Inverse Problems in Heat Exchange Processes

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Abstract: This paper deals with an inverse problem concerning the identification of the heat exchange coefficient H (assumed depending on the temperature) between a certain material with the external environment (see, e.g., [2], [4] for real applications modelled with equations involving this coefficient). Only experimental measurements of the temperature are supposed to be known. The goal is to identify H in order to get a solution for the corresponding model, approximating same given temperature measurements. We begin by setting several scenarios for the inverse problem. For each scenario, we know the initial and ambient temperatures, identify function H through different methods and obtain error bounds in adequate norms (uniform and square integrable). Finally, we study the inverse problem in the framework of the classical theory for Hilbert spaces. Several methods are used (Tikhonov, Morozov, Landweber,. . .) and the approximations obtained, as well as the one provided by the previous algorithm, are shown.

Key–Words: Model identification, Heat exchange, Regularization strategies.

1 Description of the inverse problem and their physical motivation.

Let us suppose we have a homogeneous sample of a material that is getting warm (respectively, cool) due to heat exchange with the external environment. For simplicity, let us suppose that the sample is small enough to be able to assume that the temperature gradient inside it is negligible. The Newton Cooling Law provides a simple mathematical model describing this phenomenon through the following initial value problem (*direct problem*):

$$\begin{cases} T'(t) = H(T(t))(T^e - T(t)),\ t \geq t_0 \\ T(t_0) = T_0, \end{cases} \qquad (1)$$

where $T(t)$ is the temperature of the sample at time t, T^e is the external environment temperature, T_0 is the temperature at the initial time t_0 and H is the temperature dependent heat exchange coefficient. To solve problem (1) we need to know the model data: constants $T_0, T^e \in \mathbb{R}$ and function $H(\cdot) : (T_a, T_b) \to \mathbb{R}$, where (T_a, T_b) is a range of temperatures suitable for the problem we are considering.

In real cases, the values of T_0 and T^e can be obtained through simple devices measuring temperature. However, obtaining function $H(\cdot)$ is not so easy by experimental methods. The goal of this work is to solve the *inverse problem* of identify $H(\cdot)$, knowing just certain experimental measurements of temperature.

In certain contexts, and under certain conditions, it can be assumed that H has a known expression (e.g., H constant or a function with a few real parameters to identify). The challenge that we face in this work is to identify function H when continuity and positivity are the only information available about H.

2 Scenarios of the inverse problem.

The model is not very sensitive to changes in $H(s)$ for s close to T^e in the following sense: if for some t_μ, $T(t_\mu) = T^e - \mu$ then, monotonicity of T implies that T remains in the interval $[T^e - \mu, T^e]$ for every $t \geq t_\mu$ and arbitrary values of H. For this reason, it is unrealistic (and unnecessary) pretend to identify H near T^e. These considerations lead us to pose the problem of identifying function H as follows:

i) A *threshold* $\mu > 0$, depending on the admissible error in the approximation of the temperature, is fixed so that the identification of H in the interval $[T^e - \mu, T^e]$ is not part of our goal. From this threshold, a time $t_f = t_f(\mu, T_0, T^e, H)$ is determined (by arguments explained later) such that

$$|T^e - T(t)| < \mu,\ t \geq t_f.$$

Thus, the error in the temperature will be smaller than μ for $t \geq t_f$.

ii) We use model (1) in $[t_0, t_f]$ and identify H in $[T_0, T(t_f)] \supset [T_0, T^e - \mu]$.

According to the available information about $T(t)$ in $[t_0, t_f]$ we set the inverse problem in several scenarios:

- The trivial (and unrealistic) case is to suppose that functions $T(t)$ and $T'(t)$ are known in $[t_0, t_f]$. Then, assuming $H \in \mathcal{C}([T_0, T(t_f)])$ and positive, we can identify H in a direct way from

$$H(s) = \frac{T'(T^{-1}(s))}{T^e - s}. \qquad (2)$$

- If function T can be **evaluated without error** in a **finite number** of **arbitrary instants** $t \in [t_0, t_f]$, the identification of H in $[T_0, T(t_f)]$ becomes a standard problem of numerical differentiation (in order to approximate $T'(t)$ from data).

- Next scenario arises when a function $\widetilde{T}$, representing an **approximate value** of the temperature **in every instant**, is supposed to be known.

- However, in a realistic context, only **discrete values of $\widetilde{T}_k$ approximating the temperature** are available.

Last two scenarios need a "stable" method to approach $T'(t)$ from data. Then, formula (2) provides discrete values approximating H in points of interval $[T_0, T(t_f)]$.

Let us see how to determine t_f in the non trivial situations described before:

a) In the second scenario, given $p + 1$ exact values $\{T_0, T_1, \ldots, T_p\}$ of the temperature at instants $\{\tau_0 = t_0, \tau_1, \ldots, \tau_p\}$, we consider $\mu_k = T^e - T_k$. Then μ is chosen as one of the values μ_k or any number smaller than them. We take $t_f = \tau_m$, where

$$m = \begin{cases} p, \text{ if } \mu < \mu_k \text{ for all } k. \\ \min\limits_k \{\mu = \mu_k\}, \text{ otherwise.} \end{cases} \qquad (3)$$

b) The assumptions in the third scenario are that function $\widetilde{T}$ is known in some interval $[t_0, t^*]$ and

$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([t_0,t^*])} < \delta,$$

where $\delta < \mu$ (if $\mu \leq \delta$ it is necessary to increase the value of μ). Then, we consider t_f as

$$t_f = \begin{cases} t^*, \text{ if } \widetilde{T}(t) < T^e - \mu + \delta \text{ for all } t \leq t^* \\ \min\limits_t \{\widetilde{T}(t) = T^e - \mu + \delta\}, \text{ otherwise.} \end{cases} \qquad (4)$$

c) Finally, in the fourth scenario, t_f is defined in a more sophisticated way. Measurements $\{\widehat{T}_k\}_{k=0}^p$ such that $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, with $\widehat{\delta} > 0$, are available. Let $\widetilde{T}$ be an interpolation function of values $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ in $\{\tau_0, \tau_1, \ldots, \tau_p\}$, let $\delta > 0$ such that

$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([\tau_0,\tau_p])} < \delta$$

and take $\mu_k = T^e - \widehat{T}_k + \delta$ for $k = 1, 2, \ldots, p$. Now, we assume that $\mu > 3\delta$ (otherwise, the value of μ will be increased) and that μ is lower or equal than all previous values μ_k. Then, taking m as in (3), we define

$$t_f = \tau_m. \qquad (5)$$

3 A first approach to the inverse problem.

3.1 Identifying from a finite amount of exact values of temperature.

Given $n \in \mathbb{N}$, the values of the temperature T at $t_k = t_0 + kh$ for $k = 0, 1, \ldots, n$, are supposed to be known, where $h = \dfrac{t_f - t_0}{n}$. Lets denote $T_k = T(t_k)$, $k = 0, 1, \ldots, n$. The differential equation of problem (1) can be rewritten as

$$\frac{T'(t)}{T^e - T(t)} = H(T(t)),\ t_0 < t < t_f. \qquad (6)$$

Therefore, our goal is to find, for $k = 0, 1, \ldots, n$, an approximation $\widetilde{H}_k$ of

$$\frac{T'(t_k)}{T^e - T(t_k)},$$

which is also an approximation of $H(T_k)$. Considering the first order approximate differentiation operator

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$R_h : \mathcal{C}([t_0, t_f]) \to \mathcal{C}([t_0, t_f])$ given by

$$R_h(v)(t) = \begin{cases} \Phi_h(v)(t), & t \in [t_0, \hat{t}] \\ \Psi_h(v) + \Phi_h(v)(t-h), & t \in [\hat{t}, t_f] \end{cases}$$

where $\hat{t} = t_f - h$,

$$\Phi_h(v)(t) = \frac{v(t+h) - v(t)}{h}$$

and

$$\Psi_h(v) = \frac{v(t_f) - 2v(t_f - h) + v(t_f - 2h)}{h}.$$

Let us denote by $||\cdot||$ the norm in $\mathcal{C}([t_0, t_f])$. The following result holds:

Lemma 1 *If* $v \in \mathcal{C}^2([t_0, t_f])$ *then*

$$||v' - R_h(v)|| \le \frac{7h}{2} ||v''|| .$$

In order to approach $H(T_k)$ we take

$$\widetilde{H}_k = \frac{R_h(T)(t_k)}{T^e - T_k},$$

for $k = 0, 1, \ldots, n$. Thus, the following bound for the error is obtained:

Proposition 2 *If* $T \in \mathcal{C}^2([t_0, t_f])$ *then*

$$\max_{k=0,1,\ldots,n} \left| H(T_k) - \widetilde{H}_k \right| \le \frac{7M_2}{2\mu} h, \quad (7)$$

where $M_2 = ||T''||_{\mathcal{C}([t_0, t_f])}$.

Remark 3 Note that this estimate for the error in H has the same order as the approximate differentiation method used. Thus, if an upper order method is chosen, the estimate (7) will be better. □

Remark 4 As noted at the beginning of Section 2, t_f is fixed, *a priori*, from the value of μ. Then, the bound in estimate (7) does not blow up. □

3.2 Identifying from a function that approximates the temperature.

In this context, we suppose to know a function $\widetilde{T} \in \mathcal{C}([t_0, t_f])$, where t_f is chosen according (4) and

$$\left|\left| T - \widetilde{T} \right|\right| < \delta \quad (8)$$

for $0 < \delta < \mu$. For the sake of simplicity and consistency with the properties of T, we assume that $\widetilde{T}(t) \ge T_0,\ t \in [t_0, t_f]$. From (6), we define

$$u(t) = \frac{T'(t)}{T^e - T(t)},\ t_0 < t < t_f$$

and the approximation

$$\widetilde{u}_h(t) = \frac{R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)},\ t_0 < t < t_f.$$

Next, an error estimate is obtained:

Proposition 5 *If* $T \in \mathcal{C}^2([t_0, t_f])$ *and* $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ *satisfies* (8) *with* $0 < \delta < \frac{\mu}{3}$, *then*

$$||u - \widetilde{u}_h|| \le \frac{1}{\mu - 2\delta} \left(\frac{7M_2}{2} h + \frac{3\delta}{h} \frac{T^e - T_0 + \mu - 2\delta}{\mu - 3\delta} \right). \quad (9)$$

The following result determines how to optimize the above estimate by choosing a suitable step time h:

Proposition 6 *Under the assumptions of Proposition 5, the minimum value for the right hand side in* (9) *is obtained for*

$$h^* = \sqrt{\frac{6(T^e - T_0 + \mu - 2\delta)}{7(\mu - 3\delta)M_2}\,\delta}. \quad (10)$$

In this case, estimate (9) *becomes*

$$||u - \widetilde{u}_{h^*}|| \le \frac{1}{\mu - 2\delta} \sqrt{\frac{42M_2(T^e - T_0 + \mu - 2\delta)}{\mu - 3\delta}\,\delta}.$$

From Proposition 6, choosing h^* as in (10), taking n as the entire part of $\frac{t_f - t_0}{h^*}$, denoting $t_k = t_0 + kh^*$, $\widetilde{T}_k = \widetilde{T}(t_k)$ and

$$\widetilde{H}_k = \widetilde{u}_{h^*}(t_k) = \frac{R_{h^*}(\widetilde{T})(t_k)}{T^e - \widetilde{T}_k} \quad (11)$$

for $k = 0, 1, \ldots, n$, we obtain the main result of this section:

Theorem 7 *If* $H \in \mathcal{C}^1([T_0, T^e])$ *and* $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ *satisfies* (8) *with* $0 < \delta < \frac{\mu}{3}$, *then*

$$\max_{k=0,1,\ldots,n} \left| H(\widetilde{T}_k) - \widetilde{H}_k \right| \le \delta \, ||H'||_{\mathcal{C}([T_0, T^e])} + \frac{1}{\mu - 2\delta} \sqrt{\frac{42M_2(T^e - T_0 + \mu - 2\delta)}{\mu - 3\delta}\,\delta} = O(\sqrt{\delta}).$$

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The above result can be rewrite in terms of the second derivative of T and the extreme values of H in $[T_0, T^e]$:

Corollary 8 *The assumptions of Theorem 7 imply*

$$\max_{k=0,1,\dots,n} \left| H(\widetilde{T}_k) - \widetilde{H}_k \right| \le \left(\frac{M_2}{m(\mu - 3\delta)} + M \right) \frac{\delta}{\mu - 3\delta} + \frac{1}{\mu - 2\delta} \sqrt{\frac{42 M_2 (T^e - T_0 + \mu - 2\delta)}{\mu - 3\delta}}\, \delta,$$

where $m = \min_{z \in [T_0, T^e]} H(z)$ *and* $M = \max_{z \in [T_0, T^e]} H(z)$.

3.3 Identifying from a finite number of values of the temperature.

We assume that the interpolation method used is such that the error δ between T and $\widetilde{T}$, and the measurement error $\widehat{\delta}$, are of the same order, i.e., $\delta = C\widehat{\delta}$.

For example, if $\widetilde{T}$ is the piecewise linear interpolation of measurements $\{\widehat{T}_0, \widehat{T}_1, \dots, \widehat{T}_p\}$ and we denote T_{int} the piecewise linear interpolation of values of T at points τ_k, the monotonicity of T provides

$$\begin{aligned} \left|\left| T - \widetilde{T} \right|\right| &\le ||T - T_{\text{int}}|| + \left|\left| T_{\text{int}} - \widetilde{T} \right|\right| \\ &\le \max_{1 \le k \le p} |T(\tau_k) - T(\tau_{k-1})| + \widehat{\delta} \\ &\le \max_{1 \le k \le p} \left(|\widetilde{T}(\tau_k) - \widetilde{T}(\tau_{k-1})| + 2\widehat{\delta} \right) + \widehat{\delta} \\ &= \max_{1 \le k \le p} |\widehat{T}_k - \widehat{T}_{k-1}| + 3\widehat{\delta}. \end{aligned}$$

Therefore, when the interpolation considered is the piecewise linear interpolation, if the difference between consecutive measurements is of order $\widehat{\delta}$, then δ and $\widehat{\delta}$ are of the same order. The number of measurements will be increased if needed.

3.3.1 Algorithm for determining H

The input data are: $\{\widehat{T}_k\}_{k=0}^p$, $\widehat{\delta} > 0$ and the admissible threshold $\mu > 0$. First of all, we construct a function $\widetilde{T}(t)$ interpolating $\{\widehat{T}_k\}_{k=0}^p$. Then, we estimate the error $\delta > 0$ due to the interpolation. Next, t_f is fixed from δ and μ.

The algorithm is based on an iterative process beginning from an initial guest Λ_2 for M_2. From this value, the time step is calculated by

$$h = \sqrt{\frac{6(T^e - T_0 + \mu - 2\delta)}{7(\mu - 3\delta)\Lambda_2}}\, \delta, \qquad (12)$$

according to (10). With this election of h, the corresponding values $\widetilde{H}_k$ are obtained. Approximating T'' by (13) in nodes t_k and taking the absolute maximum, a new Λ_2 (and a new h) is obtained, and so on. This iterative process finishes when h stabilizes. Since

$$T'' = \Big(H'(T)(T^e - T) - H(T) \Big) H(T)(T^e - T),$$

we approximate $T''(t_k)$ as:

$$\begin{cases} \left(\dfrac{\widetilde{H}_{k+1} - \widetilde{H}_k}{\widetilde{T}_{k+1} - \widetilde{T}_k} (T^e - \widetilde{T}_k) - \widetilde{H}_k \right) \widetilde{H}_k (T^e - \widetilde{T}_k), \\ \hfill k = 0, 1, \dots, n-1 \\ \left(\dfrac{\widetilde{H}_n - \widetilde{H}_{n-1}}{\widetilde{T}_n - \widetilde{T}_{n-1}} (T^e - \widetilde{T}_n) - \widetilde{H}_n \right) \widetilde{H}_n (T^e - \widetilde{T}_n). \end{cases} \qquad (13)$$

Algorithm

DATA	$\{\widehat{T}_k\}_{k=0}^p$: measurements of $T(t_k)$.
	$\widehat{\delta} > 0$: bound for the measurement errors.
	$\mu > 0$: threshold.
	ε: stopping test precision.
	Λ_2: initial guest for M_2.
Step 1:	Determine δ according to $\widehat{\delta}$.
Step 2:	Fix t_f from (5) adapting μ if needed.
Step 3:	Initialize h using (12).
Step 4:	While the relative error in h is bigger than ε:
	a) Calculate $\widetilde{H}_k$ from (11).
	b) Set Λ_2 as the maximum of the absolute value of (13).
	c) Set h using (12).

4 Functional framework of the inverse problem. Classical theory.

Suppose the fourth scenario (the more general one). Once t_f is determined, we consider the initial value problem (1) over the interval $[t_0, t_f]$. By denoting $u(t) = H(T(t))$, $t \in [t_0, t_f]$, we have that

$$\int_{t_0}^{t} u(s)\, ds = \int_{t_0}^{t} \frac{T'(s)}{T^e - T(s)}\, ds = -\ln \left(\frac{T^e - T(t)}{T^e - T_0} \right).$$

Thus, for suitable functional spaces X and Y, by defining the operator $K : X \to Y$ as

$$Kx(t) = \int_{t_0}^{t} x(s)\, ds,$$

our problem can be write $Ku = y$, where

$$y(t) = -\ln \left(\frac{T^e - T(t)}{T^e - T_0} \right), \ t \in [t_0, t_f].$$

Note that function y is well defined and it is positive. In order to apply the Classical Regularization Theory in Hilbert spaces (see, e.g., [1], [3]), we choose $X = Y = L^2(t_0, t_f)$.

Proposition 9 *$K : L^2(t_0,t_f) \to L^2(t_0,t_f)$ is a linear and compact operator. Moreover:*

a) $Kx \in H^1(t_0,t_f)$ and $(Kx)' = x$ in $L^2(t_0,t_f)$ for every $x \in L^2(t_0,t_f)$.

b) K is an injective operator and has dense rank in $L^2(t_0,t_f)$.

c) The adjoint operator $K^* : L^2(t_0,t_f) \to L^2(t_0,t_f)$ is given by $K^*y(t) = \int_t^{t_f} y(s)\, ds$.

In our problem we have measurements $\widehat{T}_k$ verifying $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, and an interpolation function $\widetilde{T}$ such that $\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([\tau_0,\tau_p])} < \delta$. This provides a right hand term

$$y_\delta(t) = -\ln\left(\frac{T^e - \widetilde{T}(t)}{T^e - T_0}\right)$$

and the approximate problem $Ku_\delta = y_\delta$. Next proposition estimates the error between y_δ and y in terms of error between $\widetilde{T}$ and T (given by δ).

Proposition 10 *Lets denote $e(\delta) = \dfrac{\sqrt{t_f - t_0}}{\mu - 3\delta}\delta$. Then $||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta)$.*

4.1 Tikhonov's method

The *Tikhonov's strategy* to solve $Ku_\delta = y_\delta$, consists of minimizing the Tikhonov functional

$$J_\alpha(x) = ||Kx - y_\delta||^2_{L^2(t_0,t_f)} + \alpha\, ||x||^2_{L^2(t_0,t_f)}\,, \quad (14)$$

where $\alpha = \alpha(\delta) > 0$. Theorem 2.11 of [3], guarantees uniqueness of the minimum $u_{\alpha,\delta}$ of (14), which is also the unique solution of the *normal equation*

$$(\alpha + K^*K)x = K^*y_\delta. \quad (15)$$

The regularization strategy is given for the linear operators $R_\alpha : L^2(t_0,t_f) \to L^2(t_0,t_f)$ defined by

$$R_\alpha y = (\alpha + K^*K)^{-1}K^*y.$$

For $\alpha = 0$ this becomes the normal equation of K. Since minimizing operator J_0 is an ill–posed problem (see [3], Lemma 2.1), a penalty term is added.

Proposition 11 *The solution $u_{\alpha,\delta}$ of* (15) *is the solution of the boundary problem*

$$\begin{cases} -\alpha x''(t) + x(t) = y'_\delta(t), \; t \in (t_0,t_f) \\ x'(t_0) = 0, \; x(t_f) = 0. \end{cases}$$

Moreover, denoting $\gamma(r) = \dfrac{t_f - r}{\sqrt{\alpha}}$, the solution is

$$u_{\alpha,\delta}(t) = \frac{1}{\sqrt{\alpha}}\left(\varphi_{\alpha,\delta}(t)\cosh\gamma(t) + \psi_{\alpha,\delta}(t)\sinh\gamma(t)\right)$$

where

$$\varphi_{\alpha,\delta}(t) = \int_t^{t_f} y'_\delta(s)\sinh\gamma(s)\, ds$$

and

$$\psi_{\alpha,\delta}(t) = \int_{t_0}^{t} y'_\delta(s)\cosh\gamma(s)\, ds - \tanh\gamma(t_0)\varphi_{\alpha,\delta}(t_0).$$

Remark 12 Theorem 2.12 of [3] states that if one chooses $\alpha = \alpha(\delta)$ such that $\lim_{\delta\to 0}\alpha(\delta) = 0$ and $\lim_{\delta\to 0}\dfrac{\delta^2}{\alpha(\delta)} = 0$, then the Tikhonov regularization strategy is admissible, i.e.,

$$\lim_{\delta\to 0}\left|\left|u_{\alpha(\delta),\delta} - u\right|\right|_{L^2(t_0,t_f)} = 0,$$

since $||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta)$ (see Proposition 10). □

4.1.1 Morozov's discrepancy principle.

This principle provides a way to choose the parameter $\alpha = \alpha(\delta)$ for the Tikhonov's regularization strategy: it is chosen so that the solution $u_{\alpha(\delta),\delta}$ of (15) satisfies

$$\left|\left|Ku_{\alpha(\delta),\delta} - y_\delta\right|\right|_{L^2(t_0,t_f)} = e(\delta),$$

supposing that

$$||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta) < ||y_\delta||_{L^2(t_0,t_f)}\,.$$

Theorem 2.17 of [3] assures that regularization strategy associated to this choice of $\alpha(\delta)$ is admissible.

4.2 Landweber's iterative method.

Landweber's iterative method is defined as

$$\begin{cases} x_0 = 0 \\ x_m = (I - aK^*K)x_{m-1} + aK^*y, \; m = 1, 2, \ldots, \end{cases}$$

where $a > 0$. Using Theorem 2.19 of [3], we choose a such that $0 < a < \frac{1}{||K||^2}$ and we consider the stopping test

$$||Kx_m - y_\delta||^2_{L^2(t_0,t_f)} \le r(e(\delta))^2$$

for some $r > 0$ satisfying

$$||y_\delta|| \ge re(\delta), \; \delta \in (0, \delta_0).$$

4.3 Comparison between the methods.

Consider the test problem

$$\begin{cases} T'(t) = \big(2 + \sin(14T(t))\big)\big(1 - T(t)\big), \ t \in (0, 0.48) \\ T(0) = 0. \end{cases}$$

We take $t_f = 0.48$ corresponding to the threshold $\mu = 0.4$. The goal is to identify $H(s) = 2 + \sin(14s)$ in $(0, T(t_f)) \simeq (0, 0.6)$. We consider a uniform partition of $(0, 0.48)$ with step $h = 0.01$. At these instants, approximate measurements of temperature with error $\delta = 0.001$ are supposed to be known. We use linear piecewise interpolation for $\widetilde{T}$. Every definite integral is approximated by means of the trapezoidal rule by using only points at which measurements of the temperature are available. Thus, these calculations are "independent" of the interpolation method used to compute $\widetilde{T}$.

First, we consider the algorithm described in Section 3.3 and obtain the results shown in Figure 1.

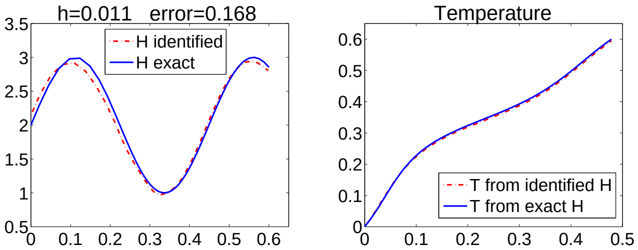

Figure 1: Algorithm of Section 3.3.1.

Computing $u_{\alpha,\delta}$ as states in Proposition 11 for $\alpha(\delta) = \delta^\gamma$, with $0 < \gamma < 2$, the exponent with lower error in L^2–norm is attained when $\gamma = 1.05$, which corresponds to $\alpha(\delta) = 7.0795 \times 10^{-4}$. For this α, Figure 2 shows the computed approximations for H and T.

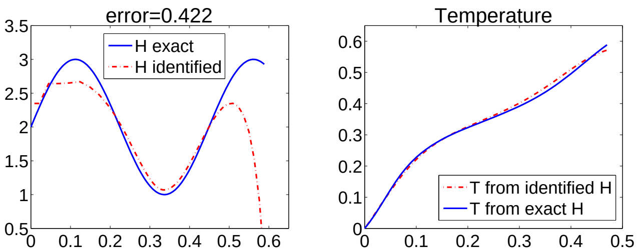

Figure 2: Tikhonov's method with the best exponent.

Figure 3 shows the results for the Morozov's discrepancy principle by applying secant method to

$$F(\alpha) = ||Ku_{\alpha,\delta} - y_\delta||^2_{L^2(t_0,t_f)} - (e(\delta))^2.$$

The obtained value is $\alpha = 0.001$.

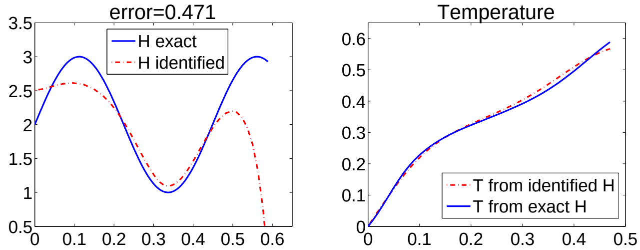

Figure 3: Morozov's discrepancy principle.

Finally, Landweber's iterative method with $a = 10$ and the stopping criterium of Section 4.2 for $r = 1$, after 620 iterations provides a residual norm of 0.0017 and the results are shown in Figure 4.

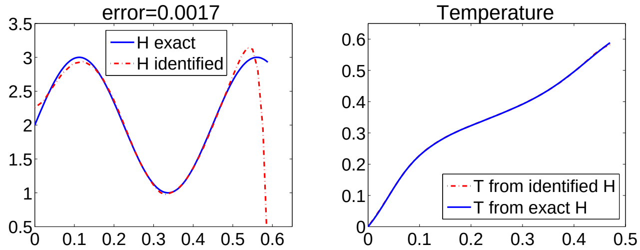

Figure 4: Landweber's iterative method.

5 Conclusions.

The bad behavior obtained with the Classical Theory for function H near t_f (see Figures 2, 3 and 4) is due to the $x(t_f) = 0$ condition needed with the square integrable approximation. This problem is corrected in Figure 1 by the uniform approach because the regularization has been chosen in order to express the qualitative behavior of the direct problem solution.

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