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Dehn surgeries on knots which yield lens spaces and genera of knots

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1. *Introduction*

It is an interesting open question when Dehn surgery on a knot in the 3-sphere S^3 can produce a lens space (see [**10**, **12**]). Some studies have been made for special knots; in particular, the question is completely solved for torus knots [**21**] and satellite knots [**3**, **29**, **31**]. It is known that there are many examples of hyperbolic knots which admit Dehn surgeries yielding lens spaces. For example, Fintushel and Stern [**8**] have shown that 18- and 19-surgeries on the $(-2,3,7)$-pretzel knot give lens spaces $L(18,5)$ and $L(19,7)$, respectively. However, there seems to be no essential progress on hyperbolic knots. It might be a reason that some famous classes of hyperbolic knots, such as 2-bridge knots [**26**], alternating knots [**5**], admit no surgery yielding lens spaces.

In this paper we focus on the genera of knots to treat the present condition methodically and show that there is a constraint on the order of the fundamental group of the resulting lens space obtained by Dehn surgery on a hyperbolic knot. Also, this new standpoint enables us to present a conjecture concerning such a constraint, which holds for all known examples.

Let K be a hyperbolic knot in S^3. The exterior of K, denoted by $E(K)$, is the complement of an open tubular neighbourhood of K. Let r be a *slope* on $\partial E(K)$, that is, the isotopy class of an essential simple closed curve in $\partial E(K)$, and let $K(r)$ be the closed 3-manifold obtained by r-Dehn surgery on K. Thus $K(r) = E(K) \cup V_r$, where V_r is a solid torus attached to $\partial E(K)$ along their boundaries in such a way that r bounds a meridian disc in V_r. Slopes on $\partial E(K)$ are parameterized as $m/n \in \mathbb{Q} \cup \{1/0\}$ in the usual way [**23**].

If $K(r)$ is a lens space, then r is an integer by the Cyclic Surgery Theorem [**4**] and $\pi_1 K(r)$ has the order $|r|$. Furthermore, there are at most two such slopes and if there are two then they are consecutive.

Theorem 1·1. *Let K be a hyperbolic knot in S^3. If $K(r)$ is a lens space, then $|r| \leqslant 12g - 7$, where g is the genus of K.*

In [2], Berge introduced 'double-primitive' knots and showed that such knots admit integral surgeries which yield lens spaces. Furthermore, he gave a list of double-primitive knots, including all known knots with surgeries yielding lens spaces.

As he wrote there, there is still a possibility that his list of double-primitive knots is not complete, but he has suggested that the list is complete and if a knot has a surgery yielding a lens space then the knot appears in his list (see [**10**, **12**, **20**]).

All knots in Berge's list can be expressed as closed positive (or negative) braids and therefore they are fibred [**25**]. Then it is easy to calculate their genera, since Seifert's algorithm gives fibre surfaces, that is, minimal genus Seifert surfaces for such knots.

On the basis of the above remark and the calculation for Berge's knots, we suggest the following.

Conjecture. *Let K be a hyperbolic knot in S^3. If $K(r)$ is a lens space, then K is fibred and $2g + 8 \leqslant |r| \leqslant 4g - 1$, where g is the genus of K.*

Since the $(-2, 3, 7)$-pretzel knot has genus 5, it is expected that this estimate would be best possible. Surprisingly, the $(-2, 3, 7)$-pretzel knot is the only hyperbolic knot that has genus 5 in Berge's list.

In [**3**], it is conjectured that for a hyperbolic knot K, if $K(r)$ is a lens space then $|r| \geqslant 18$. This implies that no lens space L with $|\pi_1(L)| < 5$ can arise from a non-trivial knot [**3**, **10**, **30**]. An affirmative answer to the above conjecture would imply that a hyperbolic knot, admitting a surgery which yields a lens space, has genus at least 5 and therefore that 18 is the minimal order of the fundamental groups of lens spaces obtained by surgery on hyperbolic knots.

For the case of genus one we have the complete answer.

Theorem 1·2. *No Dehn surgery on a genus one, hyperbolic knot in S^3 gives a lens space.*

Combining this with known facts, we can completely determine Dehn surgeries on genus one knots which yield lens spaces.

Theorem 1·3. *A genus one knot K in S^3 admits Dehn surgery yielding a lens space if and only if K is the $(\pm 3, 2)$-torus knot and the surgery slope is $(\pm 6n + \varepsilon)/n$ for $n \neq 0$, $\varepsilon = \pm 1$.*

As in earlier results, we have proved that no Dehn surgery on a genus one knot gives $L(2, 1)$ [**27**] (see also [**6**]) or $L(4k, 2k \pm 1)$ for $k \geqslant 1$ [**28**]. It was also known that if a genus one knot has a non-trivial Alexander polynomial, then the knot has no cyclic surgery of even order [**22**, corollary 2]. Recently, [**19**] showed that the lens space $L(2k, 1)$ cannot be obtained by surgery on a strongly invertible knot.

To prove Theorems 1·1 and 1·2, we analyze the graphs of the intersection of surfaces properly embedded in a knot exterior. One comes from a Heegaard torus of a lens space and the other is a minimal genus Seifert surface for the knot. By virtue of the use of a Seifert surface, instead of a level sphere in a thin position of the knot, the graphs can include the information on the order of the fundamental group of the resulting lens space after Dehn surgery. In Section 2, it is found out that there are some constraints on Scharlemann cycles. The proof of Theorem 1·1 is divided into two cases according to the number t of points of intersection between the Heegaard torus and the core of the attached solid torus. In Section 3, the case that $t \geqslant 4$ is dealt with, and the special case that $t = 2$ is discussed in Section 4 and the proof of Theorem 1·1 is completed. Finally in Sections 5 and 6, we specialize to the case that K has genus one and prove Theorems 1·2 and 1·3.

2. *Preliminaries*

Throughout this paper, K will be assumed to be a hyperbolic knot in S^3. For a slope r, suppose that $K(r) = E(K) \cup V_r$ is a lens space. Since K is not a torus knot, the Cyclic Surgery Theorem [**4**, corollary 1] implies that the slope r must be integral. We may assume that $r > 1$. Thus $\pi_1 K(r)$ has the order r. For simplicity, we denote V_r by V. Let K^* be the core of V.

Let $\widehat{T}$ be a Heegaard torus in $K(r)$. Then $K(r) = U \cup W$, where U and W are solid tori. We can assume that $\widehat{T}$ meets K^* transversely in t points and that $\widehat{T} \cap V$ consists of t mutually disjoint meridian discs of V. Then $T = \widehat{T} \cap E(K)$ is a punctured torus with t boundary components, each having slope r on $\partial E(K)$.

Let $S \subset E(K)$ be a minimal genus Seifert surface of K. Then S is incompressible and boundary-incompressible in $E(K)$.

By an isotopy of S, we may assume that S and T intersect transversely and ∂S meets each component of ∂T in exactly r points. We choose $\widehat{T}$ so that the next condition $(*)$ is satisfied:

$(*)$ $\widehat{T} \cap K^* \neq \emptyset$ and each arc component of $S \cap T$ is essential in S and in T.

This can be achieved if K^* is put in thin position with respect to $\widehat{T}$ [**9**, **11**, **14**]. (Note that if K^* can be isotoped to lie on $\widehat{T}$, then K would be a torus knot.) Furthermore, we may assume that $\widehat{T}$ is chosen so that t is minimal over all Heegaard tori in $K(r)$ satisfying $(*)$. This minimality of $\widehat{T}$ will be crucial in this paper.

Since S is incompressible in $E(K)$ and $E(K)$ is irreducible, it can be assumed that no circle component of $S \cap T$ bounds a disc in T. But it does not hold for S in general. We further assume that the number of loop components of $S \cap T$ is minimal up to an isotopy of S.

The arc components of $S \cap T$ define graphs G_S in $\widehat{S}$ and G_T in $\widehat{T}$ as follows [**4**, **16**], where $\widehat{S}$ is the closed surface obtained by capping ∂S off by a disc. Let G_S be the graph in $\widehat{S}$ obtained by taking as the (fat) vertex the disc $\widehat{S} - \operatorname{Int} S$ and as edges the arc components of $S \cap T$ in $\widehat{S}$. Similarly, G_T is the graph in $\widehat{T}$ whose vertices are the discs $\widehat{T} - \operatorname{Int} T$ and whose edges are the arc components of $S \cap T$ in $\widehat{T}$. Number the components of ∂T, $1, 2, \ldots, t$ in sequence along $\partial E(K)$. Let $\partial_i T$ denote the component of ∂T with label i. This induces a numbering of the vertices of G_T. Let u_i be the vertex of G_T with the label i for $i = 1, 2, \ldots, t$. Let $H_{x,x+1}$ be the part of V between consecutive fat vertices u_x and u_{x+1} of G_T. When $t = 2$, V is considered to be the union $H_{1,2} \cup H_{2,1}$. Each endpoint of an edge in G_S at the unique vertex v has a label, namely the label of the corresponding component of ∂T. Thus the labels $1, 2, \ldots, t$ appear in order around v repeated r times.

The graphs G_S and G_T satisfy the parity rule [**4**] which can be expressed as the following: the labels at the endpoints of an edge of G_S have distinct parities.

A *trivial loop* in a graph is a length one cycle which bounds a disc face. By $(*)$, neither G_S nor G_T contains trivial loops.

A family of edges $\{e_1, e_2, \ldots, e_p\}$ in G_S is a *Scharlemann cycle* (of length p) if it bounds a disc face of G_S and all the edges have the same pair of labels $\{x, x+1\}$, for some x, at their two endpoints, which is called the *label pair* of the Scharlemann cycle. Note that each edge e_i connects the vertex u_x with u_{x+1} in G_T. A Scharlemann cycle of length two is called an *S-cycle* for short. Remark that the interior of the face bounded by a Scharlemann cycle may meet $\widehat{T}$, since T is not necessarily incompressible in $E(K)$.

Let σ be a Scharlemann cycle in G_S with label pair $\{x, x+1\}$. If the edges of σ (and vertices u_x and u_{x+1}) are contained in an annulus in $\widehat{T}$, and if they do not lie in a disc in $\widehat{T}$, then we say that the edges of σ *lie in an essential annulus* in $\widehat{T}$.

Lemma 2·1. *Let σ be a Scharlemann cycle in G_S of length p with label pair $\{x, x+1\}$, where p is 2 or 3. Let f be the face of G_S bounded by σ. If the edges of σ do not lie in a disc in $\widehat{T}$, then they lie in an essential annulus A in $\widehat{T}$. Furthermore, if* $\mathrm{Int}\, f \cap \widehat{T} = \emptyset$, *then $M = N(A \cup H_{x,x+1} \cup f)$ is a solid torus such that the core of A runs p times in the longitudinal direction of M.*

Proof. If $p = 2$, then it is obvious that the edges of σ lie in an essential annulus in $\widehat{T}$.

Assume $p = 3$. Let $\sigma = \{e_1, e_2, e_3\}$. If the endpoints of e_1, e_2, e_3 appear in this order when one travels around u_x clockwise, say, then those of e_1, e_2, e_3 appear in the same order when one travels around u_{x+1} anticlockwise, since u_x and u_{x+1} have distinct parities. This observation implies that the edges of σ lie in an essential annulus in $\widehat{T}$.

Consider the genus two handlebody $N(A \cup H_{x,x+1})$. Then M is obtained by attaching a 2-handle $N(f)$. Since there is a meridian disc of $N(A)$ which intersects ∂f once, ∂f is primitive and therefore M is a solid torus. It is not hard to see that the core of A runs p times in the longitudinal direction of M. (See also [**17**, lemma 3·7].)

Lemma 2·2. *Let ξ be a loop in $S \cap T$. Suppose that ξ bounds a disc δ in S with* $\mathrm{Int}\, \delta \cap \widehat{T} = \emptyset$. *If ξ is inessential in $\widehat{T}$, then all vertices of G_T must lie in the disc bounded by ξ in $\widehat{T}$.*

Proof. Let δ' be the disc bounded by ξ in $\widehat{T}$. Then $\delta' \cap V \neq \emptyset$, since ξ is essential in T by the assumption on $S \cap T$. If both sides of ξ on $\widehat{T}$ meet V, replace $\widehat{T}$ by $\widehat{T'} = (\widehat{T} - \delta') \cup \delta$. Then $\widehat{T'}$ gives a new Heegaard torus of $K(r)$ satisfying (∗). However this contradicts the choice of $\widehat{T}$, since $|\widehat{T'} \cap K^*| < |\widehat{T} \cap K^*|$. Hence all vertices of G_T lie in δ'.

Lemma 2·3. *Let σ be a Scharlemann cycle in G_S of length p with label pair $\{x, x+1\}$ and let f be the face of G_S bounded by σ. Suppose that $p \neq r$. Then the edges of σ cannot lie in a disc in $\widehat{T}$ and* $\mathrm{Int}\, f \cap \widehat{T} = \emptyset$.

Proof. Assume for contradiction that the edges of σ lie in a disc D in $\widehat{T}$. Let Γ be the subgraph of G_T consisting of two vertices u_x and u_{x+1} along with the edges of σ.

First, suppose that $\mathrm{Int}\, f \cap D \neq \emptyset$. Then all components in $\mathrm{Int}\, f \cap D$ are parallel to ∂D in $D - \Gamma$ by the minimality of $S \cap T$. Thus we can replace D by a subdisc which does not meet $\mathrm{Int}\, f$. We may now assume that $\mathrm{Int}\, f \cap D = \emptyset$. Then $N(D \cup H_{x,x+1} \cup f)$ gives a punctured lens space. Since a lens space $K(r)$ is irreducible, this means that $K(r)$ is a lens space whose fundamental group has order p. This contradicts the assumption that $p \neq r$. Thus the edges of σ cannot lie in a disc in $\widehat{T}$.

Assume that $\mathrm{Int}\, f \cap \widehat{T} \neq \emptyset$. Let μ be an innermost component of $\mathrm{Int}\, f \cap \widehat{T}$ on f.

Since the edges of σ do not lie in a disc in $\widehat{T}$, it follows from Lemma 2·2 that μ is essential in $\widehat{T}$. Then it can be assumed that the disc δ bounded by μ on f is contained in W, say, one of the solid tori bounded by $\widehat{T}$ in $K(r)$. Thus δ is a meridian disc of W.

In W, compress $\widehat{T}$ along δ to obtain a 2-sphere Q. There is a disc E in Q which contains the edges of σ and two vertices u_x and u_{x+1}. Even if $\mathrm{Int}\, f \cap E \neq \emptyset$, the cut-and-paste operation gives a new f with $\mathrm{Int}\, f \cap E = \emptyset$. Thus $N(E \cup H_{x,x+1} \cup f)$ gives a punctured lens space whose fundamental group has order p, which contradicts the assumption again.

When there exist two Scharlemann cycles with disjoint label pairs, the assumption on the length in the statement of Lemma 2·3 is not necessary.

Lemma 2·4. *Let σ_1 and σ_2 be Scharlemann cycles in G_S with disjoint label pairs and let f_1 and f_2 be the faces of G_S bounded by σ_1 and σ_2 respectively. Then the edges of σ_i lie in an essential annulus A_i in $\widehat{T}$ with $A_1 \cap A_2 = \emptyset$ and $\mathrm{Int}\, f_i \cap \widehat{T} = \emptyset$ for $i = 1, 2$.*

Proof. Let $\{x_i, x_i + 1\}$ be the label pair of σ_i. Assume that the edges of σ_1 lie in a disc D_1 in $\widehat{T}$ for contradiction. By the same argument in the proof of Lemma 2·3, we may assume that $\mathrm{Int}\, f_1 \cap D_1 = \emptyset$. If $\mathrm{Int}\, f_1 \cap \widehat{T} = \emptyset$, then $N(D_1 \cup H_{x_1,x_1+1} \cup f_1)$ gives a punctured lens space in a solid torus, which is impossible. Therefore $\mathrm{Int}\, f_1 \cap \widehat{T} \neq \emptyset$.

Choose an innermost component ξ of $\mathrm{Int}\, f_1 \cap \widehat{T}$ on f_1. Let δ be the disc bounded by ξ on f_1.

Assume that ξ is inessential in $\widehat{T}$. By Lemma 2·2, G_T lies in the disc bounded by ξ. Then the edges of σ_2 also lie in a disc D_2 in $\widehat{T}$. We remark that one of D_1 and D_2 may be contained in the other, possibly. As above, we can assume that $\mathrm{Int}\, f_2 \cap D_2 = \emptyset$.

If $D_1 \cap D_2 = \emptyset$, then we can assume that $\mathrm{Int}\, f_i \cap D_j = \emptyset$ for $i, j \in \{1, 2\}$ by the cut-and-paste operation of f_i.

Otherwise, $D_2 \subset D_1$, say. Clearly, $\mathrm{Int}\, f_1 \cap D_j = \emptyset$ for $j = 1, 2$. If $\mathrm{Int}\, f_2 \cap D_1 \neq \emptyset$, then it can be assumed that each component of $\mathrm{Int}\, f_2 \cap D_1$ is parallel to ∂D_2 in $D_1 - \Gamma$, where Γ is the subgraph of G_T, consisting of the vertices u_{x_i}, u_{x_i+1} along with the edges of σ_i for $i = 1, 2$. But this contradicts Lemma 2·2. Therefore, we can assume that $\mathrm{Int}\, f_i \cap D_j = \emptyset$ for $i, j \in \{1, 2\}$ in either case.

Then $N(D_1 \cup H_{x_1,x_1+1} \cup f_1)$ and $N(D_2 \cup H_{x_2,x_2+1} \cup f_2)$ give two disjoint punctured lens spaces in $K(r)$, which is impossible. (When $D_2 \subset D_1$, say, we have to push D_2 into a suitable direction away from D_1.)

Therefore ξ is essential in $\widehat{T}$. Then δ is a meridian disc of the solid torus W, say. Compressing $\widehat{T}$ along δ gives a 2-sphere Q on which there are two disjoint discs E_1, E_2 each containing the edges of σ_1, σ_2, respectively. Then the same argument as above gives a contradiction.

Therefore the edges of σ_i cannot lie in a disc in $\widehat{T}$ for $i = 1, 2$, and then there are disjoint essential annuli A_i in $\widehat{T}$ in which the edges of σ_i lie for $i = 1, 2$, respectively.

Suppose that $\mathrm{Int}\, f_1 \cap \widehat{T} \neq \emptyset$. Consider an innermost component η of $\mathrm{Int}\, f_1 \cap \widehat{T}$ in f_1. By Lemma 2·2, η is essential in $\widehat{T}$. As above, there are two disjoint punctured lens spaces in $K(r)$, which is impossible again. Similarly for f_2. Therefore $\mathrm{Int}\, f_i \cap \widehat{T} = \emptyset$ for $i = 1, 2$.

Let f be a face of G_S. Although $\mathrm{Int}\, f \cap \widehat{T} \neq \emptyset$ in general, a small collar neigh-

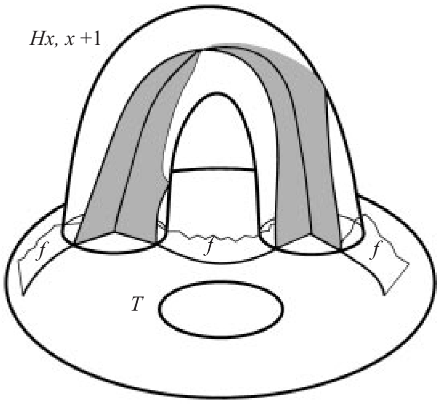

Fig. 1

bourhood of ∂f in f is contained in one side of $\widehat{T}$. Then we say that f *lies on that side of* $\widehat{T}$.

The next two lemmas deal with the situation where G_S has two Scharlemann cycles of length two and three simultaneously.

LEMMA 2·5. *Let σ be an S-cycle in G_S and let τ be a Scharlemann cycle in G_S of length three. Let f and g be the faces of G_S bounded by σ and τ respectively. If σ and τ have disjoint label pairs, then σ and τ lie on opposite sides of $\widehat{T}$ and $r \not\equiv 0 \pmod 3$.*

Proof. Let $\{x, x+1\}$, $\{y, y+1\}$ be the label pairs of σ and τ, respectively. By Lemma 2·4, the edges of σ give an essential cycle in $\widehat{T}$ after shrinking two fat vertices u_x and u_{x+1} to points and $\text{Int}\, f \cap \widehat{T} = \emptyset$. Then f is contained in the solid torus, W say, and the union $H_{x,x+1} \cup f$ gives a Möbius band B properly embedded in W, after shrinking $H_{x,x+1}$ to its core radially. (See Fig. 1.)

Similarly, by Lemma 2·4, the edges of τ lie in an essential annulus A in $\widehat{T}$ which is disjoint from the edges of σ and $\text{Int}\, g \cap \widehat{T} = \emptyset$.

Suppose that $g \subset W$. If a solid torus J is attached to W along their boundaries so that the slope of ∂B bounds a meridian disc of J, then the resulting manifold $N = J \cup W$ contains a projective plane and therefore $N = L(2, 1)$. However, in N, the edges of τ are contained in a disc D obtained by capping a boundary component of A off by a meridian disc of J. Then $N(D \cup H_{y,y+1} \cup g)$ gives a punctured lens space whose fundamental group has order three in N, which is impossible. Thus f and g lie on opposite sides of $\widehat{T}$.

Next, assume that $r \equiv 0 \pmod 3$ for contradiction. We may assume that $f \subset W$ and $g \subset U$.

By Lemma 2·1, $M = N(A \cup H_{y,y+1} \cup g)$ is a solid torus, and A runs three times in the longitudinal direction on ∂M. The annulus $A' = \text{cl}\,(\partial M - A)$ is properly embedded in U and so A' is parallel to $\text{cl}\,(\widehat{T} - A)$. Therefore, A runs three times in the longitudinal direction of U.

The slope determined by ∂B on ∂W meets a meridian of W twice. On ∂U, the slope can be expressed $a/3$ and the meridian of W defines a slope b/r for some integers a, b. Then $\Delta(a/3, b/r) = |ar - 3b| = 3|ar/3 - b| \neq 2$, which is a contradiction.

LEMMA 2·6. *Let σ, τ, f, g be as in Lemma 2·5. Suppose that σ and τ lie on opposite* *sides of $\widehat{T}$ and have the same label pair and that $r \neq 2, 3$. If there is an essential annulus A in $\widehat{T}$ in which the edges of σ and τ lie, then* $r \not\equiv 0 \pmod 3$.

Proof. By Lemma 2·3, $\operatorname{Int} f \cap \widehat{T} = \emptyset$ and $\operatorname{Int} g \cap \widehat{T} = \emptyset$. We remark that $t = 2$. Hence σ and τ have the label pair $\{1, 2\}$.

We may assume that $H_{1,2} \subset W$ and $f \subset W$. Then $M_1 = N(A \cup H_{1,2} \cup f)$ is a solid torus and A runs twice in the longitudinal direction on ∂M_1 by Lemma 2·1. Furthermore, the annulus $A_1' = \operatorname{cl}(\partial M_1 - A)$ is parallel to $\operatorname{cl}(\widehat{T} - A)$ in W. Similarly, $M_2 = N(A \cup H_{2,1} \cup g)$ is a solid torus and A runs three times in the longitudinal direction on ∂M_2 by Lemma 2·1. The annulus $A_2' = \operatorname{cl}(\partial M_2 - A)$ is also parallel to $\operatorname{cl}(\widehat{T} - A)$ in U. Then the same argument as in the proof of Lemma 2·5 gives the desired result.

3. *The generic case*

In this section we prove Theorem 1·1 under the hypothesis $t \geqslant 4$. The case $t = 2$ will be dealt with separately in the next section.

LEMMA 3·1. *Let $\{e_1, e_2, \ldots, e_t\}$ be mutually parallel edges in G_S numbered successively. Then $\{e_{t/2}, e_{t/2+1}\}$ is an S-cycle.*

Proof. We may assume that e_i has the label i at one endpoint for $1 \leqslant i \leqslant t$. If e_t has the label 1 at the other endpoint, then $\{e_{t/2}, e_{t/2+1}\}$ is an S-cycle. Therefore we may suppose that e_{2j} has the label 1 at the other endpoint for some $j < t/2$ by the parity rule. Then $\sigma_1 = \{e_j, e_{j+1}\}$ and $\sigma_2 = \{e_{t/2+j}, e_{t/2+j+1}\}$ form S-cycles with disjoint label pairs.

Let f_i be the face of G_S bounded by σ_i. By Lemma 2·4, the edges of σ_i lie in an essential annulus in $\widehat{T}$ and $\operatorname{Int} f_i \cap \widehat{T} = \emptyset$. Then, as in the proof of Lemma 2·5, we obtain two disjoint Möbius band B_1 and B_2 from $H_{j,j+1} \cup f_1$ and $H_{t/2+j,t/2+j+1} \cup f_2$ by shrinking $H_{j,j+1}$ and $H_{t/2+j,t/2+j+1}$ to their cores radially.

Since ∂B_1 and ∂B_2 are parallel in $\widehat{T}$, they divide $\widehat{T}$ into two annuli A_1 and A_2.

In G_T, u_k and u_{2j-k+1} lie in the same annulus for $1 \leqslant k \leqslant j-1$, since the edge e_k connects the two vertices in G_T. Similarly, $u_{2j+\ell}$ and $u_{t+1-\ell}$ for $1 \leqslant \ell \leqslant t/2 - j - 1$ lie in the same annulus. Therefore we see that $\operatorname{Int} A_i$ contains an even number of vertices for $i = 1, 2$. Let $F = B_1 \cup B_2 \cup A_1$. Then F meets K^* in an even number of points (after a perturbation). Then $F' = F \cap E(K)$ gives a punctured Klein bottle properly embedded in $E(K)$ having an even number of boundary components. By attaching suitable annuli in $\partial E(K)$ to F' along boundaries, we have a closed non-orientable surface in $E(K)$, which is impossible.

LEMMA 3·2. *G_S does not contain more than t mutually parallel edges.*

Proof. Let $e_1, e_2, \ldots, e_t, e_{t+1}$ be mutually parallel edges in G_S numbered successively. By Lemma 3·1, $\{e_{t/2}, e_{t/2+1}\}$ is an S-cycle. Furthermore, $\{e_t, e_{t+1}\}$ forms another S-cycle and these two S-cycles have disjoint label pairs. Then the same argument as in the proof of Lemma 3·1 gives a contradiction.

The reduced graph $\overline{G}_S$ of G_S is defined to be the graph obtained from G_S by amalgamating each set of mutually parallel edges of G_S to a single edge. If an edge $\overline{e}$ of $\overline{G}_S$ corresponds to s mutually parallel edges of G_S, then the *weight* of $\overline{e}$ is defined to be s and we denote by $w(\overline{e}) = s$. If $w(\overline{e}) = t$, then e is called a *full edge*.

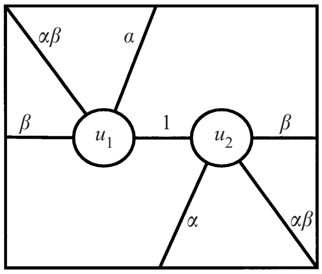

Fig. 2

Proposition 3·3. *If $t \geqslant 4$, then $r \leqslant 12g-7$, where g is the genus of K.*

Proof. Since G_S does not contain trivial loops, the unique vertex v has valency at most $12g-6$ in $\overline{G}_S$ (see [**15**, lemma 6·2]). Therefore the edges of G_S are partitioned into at most $6g-3$ families of parallel edges.

By Lemma 3·2, $w(\overline{e}) \leqslant t$ for any edge $\overline{e}$ of $\overline{G}_S$. Recall that the vertex v has valency rt in G_S. Then $rt \leqslant (12g-6)t$, hence $r \leqslant 12g-6$.

Finally, suppose that $r = 12g-6$. Then any edge of $\overline{G}_S$ is full and each face of $\overline{G}_S$ is a 3-sided disc. By Lemma 3·1, we may assume that G_S contains an S-cycle with label pair $\{t/2, t/2+1\}$ and a Scharlemann cycle of length three with label pair $\{t, 1\}$. Then $r \not\equiv 0 \pmod 3$ by Lemma 2·5, which is a contradiction. Therefore $r \leqslant 12g-7$.

4. *The case that* $t = 2$

By the parity rule, each edge of G_T connects different vertices u_1 and u_2. Then there are four *edge classes* in G_T, i.e. isotopy classes of edges of G_T in $\widehat{T}$ rel $u_1 \cup u_2$. They are called $1, \alpha, \beta, \alpha\beta$ as illustrated in Fig. 2 (see [**17**, fig. 7·1]).

We label an edge of e of G_S by the class of the corresponding edge of G_T and we call the label the *edge class label* of e.

For a face f of G_S, if a small collar neighbourhood of ∂f in f is contained in U (W), then f is said to be *black* (resp. *white*).

Lemma 4·1. *Suppose that $r \neq 2$. Then any two black* (*white*) *bigons in G_S have the same pair of edge class labels.*

Proof. By Lemma 2·3, the interior of a black (white) bigon is disjoint from $\widehat{T}$. Then the proof of [**18**, lemma 5·2] remains valid. Remark that a final contradiction comes from the fact that a Klein bottle will be found in a solid torus U or W.

Lemma 4·2. *Let e and e' be edges of G_S. If e and e' are parallel in G_S, then they have distinct edge class labels.*

Proof. If e and e' are parallel in G_S and have the same edge class label, then they are also parallel in G_T. Then $E(K)$ contains a Möbius band by [**13**, lemma 2·1], which contradicts the fact that K is hyperbolic.

Lemma 4·3. *If $r \neq 2$, then G_S cannot contain more than 3 mutually parallel edges.*

Proof. Suppose that there are 4 mutually parallel edges. Then there are two bigons with the same colour among these 4 parallel edges. By Lemma 4·1, these two bigons have the same pair of edge class labels. This contradicts Lemma 4·2.

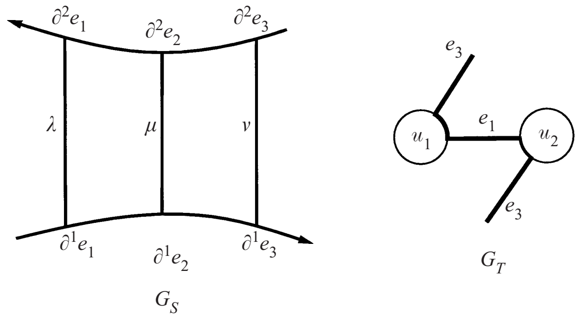

Fig. 3

LEMMA 4·4. *Suppose that $r \neq 2$. If G_S contains a black bigon and a white bigon which have an edge in common, then the other faces of G_S are not bigons.*

Proof. Let e_1, e_2, e_3 be adjacent parallel edges of G_S. By Lemma 4·2, these three edges have distinct edge class labels. Let λ, μ, ν be the edge class labels of e_1, e_2, e_3 respectively. Let us denote the endpoints of e_i by $\partial^j e_i$ for $j = 1, 2$. (See Fig. 3.)

Note that $\partial^1 e_1$ and $\partial^1 e_3$ appear consecutively around the vertex u_1 in the order, when travelling around ∂u_1 anticlockwise, say. Then $\partial^2 e_3$ and $\partial^2 e_1$ appear consecutively around u_2 in the order, when traveling around ∂u_2 clockwise. These come from the facts that r is integral and that u_1 and u_2 have distinct parities. Then there is no other edge of edge class λ (ν) than e_1 (e_3) in G_T. The conclusion follows from Lemma 4·1.

PROPOSITION 4·5. *If $t = 2$, then $r \leqslant 12g - 7$.*

Proof. The unique vertex v has valency at most $12g - 6$ in $\overline{G}_S$ and the edges of G_S are partitioned into at most $6g - 3$ families of parallel edges. Recall that v has valency $2r$ in G_S.

By Lemma 4·3, G_S cannot contain 4 mutually parallel edges. If G_S contains 3 mutually parallel edges, then we have $r \leqslant (6g - 3) + 2 = 6g - 1$ by Lemma 4·4.

If G_S does not contain 3 mutually parallel edges, then each edge of $\overline{G}_S$ has weight 1 or 2. Hence $r \leqslant 2(6g - 3) = 12g - 6$.

Suppose that $r = 12g - 6$. Then any edge of $\overline{G}_S$ is full and hence G_S has $6g - 3$ black, say, bigons and each white face of G_S is a 3-sided disc. Therefore there are an S-cycle σ and a Scharlemann cycle τ of length three in G_S with the same label pair $\{1, 2\}$. By Lemma 4·1, all black bigons have the same pair of edge class label $\{\lambda, \mu\}$, say. Then the edges of τ have the same edge class labels λ, μ by Lemma 2·3. This means that there is an essential annulus A in $\widehat{T}$ which contains the edges of σ and τ. By Lemma 2·6, we have $r \not\equiv 0 \pmod 3$, which is a contradiction. Therefore $r \leqslant 12g - 7$.

Proof of Theorem 1·1. This follows immediately from Propositions 3·3 and 4·5.

5. *Genus one case*: *the case $t \geqslant 4$*

In the remainder of this paper, K is assumed to be a genus one, hyperbolic knot in S^3 in order to prove Theorem 1·2. First, we deal with the case $t \geqslant 4$ in this section.

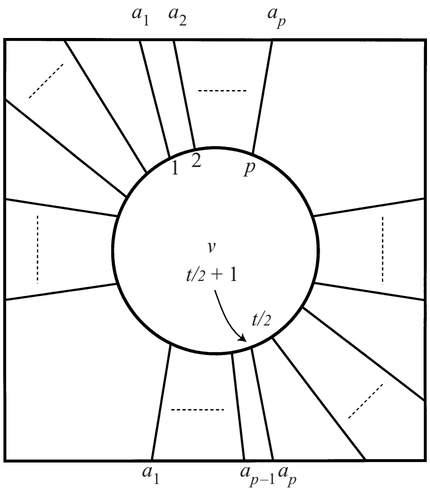

Fig. 4

Theorem 5·1. *If K has genus one, then $K(r)$ is neither $L(2,1)$ nor $L(4,1)$.*

Proof. This follows from [**6**, **27**].

Lemma 5·2. *If r is odd, then G_S cannot have more than $t/2$ mutually parallel edges.*

Proof. The vertex v has valency rt in G_S. Recall that the edges of G_S are partitioned into at most three families of mutually parallel edges. Let A be a family of mutually parallel edges in G_S and suppose that A consists of more than $t/2$ edges, $a_1, a_2, \ldots, a_p$ numbered consecutively. Note that $p \leqslant t$ by Lemma 3·2. We may assume that a_i has the label i at one endpoint for $1 \leqslant i \leqslant p$. Then a_p has the label $t/2+1$ at the other endpoint, since r is odd. (See Fig. 4.)

By the parity rule, $p \neq t/2+1$. Thus $p > t/2+1$. Then $\{a_{(t/2+p)/2}, a_{(t/2+p)/2+1}\}$ forms an S-cycle. Furthermore, some edge between a_2 and $a_{t/2}$ has the label 1 at the other endpoint. Therefore, there is another S-cycle whose label pair is disjoint from that of the above S-cycle. Then the same construction as in the proof of Lemma 3·1 gives a contradiction.

By Proposition 3·3, we have that $r \leqslant 5$. In fact, the cases that $r = 3$ and 5 remain by Theorem 5·1.

Lemma 5·3. *The case that $r = 3$ is impossible.*

Proof. The vertex v has valency $3t$ in G_S. By Lemma 5·2, G_S consists of three families of mutually parallel edges, each containing exactly $t/2$ edges. Then there is no S-cycle in G_S, but there are two Scharlemann cycles τ_1 and τ_2 of length three in G_S. Let g_i be the face of G_S bounded by τ_i for $i = 1, 2$. We may assume that g_1 has the label pair $\{t, 1\}$ and g_2 has $\{t/2, t/2+1\}$.

By Lemma 2·4, there are disjoint essential annuli A_i in $\widehat{T}$ in which the edges of τ_i lie and $\mathrm{Int}\, g_i \cap \widehat{T} = \emptyset$ for $i = 1, 2$.

Claim 5·4. The faces g_1 and g_2 lie on opposite sides of $\widehat{T}$.

Proof of Claim 5·4. Suppose that $g_i \subset W$, say, for $i = 1, 2$. Let s be the slope on ∂W determined by the essential annuli A_i. Performing s-Dehn filling on W, that is, attaching a solid torus J to W along their boundaries so that s bounds a meridian disc of J, we obtain a closed 3-manifold M, which is either S^3, $S^2 \times S^1$ or a lens space. However, there are two disjoint discs D_1 and D_2, which contain the edges of τ_1 and τ_2, respectively, on the 2-sphere Q obtained by compressing $\widehat{T}$ along s by a meridian disc of J. Then $N(D_1 \cup H_{t,1} \cup g_1)$ and $N(D_2 \cup H_{t/2,t/2+1} \cup g_2)$ give two punctured lens spaces in M, which is impossible.

Therefore, $t/2$ and t must have opposite parities, and so $t/2$ is odd. In particular, $t \geqslant 6$.

In G_S, there are exactly three edges whose endpoints have the pair of labels $\{j, t+1-j\}$ for $j = 1, 2, \ldots, t/2$. Therefore, G_T consists of $t/2$ components, each consisting two vertices u_j and u_{t+1-j} along with three edges connecting them.

Claim 5·5. Each component of G_T does not lie in a disc in $\widehat{T}$.

Proof of Claim 5·5. If there is a component of G_T which lie in a disc in $\widehat{T}$, then we can take an innermost one Λ. That is, Λ lies in a disc D in $\widehat{T}$ and there is no other component of G_T in D. Consider the intersection between D and S. Then S is divided into two discs g_3 and g_4 by the edges of Λ. By the cut-and-paste operation of g_3 or g_4, and taking D by a smaller one, if necessary, we can assume that $\operatorname{Int} g_3$ and $\operatorname{Int} g_4$ do not meet D. Then $N(D \cup V \cup g_3 \cup g_4)$, where V is the attached solid torus, gives a connected sum of two lens spaces minus an open 3-ball in $K(r)$, which is impossible.

Thus, we may assume that A_i contains only the edges and vertices of τ_i for $i = 1, 2$.

Assume that $g_1 \subset W$ and $g_2 \subset U$. Let $M_1 = N(A_1 \cup H_{t,1} \cup g_1)$ and $M_2 = N(A_2 \cup H_{t/2,t/2+1} \cup g_2)$. Let $A'_i = \operatorname{cl}(\partial M_i - A_i)$ for $i = 1, 2$. Then A'_1 is a properly embedded annulus in W and A'_2 is a properly embedded annulus in U. By Lemma 2·1, M_i is a solid torus such that the core of A_i runs three times in the longitudinal direction of M_i for $i = 1, 2$. Therefore, A'_1 is parallel to the annulus $\operatorname{cl}(\widehat{T} - A_1)$ in W and A'_2 is parallel to $\operatorname{cl}(\widehat{T} - A_2)$ in U. Let $\widehat{T}' = (\widehat{T} - (A_1 \cup A_2)) \cup A'_1 \cup A'_2$. Then it is easy to see that $\widehat{T}'$ is a new Heegaard torus in $K(r)$ such that $|\widehat{T}' \cap V| = t - 4$ (> 0). Furthermore, $\widehat{T}'$ satisfies $(*)$, which contradicts the choice of $\widehat{T}$.

LEMMA 5·6. *The case that $r = 5$ is impossible.*

Proof. Since the vertex v has valency $5t$ in G_S, there are more than $t/2$ mutually parallel edges in G_S, which contradicts Lemma 5·2.

Proof of Theorem 1·2 *when* $t \geqslant 4$. By Proposition 3·3, $r \leqslant 5$, and in fact, the remaining cases are $r = 3, 5$ by Theorem 5·1. But these cases are impossible by Lemmas 5·3 and 5·6.

6. *Genus one case*: *the case* $t = 2$

In the case that $t = 2$, the following lemma plays a key role.

Recall that an *unknotting tunnel* γ for a knot or link K in S^3 is a simple arc properly embedded in the exterior $E(K)$ such that $\operatorname{cl}(E(K) - N(\gamma))$ is homeomorphic to a handlebody of genus two.

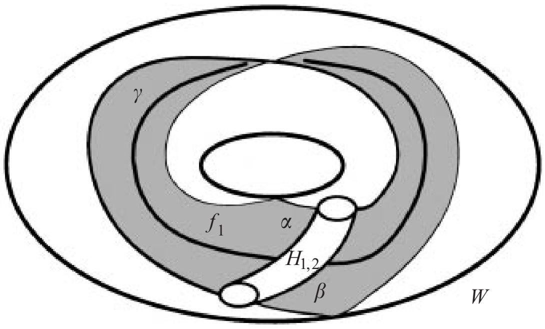

Fig. 5

Lemma 6·1. *Let K be a genus one knot in S^3 and let S be a minimal genus Seifert surface of K. If K has an unknotting tunnel γ such that $\gamma \subset S$, then K is 2-bridge.*

Proof. Take a regular neighbourhood N of γ in S. Let $F = \mathrm{cl}(S - N)$. Then F is an annulus whose boundary defines a link L in S^3. Note that F is incompressible in the exterior of L and L has an unknotting tunnel. (If F is compressible, then L is a trivial link. Since such a link has the unique unknotting tunnel [**24**], namely the obvious one, this means that K is trivial.) Then L is a 2-bridge torus link by [**7**, theorem 1]. Furthermore, an unknotting tunnel of such a link is determined by [**1**]. Then S can be restored by taking the union of F and N, showing that K is 2-bridge.

Lemma 6·2. *Suppose that K has genus one. Then G_S cannot have more than two mutually parallel edges.*

Proof. If there are three mutually parallel edges in G_S, there are two S-cycles σ_1 and σ_2 whose faces f_1 and f_2 lie on opposite sides of $\widehat{T}$. Since $K(r) \neq L(2, 1)$ by Theorem 5·1, we can assume that $\mathrm{Int}\, f_i \cap \widehat{T} = \emptyset$ for $i = 1, 2$ by Lemma 2·3. Then we may also assume that $f_1, H_{1,2} \subset W$ and $f_2, H_{2,1} \subset U$. Note that $\mathrm{cl}(W - H_{1,2})$ and $\mathrm{cl}(U - H_{2,1})$ are handlebodies of genus two, since the core of $H_{1,2}$ ($H_{2,1}$) lies on a Möbius band which is obtained from $H_{1,2} \cup f_1$ ($H_{2,1} \cup f_2$) by shrinking $H_{1,2}$ (resp. $H_{2,1}$) to its core radially.

Let α and β be the arc components of $f_1 \cap H_{1,2}$. Let γ be a simple arc in f_1 which connects a point in α with one in β. (See Fig. 5.)

Then it can be seen that $\mathrm{cl}(W - H_{1,2} - N(\gamma))$ is homeomorphic to $T \times I$, where I denotes an interval. Therefore, γ gives an unknotting tunnel of K which lies on S. By Lemma 6·1, K is 2-bridge, which contradicts the fact that a hyperbolic 2-bridge knot has no cyclic surgery [**26**].

The remaining cases are $r = 3, 5$ again by Proposition 4·5 and Theorem 5·1.

Lemma 6·3. *The case $r = 3$ is impossible.*

Proof. Recall that $\widehat{T}$ is separating in $K(r)$ and therefore the faces of G_S are partitioned into black and and white ones. This implies that G_S has no parallel edges, since G_S has just three edges. Then there are two Scharlemann cycles τ_1 and τ_2 of length three in G_S. Let g_i be the face of G_S bounded by τ_i for $i = 1, 2$. Clearly, g_1 and g_2 lie on opposite sides of $\widehat{T}$. The edges of τ_i are all edges of G_T. In particular, τ_1 and τ_2 have their edges in common.

Claim 6·4. The edges of τ_i cannot lie in a disc in $\widehat{T}$ for $i = 1, 2$.

Proof of Claim 6·4. Suppose that the edges of τ_1 (and therefore τ_2) lie in a disc D in $\widehat{T}$. By the cut-and-paste operation of g_i, we can assume that $\operatorname{Int} g_i \cap D = \varnothing$ for $i = 1, 2$. Then $N(D \cup V \cup g_1 \cup g_2)$ gives a connected sum of two lens spaces minus an open 3-ball, in $K(r)$, which is impossible.

Thus there is an essential annulus A in $\widehat{T}$ which contains G_T by Lemma 2·1. In particular, G_T has exactly one pair of parallel edges.

Claim 6·5. $\operatorname{Int} g_i \cap \widehat{T} = \varnothing$ for $i = 1, 2$.

Proof of Claim 6·5. Suppose that $\operatorname{Int} g_1 \cap \widehat{T} \neq \varnothing$. Let ξ be an innermost component of $\operatorname{Int} g_1 \cap \widehat{T}$ in g_1 and let δ be the disc bounded by ξ on g_1. By the assumption on the loops in $S \cap T$ stated in Section 2, ξ is essential in T and then ξ is parallel to ∂A. We may suppose that $\delta \subset W$. Then δ is a meridian disc of W. Let $H = V \cap W$ and let $g_j \cap H \neq \varnothing$ for some $j \in \{1, 2\}$.

Let Q be the 2-sphere obtained by compressing ∂W along δ and let B be the 3-ball bounded by Q in W. On Q, there is a disc E which contains the edges of τ_j. After the components of $\operatorname{Int} g_j \cap Q$ are removed by the cut-and-paste operation of g_j, $N(E \cup H \cup g_j)$ gives a punctured lens space in B, which is impossible. Therefore, $\operatorname{Int} g_1 \cap \widehat{T} = \varnothing$. Similarly for g_2.

Now, we may assume that $g_1 \subset W$ and $g_2 \subset U$ and that $H_{1,2} = V \cap W$ and $H_{2,1} = V \cap U$. As in the last paragraph of the proof of Lemma 5·3, let $M_1 = N(A \cup H_{1,2} \cup g_1)$ and $M_2 = N(A \cup H_{2,1} \cup g_2)$. Then M_i is a solid torus and A runs three times on M_i in the longitudinal direction for $i = 1, 2$ by Lemma 2·1. Let $A'_i = \operatorname{cl}(\partial M_i - A)$, then A'_i is parallel to the annulus $B = \operatorname{cl}(\widehat{T} - A)$ in W if $i = 1$, or U if $i = 2$.

Claim 6·6. $\operatorname{cl}(U - H_{2,1})$ is a handlebody of genus two.

Proof of Claim 6·6. The torus $A'_2 \cup B$ bounds a solid torus U' in U, which represents the parallelism of A'_2 and B. Then it can be seen that $\operatorname{cl}(U - H_{2,1})$ is obtained from U' by attaching a 1-handle $N(g_2)$. Hence we have the desired result.

Claim 6·7. Let e be one of the parallel edges in G_T. Then e is an unknotting tunnel of K.

Proof of Claim 6·7. Note that $\operatorname{cl}(W - H_{1,2})$ is homeomorphic to $\operatorname{cl}(M_1 - H_{1,2})$. Let $k = K^* \cap W$, where K^* is the core of V. Then k is a properly embedded arc in M_1 and $\operatorname{cl}(M_1 - H_{1,2}) = \operatorname{cl}(M_1 - N(k))$. Push e into W slightly. It can be assumed that $\partial e \subset \partial N(k)$. (See Fig. 6.)

Then it is not hard to see that $\operatorname{cl}(M_1 - N(k) \cup N(e))$ has a product structure $T \times I$. Since $\operatorname{cl}(U - H_{2,1})$ is a handlebody of genus two by Claim 6·6, $\operatorname{cl}(E(K) - N(e))$ is a handlebody of genus two, which gives the desired conclusion.

By Lemma 6·1, K is 2-bridge, and this means that the case $r = 3$ is impossible [**26**].

LEMMA 6·8. *The case $r = 5$ is impossible.*

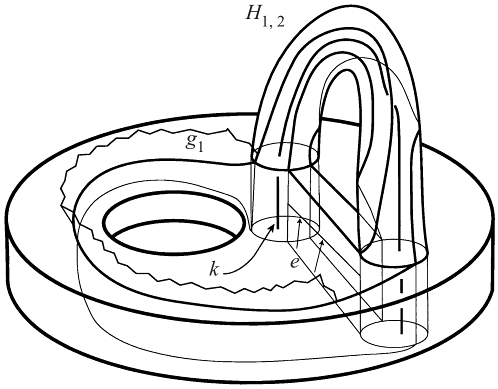

Fig. 6

Proof. G_S has exactly five edges. By Lemma 6·2, these edges of G_S are partitioned into three families, two pairs of parallel edges and one edge which is not parallel to the others. However, this configuration contradicts the fact that the faces of G_S are divided into black and white sides.

Proof of Theorem 1·2 *when* $t = 2$. By Proposition 4·5 and Theorem 5·1, the remaining cases are $r = 3, 5$. These are impossible by Lemmas 6·3, 6·8. This completes the proof of Theorem 1·2.

Proof of Theorem 1·3. Let K be a genus one knot in S^3 and suppose that $K(r)$ is a lens space. By Theorem 1·2, K is not hyperbolic and therefore it is either a satellite knot or a torus knot. If a satellite knot admits cyclic surgery, then it is a cable knot of a torus knot [**3**, **29**, **31**]. In particular, its genus is greater than 1. Thus we have that K is a torus knot and so K is the trefoil. The constraint on the slopes follows from [**21**]. The converse is obvious.

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REFERENCES

[**1**] C. Adams and A. Reid. Unknotting tunnels in two-bridge knot and link complements. *Comment. Math. Helv.* **71** (1996), 617–627.

[**2**] J. Berge. Some knots with surgeries yielding lens spaces, unpublished manuscript.

[**3**] S. Bleiler and R. Litherland. Lens spaces and Dehn surgery. *Proc. Amer. Math. Soc.* **107** (1989), 1127–1131.

[**4**] M. Culler, C. McA. Gordon, J. Luecke and P. Shalen. Dehn surgery on knots. *Ann. Math.* **125** (1987), 237–300.

[**5**] C. Delman and R. Roberts. Alternating knots satisfy strong property P. *Comment. Math. Helv.* **74** (1999), 376–397.

[**6**] M. Domergue. Dehn surgery on a knot and real 3-projective space. In *Progress in knot theory and related topics* (eds. M. Boileau, M. Domergue, Y. Mathieu and K. Millet) Travaux en cours **56** (Hermann, 1997), pp. 3–6.

[**7**] M. Eudave-Muñoz and Y. Uchida. Non-simple links with tunnel number one. *Proc. Amer. Math. Soc.* **124** (1996), 1567–1575.

[**8**] R. Fintushel and R. Stern. Constructing lens spaces by surgery on knots. *Math. Z.* **175** (1980), 33-51.

[**9**] D. Gabai. Foliations and topology of 3-manifolds, III. *J. Diff. Geom.* **26** (1987), 479–536.

[**10**] C. McA. Gordon. Dehn surgery on knots. In *Proceedings of the International Congress of Mathematicians* (Math. Soc. Japan, 1991), pp. 631–642.

[**11**] C. McA. Gordon. Combinatorial methods in Dehn surgery. In *Lectures at Knots '96* (ed. S. Suzuki) (World Sci. Publishing, 1997), pp. 263–290.

[**12**] C. McA. Gordon. Dehn filling: a survey. In *Knot theory* (Banach Center Publ., 1998), pp. 129–144.

[**13**] C. McA. Gordon. Boundary slopes on punctured tori in 3-manifolds. *Trans. Amer. Math. Soc.* **350** (1998), 1713–1790.

[**14**] C. McA. Gordon. Toroidal Dehn surgeries on knots in lens spaces. *Math. Proc. Camb. Phil. Soc.* **125** (1999), 433–440.

[**15**] C. McA. Gordon and R. Litherland. Incompressible planar surfaces in 3-manifolds. *Topology Appl.* **18** (1984), 121–144.

[**16**] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.* **2** (1989), 371–415.

[**17**] C. McA. Gordon and J. Luecke. Dehn surgeries on knots creating essential tori, I. *Comm. Anal. Geom.* **3** (1995), 597–644.

[**18**] C. McA. Gordon and J. Luecke. Toroidal and boundary-reducing Dehn fillings. *Topology Appl.*, to appear.

[**19**] M. Hirasawa and K. Shimokawa. Dehn surgeries on strongly invertible knots which yield lens spaces. *Proc. Amer. Math. Soc.*, to appear.

[**20**] J. Luecke. Dehn surgery on knots in the 3-sphere. In *Proceedings of the International Congress of Mathematicians, Zürich* (Birkhäuser Verlag, 1995), 585–594.

[**21**] L. Moser. Elementary surgery along a torus knot. *Pacific J. Math.* **38** (1971), 737–745.

[**22**] J. P. Neuzil. Elementary surgery manifolds and the elementary ideals. *Proc. Amer. Math. Soc.* **68** (1978), 225–228.

[**23**] D. Rolfsen. *Knots and links.* Mathematics Lecture Series **7** (Publish or Perish, 1976).

[**24**] J. K. Simon and K. Wolcott. Minimally knotted graphs in S^3. *Topology Appl.* **37** (1990), 163–180.

[**25**] J. R. Stallings. Constructions of fibred knots and links. In *Proc. Sympos. Pure Math.* **32** (Amer. Math. Soc., 1978), 55-60.

[**26**] M. Takahashi. *Two bridge knots have property P*. Memoirs Amer. Math. Soc. **29** (Amer. Math. Soc., 1981).

[**27**] M. Teragaito. Cyclic surgery on genus one knots. *Osaka J. Math.* **34** (1997), 145–150.

[**28**] M. Teragaito. Dehn surgery on genus one knots and Seifert fibred manifolds. In *Proc. Appl. Math. Workshop* **8**, (eds. K. H. Ko and G. T. Jin) (KAIST, 1998), pp. 315–325.

[**29**] S. Wang. Cyclic surgery on knots. *Proc. Amer. Math. Soc.* **107** (1989), 1091–1094.

[**30**] S. Wang and Q. Zhou. Symmetry of knots and cyclic surgery. *Trans. Amer. Math. Soc.* **330** (1992), 665–676.

[**31**] Y. Q. Wu. Cyclic surgery and satellite knots, *Topology Appl.* **36** (1990), 205–208.