Gain tuning and fidelity in continuous-variable quantum teleportation

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The fidelity of continuous-variable teleportation can be optimized by changing the gain in the modulation of the output field. We discuss the gain dependence of fidelity for coherent, vacuum, and one-photon inputs and propose optimal gain tuning strategies for corresponding input selections.

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I. INTRODUCTION

Continuous-variable quantum teleportation transfers unknown quantum states of a light field input from Alice (sender) to Bob (receiver) using squeezed state entanglement as a resource [1–4]. Since only finite squeezing is possible, the fidelity of this teleportation process is limited by nonmaximal entanglement [3,5,6]. However, it has been shown in Ref. [6] that the output of a single pure state teleportation always results in a pure state output conditioned by the classical information sent from Alice to Bob. Therefore, it may be possible that Bob can use this classical information to improve the fidelity if some information on the selection of possible input states is known.

In the continuous-variable quantum-teleportation experiment realized by Furusawa *et al.* [4], the classical information is a complex field amplitude β that is effectively added to the output field by a modulation process. The amplitude of this modulation process can be modified by a gain factor. In Ref. [4], the input state was a coherent state with an amplitude much larger than 1, such that the optimal fidelity was obtained at a gain of 1. However, Polkinghorne and Ralph have pointed out that a lower gain can be useful for teleporting photon entanglement [7]. Such considerations demonstrate that the optimal gain for the teleportation depends on the selection of possible input states.

In the following, the dependence of fidelity on gain is investigated for coherent states, for the vacuum state, and for a one-photon input. These results allow an optimization of fidelity for certain groups of input states, such as coherent states with constant amplitude and varying phase or qubits of zero or one photon.

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II. QUANTUM STATE TELEPORTATION WITH VARIABLE GAIN

Figure 1 shows the schematic sets of the quantum teleportation used in Ref. [4]. Alice transmits an unknown quantum state $|\psi\rangle_A$ to Bob. Alice and Bob share Einstein-Podolsky-Rosen (EPR) beams in advance. The quantum state of the EPR beams reads

$$|q\rangle_{R,B}=\sqrt{1-q^2}\sum_{n=0}^{\infty} q^n|n\rangle_R|n\rangle_B, \tag{1}$$

where R is the mode used by Alice as a quantum reference in the joint measurement of A and R [6], and B is the output mode on Bob's side. The degree of entanglement is given by the parameter q. The parameter q varies from 0 to 1, with $q=1$ for maximal entanglement and $q=0$ for no entanglement (vacuum in R and B). Experimentally, q is determined by the squeezing achieved in the entangled modes. If the

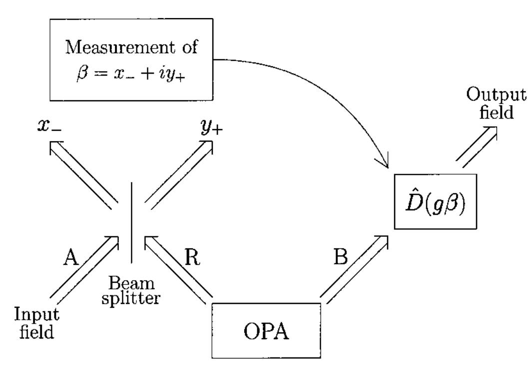

FIG. 1. Schematic representation of the quantum-teleportation setup.

variance of the squeezed quadratures is reduced by a factor of $\exp(-2r)$, the entanglement is given by $q=\tanh(r)$.

Alice mixes her input state with reference EPR beam using a 50% beam splitter and performs an entanglement measurement of the complex field value $\beta=x_-+iy_+$, where

$$\hat{x}_-=\hat{x}_A-\hat{x}_R,\qquad \hat{y}_+=\hat{y}_A+\hat{y}_R. \tag{2}$$

As has been shown previously [6,8], the output state of the teleportation process can then be obtained by applying a transfer operator to the input,

$$|\psi_{\text{out}}(\beta)\rangle_B=\hat{T}_q^g(\beta)|\psi\rangle_A. \tag{3}$$

Note that the output state is not normalized, since $\langle\psi_{\text{out}}(\beta)|\psi_{\text{out}}(\beta)\rangle$ is the probability of obtaining the measurement result β in the teleportation.

The transfer operator $\hat{T}_q^g(\beta)$ for variable gain g can be expressed using photon number states $|n\rangle$ and displacement operators $\hat{D}(\beta)$,

$$\hat{T}_q^g(\beta)=\sqrt{\frac{1-q^2}{\pi}}\sum_{n=0}^{\infty} q^n\hat{D}(g\beta)|n\rangle_{BA}\langle n|\hat{D}(-\beta). \tag{4}$$

This operator can now be applied to various input states. In the case of a coherent state, the output state is

$$\hat{T}_q^g(\beta)|\alpha\rangle=\sqrt{\frac{1-q^2}{\pi}}\, e^{-(1-q^2)|\alpha-\beta|^2/2}e^{1/2(1-qg)(\alpha\beta^*-\alpha^*\beta)} \times|q\alpha+(g-q)\beta\rangle. \tag{5}$$

This output state is also a coherent state with an amplitude given by a gain dependent superposition of α and β. Specifically, the gain factor affects the β component of the output amplitude. In the case of vacuum input state, the output state is

$$\hat{T}_q^g(\beta)|0\rangle=\sqrt{\frac{1-q^2}{\pi}}\, e^{-(1-q^2)|\beta|^2/2}|(g-q)\beta\rangle. \tag{6}$$

The vacuum is simply a coherent state with $\alpha=0$, so the output state is a coherent state with an amplitude proportional to β. The input vacuum state can be recovered by choosing $g=q$, effectively canceling the displacement [8]. In the case of a one-photon input state, the output state is

$$\hat{T}_q^g(\beta)|1\rangle=\sqrt{\frac{1-q^2}{\pi}}\, e^{-(1-q^2)|\beta|^2/2}\hat{D}[(g-q)\beta] \times[(1-q^2)\beta^*|0\rangle+q|1\rangle]. \tag{7}$$

This output state is a displaced quantum superposition of a vacuum component and a one-photon component. Even though this is also a pure state conditioned by β, it is not possible to recover the one-photon input state by varying the gain.

III. FIDELITY AND THE EFFECT OF GAIN TUNING

The success or failure of quantum teleportation can be characterized by the teleportation fidelity. It is defined by the overlap between the input state and the output state [5]. Using the results derived in the preceding section, we can obtain the gain factor dependence of teleportation fidelity for the different input states.

The total fidelity is obtained by averaging over all β, even though β is accessible classical information and the output state is really a pure state conditioned by β. In this sense, gain tuning is a method to optimize the use of the information β. In the case of a coherent-state teleportation, the fidelity is

$$\begin{aligned}F_q^\alpha(g)&=\int d^2\beta|\langle\alpha|\hat{T}_q^g(\beta)|\alpha\rangle|^2\\&=\int d^2\beta e^{-(1-q^2)|\alpha-\beta|^2}e^{-|(1-q)\alpha-(g-q)\beta|^2}\\&=\frac{1-q^2}{1-2qg+g^2}\exp\left[-\frac{1-q^2}{1-2qg+g^2}(1-g)^2|\alpha|^2\right].\end{aligned} \tag{8}$$

We note that maximal fidelity is always obtained at $g<1$. $g=1$ is optimal for $|\alpha|\to\infty$. Figure 2(a) shows the gain dependence of the teleportation fidelity for a coherent input state of amplitude $|\alpha|=1$ at different values of the entanglement parameter q. The peak of the gain dependent fidelity shifts to lower values of g and gets lower and broader as the entanglement q decreases. The peak is always at a gain value larger than $g=q$, but lower than $g=1$. Therefore, gain tuning to $g<1$ can improve the teleportation fidelity for coherent input states. However, the fidelity remains below 1 for all $q<1$. Moreover, the broadening of the peak indicates that the fidelity is less sensitive to gain tuning for low values of q. In the special case of the vacuum state ($\alpha=0$), the fidelity is

$$\begin{aligned}F_q^0(g)&=\int d^2\beta|\langle 0|\hat{T}_q^g(\beta)|0\rangle|^2\\&=\frac{1-q^2}{\pi}\int d^2\beta e^{-(1-2qg+g^2)|\beta|^2}\\&=\frac{1-q^2}{1-2qg+g^2}.\end{aligned} \tag{9}$$

Figure 2(b) shows the gain dependence of the teleportation fidelity of the vacuum state at several values of the entanglement parameter q. As in the general case of coherent states, the peak of the gain dependent fidelity shifts to lower gain values and gets broader as the entanglement parameter q decreases. However, the peak value of the fidelity is always 1 at $g=q$. The vacuum state is always teleported successfully at $g=q$ because this case corresponds to a simple attenuation at a beam splitter with reflectivity $1-q^2$ [7,8].

In the case of a one-photon state teleportation, the fidelity is

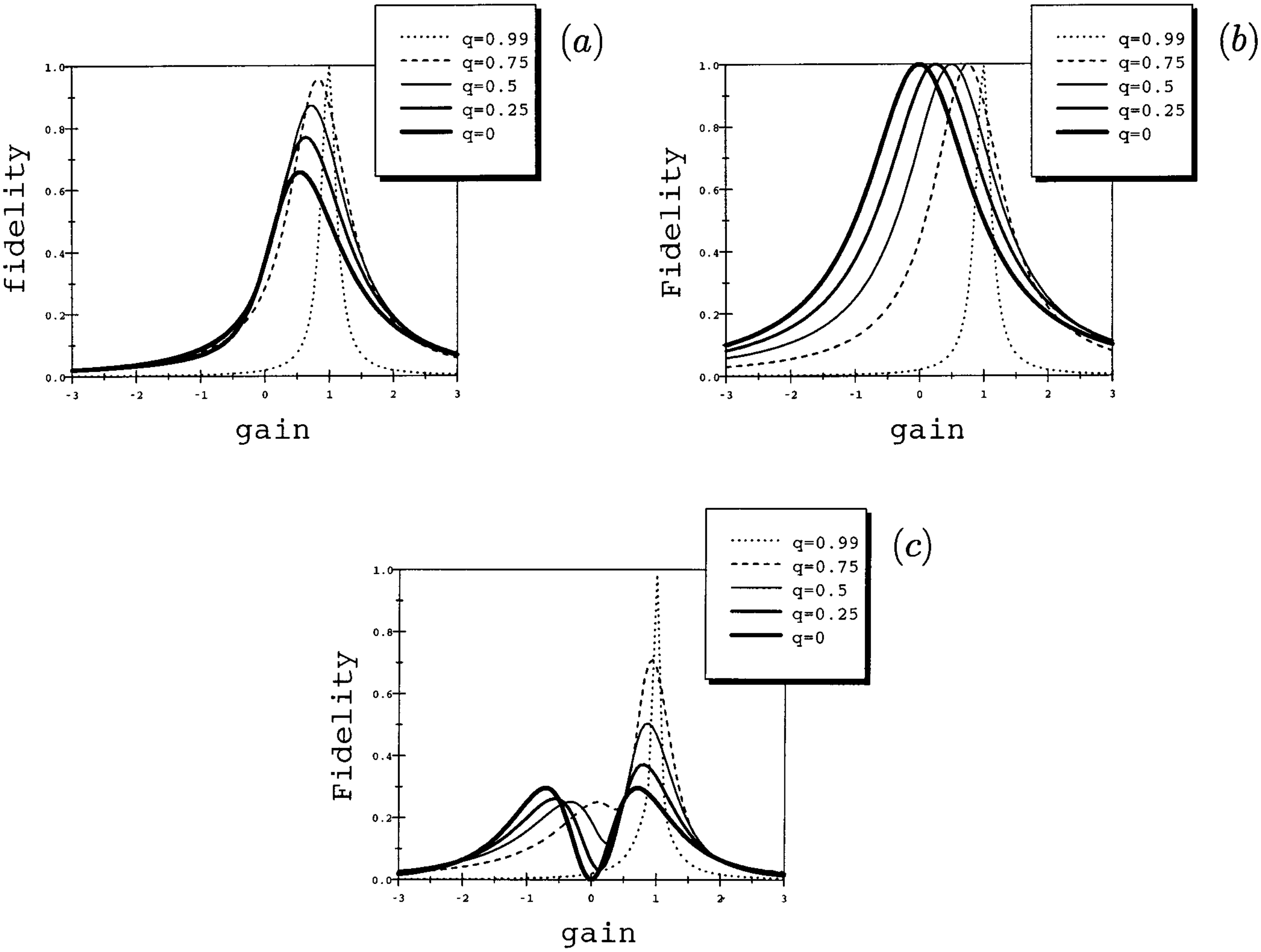

FIG. 2. The g (gain factor) dependence of fidelity for a coherent states with $|\alpha|=1$ (a), the vacuum state (b), and the one photon state (c). The coherent-state teleportation fidelity. The different curves correspond to entanglement parameter values of $q=0.99$ (dotted line), 0.75 (dashed line), 0.5 (thin line), 0.25 (thick line), 0 (thickest line), respectively. The peak positions shift to lower values of g with decreasing q.

$$F_q^1(g)=\int d^2\beta|\langle 1|\hat{T}_q^g(\beta)|1\rangle|^2,$$

$$=\frac{1-q^2}{\pi}\int d^2\beta e^{-(1-2qg+g^2)|\beta|^2}$$

$$\times[(1-qg)(g-q)|\beta|^2+q]^2,$$

$$=\frac{1-q^2}{(1-2qg+g^2)^3}[(g-q)^2(1-qg)^2$$

$$+g^2(1-q^2)^2]. \tag{10}$$

Figure 2(c) shows the gain dependence of the teleportation fidelity of the one-photon state at several values of the entanglement parameter q. Two peaks appear when $q<1$. The second peak is a result of the phase-space symmetry of the single-photon input state. This can be understood most clearly at $q=0$, where the output state is a coherent state $|g\beta\rangle$ and the overlap with the input photon number state is obviously equal for g and $-g$. The right peak of the Fig. 2(c) changes in a similar way to the peak of the Fig. 2(a) for $|\alpha|=1$. Once again the maximal fidelity is always found at $g<1$. Both peaks shift to lower gain values as the entanglement parameter q decreases. The right peak also gets broader and lower with decreasing q. The peak position is always between $g=q$ and $g=1$, with its lowest gain value at $g=1/\sqrt{2}$ for $q=0$. Some additional gain is always necessary to replace the photon losses suffered at $g=q$ [8]. Since the fidelity is always improved by lowering the gain below 1, we conclude that an optimal gain condition $g<1$ can be found for any selection of states. In the following section, we apply this result to two examples.

IV. OPTIMAL STRATEGIES FOR UNKNOWN INPUT STATES

One possible selection of states to encode quantum information are coherent states with fixed amplitude and variable phase, $|\phi\rangle=||\alpha|\exp[-i\phi]\rangle$. For example, information could

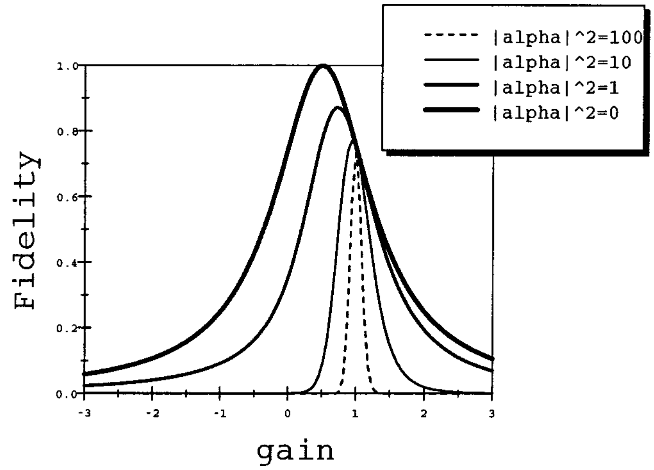

FIG. 3. The gain dependence of the coherent-state teleportation fidelity at an entanglement $q=0.5$. The curves correspond to input intensities $|\alpha|^2=0$ (thickest line), 1 (thick line), 10 (thin line), 100 (dashed line).

be encoded in the states $|\alpha\rangle, |i\alpha\rangle, |-\alpha\rangle$, and $|-i\alpha\rangle$. Since the gain dependence of fidelity for coherent states does not depend on phase, the optimized gain is the same for all such states. It is, therefore, possible to determine the optimal gain directly from Eq. (8). Figure 3 shows the gain g dependence of the fidelity for input field intensities of $|\alpha|^2=0,1,10,100$ for teleportation with an entanglement of $q=0.5$. For $|\alpha|^2=0$ (vacuum input), the peak value of the fidelity is 1 at $g=q=0.5$. The input vacuum state is perfectly recovered by a gain tuning of $g=q=0.5$. With the increase of the input field intensity $|\alpha|^2$, the fidelity peak approaches 0.75 at $g=1$. For $|\alpha|^2>0$ the input coherent state cannot be recovered fully by gain tuning. Nevertheless some improvement of the fidelity is always possible through gain tuning to an optimal gain value of $q<g_{\rm opt}(q)<1$.

The optimal gain value is found by maximizing the fidelity. See the Appendix for details of the calculation. For coherent states, the optimization condition depends on the intensity of the input fields. In its most compact form, it reads

$$|\alpha|^2=\frac{(g_{\rm opt}-q)(1-2qg_{\rm opt}+g_{\rm opt}^2)}{(1+q)(1-q)^2(1-g_{\rm opt}^2)}. \tag{11}$$

Figure 4 shows this relation between $|\alpha|^2$ and $g_{\rm opt}$ for an entanglement of $q=0.5$. The optimized gain $g_{\rm opt}$ varies from $0.5(=q)$ at $|\alpha|^2=0$ to 1 for $|\alpha|^2\to\infty$. The significance of gain tuning is already appreciable at $|\alpha|^2=12$, where a gain of $g_{\rm opt}=0.95$ is optimal, and it rapidly approaches the vacuum situation below intensities of $|\alpha|^2=4$. At $|\alpha|^2=1$, the optimized gain is already as low as $g_{\rm opt}=0.72$. The improvement of fidelity achieved by gain tuning at $|\alpha|^2=1$ is shown in Fig. 5. A substantial improvement of fidelity by gain tuning is observed for almost all entanglement values. The increase in fidelity achieved by gain tuning increases monotonously as the entanglement q decreases, with a difference between the optimized and nonoptimized fidelity greater than 0.09 for $q<0.7$. The maximal increase in fidelity is obtained in the limit of no entanglement $(q=0)$ with

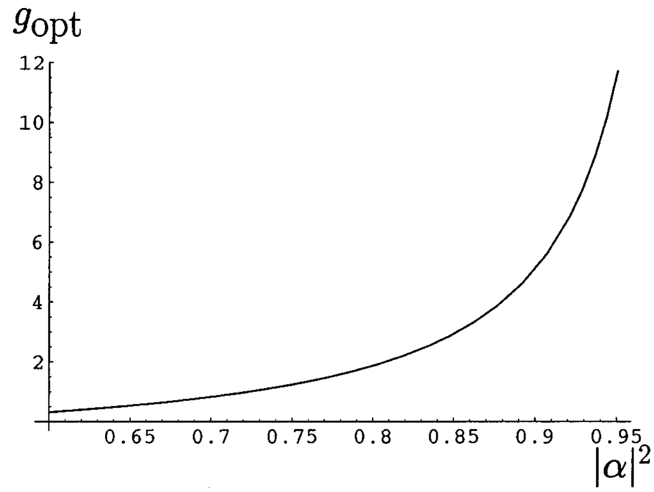

FIG. 4. Relation between optimal gain and input intensity for a coherent-state input at an entanglement parameter $q=0.5$.

$\Delta F=0.16$. The fidelity for coherent-state teleportation with known intensity can thus be significantly improved by an appropriate choice of the gain parameter $g=g_{\rm opt}$. Note that the short analysis of gain tuning given in Ref. [4] for the experimental realization of continuous-variable teleportation was only applied to the high intensity limit of $|\alpha|^2\to\infty$. As mentioned above, the maximal fidelity is then found to be extremely close to $g=1$. In this regime, a very precise measurement is necessary to reveal the slight shift of $g_{\rm opt}$ due to the finite value of $|\alpha|$ actually used in the experiment.

Another typical encoding scheme for quantum information uses the polarization states of single photons. Continuous-variable quantum teleportation can be applied to such photonic qubits by teleporting each of two orthogonal polarization modes in parallel [9]. Since successful teleportation requires that both the zero-photon component and the one-photon component of the qubit are teleported without

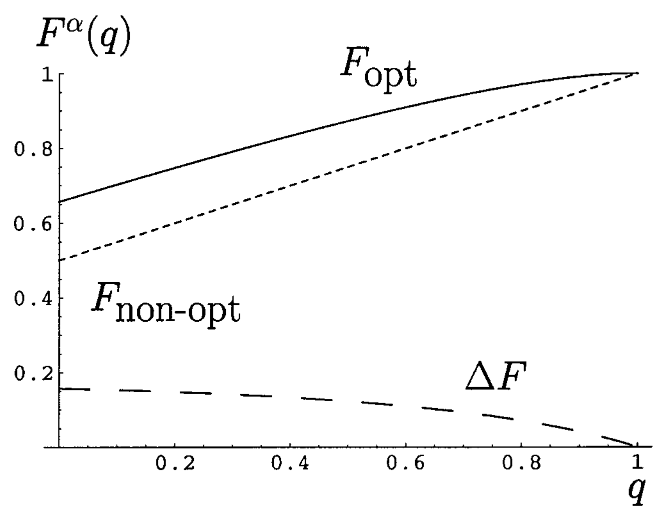

FIG. 5. The solid line shows the optimized fidelity at $g_{\rm opt}$ for the teleportation of a coherent state with $|\alpha|=1$. The dashed line shows the nonoptimized fidelity at $g=1$ for comparison. The broken line shows the difference between $\Delta F=F_{\rm opt}-F_{\rm nonopt}$ between the two.

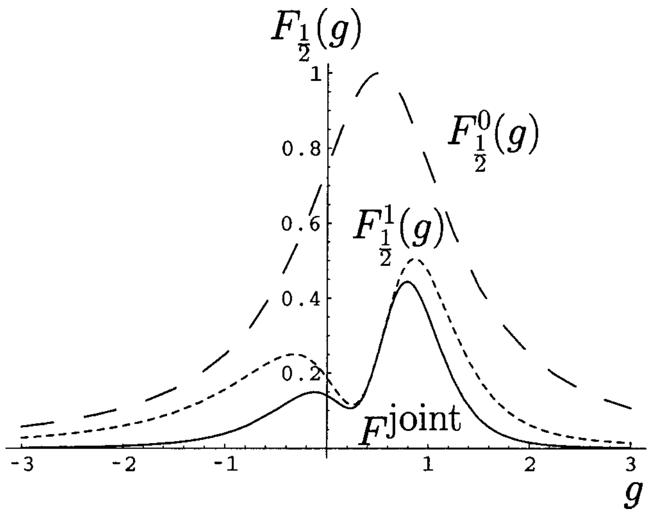

FIG. 6. The solid line shows the joint fidelity of the two-mode teleportation of a vacuum state and a single-photon state as a function of the gain factor g at an entanglement of $q=0.5$. The dashed lines show the fidelity of the single-mode teleportation of vacuum state and of a single-photon state, respectively. The joint fidelity F^{joint} is equal to the product of $F^0_{1/2}(g)$ and $F^1_{1/2}(g)$.

changes to the quantum state, the total fidelity of the process can be written as a product of the two individual fidelities for vacuum and for single-photon teleportation. As shown in Appendix B, this is even true if the polarization of the qubit is unknown. Since neither the homodyne detection nor the displacement is sensitive to the choice of polarization directions, the fidelity for a single photon of unknown polarization is always given by the joint fidelity $F_q^{\text{joint}}=F_q^0F_q^1$. The gain dependence of this joint fidelity $F_q^{\text{joint}}=F_q^0F_q^1$ reads

$$F_q^{\text{joint}}(g)=\frac{(1-q^2)^2}{(1-2qg+g^2)^4}[(g-q)^2(1-qg)^2+g^2(1-q^2)^2]. \tag{12}$$

Figure 6 shows the gain dependence of this fidelity together with the fidelities for the vacuum and for the one-photon teleportation at an entanglement of $q=0.5$. The main peak of the joint fidelity curve is found between the maxima of the vacuum and the single-photon fidelities. At $q=0.5$, $g_{\text{opt}}=0.79$ gives a maximal joint fidelity of 0.44, compared with a fidelity of 0.35 at $g=1$. The dependence of optimized gain on the entanglement parameter q can be determined by analytically maximizing the fidelity. See the Appendix for details of the calculation. Figure 7 shows the optimized gain g_{opt} as a function of entanglement q for both the photonic qubit and for a coherent state of intensity $|\alpha|^2=1$. Note that both curves are very close to each other, suggesting that the gain tuning is quite similar for both single photons and coherent states with an average photon number of one. In the case of no entanglement at $q=0$, the optimized gain $g_{\text{opt}}(q)$ is 0.544 for the coherent state and 0.577 for the photonic qubit. It increases almost linearly to 1 as the entanglement q is raised from 0 to 1. As a rule of thumb, optimal gain tuning for the teleportation of photonic qubits is obtained at

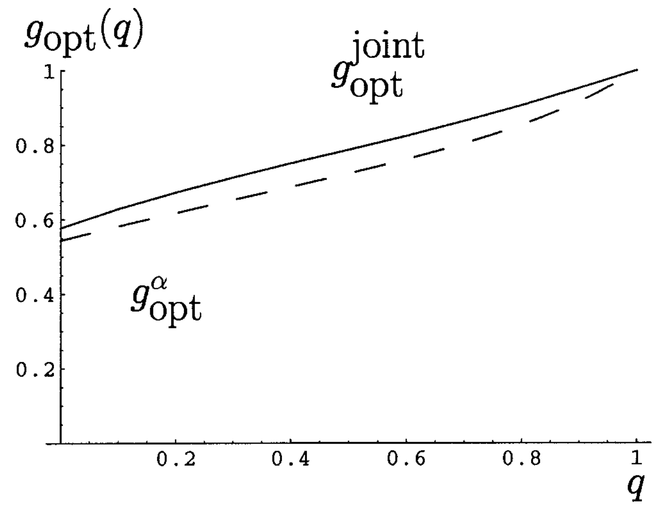

FIG. 7. Dependence of optimized gain $g_{\text{opt}}(q)$ on the entanglement parameter q for the teleportation of a photonic qubit (solid line) and for the teleportation of a coherent state with amplitude $|\alpha|=1$ (dashed line).

$$g_{\text{opt}}(q)\approx 0.6+0.4q. \tag{13}$$

For practical purposes, this simplified relation should be sufficient to achieve improved fidelities for single-photon teleportation. Note also that a similar optimization would apply if the quantum information was encoded into vacuum or one-photon states within a single mode.

The improvement of fidelity by optimized gain tuning for photonic qubit teleportation can be obtained from $g_{\text{opt}}(q)$ using Eq. (12). Figure 8 shows a comparison between the optimized fidelity F_{opt} and the nonoptimized fidelity F_{nonopt} as a function of the entanglement parameter q. F_{opt} is obtained with the optimized gain $g_{\text{opt}}(q)$, while for F_{nonopt}, the gain is fixed at $g=1$. At $q=0$, the optimized fidelity is 0.221 while the nonoptimized fidelity is 0.125. The difference of about 0.1 does not change much up to $q=0.6$, so that an

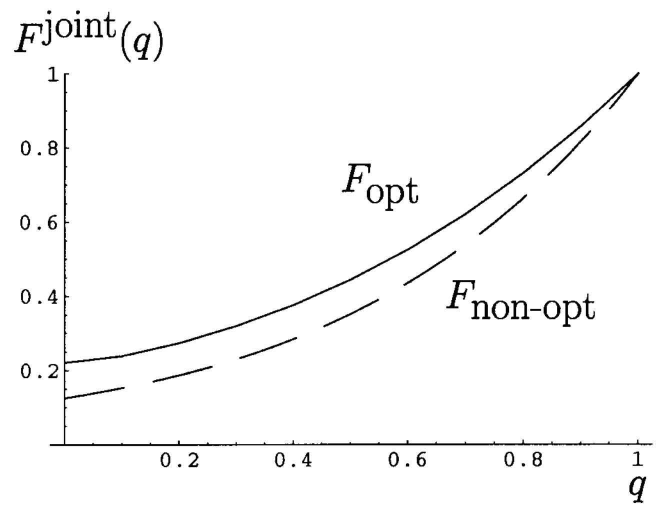

FIG. 8. The solid line shows the optimized fidelity at g_{opt} for the teleportation of a photonic qubit. The dashed line shows the nonoptimized fidelity at $g=1$ for comparison.

increase of 0.1 in fidelity is possible for most cases of photonic qubit teleportation. Again, this value is similar to the improvement of fidelity achievable for coherent states with $|\alpha|^2=1$. However, the improvement for photonic qubits appears to be even more significant, given the relatively low total fidelity of this teleportation.

V. CONCLUSION

The fidelity of continuous-variable quantum teleportation can be enhanced by varying the gain in the measurement dependent modulation on the output field. We have shown that gain tuning always maximizes the fidelity at a gain value of $g<1$. The specific results for coherent states, vacuum, and single photons have been obtained. Using these results, the optimal gain tuning for the teleportation of coherent states with known amplitude but unknown phase and for the teleportation of the polarization of a single-photon qubit have been determined. For entanglement parameters of $q<0.7$, improvements of about 0.1 are possible in the fidelity of single-photon teleportation. Similar improvements are obtained for coherent states with intensity $|\alpha|^2=1$. These results demonstrate the usefulness of gain tuning for input states with a low average photon number.

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APPENDIX A: CALCULATION FOR OPTIMAL GAIN

Maximum fidelity for a coherent-state teleportation is obtained at a gain g_{opt}, which satisfies $dF_q^{\alpha}(g)/dg=0$. The derivative of the fidelity given by Eq. (8) reads

$$\frac{dF_q^{\alpha}(g)}{dg}=2[-(g-q)(1-2qg+g^2)+(1+q)(1-q)^2 \times(1-g^2)|\alpha|^2]\frac{1-q^2}{(1-2qg+g^2)^3} \times\exp\left[-\frac{1-q^2}{1-2qg+g^2}(1-q)^2|\alpha|^2\right]. \quad \text{(A1)}$$

Therefore, the optimized gain g_{opt} is given by the polynomial

$$g_{\text{opt}}^3-(a^2+3q)g_{\text{opt}}^2+(1-2q^2)g_{\text{opt}}+(a^2-q)=0, \quad \text{(A2)}$$

where

$$a^2=(1+q)(1-q)^2|\alpha|^2. \quad \text{(A3)}$$

The solution can be written as

$$g_{\text{opt}}=\frac{1}{3}\left(A+\frac{B}{D}+D\right), \quad \text{(A4)}$$

where

$$A=(a^2+3q),$$
$$B=A^2+6q^2-3,$$
$$C=A^3+27q^2-9a^2(2-q^2),$$
$$D=(C+\sqrt{-B^3+C^2})^{1/3}. \quad \text{(A5)}$$

We can obtain the optimized fidelity $F_{\text{opt}}^{\alpha}=F^{\alpha}(g_{\text{opt}})$ from this result and Eq. (8).

Likewise, the optimal gain for the teleportation of a photonic qubit composed of a vacuum state and a single-photon state is obtained by differentiating Eq. (12),

$$\frac{dF_q^{\text{joint}}(g)}{dg}=\frac{dF_q^0(g)}{dg}F_q^1(g)+F_q^0(g)\frac{dF_q^1(g)}{dg} =-\frac{2(1-q^2)^2}{(1-2qg+g^2)^5}[2q^2g^5-5(q+q^3)g^4 +2(3+3q^2+4q^4)g^3-4q(2+2q^2+q^4)g^2 +2(-1+4q^2+2q^4)g-3q^3+q]. \quad \text{(A6)}$$

Again, a polynomial for g_{opt} is obtained. Since g_{opt} must be one at $q=1$, we express this polynomial in terms of the gain tuning parameter $h=1-g_{\text{opt}}$. It then reads

$$2q^2h^5+5q(1-q)^2h^4+2(3-10q+13q^2-10q^3+4q^4)h^3 -2(1-q)^2(9-q+8q^2-2q^3)h^2+4(1-q)^2(4-q +3q^2-2q^3)h-4(1-q)^3(1+q^2)=0. \quad \text{(A7)}$$

where $h=1-g_{\text{opt}}$. Solutions for this optimization condition have been obtained numerically. As above, we can then determine the optimized fidelity $F_{\text{opt}}^{\alpha}=F^{\text{joint}}(g_{\text{opt}})$ from this result and Eq. (12).

APPENDIX B: POLARIZATION INDEPENDENCE OF THE FIDELITY FOR SINGLE-PHOTON QUBIT TELEPORTATION

If a single photon of unknown polarization is teleported, the input photon state $|S\rangle$ is an unknown superposition of horizontal and vertical polarization states $|H\rangle=|1\rangle_H|0\rangle_V$ and $|V\rangle=|0\rangle_H|1\rangle_V$. The unknown polarization state can thus be written as

$$|S\rangle=c_H|H\rangle+c_V|V\rangle. \quad \text{(B1)}$$

Experimentally, this state is teleported by measuring β_H and β_V and applying the displacement $D(\beta_H,\beta_V)$. In this process, the experimentalist uses no information on the actual signal polarization S.

It is nevertheless possible to transform the calculation of fidelity into the S,P basis, where P is the polarization orthogonal to S. If the unitary transform $\hat{U}$ rotates the polarization so that

$$|S\rangle = \hat{U}|H\rangle,$$
$$|P\rangle = \hat{U}|V\rangle, \tag{B2}$$

the fidelity for the teleportation of the *S*-polarized state can be written as

$$F_q^{\text{joint}} = \int d^2\beta_H d^2\beta_V |\langle H|\hat{U}^\dagger \hat{T}_q(\beta_H,\beta_V)\hat{U}|H\rangle|^2. \tag{B3}$$

It is now possible to apply the unitary transformations to $\hat{T}_q(\beta_H,\beta_V)$. This is particularly simple for $\beta_H = \beta_V = 0$, because $\hat{T}_q(0,0)$ is a function of the total photon number $n_{total} = n_H + n_V = n_S + n_P$, which is independent of the mode decomposition,

$$\hat{T}_q(0,0) = \frac{1-q^2}{\pi} q^{(\hat{n}_H + \hat{n}_V)},$$
$$= \frac{1-q^2}{\pi} q^{(\hat{n}_P + \hat{n}_S)}. \tag{B4}$$

Therefore, $\hat{U}^\dagger \hat{T}_q(0,0)\hat{U} = \hat{T}_q(0,0)$. The results for all other measurement values are obtained by applying displacement operators to $\hat{T}_q(0,0)$. Since the displacement generated by $\hat{D}(\alpha_H,\alpha_V)$ is linear in the field components, its transformation reads

$$\hat{U}^\dagger \hat{D}(\alpha_H,\alpha_V)\hat{U} = \hat{D}(c_H\alpha_H + c_V\alpha_V, c_V\alpha_H - c_H\alpha_V). \tag{B5}$$

By combining these transformation properties, we obtain

$$\hat{U}^\dagger \hat{T}(\beta_H,\beta_V)\hat{U} = \hat{T}(c_H\beta_H + c_V\beta_V, c_V\beta_H - c_H\beta_V). \tag{B6}$$

It is, therefore, a straightforward matter to express the transfer operator in the basis of the unknown input state, even though this basis was not used in the experiment and all measurement data was obtained in the *H*, *V* basis. This property of the transfer operator greatly simplifies the determination of the overall fidelity. By transforming the integration using

$$\beta_S = c_H\beta_H + c_V\beta_V,$$
$$\beta_P = c_V\beta_H - c_H\beta_V, \tag{B7}$$

the fidelity for the teleportation of an unknown polarization reads

$$F_q^{\text{joint}} = \int d^2\beta_S d^2\beta_P |\langle H|\hat{T}_q(\beta_S,\beta_P)|H\rangle|^2,$$
$$= \int d^2\beta_S |\langle 1|\hat{T}_q(\beta_S)|1\rangle|^2 \int d^2\beta_P |\langle 0|\hat{T}_q(\beta_P)|0\rangle|^2. \tag{B8}$$

Once more we would like to emphasize that this formulation of fidelity does not depend on the polarization basis used in the experiment and does not relate to any actual information required for the teleportation process. It is, therefore, possible to apply the product of the single-photon fidelity and the vacuum fidelity to the teleportation of an unknown polarization state of a single photon.

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