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Analysis of Restricted Branching Programs Computational Complexity

Specialization: 01.01.09 — Discrete Math and Mathematical Cybernetic

Abstract

Candidate of sciences dissertation

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General description of work

The Relevance of Research. Analysis and classification of algorithms according to computational complexity is one of important points for understanding of problem and its algorithmic solution. Base technique for that analysis is complexity lower bounds.

The P and NP classes compare problem is one of known and important problems in computational complexity theory. In the same time there are many open problems on structure of P class, particularly on known $LSPACE$ and NC classes. These classes are subclasses of P. In the second half of XX century scientists investigated computational models with different restrictions, which allows to solve some of open complexity problems. One of such model is *Branching program*. There are known results for the model, like $LSPACE/poly = BP$, where $LSPACE/poly$ is nonuniform version of $LSPACE$ and BP is class of Boolean functions, which are computed by polynomial size Branching programs. Another important known result is $NC^1 = BP_{const}$, where NC^1 is class of Boolean functions which are computed by logarithmic-depth circuit ($NC^1 \subseteq NC$), and BP_{const} is class of Boolean functions which are computed by polynomial size constant width Branching programs.

Since 90s scientists have been exploring Boolean functions from BP_{const}, according to it many models of Branching programs have been investigated. Complexity lower bounds and complexity classes hierarchies was proved for such models.

Many scientists have explored read k times Branching programs. Borodin and others [3] got lower bound for read k times nondeterministic Branching program (k-NBP), which is explicit function. It shows that for small k program requires polynomial size. Authors considered "syntactic" and "semantic" models. In first case model reads variables k times for any path from initial note to sink nodes, and in second one model has restriction only for computational paths. Lower bound was proven for "syntactic" model.

- Main idea of technique, which is used in [3], is analysis of special case for boolean formulas. Method is based on representation of computation process in functional form and its analyzing.

In 1997 Okol'nishnikova proved [5] lower bounds for k-NBP. She showed that explicit Boolean function $f_{k \ln k/2+C}$ required exponentiation size k-NBP. Using this function Author proved hierarchy for classes of boolean functions computed by polynomial size k-NBP: NP-kBP $\subsetneq$ NP-$(k \ln k/2 + C)$BP, for $k = o(\sqrt{\ln n}/\ln\ln n)$.

- Main idea of technique, that Okol'nishnikova used in [5], is similar to technique from [3]. Method is based on representation of computation process in functional form and it's analyzing.

Thathachar improved Okol'nishnikova's lower bound in 1998 [6]. He showed that for computing of explicit Boolean function HSP_q^{k+1} nondeterministic k-BP's should be at least $exp\{n^{1/k+1}2^{-2k}k^{-4}\}$. Using this result he proved following hierarchy: NP-$(k-1)$BP $\subsetneq$ NP-kBP. This result is more tight than Okol'nishnikova's hierarchy, but it is right for $k = o(\log\log n)$.

- Author used modification of method based on results from communication complexity.

In 2005, Ajtai proved lower bound for general model [1]. He showed that size of Branching program of length kn, for $k = const$, which computes Boolean Function $N_+(X_\eta)$ is atleast $2^{n\varepsilon}$, for $\varepsilon > 0$.

- Author used both of approaches: functional and communication.

Probabilistic k-BP was investigated by Hromkovich and Sauerhoff in 2003 [4]. They proved lower bound for explicit Boolean function $m\text{-}Masked\text{-}PJ_{k,n}$. Authors showed that bounded error probabilistic k-BP should have a size at least $2^{\Omega(N^\alpha/k^3)}$, for $\alpha = 1/(1 + 2log3)$. Using that results Hromkovich and Sauerhoff got hierarchy for polynomial size bounded error probabilistic k-BP: BPP-$(k-1)$BP $\subsetneq$ BPP-kBP, for $k \leq \log n/3$.

- The technique that is used in this paper is similar to Thathachar's work [6].

Scientists explore different restrictions for Branching programs. One of the most popular is "read once" and "ordered" (variables are always read according to fixed order). That Branching program is called *Ordered Binary Decision Diagrams* (OBDD). Read-k-times OBDD (k-OBDD) is generalization for OBDD. This model has practical applications. OBDDs are extensively used in CAD software to synthesize circuits (logic synthesis), in formal verification, in analysis of streaming algorithms. The proof of lower bounds and width and k hierarchies is important question in analysis of such algorithms and programs.

Note, that k-OBDD is "syntactic" model.

Bolling, Sauerhoff, Sieling and Wegener proved lower bounds for Boolean function PJ_k (*Pointer Jumping*) for k-OBDD in 1998 [2]. They proved that $(k-1)$-OBDD, which computes PJ_k, has size at least $2^{\Omega(n^{1/2}/k)}$. Using this results authors proved hierarchy for polynomial size k-OBDD: P-$(k-1)$OBDD $\subsetneq$ P-kOBDD, for $k = o(n^{1/2}log^{3/2}n)$.

- Authors used modification of method based on results from communication complexity.

The main goal of this work is exploring of lower bounds and extension of hierarchies for k-OBDD complexity classes.

Goal of the research. Developing methods of Lower bounds proof techniques for different models of Branching program and non-uniform head-position-depends on two way automata. Proving hierarchies for k-OBDD complexity classes using the lower bounds and extension existing ones.

Researching methods. In the dissertation discrete math, computer science, probability theory and number theory methods are used.

Scientific novelty. In the dissertation two base techniques are discovered: "k-OBDD communication simulating" and "k-OBDD functional description". These two methods complement each other. Following results are proven:

1. "k-OBDD communication simulating" is technique, which based on k-OBDD representation as special communication protocol. It allows to prove new bounds for number of subfunction $N(f)$ of Boolean function, which computed by k-OBDD.

   - This techniques allows us to prove new lower bounds for complexity of Boolean function computed by deterministic, nondeterministic and probabilistic k-OBDD (k-OBDD, k-NOBDD, k-POBDD).
   - Using that lower bounds for $N(f)$, hierarchies for deterministic, nondeterministic and probabilistic k-OBDD was extended for constant, polylogorithmic, sublinear width.
   - Using analysis of $N(f)$ for specific explicit Boolean function f, I get lower bounds for deterministic and nondeterministic OBDD, which computes f. It allows to prove width hierarchy for OBDD and NOBDD and compare classes for different models.
   - Using same method width hierarchies for deterministic, nondeterministic and probabilistic k-OBDD were proven.
   - The "k-OBDD communication simulating" allows to prove lower bounds for nonuniform head-depends two way deterministic and nondeterministic automata.
   - Using these lower bounds I prove size hierarchy results for nonuniform head-depends two way deterministic and nondeterministic automata.

   The "k-OBDD communication simulating" method has following restrictions: $kw \log w < n$ for deterministic case and $kw^2 < n$ for deterministic and probabilistic cases, where w is width of k-OBDD. It means we can consider only models with sublinear width.

2. “k-OBDD functional description” is method which based on functional representation of k-OBDD computing process. Firstly, the method allows to simulate k-OBDD using nondeterministic 1-OBDD. Secondly, it allows to represent Boolean function which computed by k-OBDD (k-NOBDD) in special boolean formula.
    - Using this method, I proved lower bound for Boolean functions, which are computed by deterministic and nondeterministic k-OBDD.
    - Basing the lower bounds, existent hierarchies for polynomial width deterministic and nondeterministic k-OBDDs were extended and new hierarchies for superpolynomial and subexponential width deterministic and nondeterministic k-OBDDs were proven.

The “k-OBDD communication simulating” technique is related with techniques from papers [2], [4], [6]. The “k-OBDD functional description” technique is related with techniques from papers [3], [5].

Theoretical and practical significance. Dissertation is theoretical research, where it is explored deterministic, nondeterministic and probabilistic models complexity. The investigated techniques can be used for different models and algorithms analysis. vspace0.25cm

Results Presentations. Dissertation results was presented in Russian and international conferences and workshops: X international workshop “Discrete Math and Applications” (Moscow, 2010), XI international workshop “Discrete Math and Applications” (Moscow, 2012), XVII international conference “Problems of Theoretical Cybernetic”. (Kazan, 2014), “6th Workshop on Non-Classical Models of Automata and Applications NCMA 2014” (Germany, Kassel, 2014), “16th International Workshop on Descriptional Complexity of Formal Systems” (Finland, Turku, 2014), Kazan Federal University conferences, Kazan Federal University quantum and classic informatics workshops.

Publications. There are 10 papers published on dissertation research work, including 3 in Higher Attestation Commission's journals list.

Structure of dissertation. Dissertation contains introduction, four chapters, conclusion and references list. Value of dissertation is 135 pages.

The Main Content of the Dissertation

"k-OBDD communication simulating" method.

"k-OBDD communication simulating" method is discussed in Chapter 1. This techniques allows us represent k-OBDD as special communication protocol and to prove new lower bounds for complexity of Boolean function, which computed by deterministic, nondeterministic and probabilistic k-OBDD (k-OBDD, k-NOBDD, k-POBDD). This lower bounds are formulated in terms of inequality for number of subfunctions $N(f)$.

Let us discuss definition of number of subfunctions. Let $\pi = (X_A, X_B)$ be a partition of the set X into two parts X_A and $X_B = X \backslash X_A$. Below we will use equivalent notations $f(X)$ and $f(X_A, X_B)$. Let $f|_\rho$ be subfunction of f, where ρ is mapping $\rho : X_A \to \{0,1\}^{|X_A|}$. Function $f|_\rho$ is obtained from f by applying ρ. We denote $N^\pi(f)$ to be amount of different subfunctions with respect to partition π. Let $\Theta(n)$ be the set of all permutations of $\{1, \dots, n\}$. We say, that partition π agrees with permutation $\theta = (j_1, \dots, j_n) \in \Theta(n)$, if for some u, $1 < u < n$ the following is right: $\pi = (\{x_{j_1}, \dots, x_{j_u}\}, \{x_{j_{u+1}}, \dots, x_{j_n}\})$. We denote $\Pi(\theta)$ a set of all partitions which agrees with θ. Let $N^\theta(f) = \max_{\pi \in \Pi(\theta)} N^\pi(f)$, $\qquad N(f) = \min_{\theta \in \Theta(n)} N^\theta(f)$.

Theorem 1.4.1 Let Boolean function $f(X)$ is computed by k-OBDD P of width w. Then following statement is right: $N(f) \leq w^{(k-1)w+1}$.

Theorem 1.4.7 Let Boolean function $f(X)$ is computed by k-NOBDD P of width w. Then following statement is right: $N(f) \leq 2^{w\left((k-1)w+1\right)}$.

Theorem 1.4.10 Let Boolean function $f(X)$ is computed by k-POBDD P of width w. Then following statement is right: $N(f) \leq \left(\frac{5(\log_2 w+1+\log_2 k-0.2\log_2 \delta)}{\log_2 \sqrt[2k-1]{\frac{0.5+\delta}{0.5+\delta.2}}}\right)^{(k+1)w^2}$.

Theorem 1.4.1 is published in paper 2 and 5 from list of publications, Theorems 1.4.7 and 1.4.10 are published in paper 7 from list of publications.

Limitations of method. Note that number of subfunctions $N(f)$ cannot greats number of different inputs by definition, hence we have following restrictions: $((k-1)w+1)\log_2 w < n$ for deterministic model, $w\big((k-1)w+1\big) < n$ for nondeterministic model and $(k+1)w^2(\log_2(\log_2 k + log_2 w)) = o(n)$ for probabilistic model.

Therefore the the results from previous theorems extend lower bounds of Bollings and others for $w < n^{1/2}$, lower bounds of Okol'nishnikova, Thathachar, and Hromkovich and Sauerhoff for $w = o(n/(\log n)^2)$. We get better extension for constant w.

Hierarchies for k-OBDD. Using that lower bounds for $N(f)$, hierarchies for deterministic, nondeterministic and probabilistic k-OBDD was extended.

Let set C be one of the following sets. *poly* $= \{w : w$ is polynomial$, w > n^2\}$. It means that k-OBDD (k-NOBDD) P has polynomial size. *superpoly* $= \{w : w = O(n^{log^\alpha n})\}$, $\alpha > 0$ and it means k-OBDD (k-NOBDD) P has super polynomial size. *subexp* $= \{w : w = O(2^{n^\alpha})\}$, $0 < \alpha < 0.5$, and it means k-OBDD (k-NOBDD) P has subexponential size. Let $\mathbf{k-OBDD_C}$ and $\mathbf{k-NOBDD_C}$ be the set of Boolean functions that have representation as k-OBDD and k-NOBDD with width $w \in C$, respectively.

Following results was proven, using lower bounds from Theorems 1.4.1, 1.4.7, 1.4.10:

Corollaries 1.4.6, 1.4.9, 1.4.12 Following statements are right:

$$\Big(\mathbf{k/\log_2\log_2 n}\Big)\mathbf{-OBDD_{const}} \subsetneq \mathbf{k-OBDD_{const}},$$

$$\Big(\mathbf{k/\log_2\log_2 n}\Big)\mathbf{-NOBDD_{const}} \subsetneq \mathbf{k-NOBDD_{const}},$$

$$\left(\mathbf{k/(\log_2 n \log_2 \log_2 n)}\right)\mathbf{-POBDD_{const}} \subsetneq \mathbf{k-POBDD_{const}},$$

for $k = o(n/\log_2 n)$.

$$\left(\mathbf{k/n^{\varepsilon}}\right)\mathbf{-OBDD_{polylog}} \subsetneq \mathbf{k-OBDD_{polylog}},$$

$$\left(\mathbf{k/n^{\varepsilon}}\right)\mathbf{-NOBDD_{polylog}} \subsetneq \mathbf{k-NOBDD_{polylog}},$$

$$\left(\mathbf{k/n^{\varepsilon}}\right)\mathbf{-POBDD_{polylog}} \subsetneq \mathbf{k-POBDD_{polylog}},$$

for $\varepsilon > 0, k = o(n^{1-\varepsilon}), n^{\varepsilon} < k$.

$$\left(\mathbf{k/(n^{\alpha}(\log_2 n)^2)}\right)\mathbf{-OBDD_{sublinear_{\alpha}}} \subsetneq \mathbf{k-OBDD_{sublinear_{\alpha}}},$$

for $0 < \alpha < 0.5 - \varepsilon, \varepsilon > 0, k > n^{\alpha}(\log_2 n)^2, k = o(n^{1-\alpha}/\log_2 n)$,

$$\left(\mathbf{k/(n^{2\alpha}(\log_2 n)^2)}\right)\mathbf{-NOBDD_{sublinear_{\alpha}}} \subsetneq \mathbf{k-NOBDD_{sublinear_{\alpha}}},$$

for $0 < \alpha < 1/3 - \varepsilon, \varepsilon > 0, k > n^{2\alpha}(\log_2 n)^2, k = o(n^{1-\alpha}/\log_2 n)$.

$$\left(\mathbf{k/(n^{2\alpha}(\log_2 n)^3)}\right)\mathbf{-POBDD_{sublinear_{\alpha}}} \subsetneq \mathbf{k-POBDD_{sublinear_{\alpha}}},$$

for $0 < \alpha < 1/3 - \varepsilon, \varepsilon > 0, k > n^{2\alpha}(\log_2 n)^3, k = o(n^{1-\alpha}/\log_2 n)$.

Let us compare this results with existing ones. For deterministic k-OBDD we have extension of Bolling-Sauerhoff-Sieling-Wegener hierarchy [2] for width less than $n^{1/3}$. Better extension we have for constant width. For nondeterministic k-OBDD we have extension of Okol'nishnikova and Thathachar hierarchies [5,6] for width less than $n^{0.49}$. Better extension we have for constant width. Note, that Okol'nishnikova and Thathachar have results for general nondeterministic read-k-times Branching programs. For probabilistic k-OBDD we have extension of Hromkovich-Sauerhoff hierarchy [4] for width less than $n^{0.32}$. Better extension we also have for constant width. Note, that Hromkovich and Sauerhoff also have results for general probabilistic read-k-times Branching programs.

This results are published in paper 7 from the list of publications.

Hierarchies for OBDD and NOBDD. Lower bounds for explicit functions which are computed by 1-OBDD and 2-NOBDD are discussed in Chapter 2. This results allow to prove following width hierarchies:

Theorem 2.2.1 Let n, $d = d(n)$ are integer such that $16 \leq d \leq 2^{n/4}$, then following statements are right: $\mathbf{OBDD_{\lfloor d/8 \rfloor - 1}} \subsetneq \mathbf{OBDD_d}$ and $\mathbf{NOBDD_{\lfloor d/8 \rfloor - 1}} \subsetneq \mathbf{NOBDD_d}$.

For sublinear width $(1 < d \leq n/2)$ we have tight hierarchy

Theorem 2.1.1 Let n, $d = d(n)$ are integer such that $1 < d \leq n/2$, then following statements are right: $\mathbf{OBDD_{d-1}} \subsetneq \mathbf{OBDD_d}$ and $\mathbf{NOBDD_{d-1}} \subsetneq \mathbf{NOBDD_d}$.

And we can compare classes for different models:

Theorem 2.2.4 Let n, $d = d(n)$, and $d' = d'(n)$ are integer such that $d \leq 2^{n/4}$ and $O(\log_2^4(d+1)\log_2\log_2(d+1)) < d' < d/8 - 1$, then following statements are right: $\mathbf{NOBDD_{\lfloor \log_2(d) \rfloor}} \subsetneq \mathbf{OBDD_d}$, $\mathbf{OBDD_d}$ and $\mathbf{NOBDD_{d'}}$ are not comparable.

Theorem 2.1.4 Let n, $d = d(n)$, and $d' = d'(n)$ are integer such that $d \leq n/2$ and $O(\log_2^2 d \log_2\log_2 d) < d' \leq d - 1$, then following statements are right: $\mathbf{NOBDD_{\lfloor \log_2(d) \rfloor}} \subsetneq \mathbf{OBDD_d}$, $\mathbf{OBDD_d}$ and $\mathbf{NOBDD_{d'}}$ are not comparable.

Before this moment researchers have investigated exponential difference between models of Branching Program. Here we consider more tight hierarchy.

This results are published in papers 1 and 6 from the list of publications.

Width hierarchy for k-OBDD. Hierarchies for deterministic, nondeterministic and probabilistic k-OBDD was proved, using the lower bounds for k-OBDD

Thorems 2.3.1, 2.3.2, 2.3.3 Let $k = k(n), w = w(n)$ are integer such that $2kw(2w + \lceil \log_2 k \rceil + \lceil \log_2 2w \rceil) < n, k \geq 2, w > 20, 0 < \varepsilon < 0.5 - \delta, \delta > 0$, then following statements are right:

$$\mathbf{k-OBDD_{\lfloor w/16 \rfloor - 3}} \subsetneq \mathbf{k-OBDD_w}, \text{ for } w \geq 64$$

$$\mathbf{k-NOBDD_{\left\lfloor (w\log_2 w)^{1/2}/7 \right\rfloor}} \subsetneq \mathbf{k-NOBDD_w}, \text{ for } \left\lfloor (w\log_2 w)^{1/2}/7 \right\rfloor \geq 1$$

$$\mathbf{k-OBDD}_{\left\lfloor \frac{1}{18}(\frac{w\log_2 w}{\log_2 k})^{\varepsilon}\right\rfloor} \subsetneq \mathbf{k-POBDD_w}, \text{ for } \left\lfloor \frac{1}{18}(\frac{w\log_2 w}{\log_2 k})^{\varepsilon}\right\rfloor \geq 1$$

Before this moment researchers have investigated exponential difference between models too. Here we consider more tight hierarchy.

This results are published in papers 3, 6, 7 and 10 from the list of publications.

Hierarchy for nonuniform head-depends two way deterministic and nondeterministic automata. In Chapter 3 "k-OBDD communication simulating" method is used for nonuniform head-depends two way deterministic and nondeterministic automata. Proof of lower bounds for these models is based on this technique.

Note, that nonuniform head-depends two way automata is very close model to k-OBDD. Branching programs with different restrictions are some times called nonuniform automata.

We have following differences from classical two way automaton. The model is nonuniform and we consider family of automatas for different n; the transition function depends not only on input symbol and state, but also on head position; number of state can depends on n, for n is length of input. Additionally we consider non deterministic counterpart for the automaton. We will denote automaton $2\mathrm{DA}_n$ for deterministic case and $2\mathrm{NA}_n$ for nondeterministic one.

For nonuniform case it is equivalent to say that $2\mathrm{DA}_n(2\mathrm{NA}_n)$ D_n recognize the language L_n and computes Boolean function f_n. In setuation f computing Boolean function we can shuffle variables and it does not change the function. In sense of that fact, we consider nonuniform head-depends two way θ-automaton ($2\mathrm{DA}_n^{\Theta}$), which read input variables according to permutation θ. Additionally we consider non deterministic counterpart for the automaton $2\mathrm{NA}_n^{\Theta}$.

Let $\mathbf{2DSIZE(d(n))}$, $\mathbf{2NSIZE(d(n))}$, $\mathbf{2D\Theta SIZE(d(n))}$, $\mathbf{2N\Theta SIZE(d(n))}$ be classes of Boolean function with computed by $2\mathrm{DA}_n$, $2\mathrm{NA}_n$, $2\mathrm{DA}_n^{\Theta}$ and $2\mathrm{NA}_n^{\Theta}$ of size $d(n)$ respectively.

The "k-OBDD communication simulating" technique allows us to prove following theorems: **Theorem 3.2.1** Let Boolean function $f(X)$ be computed by $2\mathrm{DA}_n^{\Theta}$ of size d. The following inequality is right: $N(f) \leq (d+1)^{d+1}$.

Theorem 3.3.1 Let Boolean function $f(X)$ be computed by $2NA_n^\Theta$ of size d. The following inequality is right: $N(f) \leq 2^{(d+1)^2}$.

We have following restrictions: for deterministic case it is $(d+1)\log(d+1) < n$ and for nondeterministic case it is $(d+1)^2 < n$. This limitation is right due $N(f) < 2^n$.

Following hierarchies were proven:

Theorems 3.2.3, 3.3.3 Let $d = d(n)$ is integer, then following statements are right:

$\mathbf{2D\Theta SIZE}\left(\lfloor(\mathbf{d}-\mathbf{4})/\mathbf{13}\rfloor-\mathbf{4}\right) \subsetneq \mathbf{2D\Theta SIZE}(\mathbf{d})$, for $(d+1)\log(d+1) < n$,

$\mathbf{2N\Theta SIZE}\left(\left\lfloor\sqrt{\lfloor(\mathbf{d}-\mathbf{4})/\mathbf{13}\rfloor-\mathbf{4}}\right\rfloor\right) \subsetneq \mathbf{2N\Theta SIZE}(\mathbf{d})$, for $(d+1)^2 < n$.

$\mathbf{2DSIZE}\left(\lfloor(\mathbf{d}-\mathbf{4})/\mathbf{13}\rfloor-\mathbf{4}\right) \subsetneq \mathbf{2DSIZE}(\mathbf{d})$

$\mathbf{2NSIZE}\left(\left\lfloor\sqrt{\lfloor(\mathbf{d}-\mathbf{4})/\mathbf{13}\rfloor-\mathbf{4}}\right\rfloor\right) \subsetneq \mathbf{2NSIZE}(\mathbf{d})$.

Also the following fact was proven: two pairs of models $2DA_n^\Theta$ and $2DA_n$; $2NA_n^\Theta$ and $2NA_n$ are not comparable.

Theorems 3.2.4, 3.3.4 Classes $\mathbf{2DSIZE}(\mathbf{d})$ and $\mathbf{2D\Theta SIZE}(\mathbf{d'})$ are not comparable for $d^2(n) < n/2-1$, $4 \leq d'(n) \leq \lfloor(d-4)/13\rfloor - 4$. Classrs $\mathbf{2NSIZE}(\mathbf{d})$ and $\mathbf{2N\Theta SIZE}(\mathbf{d'})$ are not comparable for: $d^2(n) < n/2-1$ and $4 \leq d'(n) \leq \left\lfloor\sqrt{\lfloor(d-4)/13\rfloor-4}\right\rfloor$.

This results are published in paper 9 from the list of publications.

“k-OBDD Functional Description” method.

Method from previous part use bound for number of subfunctions. This number is easy to solve, but cannot show structure of function enough good. This was the reason to develop another method, which based on ana lysis of boolean formula for function. This method allows to get lower bound for “large” (at least polynomial) width k-OBDD.

In Chapter 4 we consider functional approach for analysis of computation process in k-OBDD and k-NOBDD. Firstly, the method allows to simulate k-OBDD using nondeterministic 1-OBDD. Secondly, it allows to represent Boolean function which computed by k-OBDD (k-NOBDD) in following form:

$$f(X) = \bigvee_{i=1}^{w^{k-1}} \bigwedge_{j=1}^{k} g_{i,j}(X)$$

where $g_{i,j}$ can be computed by OBDD (NOBDD) of width w. We can see this result in following theorem:

Theorem 4.2.1 Let $f(X)$ be Boolean function, which is computed by $k-NOBDD\ P$ of width w and order θ. Then f can be computed by NOBDD P' of width w^{2k-1} and order θ.

Following Corollaries are on previous lower bound and extends known hierarchies for k-OBDD and k-NOBDD:

Corollaries 4.1.2, 4.1.3, 4.1.4, 4.1.6, 4.1.7, 4.1.8 Let us consider following sets for n is size of input: polynomials $poly = \{w : w = O(n^t)$, for some $t = const\}$, superpolynomials $superpoly_\alpha = \{w : w = O(n^{log^\alpha n})\}$ and subexponential function $subexp_\alpha = \{w : w = O(2^{n^\alpha})\}$. We have following hierarchies for these sets:

$$\mathbf{\lfloor k/log^2 n\rfloor - NOBDD_{poly} \subsetneq k - NOBDD_{poly}}$$

$$\mathbf{\lfloor k/log^2 n\rfloor - OBDD_{poly} \subsetneq k - OBDD_{poly}},$$

for $k = o(n/\log_2 n)$.

$$\mathbf{\lfloor k/log^{\alpha+2} n\rfloor - NOBDD_{superpoly_\alpha} \subsetneq k - NOBDD_{superpoly_\alpha}},$$

$$\mathbf{\lfloor k/log^{\alpha+2} n\rfloor - OBDD_{superpoly_\alpha} \subsetneq k - OBDD_{superpoly_\alpha}},$$

for $k = o(n/\log_2^{\alpha+1} n)$, $\alpha = const$, $\alpha > 0$.

$$\mathbf{\lfloor k/(n^\alpha \log_2^2 n)\rfloor - NOBDD_{subexp_\alpha} \subsetneq k - NOBDD_{subexp_\alpha}}$$

$$\mathbf{\lfloor k/(n^\alpha \log_2^2 n)\rfloor - OBDD_{subexp_\alpha} \subsetneq k - OBDD_{subexp_\alpha}}$$

for $k = o(\frac{2^{n^{1-\alpha}}}{n^\alpha \log_2 n})$, $0 < \alpha \leq 1 - \varepsilon, \varepsilon = const, \varepsilon > 0$.

The hierarchies for deterministic case extends Bolling-Sauerhoff-Sieling-Wegener hierarchy [2]. The hierarchy for nondeterministic case extends Okol'nishnikova and Thathachar hierarchies [5,6]. Note, that Okol'nishnikova and Thathachar have results for general nondeterministic read-k-times Branching programs.

Additionally note, that this method do not allow to prove more tight hierarchy.

This results are published in papers 4 and 8 from the list of publications.

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List of Publications

There are 10 papers published on dissertation research work, including 3 in Higher Attestation Commission's journals list. Papers [1], [2] and [3] were published in journals from Higher Attestation Commission's list.

1. **Ablayev F. Very Narrow Quantum OBDDs and Width Hierarchies for Classical OBDDs. / F. Ablayev, A. Gainutdinova, K. Khadiev, A. Yakaryilmaz. // Lecture Notes in Computer Science, Springer, 2014. - V. 8614. P. 53-64.**
2. **Ablayev F. Extension of the hierarchy for k-OBDDs of small width. / F. Ablayev, K. Khadiev. // Russian Mathematics, 2013. - V. 57 (3). - P. 46–50.**
3. **K. Khadiev. Width Hierarchy for k-OBDD of Small Width // Lobachevskii Journal of Mathematics, 2015- V. 36, - I. 2, - PP. 178-183**
4. Ablayev F.M. Extension of Hierarchy for Classes of Boolean Functions, which are computed by k-OBDD / F.M. Ablayev ,K.R. Khadiev // Short papers of XI international workshop "Discrete Math and Applications", Moscow, MSU, 2012.
5. Khadiev K.R. Extension of Hierarchy for stable k-OBDD // Short papers of X international workshop "Discrete Math and Applications", Moscow, MSU, 2010.
6. Khadiev K.R. Width Hierarchies for Classes of Boolean Functions, Which are Computed by Deterministic and Nondeterministic OBDDs// Short papaers of XVII international conference "Problems of Theoretical Cybernetic", PP.285-288, Kazan, 2014.
7. Khadiev K.R. Hierarchies for Classes of Boolean Functions, Which are Computed by Deterministic, Nondeterministic and Probabilistic k-OBDDs// Short papers of IX International conference "Discrete Models in Control System Theory", Moscow, 2015
8. Khadiev K.R. Extension of Hierarchy for Classes of k-OBDD computed Boolean Functions. // Short papaers of XVII international conference "Problems of Theoretical Cybernetic", PP.288-290, Kazan, 2014.

9. Khadiev K. New Size Hierachies for Two-way Non-uniform Automata. / K. Khadiev, A. Yakaryilmaz // Sixth Workshop on Non-Classical Models of Automata and Applications (NCMA 2014) Short Papers, 2014. - P. 13-18.

10. Khadiev K. Width Hierarchy for k-OBDD of Small Width. // Electronic Colloquium on Computational Complexity (ECCC), 2015. - V. 021.