,.$$

The form a is coercive. Indeed, write k^2 in the form $k^2 = |k|^2 \exp(it)$ with $0 < t < \pi$ and choose $s \in (-\pi/2, \pi/2)$ with $t - s \in (\pi/2, \pi)$. Then

$$\mathrm{Re}\left[e^{-is} a(u,u)\right] \;=\; \cos(s)\, \|\nabla u\|^2_{L^2} - \cos(t-s)\, |k|^2 \int_W q\, |u|^2\, dx \;\geq\; c\, \|u\|^2_{H^1(\mathbb{R}^2_+)}$$

with $c = \min\{\cos(s), -|k|^2 q_0 \cos(t-s)\} > 0$. The theorem of Lax-Milgram yields uniqueness and existence of a solution u of (4). □

For real values of k a natural radiation condition would be the “upward propagating radiation condition” (UPRC) that is well known from scattering by rough surfaces (see, e.g., [3]). That is, $u(\cdot, h) \in L^\infty(\mathbb{R})$ and

$$u(x) \;=\; 2 \int_{\Gamma_h} u(y)\, \frac{\partial \Phi_k(x,y)}{\partial y_2}\, ds(y)\,, \quad x_2 > h\,, \tag{5}$$

where $\Gamma_h = \mathbb{R} \times \{h\}$. However, even with this radiation condition one can not expect uniqueness as the following example shows.

Example 2.3. Let $q = q_0 > 1$ be constant in $W = \mathbb{R} \times (0, h)$. Consider the homogeneous boundary value problem to determine $u \in C^1(\overline{\mathbb{R}^2_+}) \cap C^2(\mathbb{R}^2_+ \setminus (\mathbb{R} \times \{1\})$ with

$$\Delta u(x) \;+\; k^2 q_0\, u(x) \;=\; 0 \quad \text{for } x \in W\,,$$
$$\Delta u(x) \;+\; k^2\, u(x) \;=\; 0 \quad \text{for } x \in \mathbb{R} \times (1, \infty)\,,$$

subject to $u(x_1, 0) = 0$ for $x_1 \in \mathbb{R}$, and the upward propagating radiation condition (UPRC) for u in $x_2 > 1$.

For a real parameter $\omega \in \mathbb{R}$, we search for a solution in the form

$$u(x) \;=\; e^{i\omega x_1} v(x_2) \quad \text{for } x_2 > 0\,,$$

such that v has to satisfy

$$\begin{aligned} v''(x_2) \;+\; (k^2 q_0 - \omega^2)\, v(x_2) \;&=\; 0 \quad \text{for } 0 < x_2 < 1\,, \\ v''(x_2) \;+\; (k^2 - \omega^2)\, v(x_2) \;&=\; 0 \quad \text{for } x_2 > 1\,, \end{aligned} \tag{6}$$

subject to transmission conditions on Γ_h, homogeneous boundary condition $v(0) = 0$, and the upwards radiation condition for v. The latter conditions require v to be of the form

$$v(x_2) \;=\; \begin{cases} \alpha \exp\big[i\sqrt{k^2-\omega^2}\,x_2\big]\,, & x_2 > 1\,, \\ \beta \sin\big[\sqrt{k^2 q_0-\omega^2}\,x_2\big]\,, & 0 < x_2 < 1\,, \end{cases}$$

where the square root is chosen such that the real and imaginary parts are non-negative. We hence seek for constants α and β that solve the 2×2-system

$$\begin{aligned} \alpha \exp\big[i\sqrt{k^2-\omega^2}\big] \;-\; \beta \sin\sqrt{k^2 q_0-\omega^2} \;&=\; 0\,, \\ \alpha\, i\sqrt{k^2-\omega^2}\exp\big[i\sqrt{k^2-\omega^2}\big] \;-\; \beta\sqrt{k^2 q_0-\omega^2}\cos\sqrt{k^2 q_0-\omega^2} \;&=\; 0\,, \end{aligned}$$

which are equivalent to the transmission conditions on Γ_h. This system possesses non trivial solutions $(\alpha,\beta)^\top$ if and only if the determinant of the associated matrix vanishes; that is, if

$$d(\omega^2) \;:=\; i\sqrt{k^2-\omega^2}\,\sin\sqrt{k^2 q_0-\omega^2} \;-\; \sqrt{k^2 q_0-\omega^2}\,\cos\sqrt{k^2 q_0-\omega^2} \;=\; 0\,.$$

For $|\omega| < k$ or $|\omega| \geq \sqrt{q_0}k$ the only solution is the trivial one. Indeed, if $|\omega| < k$ then, taking real and imaginary part of this equation, yields $\cos\sqrt{k^2 q_0-\omega^2} = 0$ and $\sin\sqrt{k^2 q_0-\omega^2} = 0$ which is impossible. If $|\omega| > \sqrt{q_0}\,k$ then the equation takes the form

$$-\sqrt{\omega^2-k^2}\,\sinh\sqrt{\omega^2-k^2 q_0} \;-\; \sqrt{\omega^2-k^2 q_0}\,\cosh\sqrt{\omega^2-k^2 q_0} \;=\; 0$$

which is also not solvable because all four factors are positive. A separate argument shows that the case $|\omega| = \sqrt{q_0}\,k$ does not yield a vanishing determinant either. Therefore, nontrivial solutions can only exist for zeros of d in the interval $[k^2, k^2 q_0)$ – and, indeed, such zeros exist in general. Since the function d is analytic in the interior $(k^2, k^2 q_0)$ and $k^2 q_0$ is not a zero of d we conclude that $t = k^2$ is the only possible accumulation point of an infinite sequence $t_j \in (k^2, k^2 q_0)$ of zeros of d. The facts that $d(t_j)/(t_j - k^2) = 0$; that is,

$$\frac{1}{\sqrt{t_j-k^2}}\,\sin\sqrt{k^2 q_0-t_j} \;=\; -\;\sqrt{k^2 q_0-t_j}\,\frac{\cos\sqrt{k^2 q_0-t_j}}{t_j-k^2}\,,$$

and the analyticity of $t \mapsto \sin\sqrt{k^2 q_0-t}$ and $t \mapsto \cos\sqrt{k^2 q_0-t}$ at $t = k^2$ yields a contradiction because the right hand side is bounded as j tends to infinity while the left hand side is unbounded. (Note that in this case $\cos\sqrt{k^2 q_0-k^2}$ has to vanish and $|\sin\sqrt{k^2 q_0-k^2}| = 1$.) This shows that there exist only finitely many zeros of d. Therefore, if $\hat{\omega}^2 \in [k^2, k^2 q_0)$ is such that $d(\hat{\omega}^2) = 0$, then there exist nontrivial α and β such that

$$u(x_1,x_2) \;=\; \begin{cases} \alpha \exp\big[i\hat{\omega}\,x_1 - \sqrt{\hat{\omega}^2-k^2}\,x_2\big]\,, & x_2 > 1\,, \\ \beta \exp[i\hat{\omega}\,x_1]\sin\big[\sqrt{k^2 q_0-\hat{\omega}^2}\,x_2\big]\,, & 0 < x_2 < 1\,, \end{cases}$$

is a solution of (6). We note that this u satisfies also the upward propagating radiation condition (UPRC) because for $x \in \mathbb{R}^2$ with $x_2 > 1$ and $\omega > k$ it holds that

$$
\begin{aligned}
2 \int\limits_{-\infty}^{\infty} \frac{\partial \Phi_k(x,y)}{\partial y_2}\bigg|_{y_2=1} e^{i\omega y_1}\, dy_1 &= -\frac{ik}{2} \int\limits_{-\infty}^{\infty} H_1^{(1)}(k|x-y|)\, \frac{y_2 - x_2}{|x-y|}\bigg|_{y_2=1} e^{i\omega y_1}\, dy_1 \\
&= \frac{ik(x_2-1)}{2} \int\limits_{-\infty}^{\infty} \frac{H_1^{(1)}\big(k\sqrt{(x_1-y_1)^2+(x_2-1)^2}\big)}{\sqrt{(x_1-y_1)^2+(x_2-1)^2}}\, e^{i\omega y_1}\, dy_1 \\
&= -\frac{i}{2}\, e^{i\omega x_1} \frac{\partial}{\partial z} \int\limits_{-\infty}^{\infty} H_0^{(1)}\big(k\sqrt{y_1^2+z^2}\big)\, e^{i\omega y_1}\, dy_1 \bigg|_{z=x_2-1} \\
&= -i\, e^{i\omega x_1} \frac{\partial}{\partial z} \int\limits_{0}^{\infty} H_0^{(1)}\big(k\sqrt{y_1^2+z^2}\big)\, \cos(\omega y_1)\, dy_1 \bigg|_{z=x_2-1} \\
&= e^{i\omega x_1}\, e^{-(x_2-1)\sqrt{\omega^2-k^2}}
\end{aligned}
$$

by standard formulas on Bessel functions (see [6], formulas 6.677, 3. and 4.). This formula holds also for $\omega = k$ by continuity.

In the remaining sections we prove the limiting absorption principle; that is, we prove convergence of the (unique) solution $u = u_k \in H^1(\mathbb{R}^2_+)$ of the problem (4) for Im $k > 0$ to some function u as Im k tends to zero. Convergence will be shown in some topology to be specified later.

3. The Floquet-Bloch Transform and Operator Equations for the Scattering Problem

We go back to the problem of Definition 2.1 and use the Floquet-Bloch transform to reformulate the problem as a family of (quasi-)periodic problems. Recall that the (periodic) Floquet-Bloch transform $T_{per} : L^2(\mathbb{R}) \to L^2\big((0,2\pi) \times (-1/2, 1/2)\big)$ is defined by

$$
(T_{per} f)(t,\alpha) \;=\; \tilde{f}(t,\alpha) \;=\; \sum_{m \in \mathbb{Z}} f(t + 2\pi m)\, e^{-i\alpha(t+2\pi m)}\,.
$$

The latter formula directly shows that for smooth functions f and fixed α the transformed function $t \mapsto T_{per} f(t,\alpha) = \tilde{f}(t,\alpha)$ is $2\pi-$periodic while for fixed t the function $\alpha \mapsto T_{per} f(t,\alpha) = \tilde{f}(t,\alpha)$ is $t-$quasi-periodic; that is, $\tilde{f}(t,\alpha+1) = e^{-it}\tilde{f}(t,\alpha)$. It is hence sufficient to consider $L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ as image space of T_{per}. The inverse transform is given by

$$
(T_{per}^{-1} g)(t) \;=\; \int_{-1/2}^{1/2} g(t,\alpha)\, e^{i\alpha t} d\alpha\,, \quad t \in \mathbb{R}\,,
$$

where we extended $g(\cdot,\alpha)$ to a $2\pi-$periodic function in $\mathbb{R}$. In view of our scattering problem, we apply the Floquet-Bloch transform to the variable x_1 and consider x_2 as a parameter. Setting $W_I := \mathbb{R} \times I$ as a horizontal strip and $Q_I := (0,2\pi) \times I$ for any open interval $I \subset \mathbb{R}_{>0}$ one can then show that T_{per} is an isometry from $L^2(W_I)$ onto $L^2\big(Q_I \times (-1/2, 1/2)\big)$,

$$\|\tilde{f}\|^2_{L^2(Q_I\times(-1/2,1/2))} \;=\; \int_{-1/2}^{1/2} \int_{Q_I} |\tilde{f}(x,\alpha)|^2 \, dx \, d\alpha \;=\; \int_{W_I} |f(x)|^2 \, dx \;=\; \|f\|^2_{L^2(W_I)} \, .$$

Further, the restriction of T_{per} to $H^1(W_I)$ is an isomorphism from $H^1(W_I)$ onto functions $\tilde{f} \in L^2\big((-1/2,1/2), H^1(Q_I)\big)$ such that $\tilde{f}(\cdot,\alpha)$ is 2π–periodic in x_1, see [11, Section 6].

For $k \in \mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k > 0$, the Floquet-Bloch transformations of the compactly supported right-hand side and the variational solution to (3) in $H^1_0(\mathbb{R}^2_+)$,

$$\begin{aligned} \tilde{f}(x,\alpha) &= \sum_{m\in\mathbb{Z}} f(x + 2\pi m e^{(1)})\, e^{-i\alpha(x_1+2\pi m)} \quad \text{and} \\ \tilde{u}(x,\alpha) &= \sum_{m\in\mathbb{Z}} u(x + 2\pi m e^{(1)})\, e^{-i\alpha(x_1+2\pi m)} \quad \text{for } x \in \mathbb{R}^2_+ \text{ and } \alpha \in \mathbb{R}\, , \end{aligned}$$

are hence well-defined in $L^2((-1/2,1/2), L^2(Q_I))$ and $L^2((-1/2,1/2), H^1(Q_I))$, respectively. As one straightforwardly computes that

$$\left(\frac{\partial}{\partial x_j} + i\alpha\right) \tilde{u}(x,\alpha) \;=\; T_{per}\left(\frac{\partial u}{\partial x_j}\right)(x,\alpha) \quad \text{for } x \in \mathbb{R}^2_+,\ \alpha \in \mathbb{R},$$

the transformed $2\pi-$periodic field $\tilde{u}(\cdot,\alpha)$ hence is for fixed α a variational solution to

$$\text{(7)} \qquad \begin{cases} \Delta \tilde{u}(\cdot,\alpha) \;+\; 2i\alpha \dfrac{\partial \tilde{u}(\cdot,\alpha)}{\partial x_1} \;+\; (k^2 q - \alpha^2)\, \tilde{u}(\cdot,\alpha) \;=\; \tilde{f}(\cdot,\alpha) \quad \text{in } \mathbb{R}\times(0,\infty)\, , \\ \tilde{u}(\cdot,\alpha) = 0 \quad \text{for } x_2 = 0\, . \end{cases}$$

To tackle the last problem variationally in a bounded domain, we set $Q^\infty = (0,2\pi)\times(0,\infty)$ to be the periodicity cell and define $H^1_{per}(Q^\infty)$ as subspace of $H^1(Q^\infty)$ consisting of all functions v such that $v(0,x_2) = v(2\pi,x_2)$; that is, their 2π-periodic extension with respect to x_1 is in $H^1_{loc}(\mathbb{R}^2_+)$. Further, $H^1_{per,loc}(Q^\infty)$ is the corresponding function space gained from $H^1_{loc}(Q^\infty)$, and $H^1_{0,per}(Q^\infty)$ is the subspace of functions in $H^1_{per}(Q^\infty)$ that vanish on $(0,2\pi)\times\{0\}$.

For $k \in \mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k > 0$, the theorem of Lax-Milgram shows just as in Theorem 2.2 that for every $\alpha \in \mathbb{R}$ there exists a unique variational solution $\tilde{u}(\cdot,\alpha) \in H^1_{0,per}(Q^\infty)$ that solves (7). It is also well known that $\tilde{u}(\cdot,\alpha)$ has a so-called Rayleigh expansion of the form

$$\text{(8)} \qquad \tilde{u}(x,\alpha) \;=\; \sum_{n\in\mathbb{Z}} u_n(\alpha)\, e^{inx_1 + i\sqrt{k^2-(n+\alpha)^2}\,(x_2-h)} \quad \text{for } x_2 > h\, ,$$

where $u_n(\alpha) = (2\pi)^{-1}\int_0^{2\pi} \tilde{u}(x_1,h,\alpha)\, \exp(-inx_1)\, dx_1$, $n \in \mathbb{Z}$, are the Fourier coefficients of $\tilde{u}(\cdot,h,\alpha)$. The branch of the square root $\sqrt{z}$ for $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ is taken as the one with positive imaginary part, that is, via a branch cut along the negative imaginary axis. The series in (8) converges uniformly with all of its derivatives in any region $x_2 \geq H$ for $H > h$. Taking the same branch of the square root also in case that $k = \hat{k} > 0$, we note that the Rayleigh expansion is still well defined for real and positive values of k and provides a solution of the Dirichlet problem

$$\Delta\tilde{u} \;+\; 2i\alpha\frac{\partial\tilde{u}}{\partial x_1} \;+\; (\hat{k}^2-\alpha^2)\,\tilde{u} \;=\; 0 \text{ for } x_2>h\,, \quad \tilde{u}(x_1,h) \;=\; \sum_{n\in\mathbb{Z}} u_n\, e^{inx_1}\,,\; x_1\in\mathbb{R}\,,$$

in the half plane $\mathbb{R}^2_{x_2>h}$. Here we dropped the dependence on α that will be a fixed parameter, by quasiperiodicity of $\tilde{u}$ in α in $(-1/2,1/2]$, in the rest of this section.

We reformulate the periodic scattering problem (7) for $k\in\mathbb{C}$ with $\operatorname{Re} k>0$ and $\operatorname{Im} k>0$ variationally in the truncated periodicity cell

$$Q \;:=\; (0,2\pi)\times(0,h)$$

which is a bounded Lipschitz domain. To this end, we introduce first the periodic function space $H^1_{0,per}(Q)=\{u\in H^1(Q)\,:\; u=U|_Q \text{ for some } U\in H^1_{0,per}(Q^\infty)\}$ that is equipped with the $H^1(Q)$-norm. Second we define for parameters k and α the Dirichlet-Neumann operator $\Lambda_{k,\alpha}\,:\; H^{1/2}_{per}(\hat{\Gamma}_h)\to H^{-1/2}_{per}(\hat{\Gamma}_h)$. For simplicity, we identify $\hat{\Gamma}_h=(0,2\pi)\times\{h\}$ with $(0,2\pi)$, such that

$$\Lambda_{k,\alpha}\phi(x_1) \;=\; i\sum_{n\in\mathbb{Z}}\phi_n\,\sqrt{k^2-(n+\alpha)^2}\,e^{inx_1} \quad \text{for } x_1\in(0,2\pi)\,,$$

where, again, $\phi_n=(2\pi)^{-1}\int_0^{2\pi}\phi(x_1)\,\exp(-inx_1)\,dx_1$ are the Fourier coefficients of ϕ. Then we seek $\tilde{u}_{k,\alpha}\in H^1_{0,per}(Q)$ as a solution to the variational formulation

$$a_{k,\alpha}(\tilde{u}_{k,\alpha},\psi) \;=\; -\int_Q \tilde{f}(\cdot,\alpha)\,\overline{\psi}\,dx \quad \text{for all } \psi\in H^1_{0,per}(Q) \tag{9}$$

where

$$\begin{aligned} a_{k,\alpha}(v,\psi) \;:=\;& \int_Q\left[\nabla v\cdot\nabla\overline{\psi}+i\alpha\left(v\,\frac{\partial\overline{\psi}}{\partial x_1}-\overline{\psi}\,\frac{\partial v}{\partial x_1}\right)+(\alpha^2-k^2q)\,v\,\overline{\psi}\right]dx-\int_{\hat{\Gamma}_1}\overline{\psi}\,\Lambda_{k,\alpha}v\,ds\\ =\;& \int_Q\left[\nabla v\cdot\nabla\overline{\psi}+i\alpha\left(v\,\frac{\partial\overline{\psi}}{\partial x_1}-\overline{\psi}\,\frac{\partial v}{\partial x_1}\right)+(\alpha^2-k^2q)\,v\,\overline{\psi}\right]dx\\ &-2\pi\,i\sum_{n\in\mathbb{Z}} v_n\overline{\psi_n}\sqrt{k^2-(n+\alpha)^2}\,. \end{aligned}$$

Again, v_n and ψ_n are the Fourier coefficients of v and ψ, respectively. The last problem (9) is actually well defined for arbitrary $k\in\mathbb{C}$ with $\operatorname{Re} k>0$ and $\operatorname{Im} k\geq 0$. The proof of the following theorem is simple and left to the reader.

Theorem 3.1. *(a) If $\tilde{u}(\cdot,\alpha)\in H^1_{per}(Q^\infty)$ is a solution to the scattering problem (7) for wave number $k\in\mathbb{C}$ with* $\operatorname{Re} k>0$ *and* $\operatorname{Im} k>0$*, then the restriction $\tilde{u}_{k,\alpha}:=\tilde{u}(\cdot,\alpha)|_Q\in H^1_{0,per}(Q)$ solves (9).*
(b) If $\tilde{u}_{k,\alpha}\in H^1_{0,per}(Q)$ is a solution of (9) for wave number $k\in\mathbb{C}$ with $\operatorname{Re} k>0$ *and* $\operatorname{Im} k\geq 0$ *then we extend $\tilde{u}_{k,\alpha}$ by*

$$\tilde{u}(\cdot,\alpha) \;=\; \begin{cases} \tilde{u}_{k,\alpha}\,, & \text{in } Q\,,\\ E_{k,\alpha}(\tilde{u}_{k,\alpha}|_{\hat{\Gamma}_h}) & \text{in } Q^\infty\setminus Q\,,\end{cases}$$

where the extension operator $E_{k,\alpha} : H^{1/2}_{per}(\hat\Gamma_h) \to H^1_{per,loc}\big((0,2\pi)\times(h,\infty)\big)$ *is defined by*

$$E_{k,\alpha}\phi(x) \;=\; \sum_{n\in\mathbb{Z}} \phi_n\, e^{inx_1 + i\sqrt{k^2-(n+\alpha)^2}(x_2-h)} \quad \textit{for } x_2 > h\,, \tag{10}$$

with Fourier coefficients $\phi_n = (2\pi)^{-1}\int_0^{2\pi}\phi(t)\exp(-int)\,dt$ *of* ϕ. *Then* $\tilde u(\cdot,\alpha)$ *belongs to* $H^1_{per,loc}(Q^\infty)$ *and solves the problem (7-8) in the weak sense. If* $\operatorname{Im} k > 0$ *then* $\tilde u(\cdot,\alpha) \in H^1_{per}(Q^\infty)$.

For any $\alpha \in (-1/2, 1/2]$ and $k \in \mathbb{C}$ we decompose $\int_{\hat\Gamma_h}(\Lambda_{k,\alpha}v)\,\overline{\psi}\,ds$ into three sums,

$$\begin{aligned}
\int_{\hat\Gamma_h}(\Lambda_{k,\alpha}v)\,\overline{\psi}\,ds \;&=\; 2\pi\, i \sum_{n\in\mathbb{Z}} v_n\overline{\psi_n}\sqrt{k^2-(n+\alpha)^2} \\
&=\; -2\pi\sum_{n\in\mathbb{Z}}\sqrt{n^2+1}\,v_n\overline{\psi_n} \\
&\quad + 2\pi\, i \sum_{|n+\alpha|\le k} v_n\overline{\psi_n}\big[\sqrt{k^2-(n+\alpha)^2} - i\sqrt{n^2+1}\big] \\
&\quad + 2\pi \sum_{|n+\alpha|>k} v_n\overline{\psi_n}\big[\sqrt{n^2+1} - \sqrt{(n+\alpha)^2-k^2}\big]\,,
\end{aligned}$$

and equip $H^1_{0,per}(Q)$ with the inner product

$$(u,v)_* \;=\; \int_Q \nabla u\cdot\nabla\overline{v}\,dx \;+\; 2\pi\sum_{n\in\mathbb{Z}}\sqrt{n^2+1}\,u_n\overline{v_n}\,.$$

Then we can rewrite the variational equation (9) as

$$(\tilde u_{k,\alpha},\psi)_* \;-\; b_{k,\alpha}(\tilde u_{k,\alpha},\psi) \;=\; -\int_Q \tilde f(\cdot,\alpha)\,\overline{\psi}\,dx \quad \text{for all } \psi\in H^1_{0,per}(Q)\,, \tag{11}$$

where

$$\begin{aligned}
b_{k,\alpha}(v,\psi) \;:=\; & -\int_Q\left[i\alpha\left(v\,\frac{\partial\overline{\psi}}{\partial x_1} - \overline{\psi}\,\frac{\partial v}{\partial x_1}\right) + (\alpha^2 - k^2 q)\,v\,\overline{\psi}\right]dx \\
& + 2\pi\, i \sum_{|n+\alpha|\le k} v_n\overline{\psi_n}\big[\sqrt{k^2-(n+\alpha)^2} - i\sqrt{n^2+1}\big] \\
& + 2\pi \sum_{|n+\alpha|>k} v_n\overline{\psi_n}\big[\sqrt{n^2+1} - \sqrt{(n+\alpha)^2-k^2}\big] \quad \text{for } v,\psi\in H^1_{0,per}(Q).
\end{aligned} \tag{12}$$

We note that the source term $\tilde f(\cdot,\alpha)$ depends also on k. Let again $k\in\mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k \ge 0$. By the theorem of Riesz, the compact embedding of $H^1_{0,per}(Q)$ in $L^2(Q)$, and the boundedness of the sequence $n \mapsto \sqrt{n^2+1} - \sqrt{(n+\alpha)^2-k^2}$ there exists a compact operator $K_{k,\alpha}$ from $H^1_{0,per}(Q)$ into itself with $b_{k,\alpha}(u,\psi) = (K_{k,\alpha}u,\psi)_*$ for all $u,\psi\in H^1_{0,per}(Q)$. Furthermore, there exists $f_{k,\alpha}\in H^1_{0,per}(Q)$ with $-\int_Q \tilde f(\cdot,\alpha)\,\overline{\psi}\,dx = (f_{k,\alpha},\psi)_*$ for all $\psi\in H^1_{0,per}(Q)$. Then we can rewrite the variational equation (11) as an operator equation,

$$\tilde u_{k,\alpha} \;-\; K_{k,\alpha}\tilde u_{k,\alpha} \;=\; f_{k,\alpha} \quad \text{in } H^1_{0,per}(Q)\,. \tag{13}$$

So far, we have assumed that Im $k > 0$. For these values of k we have uniqueness and existence:

Theorem 3.2. *For* Im $k > 0$ *equation (13) is uniquely solvable in* $H^1_{0,per}(Q)$ *for all* $f_{k,\alpha} \in H^1_{0,per}(Q)$ *and all* $\alpha \in (-1/2, 1/2]$.

Proof: This follows from the fact that $a_{k,\alpha}$ is coercive for Im $k > 0$, compare Theorem 2.2. □

The operator equation (13) is also well defined for positive $k > 0$. However, due to Example 2.3, we expect non-uniqueness for some real $k = \hat{k} > 0$ at certain values of α that we call *exceptional values*. In other words, we expect that for some $\alpha \in (-1/2, 1/2]$ there is an eigenvalue $\lambda = 1$ of the non-selfadjoint operator $K_{\hat{k},\alpha}$.

4. The Operator Equation at Exceptional Values

In this section we fix an arbitrary wave number $\hat{k} \in \mathbb{R}_{>0}$ and investigate the operator equation (13) in a neighborhood of an arbitrary exceptional value $\alpha = \alpha(\hat{k})$. (Of course, such exceptional values do not need to exist for every $\hat{k} > 0$.) As it is well known from periodic scattering theory, the values of α where $(n + \alpha)^2 = \hat{k}^2$ for some $n \in \mathbb{Z}$ are difficult to treat analytically (see, e.g., the proof of Theorem 5.2 below). For this reason we introduce the corresponding set

$$A(\hat{k}) \;=\; \left\{\alpha \in (-1/2, 1/2] : |n + \alpha| = \hat{k} \text{ for some } n \in \mathbb{Z}\right\}.$$

If we write $\hat{k}$ in the form $\hat{k} = m + \rho$ with $m \in \mathbb{N}$ and $\rho \in [0, 1)$ then $A(\hat{k}) = \{\rho, -\rho\}$ or $A(\hat{k}) = \{1 - \rho, \rho - 1\}$ if $\rho \neq 1/2$ and $A(\hat{k}) = \{1/2\}$ if $\rho = 1/2$. Thus, $(-1/2, 1/2] \setminus A(\hat{k})$ consists of at most three intervals.

The following assumption ensures from now on that the operator $I - K_{\hat{k},\hat{\alpha}}$ is always invertible for $\alpha \in A(\hat{k})$; this assumption is quite common for periodic scattering problems.

Assumption 4.1. *The operator* $I - K_{\hat{k},\hat{\alpha}}$ *is one-to-one for the elements* $\hat{\alpha}$ *of* $A(\hat{k})$*; that is,* $\lambda = 1$ *is not an eigenvalue of* $K_{\hat{k},\hat{\alpha}}$ *for* $\hat{\alpha} \in A(\hat{k})$.

Remark: According to Theorem 3.1 the equation $(I - K_{\hat{k},\hat{\alpha}})\phi = 0$ is equivalent to the homogeneous problem

$$\Delta\tilde{\phi} + \hat{k}^2 q\, \tilde{\phi} = 0 \text{ in } (0, 2\pi) \times (0, \infty)\,, \quad \tilde{\phi} = 0 \text{ for } x_2 = 0\,,$$

and $\tilde{\phi}(x) := \phi(x)e^{i\hat{\alpha}x_1}$ is $\hat{\alpha}-$quasi periodic and has a Rayleigh expansion (8). Therefore, Assumption 4.1 transforms into the requirement that for $\hat{k} \pm \hat{\alpha} \in \mathbb{N}_0$ this homogeneous boundary values admits only the trivial solution $\tilde{\phi} = 0$. We will see below in Lemma 4.2 that this implies an exponential decay of the solutions to this homogeneous problem.[1]

We next study the dependence of $K_{k,\alpha}$ and $f_{k,\alpha}$ on k and α.

Lemma 4.2. *Let again* $A(\hat{k}) = \{\alpha \in (-1/2, 1/2] : |n + \alpha| = \hat{k}$ *for some* $n \in \mathbb{Z}\}$ *and assume that Assumption 4.1 holds.*

[1]Otherwise, the component $e^{\hat{k}x_1}$ appears in the Rayleigh expansion which does not decay.

(a) *If* $\hat\alpha \in (-1/2, 1/2]$ *is an exceptional value; that is,* $I - K_{\hat k,\hat\alpha}$ *fails to be one-to-one, and* $(I - K_{\hat k,\hat\alpha})v = 0$ *then* v *is an evanescent solution of (7); that is, there exists* $c > 0$ *and* $\delta > 0$ *such that the extension satisfies* $\left|E_{\hat k,\hat\alpha} v(x)\right| \le c\, e^{-\delta(x_2-h)}$ *for all* $x_2 > h$ *where* $E_{\hat k,\hat\alpha}$ *is the extension operator from (10).*

(b) *If* $\hat\alpha \in (-1/2, 1/2]$ *is an exceptional value and* $(I - K_{\hat k,\hat\alpha})v = 0$ *then also* $-\hat\alpha \in [-1/2, 1/2)$ *is an exceptional value with eigenfunction* $\overline{v}$*.*

(c) *There exists an open set* $D \subset \mathbb{C} \times \mathbb{C}$ *with* $\{\hat k\} \times \big([-1/2, 1/2] \setminus A(\hat k)\big) \subset D$ *such that* D *is the union of at most three connected sets, and the operator* $K_{k,\alpha}$ *and the right-hand side* $f_{k,\alpha}$ *of (13) depend holomorphicly on* $(k, \alpha) \in D$*.*

(d) *There exist at most finitely many exceptional values* $\{\hat\alpha_j : j \in J\} \subset (-1/2, 1/2]$ *for some finite index set* $J \subset \mathbb{Z}$*. By part (b) we can assume that* J *is symmetric with respect to the origin and* $\hat\alpha_{-j} = -\hat\alpha_j$ *for all* $j \in J$*.*

Remark 4.3. *It may happen that the boundary point* $\alpha = 1/2$ *of* $(-1/2, 1/2]$ *is an exceptional value. As the proof of Lemma 4.2 below uses open sets around exceptional values, we implicitly exploit that all operators and functions introduced so far depend* x_1*-quasi-periodically on* $\alpha \in \mathbb{R}$ *(such that, e.g.,* $f_{k,1/2} = \exp(-ix_1)\, f_{k,-1/2}$*) and can be extended quasi-periodically in* α *via their original definition into a neighborhood of the real axis. The choice of an open ball in* $\mathbb{C}$ *around* $\alpha = 1/2$ *is then no problem anymore. Note that we omit to note this in most of the subsequent proofs.*

Proof: (a) If $(I - K_{\hat k,\hat\alpha})v = 0$ then we substitute $\psi = v$ in the homogeneous form of (9) (for v instead of $v_{k,\alpha}$) and take the imaginary part. Thus, $\sum_{|n+\alpha|<\hat k} |v_n|^2 \sqrt{\hat k^2 - (n+\alpha)^2} = 0$ and thus $v_n = 0$ for all $|n + \alpha| < \hat k$. Therefore,

$$E_{\hat k,\hat\alpha} v(x) \;=\; \sum_{|n+\alpha|>\hat k} v_n\, e^{inx_1 - \sqrt{(n+\alpha)^2 - \hat k}\,(x_2-h)} \quad \text{for } x_2 > h\,,$$

which yields the assertion.

(b) If $(I - K_{\hat k,\hat\alpha})v = 0$ then $a_{\hat k,\hat\alpha}(v, \overline{\psi}) = 0$ for all $\psi \in H^1_{0,per}(Q)$. Taking the complex conjugate of this equation yields $a_{\hat k,-\hat\alpha}(\overline{v}, \psi) = 0$ for all $\psi \in H^1_{0,per}(Q)$.

(c) Let $\hat\alpha \in (-1/2, 1/2] \setminus A(\hat k)$; that is, $|n + \hat\alpha| \ne \hat k$ for all $n \in \mathbb{Z}$. We show that the operator $K_{k,\alpha}$ depends holomorphicly on both variables $(k, \alpha) \in \mathbb{C}^2$ in a neighborhood of $(\hat k, \hat\alpha)$. Set $N := \big\{n \in \mathbb{Z} : |n + \hat\alpha| < \hat k\big\}$. Then there exists $\rho > 0$ and open discs $B(\hat\alpha, \delta_0) \in \mathbb{C}$ and $B(\hat k, \delta_0) \in \mathbb{C}$ such that

- $|n + \alpha| < |k|\sqrt{1-\rho}$ for all $(k, \alpha) \in D(\hat\alpha) := B(\hat k, \delta_0) \times B(\hat\alpha, \delta_0)$ and $n \in N$ and
- $|n + \alpha| > |k|\sqrt{1+\rho}$ for $(k, \alpha) \in D(\hat\alpha)$ and $n \notin N$.

Note that δ_0 depends on $\hat\alpha$. For $k = k_1 + ik_2 \in B(\hat k, \delta_0)$, $\alpha = \alpha_1 + i\alpha_2 \in B(\hat\alpha, \delta_0)$ and $n \in N$ we have that $(n + \alpha_1)^2 + \alpha_2^2 < (1 - \rho)(k_1^2 + k_2^2)$ and thus

$$\begin{aligned}
\mathrm{Re}\left[k^2 - (n+\alpha)^2\right] &= k_1^2 - k_2^2 - (n+\alpha_1)^2 + \alpha_2^2 \\
&> k_1^2 - k_2^2 - \left[(1-\rho)(k_1^2 + k_2^2) - \alpha_2^2\right] + \alpha_2^2 \\
&\ge \rho k_1^2 - (2-\rho)k_2^2 \;>\; 0 \quad \text{for} \quad k_2^2 < \frac{\rho}{2-\rho} k_1^2\,.
\end{aligned}$$

We choose δ_0 so small such that $k_2^2 < \frac{\rho}{2-\rho} k_1^2$ for $k \in B(\hat{k}, \delta_0)$. Then $\mathrm{Re}\left[k^2 - (n+\alpha)^2\right] > 0$ for $(k, \alpha) \in D(\hat{\alpha})$ and $n \in N$. Analogously, for $(k, \alpha) \in D(\hat{\alpha})$ and $n \notin N$; that is, $|n + \hat{\alpha}| > \hat{k}$ we have

$$\begin{aligned} \mathrm{Re}\left[(n+\alpha)^2 - k^2\right] &= (n+\alpha_1)^2 - \alpha_2^2 - k_1^2 + k_2^2 \\ &> (1+\rho)(k_1^2 + k_2^2) - 2\alpha_2^2 - k_1^2 + k_2^2 \\ &\geq \rho k_1^2 - 2\alpha_2^2 \; > \; 0 \quad \text{for} \quad \alpha_2^2 < \frac{\rho}{2} k_1^2 \end{aligned}$$

which holds for δ_0 sufficiently small. Therefore, also in this case we have $\mathrm{Re}\left[k^2 - (n+\alpha)^2\right] > 0$. Summarizing, we can choose δ_0 so small such that $(k, \alpha) \mapsto \sqrt{k^2 - (n+\alpha)^2}$ and $(k, \alpha) \mapsto \sqrt{(n+\alpha)^2 - k^2}$ are holomorphic in $D(\hat{\alpha})$ and $n \in N$ and $n \notin N$, respectively. Since the set $D := \cup\{D(\hat{\alpha}) : \hat{\alpha} \in (-1/2, 1/2] \setminus A(\hat{k})\}$ is open and the union of finitely many (at most three) domains and covers $\{\hat{k}\} \times ((-1/2, 1/2] \setminus A(\hat{k}))$ we have shown that $(k, \alpha) \mapsto K_{k,\alpha}$ is holomorphic in D. The proof for the right-hand side $f_{k,\alpha}$ follows from the fact that f depends holomorphically on k and has compact support which implies that the infinite series in the definition of the Floquet-Bloch transform is merely a finite one.

(d) Assume on the contrary that there exists an (infinite) sequence $(\hat{\alpha}_j)$ in $(-1/2, 1/2)$ and a sequence (w_j) in $H^1_{0,per}(Q)$ of corresponding normalized functions such that $(I - K_{\hat{k},\hat{\alpha}_j}) w_j = 0$ for all j. We can assume that the sequence belongs to one of the at most three open intervals of $(-1/2, 1/2) \setminus A(\hat{k})$, say to $\mathcal{I}$. Since the mapping $\alpha \mapsto \tilde{K}_{\hat{k},\alpha}$ is analytic from a neighborhood of $\mathcal{I}$ into $\mathcal{L}(H^1_{0,per}(Q))$ it follows from [5, Th. 5.1] that the equation $(I - \tilde{K}_{\hat{k},\alpha})w = 0$ has the same number of linearly independent solutions at every parameter $\alpha \in \mathcal{I}$ except for finitely many. Since for the infinite sequence $\hat{\alpha}_j$ this number is at least one, it has to be at least one for all $\hat{\alpha} \in \mathcal{I}$ except for finitely many. From the continuity of $\alpha \mapsto \tilde{K}_{\hat{k},\alpha}$ one shows easily that also $I - K_{\hat{k},\alpha}$ cannot be one-to-one at the boundary points of $\mathcal{I}$. One of these boundary points must belong to $A(\hat{k})$. This actually contradicts Assumption 4.1, which supposes the latter operator to be one-to-one for all exceptional values, and hence ends the proof. □

The last result introduced a finite index set $J \subset \mathbb{Z}$ that depends on $\hat{k}$. Since $I - K_{\hat{k},\alpha}$ is an isomorphism for $\alpha \notin \{\hat{\alpha}_j : j \in J\}$ we have convergence of $\tilde{u}_{k,\alpha}$ as k tends to $\hat{k}$. Before stating this, recall that $B(\hat{\alpha}, \delta) \subset \mathbb{C}$ is the disc with center $\hat{\alpha}$ and radius δ.

Lemma 4.4. *For any fixed $\delta > 0$ the functions $\tilde{u}_{k,\alpha}$ converge to $\tilde{u}_{\hat{k},\alpha}$ in $H^1_{0,per}(Q)$ as $k \to \hat{k}$ uniformly with respect to $\alpha \in [-1/2, 1/2] \setminus \bigcup_{j \in J} B(\hat{\alpha}_j, \delta)$.*

It remains to study the convergence of $\tilde{u}_{k,\alpha}$ in neighborhoods of the exceptional values $\hat{\alpha}_j$.

To this end, we recall the following results from abstract functional analysis. For any eigenvalue $\lambda \neq 0$ of a compact operator $K : H \to H$ in a Hilbert space there exists $r = r(\lambda) \in \mathbb{N}$ (sometimes called Riesz number) with

$$\{0\} \;=\; \mathcal{N}\big((\lambda I - K)^0\big) \;\subset\; \mathcal{N}\big((\lambda I - K)^1\big) \;\subset \cdots \subset\; \mathcal{N}\big((\lambda I - K)^r\big) \;=\; \mathcal{N}\big((\lambda I - K)^{r+1}\big)$$

and

$$H \;=\; (\lambda I - K)^0(H) \;\supset\; (\lambda I - K)^1(H) \;\supset \cdots \supset\; (\lambda I - K)^r(H) \;=\; (\lambda I - K)^{r+1}(H)\,.$$

The dimension of $\mathcal{N}\big((\lambda I - K)^r\big)$ is called the algebraic multiplicity of λ and elements of $\mathcal{N}\big((\lambda I - K)^r\big)$ the generalized eigenvectors (called root vectors in [5]) of λ.

Lemma 4.5. *Let Assumption 4.1 hold and $\hat\alpha = \hat\alpha_j \in (-1/2, 1/2]$ be an exceptional value for some $j \in J$. Then the Riesz number of the eigenvalue $\lambda = 1$ of $K_{\hat k,\hat\alpha}$ is one such that $\lambda = 1$ is a semi-simple eigenvalue; that is, its geometric and the algebraic multiplicities coincide.*

Proof: Let v be such that $(I - K_{\hat k,\hat\alpha})^2 v = 0$. Then $u := (I - K_{\hat k,\hat\alpha})v$ is an eigenfunction of $K_{\hat k,\hat\alpha}$ and therefore an evanescent wave by part (a) of Lemma 4.2. Therefore,

$$\begin{aligned}
(u,u)_* \;&=\; \big((I - K_{\hat k,\hat\alpha})v, u\big)_* \;=\; a_{\hat k,\hat\alpha}(v,u) \\
&=\; \int_Q \left[\nabla v \cdot \nabla \overline{u} + i\hat\alpha \left(v\,\frac{\partial \overline{u}}{\partial x_1} - \overline{u}\,\frac{\partial v}{\partial x_1}\right) + (\hat\alpha^2 - \hat k^2 q)\, v\, \overline{u}\right] dx \\
&\quad -2\pi \sum_{|n+\hat\alpha| > \hat k} v_n \overline{u_n} \sqrt{(n+\hat\alpha)^2 - \hat k^2} \\
&=\; \overline{a_{\hat k,\hat\alpha}(u,v)} \;=\; 0
\end{aligned}$$

because u is an eigenfunction. Therefore, $u = 0$; that is $(I - K_{\hat k,\hat\alpha})v = 0$. □

Recall that we have fixed the wave number $\hat k > 0$ in the beginning of this section. Now we further fix $j \in J$, write $\hat\alpha$ for $\hat\alpha_j$, and study certain projections of the operator equation (13) for $(k,\alpha) \in B_{\delta_1} := B(\hat k, \delta_1) \times B(\hat\alpha, \delta_1)$ where δ_1 is chosen such that $K_{k,\alpha}$ and $f_{k,\alpha}$ of (13) depend holomorphicly on $(k,\alpha) \in B_{\delta_1}$. We define a projection onto the algebraic eigenspace of $K_{k,\alpha}$ as follows: Choose a small circle C with center one such that no other eigenvalue of $\hat K := K_{\hat k,\hat\alpha}$ lies inside of C and define

$$P_{k,\alpha} u \;=\; -\frac{1}{2\pi i} \int_C (z - K_{k,\alpha})^{-1} u \, dz \quad \text{for } u \in H^1_{0,per}(Q) \tag{14}$$

and $(k,\alpha) \in B_{\delta_2} \subset \mathbb{C} \times \mathbb{C}$ where $\delta_2 \le \delta_1$ is chosen that $z - K_{k,\alpha}$ is invertible for all $z \in C$. (This follows from a Neumann series argument.) Then it is well known that $P_{k,\alpha}$ is the projection operator onto the sum $\bigoplus_\ell \mathcal{N}\big((\lambda_\ell - K_{k,\alpha})^{r_\ell}\big)$ of algebraic eigenspaces, where $\lambda_\ell = \lambda_\ell(k,\alpha)$ are the eigenvalues of $K_{k,\alpha}$ inside the curve C, with Riesz numbers r_ℓ, see, e.g., [9, Section II.1.4]. The sum of the algebraic multiplicities is constant; that is, for all $(k,\alpha) \in B_{\delta_2}$ there holds that

$$\sum_\ell \dim \mathcal{N}\big((\lambda_\ell - K_{k,\alpha})^{r_\ell}\big) \;=\; \dim \mathcal{N}\big(1 - \hat K\big) \;=\; m \in \mathbb{N}\,,$$

which is the dimension of the (geometric and algebraic) eigenspace of $\hat K = K_{\hat k,\hat\alpha}$ corresponding to the eigenvalue one. From the explicit representation of the projection operator and the analytic dependence of $K_{k,\alpha}$ on $(k,\alpha) \in B_{\delta_2}$ we note that also the projection operator $P_{k,\alpha}$ depends holomorphicly on $(k,\alpha) \in B_{\delta_2}$.

5. A Finite-dimensional Perturbation Problem

We are next going to project the operator equation (13) onto the ranges of the spectral projection $P_{k,\alpha}$ and of $I-P_{k,\alpha}$, respectively, that we introduced in (14) for (k,α) sufficiently close to $(\hat{k},\hat{\alpha})$ for some exceptional value $\hat{\alpha}$. We call the solutions to the resulting two projected equations $\tilde{v}^{(1)}_{k,\alpha}$ and $\tilde{v}^{(2)}_{k,\alpha}$, which naturally belong to k- and α-dependent spaces $\mathcal{N}(I-K_{k,\alpha})$ and $\mathcal{R}(I-K_{k,\alpha})$, respectively. Using a so-called similarity transform we then map the projected equations into $\mathcal{N}(I-K_{\hat{k},\hat{\alpha}})$ and $\mathcal{R}(I-K_{\hat{k},\hat{\alpha}})$, which are spaces independent of k and α. By an additional linearization argument, we will then be able to derive the behavior of a solution to the scattering problem for complex-valued k in the neighborhood of a critical value.
Now we carry out these steps in detail. Since $K_{k,\alpha}$ commutes with $P_{k,\alpha}$ the projection of equation (13) onto $\mathcal{N}(I-K_{k,\alpha}) = P_{k,\alpha}\big(H^1_{0,per}(Q)\big)$ and $\mathcal{R}(I-K_{k,\alpha}) = (I-P_{k,\alpha})\big(H^1_{0,per}(Q)\big)$ yields

$$\tilde{v}^{(1)}_{k,\alpha} \;-\; P_{k,\alpha}K_{k,\alpha}\tilde{v}^{(1)}_{k,\alpha} \;=\; P_{k,\alpha}f_{k,\alpha} \quad \text{in } \mathcal{N}(I-K_{k,\alpha}) \tag{15}$$

where $\tilde{v}^{(1)}_{k,\alpha} = P_{k,\alpha}\tilde{u}_{k,\alpha} \in \mathcal{N}(I-K_{k,\alpha})$, and

$$\tilde{v}^{(2)}_{k,\alpha} \;-\; (I-P_{k,\alpha})K_{k,\alpha}\tilde{v}^{(2)}_{k,\alpha} \;=\; (I-P_{k,\alpha})f_{k,\alpha} \quad \text{in } \mathcal{R}(I-K_{k,\alpha}) \tag{16}$$

where $\tilde{v}^{(2)}_{k,\alpha} = (I-P_{k,\alpha})\tilde{u}_{k,\alpha} \in \mathcal{R}(I-K_{k,\alpha})$. Since the spaces $\mathcal{N}(I-K_{k,\alpha})$ and $\mathcal{R}(I-K_{k,\alpha})$ depend on k and α we use a similarity transformation to transform the equations into the spaces $\mathcal{N}(I-K_{\hat{k},\hat{\alpha}}) = \mathcal{N}(I-\hat{K})$ and $\mathcal{R}(I-K_{\hat{k},\hat{\alpha}}) = \mathcal{R}(I-\hat{K})$ that are independent of k and α. (Recall that $\hat{K} = K_{\hat{k},\hat{\alpha}}$ for brevity.)

Lemma 5.1. *For all $(k,\alpha) \in B_{\delta_2} := B(\hat{k},\delta_2)\times B(\hat{\alpha},\delta_2)$ there exists a linear bounded mapping $U_{k,\alpha}$ from $H^1_{0,per}(Q)$ onto itself with the properties*

1. *the inverse $U^{-1}_{k,\alpha} : H^1_{0,per}(Q) \to H^1_{0,per}(Q)$ exists and $U_{k,\alpha}$ and $U^{-1}_{k,\alpha}$ are holomorphic for $(k,\alpha) \in B_{\delta_2}$,*
2. *$U_{k,\alpha}P_{k,\alpha}U^{-1}_{k,\alpha} = \hat{P} := P_{\hat{k},\hat{\alpha}}$ for all $(k,\alpha) \in B_{\delta_2}$.*

Proof: The theorem is proven in [9], Section II.4.2, for the case when P depends only on one variable. Keeping α fixed and applying this result to the parameter k yields the existence of $V_{k,\alpha} : H^1_{0,per}(Q) \to H^1_{0,per}(Q)$ with $V_{k,\alpha}P_{k,\alpha}V^{-1}_{k,\alpha} = P_{\hat{k},\alpha}$. A careful study of the proof shows that $V_{k,\alpha}$ depends also holomorphically on the parameter α. Now we keep $\hat{k}$ fixed and apply Kato's result to the parameter α. This yields the existence of $W_\alpha : H^1_{0,per}(Q) \to H^1_{0,per}(Q)$ with $W_\alpha P_{\hat{k},\alpha}W^{-1}_\alpha = P_{\hat{k},\hat{\alpha}}$. Substituting $P_{\hat{k},\alpha}$ from the first equation into the second yields the assertion for $U_{k,\alpha} = W_\alpha V_{k,\alpha}$. □

Therefore, we can transform (15) and (16) onto the equations

$$w^{(1)}_{k,\alpha} \;-\; \hat{P}U_{k,\alpha}K_{k,\alpha}U^{-1}_{k,\alpha}w^{(1)}_{k,\alpha} \;=\; \hat{P}U_{k,\alpha}f_{k,\alpha} \quad \text{in } \mathcal{N}(I-\hat{K}) \tag{17}$$

with $w^{(1)}_{k,\alpha} = U_{k,\alpha}\tilde{v}^{(1)}_{k,\alpha} = U_{k,\alpha}P_{k,\alpha}\tilde{u}_{k,\alpha}$ and

$$w^{(2)}_{k,\alpha} \;-\; (I-\hat{P})U_{k,\alpha}K_{k,\alpha}U^{-1}_{k,\alpha}w^{(2)}_{k,\alpha} \;=\; (I-\hat{P})U_{k,\alpha}f_{k,\alpha} \quad \text{in } \mathcal{R}(I-\hat{K}) \tag{18}$$

with $w^{(2)}_{k,\alpha} = U_{k,\alpha}\tilde{v}^{(2)}_{k,\alpha} = U_{k,\alpha}(I-P_{k,\alpha})\tilde{u}_{k,\alpha}$.

For $(k,\alpha) = (\hat{k},\hat{\alpha})$ equation (18) reduces to

$$w^{(2)}_{\hat{k},\hat{\alpha}} \;-\; (I-\hat{P})\hat{K}w^{(2)}_{\hat{k},\hat{\alpha}} \;=\; (I-\hat{P})f_{\hat{k},\hat{\alpha}} \quad \text{in } \mathcal{R}(I-\hat{K})$$

because $U_{\hat{k},\hat{\alpha}} = I$. It is easily seen that $[I-(I-\hat{P})\hat{K}]\big|_{\mathcal{R}(I-\hat{K})}$ is an isomorphism from $\mathcal{R}(I-\hat{K})$ onto itself. By a perturbation argument also $\big[I-(I-\hat{P})U_{k,\alpha}K_{k,\alpha}U^{-1}_{k,\alpha}\big]\big|_{\mathcal{R}(I-\hat{K})}$ is an isomorphism from $\mathcal{R}(I-\hat{K})$ onto itself for all $(k,\alpha)\in B_{\delta_3}$ for some $\delta_3 \le \delta_2$, and the solution $w^{(2)}_{k,\alpha}$ of (18) tends to $w^{(2)}_{\hat{k},\alpha}$ as k tends to $\hat{k}$, uniformly with respect to $\alpha \in B(\hat{\alpha},\delta_3)$. This implies that $(I-P_{k,\alpha})\tilde{u}_{k,\alpha}$ tends to $(I-P_{\hat{k},\alpha})\tilde{u}_{\hat{k},\alpha}$ in $H^1_{0,per}(Q)$, uniformly with respect to $\alpha \in B(\hat{\alpha},\delta_3)$,

$$\text{(19)}\quad \tilde{v}^{(2)}_{k,\alpha} = (I-P_{k,\alpha})\tilde{u}_{k,\alpha} = U^{-1}_{k,\alpha}w^{(2)}_{k,\alpha} \xrightarrow{k\to\hat{k}} U^{-1}_{\hat{k},\alpha}w^{(2)}_{\hat{k},\alpha} = (I-P_{\hat{k},\alpha})\tilde{u}_{\hat{k},\alpha} = \tilde{v}^{(2)}_{\hat{k},\alpha} \quad \text{in } H^1_{0,per}(Q).$$

Now we consider equation (17) and recall that $I-\hat{K} = I-K_{\hat{k},\hat{\alpha}}$ has Riesz number one, such that we have reduced the equation to a problem in the finite-dimensional space $X := \mathcal{N}(I-\hat{K})$. We abbreviate $M(k,\alpha) = \hat{P}U_{k,\alpha}K_{k,\alpha}U^{-1}_{k,\alpha} - I \in \mathcal{L}(X)$ and $g_{k,\alpha} = -\hat{P}U_{k,\alpha}f_{k,\alpha} \in X$; that is,

$$\text{(20)}\qquad M(k,\alpha)\,w^{(1)}_{k,\alpha} \;=\; g_{k,\alpha} \quad \text{in } X = \mathcal{N}(I-\hat{K})\,.$$

Theorem 5.2. *The partial derivatives* $\frac{\partial}{\partial k}M(\hat{k},\hat{\alpha})$ *and* $\frac{\partial}{\partial \alpha}M(\hat{k},\hat{\alpha})$ *are self-adjoint operators on* X *and* $\frac{\partial}{\partial k}M(\hat{k},\hat{\alpha})$ *is a positive definite operator; that is,* $\left(\frac{\partial}{\partial k}M(\hat{k},\hat{\alpha})w,w\right)_* > 0$ *for all* $w\in X$ *different from zero.*

Proof: First we note that $U_{\hat{k},\hat{\alpha}} = U^{-1}_{\hat{k},\hat{\alpha}} = I$. Therefore, by the product rule $0 = \frac{\partial}{\partial k}\big[U_{k,\alpha}\,U^{-1}_{k,\alpha}\big]\big|_{(k,\alpha)=(\hat{k},\hat{\alpha})} = \frac{\partial}{\partial k}U_{\hat{k},\hat{\alpha}} + \frac{\partial}{\partial k}U^{-1}_{\hat{k},\hat{\alpha}}$ and

$$\frac{\partial}{\partial k}M(k,\alpha) = \hat{P}\left(\frac{\partial}{\partial k}U_{k,\alpha}\right)K_{k,\alpha}U^{-1}_{k,\alpha} \;+\; \hat{P}U_{k,\alpha}\left(\frac{\partial}{\partial k}K_{k,\alpha}\right)U^{-1}_{k,\alpha} + \;\hat{P}U_{k,\alpha}K_{k,\alpha}\left(\frac{\partial}{\partial k}U^{-1}_{k,\alpha}\right).$$

For $(k,\alpha) = (\hat{k},\hat{\alpha})$ this reduces to

$$\begin{aligned}\frac{\partial}{\partial k}M(k,\alpha)\bigg|_{(k,\alpha)=(\hat{k},\hat{\alpha})} &= \hat{P}\left(\frac{\partial}{\partial k}U_{\hat{k},\hat{\alpha}}\right)\hat{K} \;+\; \hat{P}\frac{\partial}{\partial k}K_{\hat{k},\hat{\alpha}} \;+\; \hat{P}\hat{K}\left(\frac{\partial}{\partial k}U^{-1}_{\hat{k},\hat{\alpha}}\right)\\ &= \hat{P}(I-\hat{K})\frac{\partial}{\partial k}U_{\hat{k},\hat{\alpha}} \;+\; \hat{P}\frac{\partial}{\partial k}K_{\hat{k},\hat{\alpha}}\\ &= \hat{P}\frac{\partial}{\partial k}K_{\hat{k},\hat{\alpha}}\end{aligned}$$

because $\hat{P}(I-\hat{K}) = 0$ and $\hat{K} = I$ on X. The partial derivative with respect to α has the same form. For $w,\psi \in X$ we compute from (12)

$$\begin{aligned}\left(\frac{\partial}{\partial k}K_{k,\alpha}w,\psi\right)_* &= 2k\int_Q q\,w\,\overline{\psi}\,dx \;+\; 2k\pi\sum_{|n+\alpha|>|k|} w_n\overline{\psi_n}\frac{1}{\sqrt{(n+\alpha)^2-k^2}}\\ &\quad + 2k\pi i\sum_{|n+\alpha|\le|k|} w_n\overline{\psi_n}\frac{1}{\sqrt{k^2-(n+\alpha)^2}}\end{aligned}$$

and thus for $(k,\alpha)=(\hat k,\hat\alpha)$ there holds

$$\begin{aligned}\left(\frac{\partial}{\partial k}M(\hat k,\hat\alpha)w,\psi\right)_* &= \left(\hat P\frac{\partial}{\partial k}K_{\hat k,\hat\alpha}w,\psi\right)_* = \left(\frac{\partial}{\partial k}K_{\hat k,\hat\alpha}w,\psi\right)_* \\ &= 2\hat k\int_Q q\,u\,\overline{\psi}\,dx \;+\; 2\hat k\pi\sum_{|n+\hat\alpha|>\hat k} w_n\overline{\psi_n}\frac{1}{\sqrt{(n+\hat\alpha)^2-\hat k^2}} \\ &= 2\hat k\int_{Q^\infty} q\,u\,\overline{\psi}\,dx\end{aligned}$$

because $w_n = 0$ for $|n+\hat\alpha|\le|\hat k|$ and $w\in X=\mathcal{N}(I-\hat K)$. Here, we have identified $w,\psi\in X$ with their extensions into $Q^\infty=(0,2\pi)\times(0,\infty)$ as evanescent solutions of the Helmholtz equation. This proves that $\frac{\partial}{\partial k}M(\hat k,\hat\alpha)=\hat P\frac{\partial}{\partial k}K_{\hat k,\hat\alpha}$ is self-adjoint on X and positive. Analogously we have

$$\begin{aligned}\left(\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)w,\psi\right)_* &= \left(\frac{\partial}{\partial\alpha}K_{\hat k,\hat\alpha}w,\psi\right)_* \\ &= \int_Q\left[i\left(\overline{\psi}\,\frac{\partial w}{\partial x_1}-w\,\frac{\partial\overline{\psi}}{\partial x_1}\right)-2\hat\alpha\,w\,\overline{\psi}\right]dx \;-\; 2\pi\sum_{|n+\hat\alpha|>\hat k} w_n\overline{\psi_n}\frac{n+\hat\alpha}{\sqrt{(n+\hat\alpha)^2-\hat k^2}} \\ &= 2i\int_{Q^\infty}\left(\frac{\partial w}{\partial x_1}+i\hat\alpha\,w\right)\overline{\psi}\,dx \qquad (21)\end{aligned}$$

which shows that also $\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)=\hat P\frac{\partial}{\partial\alpha}K_{\hat k,\hat\alpha}$ is self-adjoint on X. □

For the rest of the paper we make the following assumption:

Assumption 5.3. *The operator $\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)$ is one-to-one on X.*

Since $M(\hat k,\hat\alpha)=0$ we next compare the solution $w^{(1)}_{k,\alpha}$ to (20) with the solution $\tilde w_{k,\alpha}$ to the linearized equation

$$\left[(k-\hat k)\frac{\partial}{\partial k}M(\hat k,\hat\alpha) \;+\; (\alpha-\hat\alpha)\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)\right]\tilde w_{k,\alpha} \;=\; g_{\hat k,\hat\alpha} \quad \text{in } X=\mathcal{N}(I-\hat K)\,. \qquad (22)$$

To this end, we set $k=\hat k+i\varepsilon$ with $\varepsilon>0$ und write $w^{(1)}_{\varepsilon,\alpha}$ instead of $w^{(1)}_{\hat k+i\varepsilon,\alpha}$, etc. To study this equation we introduce an eigensystem $(d_\ell,\phi_\ell)_{\ell=1}^m$ of the generalized self-adjoint eigenvalue problem

$$-\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)\phi \;=\; d\,\frac{\partial}{\partial k}M(\hat k,\hat\alpha)\phi\,, \qquad (23)$$

that is, $d_\ell\in\mathbb{R}$ and $\{\phi_1,\ldots,\phi_m\}\subset X$ forms a basis and

$$-\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)\phi_\ell \;=\; d_\ell\,\frac{\partial}{\partial k}M(\hat k,\hat\alpha)\phi_\ell\,,\quad \ell=1,\ldots,m\,. \qquad (24)$$

The functions ϕ_ℓ are orthonormalized such that

$$\big(\phi_\ell,\phi_{\ell'}\big)_X \;:=\; \left(\frac{\partial}{\partial k}M(\hat k,\hat\alpha)\phi_\ell,\phi_{\ell'}\right)_* \;=\; \delta_{\ell,\ell'}\,,\quad \ell,\ell'\in\{1,\ldots,m\}\,. \qquad (25)$$

As we assumed that $\frac{\partial}{\partial\alpha}M(\hat k,\hat\alpha)$ is injective, none of the eigenvalues $d_\ell\in\mathbb{R}$ can vanish.

Remark 5.4. *The eigenpairs* (d_ℓ, ϕ_ℓ) *do later on explicitly determine the propagating modes of the open waveguide, which is obvious by noting that* $\phi_\ell \in X$ *is already a periodic solution to the transformed, homogeneous Helmholtz equation* (7)*. By the periodic Lippmann-Schwinger equation this implies that* ϕ_ℓ *belongs to* $C^1(\overline{Q})$*. The sign of* $d_\ell \neq 0$ *moreover determines whether that mode propagates to the left or to the right, as we see in the next section.*

Lemma 5.5. *Let* $w^{(1)}_{\varepsilon,\alpha}, \tilde{w}_{\varepsilon,\alpha} \in X$ *be the solutions of (20) and (22), respectively, for* $\varepsilon \in (0, \delta_3)$ *and* $\alpha \in (\hat{\alpha}-\delta_3, \hat{\alpha}+\delta_3)$*. Then there exists* $c > 0$ *and* $\delta_4 \leq \delta_3$ *such that for* $\varepsilon \in (0, \delta_4)$ *and* $\alpha \in (\hat{\alpha} - \delta_4, \hat{\alpha} + \delta_4)$*:*

(a) $\|w^{(1)}_{\varepsilon,\alpha} - \tilde{w}_{\varepsilon,\alpha}\|_X \leq c$ *and*
(b) $\|\tilde{v}^{(1)}_{\varepsilon,\alpha} - \tilde{w}_{\varepsilon,\alpha}\|_X \leq c$ *where* $\tilde{v}^{(1)}_{\varepsilon,\alpha}$ *is the solution of (15) for* $k = \hat{k} + i\varepsilon$.

Proof: First we show that there exists $\gamma > 1$ such that

$$\frac{1}{\gamma\sqrt{\varepsilon^2+\alpha^2}} \;\leq\; \left\| \left[i\varepsilon \frac{\partial}{\partial k} M(\hat{k},\hat{\alpha}) + \alpha \frac{\partial}{\partial \alpha} M(\hat{k},\hat{\alpha}) \right]^{-1} \right\| \;\leq\; \frac{\gamma}{\sqrt{\varepsilon^2+\alpha^2}} \tag{26}$$

for all $\varepsilon > 0$, $\alpha \in \mathbb{R}$. Indeed, this is equivalent to

$$\frac{1}{\gamma}\sqrt{\varepsilon^2+\alpha^2}\,\|w\|_X \;\leq\; \left\| i\varepsilon\,\hat{M}_k w + \alpha\,\hat{M}_\alpha w \right\|_X \;\leq\; \gamma\sqrt{\varepsilon^2+\alpha^2}\,\|w\|_X \tag{27}$$

for all $w \in X$, $\varepsilon > 0$, and $\alpha \in \mathbb{R}$, where we have set $\hat{M}_k = \frac{\partial}{\partial k} M(\hat{k},\hat{\alpha})$ and $\hat{M}_\alpha = \frac{\partial}{\partial \alpha} M(\hat{k},\hat{\alpha})$ for abbreviation. Expanding w in the form $w = \sum_{\ell=1}^m w_\ell \phi_\ell$ we estimate for any $\ell' \in \{1,\ldots,m\}$

$$\begin{aligned}
\left\| i\varepsilon\,\hat{M}_k w + \alpha\,\hat{M}_\alpha w \right\|_X \;&\geq\; \left| \left(i\varepsilon\,\hat{M}_k w + \alpha\,\hat{M}_\alpha w, \phi_{\ell'} \right)_X \right| \\
&=\; \left| \sum_{\ell=1}^m w_\ell \left[i\varepsilon\,(\hat{M}_k \phi_\ell, \phi_{\ell'})_X + \alpha\,(\hat{M}_\alpha \phi_\ell, \phi_{\ell'})_X \right] \right| \\
&=\; |w_{\ell'}|\,\left| i\varepsilon - \alpha\,d_{\ell'} \right| \;=\; |w_{\ell'}|\sqrt{\varepsilon^2 + \alpha^2\,d_{\ell'}^2} \\
&\geq\; \tilde{c}\sqrt{\varepsilon^2+\alpha^2}\,|w_{\ell'}|
\end{aligned}$$

where $\tilde{c} \in (0,1]$ is chosen such that $\tilde{c} \leq |d_\ell|$ for all $\ell = 1,\ldots,m$. This yields the lower estimate of (27) since ℓ' was arbitrary. The upper estimate is obvious.

Now we start with the actual proof of the lemma.

(a) By (26) we have

$$\|\tilde{w}_{\varepsilon,\alpha}\|_X \;\leq\; \frac{\gamma}{\sqrt{\varepsilon^2+(\alpha-\hat{\alpha})^2}}\,\|g_{\varepsilon,\alpha}\|_X\,. \tag{28}$$

Now we consider the difference $\hat{w} := \tilde{w}_{\varepsilon,\alpha} - w^{(1)}_{\varepsilon,\alpha}$ which solves

$$\left[i\varepsilon\,\hat{M}_k \;+\; (\alpha-\hat{\alpha})\,\hat{M}_\alpha \right]\hat{w} \;=\; R(\varepsilon,\alpha) w^{(1)}_{\varepsilon,\alpha} \;+\; g_{0,\hat{\alpha}} - g_{\varepsilon,\alpha}$$

where

$$R(\varepsilon,\alpha) :=\; M(\hat{k}+i\varepsilon,\alpha) \;-\; \left[i\varepsilon\,\hat{M}_k \;+\; (\alpha-\hat{\alpha})\,\hat{M}_\alpha \right]$$

satisfies an estimate of the form $\|R(\varepsilon,\alpha)\| \leq c[\varepsilon^2 + (\alpha-\hat{\alpha})^2]$. Now we use (26) again which yields

$$\begin{aligned}\|\hat{w}\|_X &\leq \frac{\gamma}{\sqrt{\varepsilon^2+(\alpha-\hat{\alpha})^2}}\, c[\varepsilon^2+(\alpha-\hat{\alpha})^2]\|w^{(1)}_{\varepsilon,\alpha}\|_X \;+\; \|g_{0,\hat{\alpha}} - g_{\varepsilon,\alpha}\|_X \\ &\leq c\gamma\sqrt{\varepsilon^2+(\alpha-\hat{\alpha})^2}\,\big[\|\hat{w}\|_X + \|\tilde{w}_{\varepsilon,\alpha}\|_X + \tilde{c}\big]\end{aligned}$$

for some $\tilde{c} > 0$. Combining the terms involving $\|\hat{w}\|_X$ and using (28) yields the assertion.

(b) By (a) it is sufficient to estimate $\|\tilde{v}^{(1)}_{\varepsilon,\alpha} - \tilde{w}^{(1)}_{\varepsilon,\alpha}\|_X$.

$$\begin{aligned}\|\tilde{v}^{(1)}_{\varepsilon,\alpha} - \tilde{w}^{(1)}_{\varepsilon,\alpha}\|_X &= \|(U^{-1}_{\varepsilon,\alpha} - I)\tilde{w}^{(1)}_{\varepsilon,\alpha}\|_X \;\leq\; \|U^{-1}_{\varepsilon,\alpha} - I\|\,\|\tilde{w}^{(1)}_{\varepsilon,\alpha}\|_X \\ &\leq c\sqrt{\varepsilon^2+(\alpha-\hat{\alpha})^2}\|\tilde{w}^{(1)}_{\varepsilon,\alpha}\|_X \\ &\leq c\sqrt{\varepsilon^2+(\alpha-\hat{\alpha})^2}\big[\|\hat{w}\|_X + \|\tilde{w}_{\varepsilon,\alpha}\|_X\big] \;\leq\; \tilde{c}\end{aligned}$$

by (28) and part (a). □

We are now ready to study the behavior of the the solution to the linearized equation (22) with complex-values wave number $\hat{k}+i\varepsilon$ as ε tends to zero. Expanding the right-hand side side and the solution of (22) for $k = \hat{k}+i\varepsilon$ in the forms $\big(\frac{\partial}{\partial k}M(\hat{k},\hat{\alpha})\big)^{-1}g_{0,\hat{\alpha}} = \sum_{\ell=1}^m y_\ell\,\phi_\ell$ and $\tilde{w}_{\varepsilon,\alpha} = \sum_{\ell=1}^m a_\ell\,\phi_\ell$, respectively, we observe that the solution of (22) is given by

$$\tilde{w}_{\varepsilon,\alpha} = \sum_{\ell=1}^m \frac{y_\ell}{i\varepsilon - d_\ell(\alpha-\hat{\alpha})}\,\phi_\ell\,. \tag{29}$$

Before we turn to the limiting absorption principle we formulate the eigenvalue problem (23) as a variational equation for the $\hat{\alpha}-$quasi-periodic eigenfunction in $Q^\infty = (0,2\pi)\times(0,\infty)$, relying on the expressions for $\partial M/\partial k$ and $\partial M/\partial\alpha$ from the proof of Theorem 5.2.

Lemma 5.6. *We define the space*

$$\tilde{X} \;:=\; \left\{\tilde{\phi}\in H^1_{loc}(\mathbb{R}^2_+):\ \begin{array}{l}\Delta\tilde{\phi}_\ell + \hat{k}^2 q\tilde{\phi}_\ell = 0 \text{ in } \mathbb{R}^2_+,\ \tilde{\phi} = 0 \text{ for } x_2 = 0,\\ \tilde{\phi} \text{ is } \hat{\alpha}-\text{quasi-periodic}\end{array}\right\}.$$

Then $\phi\in X$ if, and only if, the $\hat{\alpha}-$quasi-periodic extension of

$$\tilde{\phi}(x) \;=\; \begin{cases}\phi(x)\,e^{i\hat{\alpha}x_1}\,, & x\in Q\,,\\ E_{\hat{k},\hat{\alpha}}\big(\phi\,e^{i\hat{\alpha}x_1}\big)(x)\,, & x\in Q^\infty\setminus Q\,,\end{cases}$$

is in $\tilde{X}$. Furthermore, if $\{\phi_\ell\in X : \ell = 1,\ldots,m\}$ is a complete system of eigenfunctions of (23) corresponding to the eigenvalues d_ℓ, $\ell = 1,\ldots,m$, then the corresponding functions $\tilde{\phi}_\ell\in\tilde{X}$ satisfy

$$-i\int_{Q^\infty}\frac{\partial\tilde{\phi}_\ell}{\partial x_1}\,\overline{\psi}\,dx \;=\; d_\ell\,\hat{k}\int_{Q^\infty} q\,\tilde{\phi}_\ell\,\overline{\psi}\,dx \quad \text{for all } \psi\in\tilde{X}\,. \tag{30}$$

The normalization (25) transforms into $\hat{k}\int_{Q^\infty} q\,\tilde{\phi}_\ell\,\overline{\tilde{\phi}_{\ell'}}\,dx = \delta_{\ell,\ell'}$ for all $\ell,\ell' = 1,\ldots,m$.

6. A Limiting Absorption Solution

The semi-explicit expression of the Floquet-Bloch transform of a solution to the linearization of the scattering problem's operator equation in (29) allows to study convergence of this quantity as the artificial absorption tends to zero. To this end, recall that for wave number k and quasi-periodicity α in a ball B_{δ_4} around $(\hat{k},\hat{\alpha})$, the solution $\tilde{u}_{k,\alpha}$ to (22) can be represented as

$$\tilde{u}_{k,\alpha} \overset{(15-16)}{=} \tilde{v}^{(1)}_{k,\alpha} + \tilde{v}^{(2)}_{k,\alpha} = U^{-1}_{k,\alpha} w^{(1)}_{k,\alpha} + U^{-1}_{k,\alpha} w^{(2)}_{k,\alpha}, \quad (k,\alpha) \in B_{\delta_4}, \tag{31}$$

with $\tilde{v}^{(1)}_{k,\alpha} = U^{-1}_{k,\alpha} w^{(1)}_{k,\alpha}$ in $\mathcal{N}(I - K_{k,\alpha})$ and $\tilde{v}^{(2)}_{k,\alpha} = U^{-1}_{k,\alpha} w^{(2)}_{k,\alpha}$ in $\mathcal{R}(I - K_{k,\alpha})$. Thus, we need to investigate convergence of the back transformations

$$\int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \tilde{v}^{(1)}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \quad \text{and} \quad \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \tilde{v}^{(2)}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \tag{32}$$

for $(k,\alpha) \in B_{\delta_2}$ and $x \in W = \mathbb{R} \times (0,h)$ in suitable function spaces as $k \to \hat{k}$.

As we already showed in (19) that $\tilde{v}^{(2)}_{k,\alpha}$ converges in $H^1_{0,per}(Q)$ to $\tilde{v}^{(2)}_{\hat{k},\alpha}$, uniformly with respect to α, we conclude that the inverse Floquet-Bloch transform converges, too,

$$\int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \tilde{v}^{(2)}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \overset{k\to\hat{k}}{\longrightarrow} \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \tilde{v}^{(2)}_{\hat{k},\alpha}(x)\, e^{i\alpha x_1} d\alpha \qquad \text{in } H^1(W).$$

It is hence sufficient to consider the first integral in (32); further, we merely consider complex wave numbers of the form $k + i\varepsilon$ and, as in the last section, abbreviate the dependence of all fields on the wave number by an index ε. Let us split the first integral as

$$\begin{aligned}
&\int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \tilde{v}^{(1)}_{\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha \\
&= \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \tilde{w}_{\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha + \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} [\tilde{v}^{(1)}_{\varepsilon,\alpha}(x) - \tilde{w}_{\varepsilon,\alpha}(x)]\, e^{i\alpha x_1} d\alpha \\
&= \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \left[\tilde{v}^{(1)}_{\varepsilon,\alpha}(x) - \tilde{w}_{\varepsilon,\alpha}(x)\right] e^{i\alpha x_1} d\alpha + \sum_{\ell=1}^{m} y_\ell\, \phi_\ell(x) \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \frac{1}{i\varepsilon - d_\ell(\alpha-\hat{\alpha})}\, e^{i\alpha x_1} d\alpha \\
&= \int_{\hat{\alpha}-\delta_4}^{\hat{\alpha}+\delta_4} \left[\tilde{v}^{(1)}_{\varepsilon,\alpha}(x) - \tilde{w}_{\varepsilon,\alpha}(x)\right] e^{i\alpha x_1} d\alpha - \sum_{\ell=1}^{m} y_\ell\, \phi_\ell(x) \int_{\delta_4<|\alpha-\hat{\alpha}|<1/2} \frac{1}{i\varepsilon - d_\ell(\alpha-\hat{\alpha})}\, e^{i\alpha x_1} d\alpha \\
&\quad + \sum_{\ell=1}^{m} y_\ell\, \phi_\ell(x) \int_{\hat{\alpha}-1/2}^{\hat{\alpha}+1/2} \frac{1}{i\varepsilon - d_\ell(\alpha-\hat{\alpha})}\, e^{i\alpha x_1} d\alpha \\
&=: v^{(1)}_\varepsilon(x) + v^{(2)}_\varepsilon(x) \qquad \text{for } x \in W = \mathbb{R} \times (0,h),
\end{aligned} \tag{33}$$

where

$$v_\varepsilon^{(2)}(x) \;=\; \sum_{\ell=1}^m y_\ell\,\phi_\ell(x) \int\limits_{\hat\alpha-1/2}^{\hat\alpha+1/2} \frac{1}{i\varepsilon - d_\ell(\alpha-\hat\alpha)}\, e^{i\alpha x_1} d\alpha$$

and obvious meaning of $v_\varepsilon^{(1)}(x)$.

Lemma 6.1. *(a) As $\varepsilon \to 0$, the functions $v_\varepsilon^{(1)}$ converge in $H^1(W)$ to*

$$(34)\quad v_0^{(1)}(x) \;=\; \int\limits_{\hat\alpha-\delta_4}^{\hat\alpha+\delta_4} \big[\tilde v_{0,\alpha}^{(1)}(x) - \tilde w_{0,\alpha}(x)\big]\, e^{i\alpha x_1} d\alpha \;-\; \sum_{\ell=1}^m \frac{y_\ell}{d_\ell}\,\phi_\ell(x) \int\limits_{\delta_4<|\alpha-\hat\alpha|<1/2} \frac{1}{\alpha-\hat\alpha}\, e^{i\alpha x_1} d\alpha\,.$$

(b) The functions $v_\varepsilon^{(2)}$ are defined in all of $\mathbb{R}^2_+$. As $\varepsilon \to 0$, they converge for all $R>0$ in $C^1\big([-R,R]\times\mathbb{R}_{\geq 0}\big)$ to

$$(35)\qquad v_0^{(2)}(x) \;=\; -i\pi \sum_{\ell=1}^m \frac{y_\ell}{|d_\ell|}\,\phi_\ell(x)\, e^{i\hat\alpha x_1} \left[1 + \mathrm{sign}(d_\ell)\,\frac{2}{\pi}\int_0^{x_1/2} \frac{\sin t}{t}\,dt\right].$$

Proof: (a) First we note that it is sufficient to consider the first term because convergence of $\sum_{\ell=1}^m y_\ell\,\phi_\ell(x)\int_{\delta_4<|\alpha-\hat\alpha|<1/2} \exp(i\alpha x_1)/[i\varepsilon - d_\ell(\alpha-\hat\alpha)]\,d\alpha$ is obvious. For simplicity, we denote the integrand of the first term of $v_\varepsilon^{(1)}$ as $z_\varepsilon(\cdot,\alpha) := \tilde v_{\varepsilon,\alpha}^{(1)} - \tilde w_{\varepsilon,\alpha}$ for $\alpha \in [\hat\alpha-\delta_4,\ \hat\alpha+\delta_4]$. As $I - K_{\hat k,\alpha}$ is an isomorphism for all α that are no exceptional values, we deduce that $\|z_\varepsilon(\cdot,\alpha) - z_0(\cdot,\alpha)\|_* \to 0$ as $\varepsilon \to 0$ holds for all $\alpha \neq \hat\alpha$. Furthermore, $\|z_\varepsilon(\cdot,\alpha) - z_0(\cdot,\alpha)\|_*$ is uniformly bounded with respect to ε and α. Indeed, for the terms of the sum this is obvious by the holomorphy of y_ℓ in both arguments and for the term in brackets this follows from Lemma 5.5(b). In case that $\hat\alpha \in (-1/2, 1/2)$ we reduce δ_4 such that $(\hat\alpha-\delta_4, \hat\alpha+\delta_4)$ is a subset of $(-1/2,1/2)$. Then, Lebesgue's theorem on dominated convergence yields $\int_{\hat\alpha-\delta_4}^{\hat\alpha+\delta_4} \|z_\varepsilon(\cdot,\alpha) - z_0(\cdot,\alpha)\|_*^2 d\alpha \to 0$ as $\varepsilon \to 0$, that is, $z_\varepsilon \to z_0$ in $L^2\big((\hat\alpha-\delta_4, \hat\alpha+\delta_4),\ H^1_{0,per}(Q)\big)$. If $\hat\alpha = 1/2$, Remark 4.3 shows how to treat this boundary case. In each case, the boundedness of the inverse Floquet-Bloch transform finally yields $v_\varepsilon^{(1)} \to v_0^{(1)}$ in $H^1(W)$.

(b) The asymptotics of the integral $\int_{\hat\alpha-1/2}^{\hat\alpha+1/2} \exp(i\alpha x_1)/[i\varepsilon - d_\ell(\alpha-\hat\alpha)]\,d\alpha$ can be determined explicitly,

$$\int\limits_{\hat\alpha-1/2}^{\hat\alpha+1/2} \frac{\exp(i\alpha x_1)}{i\varepsilon - d_\ell(\alpha-\hat\alpha)}\,d\alpha \;=\; e^{i\hat\alpha x_1}\int\limits_{-1/2}^{1/2} \frac{1}{i\varepsilon - d_\ell\alpha}\, e^{i\alpha x_1} d\alpha \;=\; e^{i\hat\alpha x_1}\int\limits_{-1/2}^{1/2} \frac{-i\varepsilon - d_\ell\alpha}{\varepsilon^2 + d_\ell^2\alpha^2}\, e^{i\alpha x_1} d\alpha$$

$$=\; -i\varepsilon\, e^{i\hat\alpha x_1}\int\limits_{-1/2}^{1/2} \frac{\cos(\alpha x_1)}{\varepsilon^2 + d_\ell^2\alpha^2}\,d\alpha \;-\; i d_\ell\, e^{i\hat\alpha x_1}\int\limits_{-1/2}^{1/2} \frac{\alpha\sin(\alpha x_1)}{\varepsilon^2 + d_\ell^2\alpha^2}\,d\alpha\,.$$

In the first integral we substitute $\alpha = t(\varepsilon/|d_\ell|)$ and in the second integral $t = \alpha x_1$. This yields

$$\int_{\hat\alpha-1/2}^{\hat\alpha+1/2} \frac{\exp(i\alpha x_1)}{i\varepsilon - d_\ell(\alpha-\hat\alpha)}\,d\alpha \;=\; e^{i\hat\alpha x_1}\left[\frac{-i}{|d_\ell|}\int_{-|d_\ell|/(2\varepsilon)}^{|d_\ell|/(2\varepsilon)} \frac{\cos(t\varepsilon x_1/|d_\ell|)}{1+t^2}\,dt - i d_\ell \int_{-x_1/2}^{x_1/2} \frac{t\sin t}{x_1^2\varepsilon^2 + d_\ell^2 t^2}\,dt\right].$$

For $\varepsilon \to 0$ the expression on the right converges to

$$e^{i\hat\alpha x_1}\left[-\frac{i}{|d_\ell|}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\,dt - \frac{2i}{d_\ell}\int_0^{x_1/2}\frac{\sin t}{t}\,dt\right] = -\frac{i\pi}{|d_\ell|}\,e^{i\hat\alpha x_1}\left[1 + \operatorname{sign}(d_\ell)\,\frac{2}{\pi}\int_0^{x_1/2}\frac{\sin t}{t}\,dt\right].$$

uniformly with respect to $|x_1| \le R$, for arbitrary $R > 0$. The derivative of the investigated parameter integral with respect to x_1 converges uniformly for $|x_1| \le R$ for every $R > 0$ as well. □

Remark 6.2. *As* $\lim_{T\to\infty}\int_0^T \sin(t)/t\,dt = \pi/2$ *we observe that* $\psi^\pm \in C^\infty(\mathbb{R})$*, defined by*

$$\psi^\pm(x_1) \;=\; \frac{1}{2}\left[1 \;\pm\; \frac{2}{\pi}\int_0^{x_1/2}\frac{\sin t}{t}\,dt\right], \quad x_1 \in \mathbb{R}, \tag{36}$$

tends to 1 *as* $x_1 \to \pm\infty$ *while it converges to* 0 *for* $x_1 \to \mp\infty$*. Thus, as* ε *tends to zero,* $v_\varepsilon^{(2)}$ *from Lemma 6.1(b) converges to*

$$v_0^{(2)}(x) \;=\; u^+(x)\,\psi^+(x_1) \;+\; u^-(x)\,\psi^-(x_1) \tag{37}$$

where

$$u^\pm(x) \;=\; -2\pi i \sum_{d_\ell \gtrless 0} \frac{y_\ell}{|d_\ell|}\,\phi_\ell(x)\,e^{i\hat\alpha x_1}. \tag{38}$$

This separates $v_0^{(2)}$ *into groups of modes propagating to the left and the right.*

The following lemma shows that the modes $\tilde\phi_\ell$ satisfy a kind of radiation condition.

Lemma 6.3. *Let* $L^\pm = \{\ell : d_\ell \gtrless 0\}$ *and* $u^\pm = \sum_{\ell\in L^\pm} a_\ell^\pm\,\tilde\phi_\ell$ *for some* $a_\ell^\pm \in \mathbb{C}$*. Then, for every* $a \in \mathbb{R}$*,*

$$2\pi\,\mathrm{Im}\int_0^\infty \overline{u^\pm}\,\frac{\partial u^\pm}{\partial x_1}\bigg|_{x_1=a}\,dx_2 \;=\; \mathrm{Im}\int_{Q^\infty}\overline{u^\pm}\,\frac{\partial u^\pm}{\partial x_1}\,dx \;\gtrless\; 0 \quad \textit{if } u^\pm \ne 0.$$

Proof: We only consider u^+. Setting $v(x) = (x_1 - a)u^+(x)$ yields $\frac{\partial v}{\partial x_1} = u^+ + (x_1 - a)\,\frac{\partial u^+}{\partial x_1}$ and $\Delta v + \hat{k}^2 q v = 2\frac{\partial u^+}{\partial x_1}$. Therefore, with $Q_a^\infty = (a, a+2\pi) \times (0,\infty) \subset \mathbb{R}^2_+$,

$$
\begin{aligned}
2 \int\limits_{Q_a^\infty} \overline{u^+}\, \frac{\partial u^+}{\partial x_1}\, dx &= \int\limits_{Q_a^\infty} \overline{u^+}\left(\Delta v + \hat{k}^2 q v\right) dx \\
&= \int\limits_{Q_a^\infty} v\left(\Delta \overline{u^+} + \hat{k}^2 q \overline{u^+}\right) dx \;+\; \int_{\partial Q_a^\infty} \left(\overline{u^+}\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial \overline{u^+}}{\partial \nu}\right) ds \\
&= -\int\limits_{x_1=a} |u^+|^2 dx_2 + \int\limits_{x_1=a+2\pi} \left[\overline{u^+}\left(u^+ + 2\pi\,\frac{\partial u^+}{\partial x_1}\right) - 2\pi u^+\,\frac{\partial \overline{u^+}}{\partial x_1}\right] dx_2 \\
&= 2\pi \int\limits_{x_1=a} \left(\overline{u^+}\,\frac{\partial u^+}{\partial x_1} - u^+\,\frac{\partial \overline{u^+}}{\partial x_1}\right) dx_2 \;=\; 4\pi\, i\, \mathrm{Im} \int\limits_{x_1=a} \overline{u^+}\,\frac{\partial u^+}{\partial x_1}\, dx_2
\end{aligned}
$$

which proves the first equality because $x_1 \mapsto \overline{u^+}\,\frac{\partial u^+}{\partial x_1}$ is $2\pi-$periodic. Furthermore,

$$
\begin{aligned}
\int_{Q^\infty} \overline{u^+}\,\frac{\partial u^+}{\partial x_1}\, dx &= \sum_{\ell,\ell' \in L^-} \overline{a_\ell}\, a_{\ell'} \int_{Q^\infty} \overline{\tilde{\phi}_\ell}\, \frac{\partial \tilde{\phi}_{\ell'}}{\partial x_1}\, dx \\
&= -i\hat{k} \sum_{\ell,\ell' \in L^-} \overline{a_\ell}\, a_{\ell'}\, d_{\ell'} \int_{Q^\infty} q\, \overline{\tilde{\phi}_\ell}\, \tilde{\phi}_{\ell'}\, dx \;=\; -i \sum_{\ell \in L^-} |a_\ell|^2\, d_\ell
\end{aligned}
$$

by the orthonormalization of $\tilde{\phi}_\ell$. Taking the imaginary part yields the assertion. □

So far, we have considered the behavior of the periodic solutions to the Floquet-Bloch-transformed Helmholtz equation with absorption in the neighborhood of one exceptional value $\hat{\alpha} = \hat{\alpha}_j$ at wave number $\hat{k} > 0$. Now we consider all exceptional values $\hat{\alpha}_j$ simultaneously. The quantities m, d_ℓ, $\tilde{X}$, $\tilde{\phi}_\ell$, $L^\pm$ and so on depend on j. Before stating the related limiting absorption result, let us recall that we work at wave number $\hat{k} > 0$, and that that for all elements in the the (possibly empty) set $\{\hat{\alpha}_j : j \in J\}$ of the corresponding exceptional values there exist m_j linearly independent $\hat{\alpha}_j$-quasi-periodic and evanescent solutions to the Helmholtz equation $\Delta\tilde{\phi}_{\ell,j} + \hat{k}^2 q \tilde{\phi}_{\ell,j} = 0$ in $\mathbb{R}^2_+$, $j \in J$, that vanish for $x_2 = 0$. These functions are chosen as the normalized eigenfunctions of (23) for every $j \in J$. Of course, we still suppose Assumptions 4.1 and 5.3 to hold. In particular, $d_{\ell,j} \neq 0$ for all $\ell = 1, \ldots, m_j$ and $j \in J$. Then we can extend Lemma 6.3 to different $\hat{\alpha}_j$.

Lemma 6.4. *Let $L_j^\pm = \{\ell : d_{\ell,j} \gtrless 0\}$ and $u^\pm = \sum_{j\in J}\sum_{\ell \in L_j^\pm} a_{\ell,j}^\pm\, \tilde{\phi}_{\ell,j}$ for some $a_{\ell,j}^\pm \in \mathbb{C}$. Then, for every $a \in \mathbb{R}$,*

$$\mathrm{Im} \int_0^\infty \overline{u^\pm}\, \frac{\partial u^\pm}{\partial x_1}\bigg|_{x_1=a} dx_2 \;\gtrless\; 0 \quad \textit{if } u^\pm \neq 0\,.$$

Proof: Set $u_j^\pm = \sum_{\ell\in L_j^\pm} a_{\ell,j}^\pm\, \tilde\phi_{\ell,j}$ for $j\in J$. Then, for $j,j'\in J$,

$$
\begin{aligned}
0 &= \int_{\partial Q_a^\infty} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial\nu} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial\nu}\right) ds \\
&= -\int_{x_1=a} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial x_1} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right) dx_2 \;+\; \int_{x_1=a+2\pi} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial x_1} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right) dx_2 \\
&= \left(e^{i(\hat\alpha_{j'}-\hat\alpha_j)2\pi} - 1\right) \int_{x_1=a} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial x_1} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right) dx_2\,.
\end{aligned}
$$

Therefore, the last integral vanishes for $j\neq j'$. Thus we have

$$
\begin{aligned}
&2\pi\,\mathrm{Im} \int_0^\infty \overline{u^\pm}\, \frac{\partial u^\pm}{\partial x_1}\bigg|_{x_1=a} dx_2 \\
&= \pi \int_{x_1=a} \left[\overline{u^\pm}\, \frac{\partial u^\pm}{\partial x_1} - u^\pm\, \frac{\partial \overline{u^\pm}}{\partial x_1}\right] dx_2 \;=\; \pi \sum_{j\in J} \int_{x_1=a} \left[\overline{u_j^\pm}\, \frac{\partial u_j^\pm}{\partial x_1} - u_j^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right] dx_2 \\
&= 2\pi \sum_{j\in J} \mathrm{Im} \int_{x_1=a} \overline{u_j^\pm}\, \frac{\partial u_j^\pm}{\partial x_1}\, dx_2 \;=\; \sum_{j\in J} \mathrm{Im} \int_{Q^\infty} \overline{u_j^\pm}\, \frac{\partial u_j^\pm}{\partial x_1}\, dx \;\gtrless\; 0
\end{aligned}
$$

where we have used Lemma 6.3. $\square$

Theorem 6.5. *The restriction of the solution $u_{\hat k+i\varepsilon}$ to (4) for $k=\hat k+i\varepsilon$ to W has a decomposition in the form $u_{\hat k+i\varepsilon} = u_\varepsilon^{(1)} + u_\varepsilon^{(2)}$ where $u_\varepsilon^{(1)}\in H^1(W)$ converges in $H^1(W)$ to some $u^{(1)}\in H^1(W)$. Further, $u_\varepsilon^{(2)}\in C_b^1(W)$ converges for every $R>0$ in $C^1\big([-R,R]\times[0,h]\big)$ to $u^{(2)}\in C_b^1(W)$ which has the form*

$$
u^{(2)}(x) \;=\; \psi^+(x_1) \sum_{j\in J} \sum_{\ell\in L_j^+} a_{\ell,j}^+\, \tilde\phi_{\ell,j}(x) \;+\; \psi^-(x_1) \sum_{j\in J} \sum_{\ell\in L_j^-} a_{\ell,j}^-\, \tilde\phi_{\ell,j}(x) \tag{39}
$$

for some $a_{\ell,j}^\pm\in\mathbb{C}$. Here, the functions $\psi^\pm$ are defined in (36).

Proof: We have to evaluate the inverse Bloch-Floquet transform $u_k(x) = \int_{-1/2}^{1/2} \tilde u_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha$ for $k=\hat k+i\varepsilon$. Defining $\mathcal{I} = (-1/2,1/2)\setminus \bigcup_{j\in J} B(\hat\alpha_j,\delta_4)$ we decompose this integral into the form

$$
\begin{aligned}
u_k(x) \;&=\; \int\limits_{-1/2}^{1/2} \tilde{u}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \\
&=\; \int\limits_{\mathcal{I}} \tilde{u}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \;+\; \sum_{j\in J} \int\limits_{\hat{\alpha}_j-\delta_4}^{\hat{\alpha}_j+\delta_4} \tilde{v}^{(2,j)}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \;+\; \sum_{j\in J} \int\limits_{\hat{\alpha}_j-\delta_4}^{\hat{\alpha}_j+\delta_4} \tilde{v}^{(1,j)}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \\
&=\; \int\limits_{\mathcal{I}} \tilde{u}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha \;+\; \sum_{j\in J} \int\limits_{\hat{\alpha}_j-\delta_4}^{\hat{\alpha}_j+\delta_4} \tilde{v}^{(2,j)}_{k,\alpha}(x)\, e^{i\alpha x_1} d\alpha + \sum_{j\in J} v^{(1,j)}_{\varepsilon}(x) \;+\; \sum_{j\in J} v^{(2,j)}_{\varepsilon}(x) \\
&=\; u^{(1)}_k(x) \;+\; \sum_{j\in J} v^{(2,j)}_{\varepsilon}(x) \qquad (40)
\end{aligned}
$$

where $\tilde{v}^{(1,j)}_{k,\alpha}$, $\tilde{v}^{(2,j)}_{k,\alpha}$, $v^{(1,j)}_{\varepsilon}$, and $v^{(2,j)}_{\varepsilon}$ denote the functions from (31) and (33), respectively, for $\hat{\alpha} = \hat{\alpha}_j$. As we have seen before, the first three terms (which form the function $u^{(1)}_k$) converge in $H^1(W)$ as ε tends to zero to the corresponding function $u^{(1)}$ for $\varepsilon = 0$. Furthermore, by Lemma 6.1 the forth term converges in $C^1\big([-R,R]\times[0,h]\big)$ to

$$
\sum_{j\in J} v^{(2,j)}_0(x) \;=\; \psi^+(x_1) \sum_{j\in J} \sum_{\ell\in L^+_j} a^+_{\ell,j}\, \tilde{\phi}_{\ell,j}(x) \;+\; \psi^-(x_1) \sum_{j\in J} \sum_{\ell\in L^-_j} a^-_{\ell,j}\, \tilde{\phi}_{\ell,j}(x)
$$

for some coefficients $a^{\pm}_{\ell,j}$ which are explicitly given through $y_{\ell,j}$ and $d_{\ell,j}$. This yields the form (39). □

As $\psi^{\pm}(x_1) \to 1$ for $x_1 \to \pm\infty$ and $\psi^{\pm}(x_1) \to 0$ for $x_1 \to \mp\infty$, we further note that we can also decompose the solution $u = u^{(1)} + u^{(2)}$ from Theorem 6.5 in the forms

$$
u \;=\; [u \;-\; u^+] \;+\; u^+ \;=\; [u \;-\; u^-] \;+\; u^-
$$

with $u^{\pm} = \sum_{j\in J} \sum_{\ell\in L^{\pm}_j} a^{\pm}_{\ell,j}\, \tilde{\phi}_{\ell,j}$. In these decompositions all terms are solutions of the Helmholtz equation; in the first form the term in the bracket decays to zero as x_1 tends to $+\infty$ while in the second form the term in the bracket decays to zero as x_1 tends to $-\infty$.

So far, we have studied the behavior of the solution inside the waveguide W. Now we will extend the solution into the half space $\mathbb{R}^2_+$. Since the functions $\tilde{\phi}_{\ell,j}$ are already (evanescent) solutions of the Helmholtz equation in $\mathbb{R}^2_+$ it is sufficient to extend $u^{(1)}$.

Theorem 6.6. *Let $\{\hat{\alpha}_j : j \in J\}$ be the (possibly empty) set of exceptional values for wave number $\hat{k} > 0$ and let Assumptions 4.1 and 5.3 hold (the latter for all $\hat{\alpha}_j$, $j \in J$). Then the solutions $u_{\hat{k}+i\varepsilon}$ of (3) for $k = \hat{k} + i\varepsilon$ converge as $\varepsilon \to 0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to some $u \in H^1_{loc}(\mathbb{R}^2_+)$ which satisfies the Helmholtz equation $\Delta u + \hat{k}^2 q u = f$ in $\mathbb{R}^2_+$. The solution u has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)} \in H^1\big(\mathbb{R}\times(0,H)\big)$ for every*

$H>h$ and $u^{(2)}$ is a linear combination of surface waves; that is, it has the form

$$u^{(2)}(x) \;=\; \psi^+(x_1)\sum_{j\in J}\sum_{\ell\in L_j^+} a^+_{\ell,j}\,\tilde\phi_{\ell,j}(x) \;+\; \psi^-(x_1)\sum_{j\in J}\sum_{\ell\in L_j^-} a^-_{\ell,j}\,\tilde\phi_{\ell,j}(x), \tag{41}$$

for $x\in\mathbb{R}^2_+$ where $a^\pm_{\ell,j}\in\mathbb{C}$ for $\ell=1,\dots,m_j$ and $j\in J$. Furthermore, u satisfies the upward propagating radiation condition (5).

Proof: Let again $k=\hat k+i\varepsilon$. For $\varepsilon>0$ the solution $u_k\in H^1(\mathbb{R}^2_+)$ satisfies the Helmholtz equation $\Delta u_k+k^2qu_k=0$ for $x_2>0$ and $u_k(x_1,0)=0$ for $x_1\in\mathbb{R}$. From (40) we recall that in W it has the decomposition $u_k=u_k^{(1)}+u_k^{(2)}$ in W where $u_k^{(1)}\in H^1(W)$ converges to $u^{(1)}\in H^1(W)$ and $u_k^{(2)}$ has the form

$$\begin{aligned}
u_k^{(2)}(x) &= \sum_{j\in J}\sum_{\ell=1}^{m_j} y_{\ell,j}\,\phi_{\ell,j}(x)\int\limits_{\hat\alpha_j-1/2}^{\hat\alpha_j+1/2}\frac{1}{i\varepsilon-d_{\ell,j}(\alpha-\hat\alpha_j)}\,e^{i\alpha x_1}d\alpha\\
&= \sum_{j\in J}\sum_{\ell=1}^{m_j}\underbrace{y_{\ell,j}\int\limits_{-1/2}^{1/2}\frac{1}{i\varepsilon-d_{\ell,j}\alpha}\,e^{i\alpha x_1}d\alpha}_{=\,\psi_{\ell,j,\varepsilon}(x_1)}\;\tilde\phi_{\ell,j}(x)\\
&= \sum_{j\in J}\sum_{\ell=1}^{m_j}\psi_{\ell,j,\varepsilon}(x_1)\,\tilde\phi_{\ell,j}(x)\,.
\end{aligned}$$

Note that $\tilde\phi_{\ell,j}(x)=\phi_{\ell,j}(x)e^{i\hat\alpha_j x_1}$. Therefore, $u_k^{(2)}$ is defined in all of $\mathbb{R}^2_+$. We have seen in part (b) of Lemma 6.1 that it converges to

$$u^{(2)}(x) \;:=\; \psi^+(x_1)\sum_{\in J}\sum_{\ell\in L_j^+} a^+_{\ell,j}\,\tilde\phi_{\ell,j}(x) \;+\; \psi^-(x_1)\sum_{\in J}\sum_{\ell\in L_j^-} a^-_{\ell,j}\,\tilde\phi_{\ell,j}(x)$$

for some coefficients $a^\pm_{\ell,j}\in\mathbb{C}$.

Now we consider $u_k^{(1)}$. Since u_k satisfies the Helmholtz equation we observe that $u_k^{(1)}$ satisfies the following inhomogeneous boundary value problem

$$\begin{aligned}
\Delta u_k^{(1)}(x)+k^2q(x)u_k^{(1)}(x) &= -\big[\Delta u_k^{(2)}(x)+k^2q(x)u_k^{(2)}(x)\big]\\
&= -\sum_{j\in J}\sum_{\ell=1}^{m_j}\big[2\,\psi'_{\ell,j,\varepsilon}(x_1)\,\partial\tilde\phi_{\ell,j}(x)/\partial x_1+\psi''_{\ell,j,\varepsilon}(x_1)\,\tilde\phi_{\ell,j}(x)\big]\\
&=: \sigma_\varepsilon(x)\,,\quad x\in W\,, &(42)\\
u_k^{(1)}(x) &= u_k(x)\;-\;u_k^{(2)}(x)\\
&=: \xi_\varepsilon(x)\,,\quad x\in\Gamma_h\,. &(43)
\end{aligned}$$

where we again used the index k on the left hand side and ε on the right hand side.[2] We compute $\psi'_{\ell,j,\varepsilon}$ as

$$\psi'_{\ell,j,\varepsilon}(x_1) \;=\; y_{\ell,j} \int\limits_{-1/2}^{1/2} \frac{i\,\alpha}{i\varepsilon - d_{\ell,j}\alpha}\, e^{i\alpha x_1}\, d\alpha$$

which converges to

$$\psi'_{\ell,j,0}(x_1) \;=\; -\frac{i\,y_{\ell,j}}{d_{\ell,j}} \int\limits_{-1/2}^{1/2} e^{i\alpha x_1}\, d\alpha \;=\; -\frac{i\,y_{\ell,j}}{d_{\ell,j}}\,\frac{\sin(x_1/2)}{x_1/2}$$

in $C^1[-R,R]$ for every $R > 0$ as ε tends to zero. Analogously, $\psi''_{\ell,j,\varepsilon}$ converges to $\psi''_{\ell,j,0}(x_1) = -\frac{i\,y_{\ell,j}}{d_{\ell,j}}\,\frac{d}{dx_1}\,\frac{\sin(x_1/2)}{x_1/2}$ in $C^1[-R,R]$ for every $R > 0$. Therefore, σ_ε converges to $\sigma_0 = -[\Delta u^{(2)} + k^2 q u^{(2)}]$ in $C^1\big([-R,R]\times[0,H]\big)$ for every $H,R>0$. As the above representations of $\psi'_{\ell,j,0}$ and $\psi''_{\ell,j,0}$ show that both functions belong to $L^2(\mathbb{R})$, the limit σ_0 belongs to $L^2(\mathbb{R}^2_+)$ and decays exponentially as x_2 tends to infinity. Furthermore, Theorem 6.5 implies convergence of ξ_ε to ξ_0 in $H^{1/2}(\Gamma_h)$ where ξ_0 is given by $\xi_0 = u - u^{(2)}$.

We still assume $\varepsilon = \text{Im}\, k > 0$ and extend $u_k^{(1)}$ into the upper half plane by solving the Dirichlet Problem

$$\Delta u_k^{(1)} + k^2 u_k^{(1)} = \sigma_\varepsilon \text{ for } x_2 > h\,, \quad u_k^{(1)} = \xi_\varepsilon \text{ for } x_2 = h\,.$$

The H^1-solution has the form

$$u_k^{(1)}(x) \;=\; -\int_{x_2>h} \sigma_\varepsilon(y)\, G_{k,h}(x,y)\, dy \;+\; 2\int_{\Gamma_h} \xi_\varepsilon(y)\,\frac{\partial \Phi_k}{\partial y_2}(x,y)\, ds(y)\,, \quad x_2 > h\,, \tag{44}$$

where $G_{k,h}$ denotes the Dirichlet Green's function of the half space $\mathbb{R}^2_{x_2>h}$; that is, $G_{k,h}(x,y) = \Phi_k(x,y) - \Phi_k(x,y_h^*)$ where $y_h^* = (y_1, 2h-y_2)^\top$. Then u_k coincides with $u_k^{(1)} + \sum_{j\in J} v_\varepsilon^{(2,j)}$ also in $\mathbb{R}^2_{x_2>h}$ because both satisfy the same coercive Dirichlet boundary value problem in $\{x_2 > h\}$. (Note that $\partial G_{k,h}(x,y)/\partial y_2$ equals $2\partial\Phi_k/\partial y_2$ for $y \in \Gamma_h$.)

As $\varepsilon \to 0$ the function $u_k^{(1)}$ converges to

$$u^{(1)}(x) \;=\; -\int_{x_2>h} \sigma_0(y)\, G_{\hat{k},h}(x,y)\, dy \;+\; 2\int_{\Gamma_h} \xi_0(y)\,\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\, ds(y) \tag{45}$$

in $H^1(K)$ for every bounded domain $K \subset \mathbb{R}^2_{x_2>h}$. Therefore, we have shown convergence of $u_k = u_k^{(1)} + u_k^{(2)}$ to u in $H^1(K)$ for every bounded domain $K \subset \mathbb{R}^2_+$ where u has the form $u = u^{(1)} + u^{(2)}$ with $u^{(1)} \in H^1\big(\mathbb{R}\times[0,H)\big)$ for every $H > h$ and

$$u^{(2)}(x) \;=\; \psi^+(x_1) \sum_{j\in J} \sum_{\ell \in L_j^+} a^+_{\ell,j}\, \tilde{\phi}_{\ell,j}(x) \;+\; \psi^-(x_1) \sum_{j\in J} \sum_{\ell\in L_j^-} a^-_{\ell,j}\, \tilde{\phi}_{\ell,j}(x)\,.$$

Finally, we have to show the radiation condition (5). From (45) and the form of σ_0 and ξ_0 we note that

$$u^{(1)}(x) \;=\; \int_{x_2>h} (\Delta + \hat{k}^2) u^{(2)}(y)\, G_{\hat{k},h}(x,y)\, dy \;+\; 2\int_{\Gamma_h} u^{(1)}(y)\,\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\, ds(y) \tag{46}$$

[2] We hope that this does not lead to any confusion. Note that always $k = \hat{k} + i\varepsilon$.

for $x_2 > h$. We set $Q^{R,H} = (-R,R)\times(h,H)$ and $Q^{\infty,H} = \mathbb{R}\times(h,H)$ for the moment and use Green's representation theorem for $u^{(2)}$ and x in $Q^{R,H}$,

$$\begin{aligned} u^{(2)}(x) &= -\int_{Q^{R,H}} (\Delta+\hat{k}^2)u^{(2)}(y)\, G_{\hat{k},h}(x,y)\,dy \\ &\quad - \int_{\partial Q^{R,H}} \left[u^{(2)}(y)\,\frac{\partial G_{\hat{k},h}}{\partial \nu}(x,y) - G_{\hat{k},h}(x,y)\,\frac{\partial u^{(2)}}{\partial \nu}(y)\right] ds(y)\,. \end{aligned}$$

When R tends to infinity the integrals over the vertical components of $\partial Q^{R,H}$ tend to zero because $u^{(2)}$ and its derivatives are bounded and $G_{\hat{k},h}$ and $\partial G_{\hat{k},h}/\partial x_1$ tend to zero uniformly in every layer of finite height. Thus,

$$\begin{aligned} u^{(2)}(x) &= -\int_{Q^{\infty,H}} (\Delta+\hat{k}^2)u^{(2)}(y)\, G_{\hat{k},h}(x,y)\,dy \;+\; \int_{\Gamma_h} u^{(2)}(y)\,\frac{\partial G_{\hat{k},h}}{\partial y_2}(x,y)\,ds(y) \\ &\quad - \int_{\Gamma_H} \left[u^{(2)}(y)\,\frac{\partial G_{\hat{k},h}}{\partial y_2}(x,y) - G_{\hat{k},h}(x,y)\,\frac{\partial u^{(2)}}{\partial y_2}(y)\right] ds(y)\,. \end{aligned}$$

Now we let H tend to infinity. The last integral tends to zero because of the estimate

$$\left| G_{\hat{k},h}(x,y)\,\frac{\partial u^{(2)}}{\partial y_2}(y)\right| \;\le\; \frac{c\,x_2}{1+|x|^{3/2}}\,e^{-\delta x_2}$$

for some $c,\delta>0$ and the same for $G_{\hat{k},h}$ and $u^{(2)}$ interchanged. In consequence,

$$u^{(2)}(x) \;=\; -\int_{x_2>h} (\Delta+\hat{k}^2)u^{(2)}(y)\, G_{\hat{k},h}(x,y)\,dy \;+\; \int_{\Gamma_h} u^{(2)}(y)\,\frac{\partial G_{\hat{k},h}}{\partial y_2}(x,y)\,ds(y)\,. \tag{47}$$

Substituting this into (46) yields

$$\begin{aligned} u^{(1)}(x) &= -u^{(2)}(x) \;+\; \int_{\Gamma_h} u^{(2)}(y)\,\frac{\partial G_{\hat{k},h}}{\partial x_2}(x,y)\,ds(y) \;+\; 2\int_{\Gamma_h} u^{(1)}(y)\,\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\,ds(y) \\ &= -u^{(2)}(x) \;+\; 2\int_{\Gamma_h} \left[u^{(1)}(y)+u^{(2)}(y)\right]\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\,ds(y)\,, \quad x_2>h\,, \end{aligned}$$

where we again exploited that $\partial G_{\hat{k},h}/\partial y_2(\cdot,y)$ equals $2\partial\Phi_{\hat{k}}/\partial y_2(\cdot,y)$ for $y\in\Gamma_h$. This proves the form (5). Note that this implies also that $u(\cdot,h)\in L^\infty(\mathbb{R})$. □

We have just seen that the limiting absorption solution satisfies the following radiation condition.

Definition 6.7. *(Radiation Condition)*
Let $\{\hat{\alpha}_j : j\in J\}$ be the (possibly empty) set of exceptional values for wave number $\hat{k}>0$ and let Assumptions 4.1 and 5.3 hold (the latter for all $\hat{\alpha}_j$, $j\in J$). For every $j\in J$ let $\{\tilde{\phi}_{\ell,j} : \ell=1,\dots,m_j\}$ be the eigenfunctions of Lemma 5.6 for $\hat{\alpha}=\hat{\alpha}_j$ and define the sets $L_j^\pm$ by

$$L_j^\pm \;:=\; \left\{\ell\in\{1,\dots,m_j\} : d_{\ell,j} \gtrless 0\right\} \;=\; \left\{\ell\in\{1,\dots,m_j\} : \mathrm{Im}\int_{Q^\infty} \frac{\partial\tilde{\phi}_{\ell,j}}{\partial x_1}\,\overline{\tilde{\phi}_{\ell,j}}\,dx \gtrless 0\right\}.$$

Then the field u has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)} \in H^1\big(\mathbb{R}\times(0,H)\big)$ for all $H > h$ and $u^{(2)}$ has the form

$$u^{(2)}(x) \;=\; \psi^+(x_1)\sum_{j\in J}\sum_{\ell\in L_j^+} a_{\ell,j}^+\,\tilde\phi_{\ell,j}(x) \;+\; \psi^-(x_1)\sum_{j\in J}\sum_{\ell\in L_j^-} a_{\ell,j}^-\,\tilde\phi_{\ell,j}(x), \quad x\in\mathbb{R}^2_+,$$

for some $a_{\ell,j}^\pm \in \mathbb{C}$ where $\psi^\pm$ are given by (36). Furthermore, u satisfies the upward propagating radiation condition (5).

This radiation condition in turn also provides uniqueness of a limiting absorption solution.

Theorem 6.8. *For every $f \in L^2(\mathbb{R}^2_+)$ with compact support in W there exists a unique solution u of (3) which vanishes on $\mathbb{R}\times\{0\}$ and satisfies the radiation condition of Definition 6.7.*

Proof: Existence has been shown in Theorem 6.6. To show uniqueness let $f = 0$ and set $u_j^\pm = \sum_{\ell\in L_j^\mp} a_{\ell,j}^\pm\,\tilde\phi_{\ell,j}$. Substituting the representation $u = u^{(1)} + u^{(2)}$ from the radiation condition into the differential equation $\Delta u + \hat k^2 q u = 0$ yields

$$\begin{aligned}\Delta u^{(1)}(x) + \hat k^2 q(x)\,u^{(1)}(x) &= -\big[\Delta u^{(2)}(x) + \hat k^2 q(x)\,u^{(2)}(x)\big]\\ &= -\sum_{j\in J}\left[u_j^+(x)\,\frac{d^2\psi^+(x_1)}{dx_1^2} \;+\; 2\,\frac{d\psi^+(x_1)}{dx_1}\,\frac{\partial u_j^+(x)}{\partial x_1}\right]\\ &\quad -\sum_{j\in J}\left[u_j^-(x)\,\frac{d^2\psi^-(x_1)}{dx_1^2} \;+\; 2\,\frac{d\psi^-(x_1)}{dx_1}\,\frac{\partial u_j^-(x)}{\partial x_1}\right].\end{aligned} \tag{48}$$

We set $\varphi^\pm = d\psi^\pm/dx_1$ and note that the right-hand side is in $L^2\big(\mathbb{R}\times(0,H)\big)$ for every $H > h$. We take the quasi-periodic Bloch transform

$$(T_{qp}v)(x,\alpha) \;=\; \hat v(x,\alpha) \;=\; \sum_{n\in\mathbb{Z}} v(x + 2\pi n e^{(1)})\,e^{-2\pi i n\alpha}$$

to both sides and note that, for any $\varphi \in L^2(\mathbb{R})$,

$$T_{qp}(u_j^\pm\,\varphi)(x,\alpha) \;=\; u_j^\pm(x)\sum_{n\in\mathbb{Z}}\varphi(x_1 + 2\pi n)\,e^{2\pi n(\hat\alpha_j-\alpha)i} \;=\; u_j^\pm(x)\,\hat\varphi(x_1,\alpha-\hat\alpha_j)\,.$$

For $x \in (0,2\pi)\times(0,\infty)$ and $\alpha \in (-1/2, 1/2]$, this yields

$$\begin{aligned}&\Delta\hat u^{(1)}(x,\alpha) + \hat k^2 q(x)\hat u^{(1)}(x,\alpha) \;=\\ &-\;\sum_{j\in J}\left[u_j^+(x)\,\frac{\partial\hat\varphi^+(x_1,\alpha-\hat\alpha_j)}{\partial x_1} \;+\; 2\,\hat\varphi^+(x_1,\alpha-\hat\alpha_j)\,\frac{\partial u_j^+(x)}{\partial x_1}\right]\\ &-\;\sum_{j\in J}\left[u_j^-(x)\,\frac{\partial\hat\varphi^-(x_1,\alpha-\hat\alpha_j)}{\partial x_1} \;+\; 2\,\hat\varphi^-(x_1,\alpha-\hat\alpha_j)\,\frac{\partial u_j^-(x)}{\partial x_1}\right].\end{aligned}$$

From the inversion formula for the quasi-periodic Floquet-Bloch transform we directly compute the Floquet-Bloch transform of $\varphi^\pm$,

$$\frac{1}{\pi}\int_{-1/2}^{1/2} e^{i\alpha x_1}d\alpha \;=\; \frac{\sin(x_1/2)}{\pi\,x_1/2} \;=\; \pm\varphi^\pm(x_1),$$

such that $\hat{\varphi}^{\pm}(x_1,\alpha) = \pm\, \exp(i\alpha x_1)/\pi$. Therefore,

$$
\begin{aligned}
\Delta \hat{u}^{(1)}(x,\alpha) + \hat{k}^2 q(x) \hat{u}^{(1)}(x,\alpha) \;&=\\
-\;& \frac{1}{\pi}\sum_{j\in J}\left[u_j^+(x)\, i\,(\alpha-\hat{\alpha}_j) \;+\; 2\,\frac{\partial u_j^+(x)}{\partial x_1}\right] e^{i(\alpha-\hat{\alpha}_j)x_1}\\
+\;& \frac{1}{\pi}\sum_{j\in J}\left[u_j^-(x)\, i\,(\alpha-\hat{\alpha}_j) \;+\; 2\,\frac{\partial u_j^-(x)}{\partial x_1}\right] e^{i(\alpha-\hat{\alpha}_j)x_1} \qquad (49)\\
=\;& -\big[\Delta w_1(x,\alpha) + \hat{k}^2 q(x) w_1(x,\alpha)\big]\,, \quad x\in Q^\infty,\; \alpha\notin\{\hat{\alpha}_j : j\in J\}\,,
\end{aligned}
$$

where

$$
w_1(x,\alpha) \;:=\; \frac{1}{\pi}\sum_{j\in J}\big[u_j^+(x) - u_j^-(x)\big]\,\frac{1}{i(\alpha-\hat{\alpha}_j)}\,e^{i(\alpha-\hat{\alpha}_j)x_1}\,, \quad x\in Q^\infty,\; \alpha\notin\{\hat{\alpha}_j : j\in J\}\,.
$$

Now we set $w = \hat{u}^{(1)} + w_1$ in Q^∞ for $\alpha\in(-1/2,1/2]$, $\alpha\notin\big\{\hat{\alpha}_j : j\in J\big\}$. Then we observe that $w(\cdot,\alpha)$ is $\alpha-$quasi-periodic and $\Delta w(\cdot,\alpha) + \hat{k}^2 q\, w(\cdot,\alpha) = 0$ in Q^∞ for $\alpha\in(-1/2,1/2]$, $\alpha\notin\big\{\hat{\alpha}_j : j\in J\big\}$ and vanishes for $x_2=0$.
Next we show that w has a Rayleigh expansion for $x_2>h$. First we rewrite (5) via (47) in the form

$$
\begin{aligned}
u^{(1)}(x) \;&=\; -u^{(2)}(x) \;+\; \int_{\Gamma_h} u^{(2)}(y)\,\frac{\partial G_{\hat{k},h}}{\partial y_2}(x,y)\,ds(y) \;+\; 2\int_{\Gamma_h} u^{(1}(y)\,\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\,ds(y)\\
&=\; \int_{x_2>h} (\Delta+\hat{k}^2)u^{(2)}(y)\,G_{\hat{k},h}(x,y)\,dy \;+\; 2\int_{\Gamma_h} u^{(1)}(y)\,\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\,ds(y)\\
&=\; -\int_{x_2>h} \sigma_0(y)\,G_{\hat{k},h}(x,y)\,dy \;+\; 2\int_{\Gamma_h} \xi_0(y)\,\frac{\partial \Phi_{\hat{k}}}{\partial y_2}(x,y)\,ds(y)) \qquad (50)
\end{aligned}
$$

with $\xi_0 = u^{(1)}\big|_{\Gamma_h} \in H^{1/2}(\Gamma_h)$ and $\sigma_0 = -(\Delta+\hat{k}^2)u^{(2)} \in L^2\big(\mathbb{R}\times(h,H)\big)$ for any $H>h$. (The latter fact follows as in the first part of the proof of Theorem 6.6.) To compute the Bloch transform of this equation we first compute the Bloch transform of a convolution:

Let $F \in L^1(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R})$:

$$\begin{aligned}
T_{qp}(F * \varphi)(t,\alpha) &= \sum_{n\in\mathbb{Z}} e^{-2\pi i n\alpha} \int_{-\infty}^{\infty} F(t+2\pi n - s)\,\varphi(s)\,ds \\
&= \sum_{n\in\mathbb{Z}} e^{-2\pi i n\alpha} \int_{-\infty}^{\infty} F(t-s)\,\varphi(s+2\pi n)\,ds \\
&= \int_{-\infty}^{\infty} F(t-s)\,\hat{\varphi}(s,\alpha)\,ds \;=\; \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} F(t-s)\,\hat{\varphi}(s,\alpha)\,ds \\
&= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} F(t-s-2\pi n)\,\hat{\varphi}(s+2\pi n,\alpha)\,ds \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n\alpha} \int_0^{2\pi} F(t-s-2\pi n)\,\hat{\varphi}(s,\alpha)\,ds \\
&= \sum_{n\in\mathbb{Z}} e^{-2\pi i n\alpha} \int_0^{2\pi} F(t+2\pi n-s)\,\hat{\varphi}(s,\alpha)\,ds \\
&= \int_0^{2\pi} \hat{F}(t-s,\alpha)\,\hat{\varphi}(s,\alpha)\,ds\,.
\end{aligned}$$

This formula extends to $\varphi \in L^2(\mathbb{R})$ by density. Now we take the Bloch transform of the representation (50). Since the integrals in (50) are convolutions, we get that

$$\hat{u}^{(1)}(x,\alpha) \;=\; -\int_{\hat{Q}_h^\infty} \hat{\sigma}_0(y,\alpha)\,\hat{G}_{\hat{k},h}(x,y,\alpha)\,dy \;+\; 2\int_{\hat{\Gamma}_h} \hat{\xi}_0(y,\alpha)\,\frac{\partial\hat{\Phi}_{\hat{k}}}{\partial y_2}(x,y,\alpha)\,ds(y) \tag{51}$$

where $\hat{G}_{\hat{k},h}$ and $\hat{\Phi}_{\hat{k}}$ are $\alpha-$quasi-periodic Green's functions (i.e., quasi-periodic with respect to x_1 and y_1), $\hat{Q}_h^\infty := (0,2\pi)\times(h,\infty)$, and again $\hat{\Gamma}_h = (0,2\pi)\times\{h\}$. As (51) shows that $\hat{\sigma}_0(\cdot,\alpha) = (\Delta+\hat{k}^2)\hat{u}^{(1)}(\cdot,\alpha) = -(\Delta+\hat{k}^2)w_1(\cdot,\alpha)$ for $\alpha \notin \{\hat{\alpha}_j : j\in J\}$ and thus by Green's representation theorem

$$\begin{aligned}
\hat{u}^{(1)}(x,\alpha) &= 2\int_{\hat{\Gamma}_h} \hat{\xi}_0(y,\alpha)\,\frac{\partial\hat{\Phi}_{\hat{k}}}{\partial y_2}(x,y,\alpha)\,ds(y) + \int_{\hat{Q}_h^\infty} (\Delta+\hat{k}^2)w_1(y,\alpha)\,\hat{G}_{\hat{k},h}(x,y,\alpha)\,dy \\
&= -w_1(x,\alpha) \;+\; \int_{\hat{\Gamma}_h} \left[2\,\hat{\xi}_0(y,\alpha)\,\frac{\partial\hat{\Phi}_{\hat{k}}}{\partial y_2}(x,y,\alpha) - w_1(y,\alpha)\,\frac{\partial\hat{G}_{\hat{k},h}}{\partial y_2}(x,y,\alpha)\right] ds(y) \\
&= -w_1(x,\alpha) \;+\; 2\int_{\hat{\Gamma}_h} \left[\hat{\xi}_0(y,\alpha) - w_1(y,\alpha)\right] \frac{\partial\hat{\Phi}_{\hat{k}}}{\partial y_2}(x,y,\alpha)\,ds(y)\,.
\end{aligned}$$

The quasi-periodic Green's function $\hat{\Phi}_{\hat{k}}$ can be written via its a Rayleigh expansion

$$\hat{\Phi}_{\hat{k}}(x,y,\alpha) \;=\; \frac{i}{4\pi\hat{k}} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{\hat{k}^2-(n+\alpha)^2}}\, e^{i(n+\alpha)(x_1-y_1)}\, e^{i\sqrt{\hat{k}^2-(n+\alpha)^2}|x_2-y_2|}$$

for $x-y \notin \{2\pi n e^{(1)} : n\in\mathbb{Z}\}$ provided $\hat{k} \neq |n+\alpha|$ for any $n\in\mathbb{Z}$, see, e.g. [12] or [2]. For the derivative with respect to x_2 this latter assumption can be removed by a continuity argument. This shows that $w(\cdot,\alpha) = \hat{u}^{(1)}(\cdot,\alpha) + w_1(\cdot,\alpha)$ is $\alpha-$quasi-periodic, satisfies $(\Delta + \hat{k}^2 q)w(\cdot,\alpha) = 0$ for $x_2 > h$, the homogeneous boundary condition $w(x,\alpha) = 0$ for $x_2 = 0$, and a Rayleigh expansion for $x_2 > h$ for every $\alpha \notin \{\hat{\alpha}_j : j \in J\}$. The trivial uniqueness result for the $\alpha-$quasi-periodic scattering problem at these non-exceptional wave numbers implies that $w(\cdot,\alpha)$ vanishes in Q for all these α. Thus,

$$\sum_{j\in J} \big[u_j^+(x) - u_j^-(x)\big] \frac{1}{i(\alpha - \hat{\alpha}_j)} e^{i(\alpha-\hat{\alpha}_j)x_1} \;=\; -\hat{u}^{(1)}(x,\alpha)\,, \quad x \in (0,2\pi)\times(0,H)\,,$$

for all $\alpha \in (-1/2, 1/2]$ with $\alpha \notin \{\hat{\alpha}_j : j \in J\}$. The right-hand side is in $L^2\big((0,2\pi)\times(0,H)\times(-1/2,1/2)\big)$ but the left-hand side is not unless $u_j^- = u_j^+$ in Q for all $j \in J$. Since u_j^+ and u_j^- are disjoint combinations of the basis functions $\tilde{\phi}_{\ell,j}$ we conclude that $u_j^\pm$ vanishes for all $j \in J$. Therefore, also $u^{(1)}$ vanishes which proves uniqueness since the coefficients for $u^{(2)}$ can be computed from $u^{(1)}$ using (48). □

Remarks 6.9. *(a) The form of the radiation condition is justified by the limiting absorption principle of Theorem 6.6. Accepting this radiation, existence of a solution can be shown more directly as through the limiting absorption principle. We sketch this approach but refer to a subsequent paper where we will carry out this in detail. One makes an ansatz of the solution in the form* $u = u^{(1)} + u^{(2)}$ *as in the radiation condition of Definition 6.7 and arrives at the inhomogeneous form of (48) (including the source* f*) and, analogously, the inhomogeneous form of (49). In order that the latter equation has a solution for* $\hat{\alpha} = \hat{\alpha}_j$ *for some fixed* $j \in J$ *the right hand side of (49) has to be orthogonal to* $\tilde{\phi}_{\ell,j}$ *for all* ℓ*. This leads to a linear system for the coefficients* $a_{\ell,j}^\pm$ *which is uniquely solvable. Then one has to show that the mapping* $\alpha \mapsto u^{(1)}(\cdot,\alpha)$ *is continuous.*
(b) Our method of proving the limiting absorption principle carries over without difficulty to the case of a closed *waveguide; that is the problem to find* u *such that*

$$\Delta u + k^2 q u = f \text{ in } W\,, \quad u = 0 \text{ on } \partial W\,,$$

where $f \in L^2(W)$ *is a given function of compact support in* W*. Also the case of a completely open layer can be treated analogously, one just has to treat the lower half-space* $\mathbb{R}^2_{x_2<0}$ *in the same way as the half-space* $\mathbb{R}^2_{x_2>h}$*.*

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