International Journal of Bifurcation and Chaos

Bifurcation Analysis of the Nagumo-Sato Model and its Coupled Systems

Daisuke Ito
Department of Electronic Systems Engineering, The University of Shiga Prefecture
2500 Hassaka-cho, Hikone-shi, Shiga 522-8533, Japan
ito.d@e.usp.ac.jp

Tetsushi Ueta
Center for Administration of Information Technology, Tokushima University
2-1 Minami-Jousanjima, Tokushima-shi, Tokushima 770-8501, Japan
ueta@tokushima-u.ac.jp

Takuji Kousaka
Department of Mechanical and Energy Systems Engineering, Oita University,
700 Dannoharu, Oita-shi, Oita 870-1192, Japan
takuji@oita-u.ac.jp

Kazuyuki Aihara
Institute of Industrial Science, The University of Tokyo,
4-6-1 Komaba Meguro-ku, Tokyo 152-8505, Japan
aihara@sat.t.u-tokyo.ac.jp

Received (to be inserted by publisher)

The Nagumo-Sato model is a simple mathematical expression of a single neuron, and it is categorized as a discrete-time hybrid dynamical system. To compute bifurcation sets in such a discrete-time hybrid dynamical system accurately, conditions for periodic solutions and bifurcations are formulated herewith as a boundary value problem, and Newton's method is implemented to solve that problem. As the results of the analysis, the following properties are obtained: border-collision bifurcations play a dominant role in dynamical behavior of the model; chaotic regions are distinguished by tangent bifurcations; and multi-stable attractors are observed in its coupled system. We demonstrate several bifurcation diagrams and corresponding topological properties of periodic solutions.

Keywords: discrete-time hybrid dynamical system, Nagumo-Sato model, border-collision bifurcation

1. Introduction

The Nagumo-Sato (NS) model is represented by a first-order difference equation and it gives a simple mathematical neuron model [Nagumo & Sato, 1972; Hata, 1982]. There is a threshold value to bind two different maps in the equation, thereby the equation should be conditional, and it is classified as a discrete-time hybrid dynamical system [Bemporad & Morari, 1999; Blondel & Tsitsiklis, 1999; Heemels *et al.*, 2001;

Bernardo *et al.*, 2008]. The chaotic neuron model derived from the NS model [Aihara *et al.*, 1990] exhibits various bifurcation phenomena as well as chaotic attractors [Aihara *et al.*, 1990; Nakagawa & Okabe, 1992; Kitajima *et al.*, 2001].

Dynamical properties of the diffusively coupled NS models [Crutchfield & Kaneko, 1988; Politi *et al.*, 1993; Kinoshita & Ueta, 2010] and synaptically coupled the NS models [Oku & Aihara, 2012] were studied. Especially in our previous papers [Kinoshita & Ueta, 2010; Oku & Aihara, 2012], results by bifurcation analyses were shown. In both papers, bifurcation structures that look like Arnold's tongues [Boyland, 1986] are confirmed in the bifurcation diagrams; however, it is experimentally shown that the border-collision (BC) bifurcations [Banerjee *et al.*, 2000] form these structures instead of tangent bifurcations. Kinoshita and Ueta (2010) computed bifurcation sets by combining the bisection method and the brute-force method. In the synaptically coupled NS models [Oku & Aihara, 2012], bifurcation diagrams are visualized with classified responses (including periodic solutions and chaos) by evaluating firing rates and periods, mainly on the basis of simulations and the brute-force method. Thereby, only bifurcation sets of stable attractors are visualized. In spite of the fact that the NS model is described by a piecewise linear equation, solvable bifurcation conditions were not specified explicitly because of its discontinuity. As far as we know, accurate bifurcation parameter values have not obtained because discontinuity in the model affects the solvability of the condition for bifurcations.

In this study, the NS model and its coupled system are analyzed rigorously by utilizing Newton's method. First, definitions of bifurcation phenomena of periodic solutions are formulated as differentiable two-point boundary value problems, and Newton's method is implemented for it [Kuznetsov, 2004]. In general, for high-order discrete-time hybrid dynamical systems, the convergence ability of Newton's method becomes worse. To overcome this, we develop a repetitive procedure of Newton's method, i.e., we introduce an extended border to allow a tolerant error evaluation. As a result of the analysis for bifurcation structures, we have found that BC bifurcations play a dominant role in dynamical behavior of the NS model, period-locking regions edged by BC bifurcation sets are computed accurately, and chaotic regions rimmed with tangent bifurcations are found. In addition, multi-stable attractors that do not exist in the single NS model are observed in the coupled system. We demonstrate several bifurcation diagrams and corresponding topological properties of periodic solutions.

2. The Nagumo-Sato model

The Nagumo-Sato model [Nagumo & Sato, 1972; Hata, 1982] is expressed as follows:

$$x_{k+1} = f(x_k),\ \ f(x_k) = \begin{cases} ax_k + b - 1 & \text{if} \quad x_k \in D, \\ ax_k + b & \text{if} \quad x_k \in \overline{D}, \end{cases} \tag{1}$$

$$D = \{x \in \boldsymbol{R} \mid x \geq c\}, \tag{2}$$

where a, b and c are parameters related to a decay factor, input and threshold of a neuron. $\overline{D}$ is the complement sub-domain of D. This dynamical system is recognized as a piecewise-affine (PA) system since the model has discontinuity such that smooth difference equations change on the boundary, and it is considered as an *interrupt dynamical system*. Furthermore, PA systems that have non-smooth characteristics are categorized as hybrid dynamical systems in control theory [Bemporad & Morari, 1999; Heemels *et al.*, 2001; Blondel & Tsitsiklis, 1999].

The detailed analysis of bifurcation parameter sets for hybrid dynamical systems is achieved by solving two-point boundary problems. However, in a preceding study [Oku & Aihara, 2012; Kinoshita & Ueta, 2010], bifurcation structures of the NS model were computed mainly by the brute-force method. Numerical methods were not used because the accurate computation of the Jacobian matrix requires complicated schemes due to discontinuity of the NS model. In the next section, we propose a numerical computation method to solve boundary value problems for periodic solutions of discrete-time hybrid systems.

3. Computation algorithm for bifurcations in a discrete-time hybrid dynamical system

A general form of the discrete-time hybrid dynamical system is described as

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_j(\boldsymbol{x}_k) \quad \text{if} \quad \boldsymbol{x}_k \in D_j, \tag{3}$$

where $\boldsymbol{x}_k \in \boldsymbol{R}^n$ is the state and $\boldsymbol{f}_j$ with $j = 1,\ 2,\ \ldots,\ m$ is a C^∞-class function defined in a domain D_j in the state space; for example, D and $\overline{D}$ of Eq. (2) correspond to D_1 and D_2 with $m = 2$, respectively. Suppose that $\boldsymbol{x}_{k+1}$ can go outside D_j by applying $\boldsymbol{f}_j$ to $\boldsymbol{x}_k$.

Assume that $\boldsymbol{x}_0$ is the initial state, solutions of Eq. (3) are represented as

$$\boldsymbol{x}_k = \boldsymbol{\varphi}(\boldsymbol{x}_0,\ k), \tag{4}$$

where $\boldsymbol{x}_0 = \boldsymbol{\varphi}(\boldsymbol{x}_0,\ 0)$. When solutions satisfy

$$\boldsymbol{x}_0 = \boldsymbol{\varphi}(\boldsymbol{x}_0,\ p), \tag{5}$$

and the points $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{p-1}\}$ are different each other, we call them p-periodic points, especially we call $\boldsymbol{x}_0$ a fixed point when $p = 1$. The Jacobian matrix is expressed as follows:

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}(\boldsymbol{x}_0,\ p) = \prod_{i=0}^{p-1} \left. \frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}} \right|_{\boldsymbol{x}=\boldsymbol{x}_i \in D_j}, \tag{6}$$

where we assume that the map $\boldsymbol{f}_j$ at each time i is the function of the domain D_j including the current state $\boldsymbol{x}_i$, thus also the Jacobian $\partial \boldsymbol{f}_j/\partial \boldsymbol{x}$ depends on the domain.

Local bifurcations are considered as follows. The characteristic equation is given as

$$\chi(\mu) = \det\left(\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}(\boldsymbol{x}_0,\ p) - \mu I\right) = 0, \tag{7}$$

where μ is a multiplier determining the stability of the periodic point, and I is the identity matrix. A local bifurcation occurs if the absolute value of the multiplier is unity; for example, $\mu = 1$ and $\mu = -1$ mean a tangent bifurcation and a period-doubling bifurcation, respectively. By combining Eqs. (5) and (7), the two-point boundary value problem is expressed as

$$\begin{cases} \boldsymbol{\varphi}(\boldsymbol{x}_0,\ p) - \boldsymbol{x}_0 = 0, \\ \chi(\mu) = 0, \end{cases} \tag{8}$$

where p is the period of the objective attractor.

It is well known that BC bifurcations are typical phenomena of hybrid dynamical systems [Kinoshita & Ueta, 2010]. They emerge when a part of the periodic point hits the border separating two sub-spaces in general, and they are independent of the stability for the attractor. Hence, the two-point boundary value problem for the BC bifurcation is described by the following equations with definition of the border (the threshold value) at which objective hybrid dynamical systems show discontinuity:

$$\begin{cases} \boldsymbol{\varphi}(\boldsymbol{x}_0,\ p) - \boldsymbol{x}_0 = 0, \\ q(\boldsymbol{x}_0,\ \theta) = 0, \end{cases} \tag{9}$$

where $q(\boldsymbol{x},\ \theta)$: $\boldsymbol{R}^{n+1} \mapsto \boldsymbol{R}$ is the function defining the border. Eq. (9) can be solved by numerical approach, i.e. Newton's method, through which the bifurcation structure of the hybrid dynamical system can be computed in detail.

4. Bifurcations of the NS model

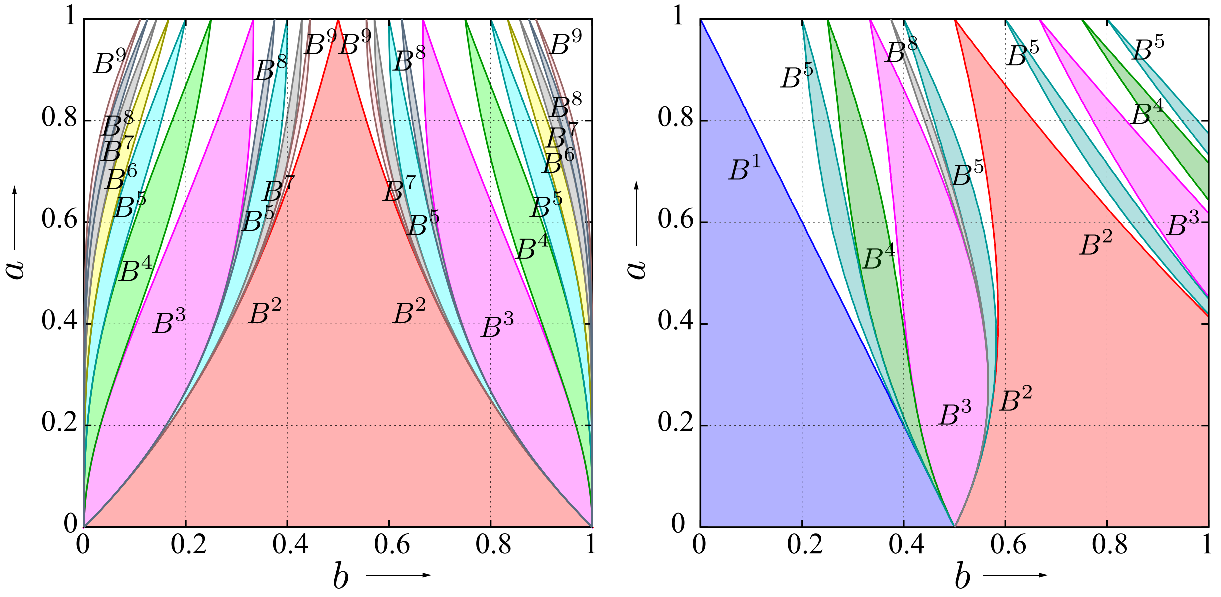
The numerical approach for analyzing the bifurcation structure of the single NS model is explained first. Bifurcation phenomena can be computed by using Newton's method. Two sub-spaces in the NS model can be defined by Eq. (2), and the differential of each map is obtained as follows:

$$\frac{\partial f_i}{\partial x} = a,\ i = 1,\ 2. \tag{10}$$

Therefore, multipliers that show the stability of a periodic point are given dependently on the parameter a and its period. The characteristic equation for the p-periodic point is shown as

$$\chi(\mu) = \mu - a^p = 0. \tag{11}$$

The multipliers are thus given as $\mu = a^p$. Bifurcation phenomena generally emerge when one of the multipliers is on a unit circle. It is thus easy to determine bifurcation parameter values by using multipliers; for example, the tangent bifurcation occurs if $a = 1$.



(a) (b)

Fig. 1. Bifurcation diagrams of the single NS model. B^i means a BC bifurcation of i-periodic solution. (a): $c = 0$, (b): $c = 0.5$.

Bifurcation diagrams of the NS model are shown in Fig. 1. These diagrams visualize periodic regions in which the period is less than ten. All bifurcation sets are BC bifurcations and they are obtained by solving Eq. (9). Figure 1(a) shows the case of $c = 0$. For all periodic regions, the upper cusp points are located at rational numbers of b and aligned along the line with $a = 1$, and the lower cusp points gather at $(a,\ b) = (0,\ 0),\ (0,\ 1)$. All regions are not overlapped, and many other longer period regions are embedded between any two regions (not visualized here). Here, p-periodic BC bifurcations are represented as B^p. It is confirmed that the model mainly generates the BC bifurcations. The tangent bifurcation occurs only on the line with $a = 1$. Thus, the BC bifurcations dominate the dynamical behavior of the NS model. If this bifurcation occurs, the concerned attractor will disappear. Note that a chaotic attractor does not exist at this parameter region ($0 < a < 1$ and $0 < b < 1$).

Figure 1(b) shows the case of $c = 0.5$, where periodic regions in which the period is less than five are visualized. Compared with Fig. 1(a), a period-1 region edged by B^1 newly emerges in the left part, and the BC bifurcation set B^1 is given analytically as $ac + b - c = 0$. For other bifurcation structures, upper cusp points still touch rational numbers of b, but period locking regions are shifted to the right.

Oku and Aihara (2012) have studied firing rates and periods in the NS model by a brute-force method. Their results are reproduced in Fig. 2. Compared with Fig. 1(a), it is easy to see that the edges of regions in Fig. 2 correspond to BC bifurcation sets. Figures 1 and 2 show that BC bifurcations characterize firing rates and periods of the NS model.

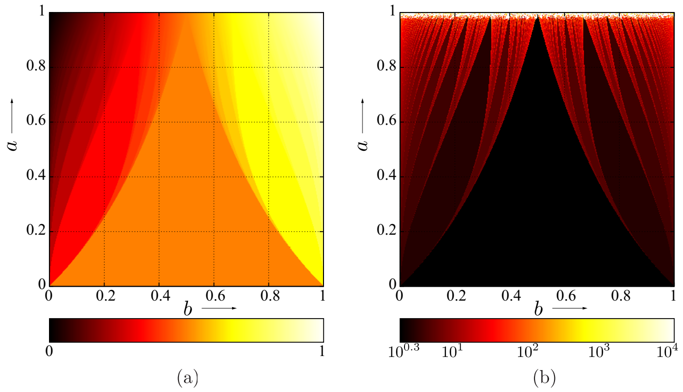


Fig. 2. (a) Firing rates and (b) firing periods in the single NS model ($c = 0$). Reproduced images by referring [Oku & Aihara, 2012].

5. Bifurcations of coupled Nagumo-Sato models

5.1. *Definitions of borders and domains*

In the previous works [Ito *et al.*, 2011; Fujii & Tsuda, 2004; Ueta *et al.*, 2004], the following diffusively coupled NS models are considered:

$$\begin{cases} x_{k+1} = f(x_k) + \delta(x_k - y_k), \\ y_{k+1} = f(y_k) + \delta(y_k - x_k), \end{cases} \tag{12}$$

where f is given in Eq. (1), and δ is called "the factor of the coupling." We do not restrict the sign of δ in this paper.

Four domains according to the border c are assigned in the state space as follows (see Fig. 3):

$$\begin{array}{l} D_1 : x < c \wedge y < c, \\ D_2 : x \geq c \wedge y < c, \\ D_3 : x < c \wedge y \geq c, \\ D_4 : x \geq c \wedge y \geq c. \end{array} \tag{13}$$

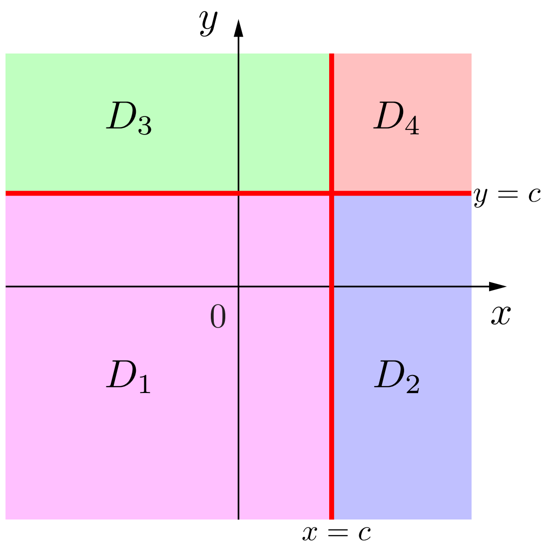


Fig. 3. Four sub-spaces of the coupled system. Red lines show the borders $x = c$ and $y = c$.

Equation (12) is rewritten as follows:

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_j(\boldsymbol{x}_k) \quad \text{if} \quad \boldsymbol{x}_k \in D_j, \tag{14}$$

where, $\boldsymbol{x} = (x, y)^\top$, and

$$\begin{aligned}
\boldsymbol{f}_1 &= \begin{pmatrix} ax + b + \delta(x - y) \\ ay + b + \delta(y - x) \end{pmatrix}, \quad \boldsymbol{x} \in D_1, \\
\boldsymbol{f}_2 &= \begin{pmatrix} ax + b - 1 + \delta(x - y) \\ ay + b + \delta(y - x) \end{pmatrix}, \boldsymbol{x} \in D_2, \\
\boldsymbol{f}_3 &= \begin{pmatrix} ax + b + \delta(x - y) \\ ay + b - 1 + \delta(y - x) \end{pmatrix}, \boldsymbol{x} \in D_3, \\
\boldsymbol{f}_4 &= \begin{pmatrix} ax + b - 1 + \delta(x - y) \\ ay + b - 1 + \delta(y - x) \end{pmatrix}, \boldsymbol{x} \in D_4.
\end{aligned} \tag{15}$$

In this system, the differentials of all the maps in Eq. (15) are the same. Thus, if the periods of some attractors are equal to each other, the Jacobian matrices for all the periodic points are also the same and determined by parameters a and δ independently of the state and interrupt characteristics as follows:

$$\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}} = \begin{pmatrix} a + \delta & -\delta \\ -\delta & a + \delta \end{pmatrix}, \ \forall\, i = 1,\ 2,\ 3,\ 4. \tag{16}$$

Therefore, the multipliers also depend only on the two parameter values and the period. The characteristic equation of the 1-periodic point is given as

$$\chi(\mu) = \mu^2 - 2(a + \delta)\mu + (a + \delta)^2 - \delta^2 = 0. \tag{17}$$

The multipliers are thus given as $\mu_1 = a + 2\delta$ and $\mu_2 = a$. As the result, it is easy to determine the bifurcation parameter values by using the multipliers. For example, the tangent bifurcation occurs if the following equation is satisfied:

$$a = 1 \quad \text{or} \quad a = 1 - 2\delta. \tag{18}$$

5.2. *Extension of Newton's method for solving Border-collision bifurcation*

Newton's method is also applicable to solve the boundary value problems of Eqs. (8) and (9) for coupled NS models, however, the method fails solving them in some reasons for hybrid dynamical systems. Let us describe the problem and show its workaround.

A 3-periodic point attractor just before the BC bifurcation is shown in Fig. 4, where (a) shows the 3-periodic point, and (b) shows the solution with small perturbation ($\Delta\boldsymbol{x} = (10^{-4}, 10^{-4})^\top$) at the initial point $\boldsymbol{x}_0$.

Figure 4(b) shows that the mapped point $\boldsymbol{f}^3(\boldsymbol{x}_0 + \Delta\boldsymbol{x})$ is far from $\boldsymbol{x}_0$, even though $\Delta\boldsymbol{x}$ on $\boldsymbol{x}_0$ is small enough. This is caused due to change the order of maps. The 3-periodic point is mapped $\boldsymbol{f}_3 \to \boldsymbol{f}_1 \to \boldsymbol{f}_2$ as shown in Fig. 4 (a), but in the solution with the small perturbation, the order of maps is changed to $\boldsymbol{f}_3 \to \boldsymbol{f}_1 \to \boldsymbol{f}_4$ because the state $\boldsymbol{f}^2(\boldsymbol{x}_0 + \Delta\boldsymbol{x})$ crosses the border $y = c$, as shown in Fig. 4(c). As a result, the small perturbation in the initial state leads to large difference between the mapped points. The parameter perturbation also causes the similar effect. Suppose that we try to compute the accurate periodic points by Newton's method for this case. If the correction $\Delta\boldsymbol{x}$ for the first guess $\boldsymbol{x}_0$ is computed by the first iteration of the method, possibly the orbit starting from the updated value $\boldsymbol{x}_0 + \Delta\boldsymbol{x}$ does not form a periodic solution any longer in this example, thus the scheme may fail. This undesirable failure could appear in discrete-time hybrid dynamical systems in general, and in fact, this is very closed to a BC bifurcation phenomenon.

In general continuous-time hybrid dynamical systems, any flow approach a border without a break as the time grows, i.e., switching of the system always occurs on the border. One can calculate when and where the flow arrives at the border accurately by controlling a tick of the time for the numerical integration. Newton's method for solving bifurcation problems can work properly as far as we consider these arrival points on the border [Kousaka *et al.*, 1999].

In contrast with this, in the discrete-time hybrid dynamical systems, an orbit may not approach the border like a flow. The border only notifies us which domain the current point locates. In Newton's method to compute boundary value problems formulated in Eq. (8) or Eq. (9), although the computation scheme is

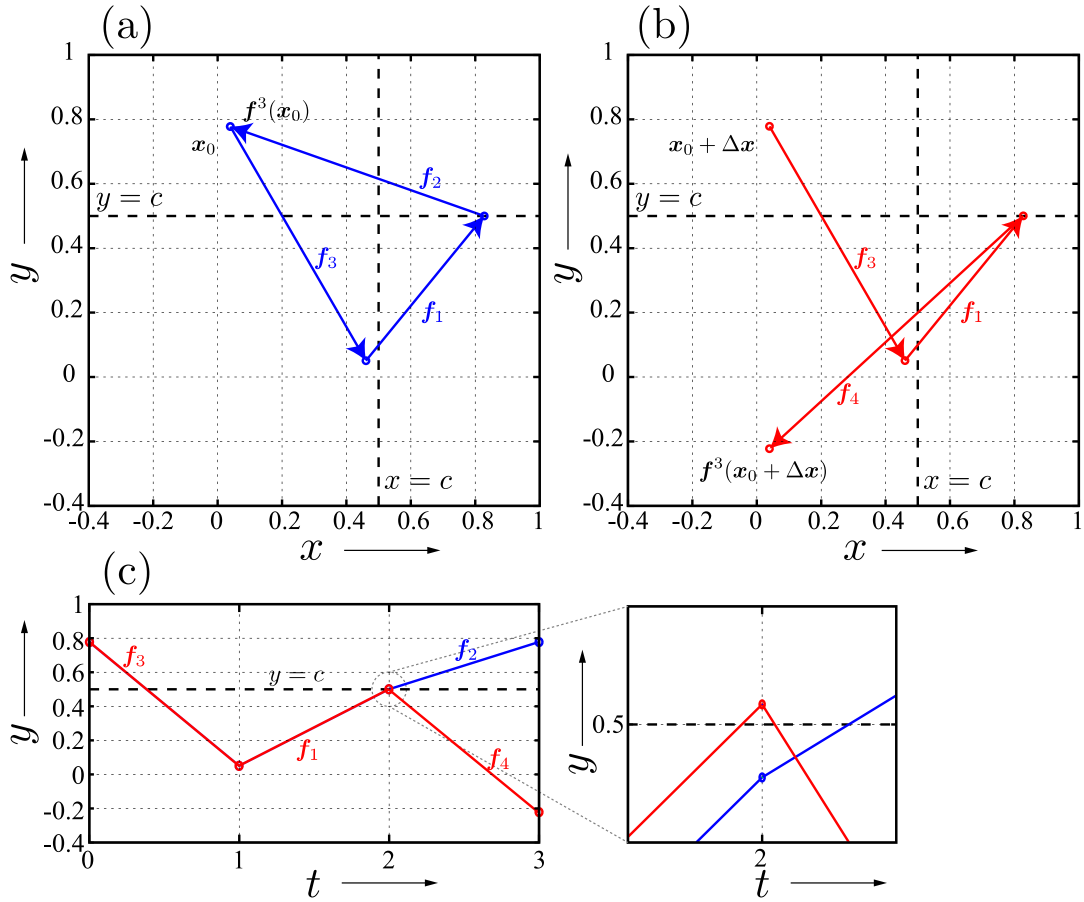


Fig. 4. The problem of the BC bifurcation analysis with Newton's method. The large error can be generated by discontinuity dynamics. This problem happens due to a small perturbation on the initial state.

independent from the original dynamics of the system, A small perturbation of the state or the parameter value possibly gives improper correction values for the condition of the periodic point. In other words, Newton's scheme yields BC bifurcation.

To complete Newton's method for boundary value problems without above mentioned failure, we propose an extension of the border. If the periodic point that we want to solve is located near the border, the failure possibly happens during the correction process of Newton's method, i.e., after the initial guess is updated by adding the correction, the new orbit starting from that initial value is mapped into the different domain. Figure 5(a) shows an example. In order to find a true periodic point $\boldsymbol{x}^*$ from the first guess $\boldsymbol{x}_0$, Newton's method generates updated points as the iteration progresses. If the convergence area is beyond the domain D_2, updated points may enter to another domain. In this example, the point $\boldsymbol{x}_a$ is out of D_2, then the method offers the next point $\boldsymbol{x}_b \in D_4$ since the Jacobian matrix defined for D_4 is chosen there. The further updated point cannot be back to the convergence area anymore.

To overcome this difficulty, we offer an extended border for Newton's method, as shown in Fig. 5(b). We virtually provide the extended domain that covers the whole convergence area of Newton's method for $\boldsymbol{x}^*$, i.e., during the iteration of the method, we keep update points into the same domain D_2. The Jacobian matrix defined for D_4 is never chosen. Even if the point $\boldsymbol{x}^*$ is unknown, the convergence area can be estimated adaptively.

As a result of this extension, large errors of mapped points with small errors in initial points are not generated even if solutions cross beyond the original border; therefore, BC bifurcations for Newton's method is suppressed. For example, in Fig. 4, the correct order of maps, namely $\boldsymbol{f}_3 \to \boldsymbol{f}_1 \to \boldsymbol{f}_2$, is applied independently of the state of trajectory, since the region on which $\boldsymbol{f}_2$ is applied is extended. Thus, if the trajectory goes over the border $y = c$, $\boldsymbol{f}_2$ rather than $\boldsymbol{f}_4$ is applied.

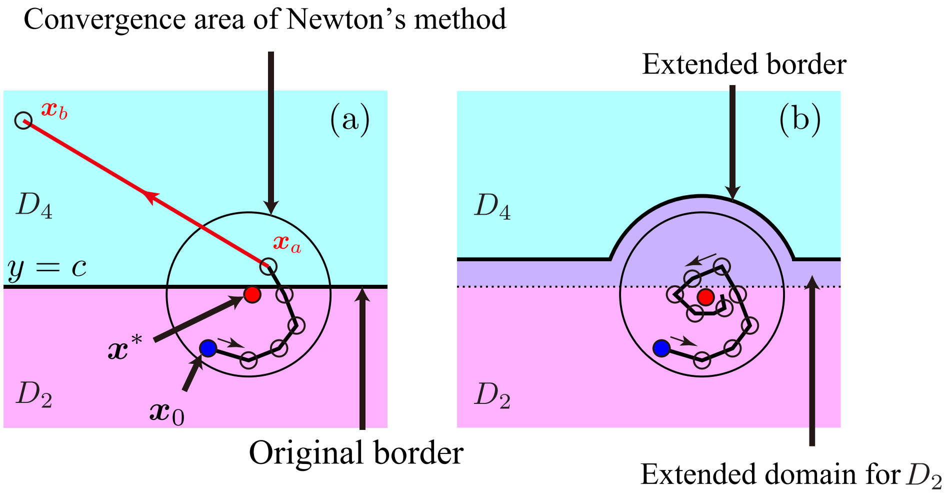


Fig. 5. A sketch of a solution to eliminate a BC bifurcation associated with Newton's method. Owing to the extended border, $\boldsymbol{x}^*$ can be found.

5.3. *Bifurcation structures of the coupled system*

The coupled system can be equivalent to a single neuron model when responses of two neurons are synchronized in-phase since the coupling terms vanish for $x_k = y_k$. Henceforth, the case that two neurons are not synchronized in-phase is considered below. To clarify the effects of coupling factor δ, the bifurcation structures are compared with Fig. 1(a) and (b).

Assume $c = 0.5$. A bifurcation diagram in the a-b plane with $\delta = -0.1$ is shown in Fig. 6. Periodic regions that the periods are less than nine are depicted. From comparison with Fig. 1(b), two conspicuous properties of the bifurcation structures emerge: First, the left 4-periodic-point area (surrounded by B_4) is clearly separated from the other regions, i.e., the bottom cusp point is isolated. For other periodic areas, the bottom cusp points are not concentrated. Second, some overlapped regions appear. This property suggests the existence of multi-stable situations; in fact, in the hatched area at the upper-center of Fig. 6, two 5-periodic attractors coexist and one of them appears depending on the initial value.

A bifurcation diagram for $\delta = 0.1$ is shown in Fig. 7. The number of overlapped periodic regions is increased in comparison with Fig.6. In the hatched regions, multiple periodic solutions coexist. Line T shows the tangent bifurcation set for all the periodic attractors, and it is given by Eq. (18). Beyond this bifurcation set, all the periodic attractors disappear, and instead of them, chaotic attractors appear. Since the Jacobian matrix Eq. (16) is independent of b, the maximum Lyapunov exponent values λ have constant values corresponding to the parameter a; e.g., for $a = 0.7$, λ is -0.105413, and for $a = 0.9$, λ is 0.095241. In the gray shaded region, BC bifurcation sets are mathematically survived, but only chaotic solutions are observed in the numerical simulation.

To visualize the coupling effects totally, a bifurcation diagram in the a-δ plane is shown in Fig. 8. Chaotic attractors appear for positive values of δ. The bottom cusp points of the periodic regions are concentrated to $\delta = 0$ and $a = 0$. In the hatched areas, multiple periodic solutions are obtained. In this figure, $\delta = 0$ (an uncoupled case) looks very special, i.e., it passes through the hatched areas. As we shown in Fig. 1(b), there is no overlapped area in bifurcation structures. In this hatched area, as $|\delta|$ decreases, different 5-periodic solutions approach each other, and the single period-5 solution exists on the line $\delta = 0$.

Firing rates and periods of the coupled system in the a-δ plane are shown in Figure 9. After the tangent bifurcation, the period of an attractor becomes long, but the firing rates of the attractors are almost unchanged.

A chaotic and the 3-periodic attractor are shown in Fig. 10, and a one-dimensional bifurcation diagram and the Lyapunov exponent values with varying the parameter δ are shown in Fig. 11. The 3-periodic attractor disappears by the tangent bifurcation at $\delta = 0.25$, and thereafter a chaotic attractor appears.

In Fig. 8, 5, 8, and 11-periodic attractors coexist as multi-stable attractors at the parameter values of

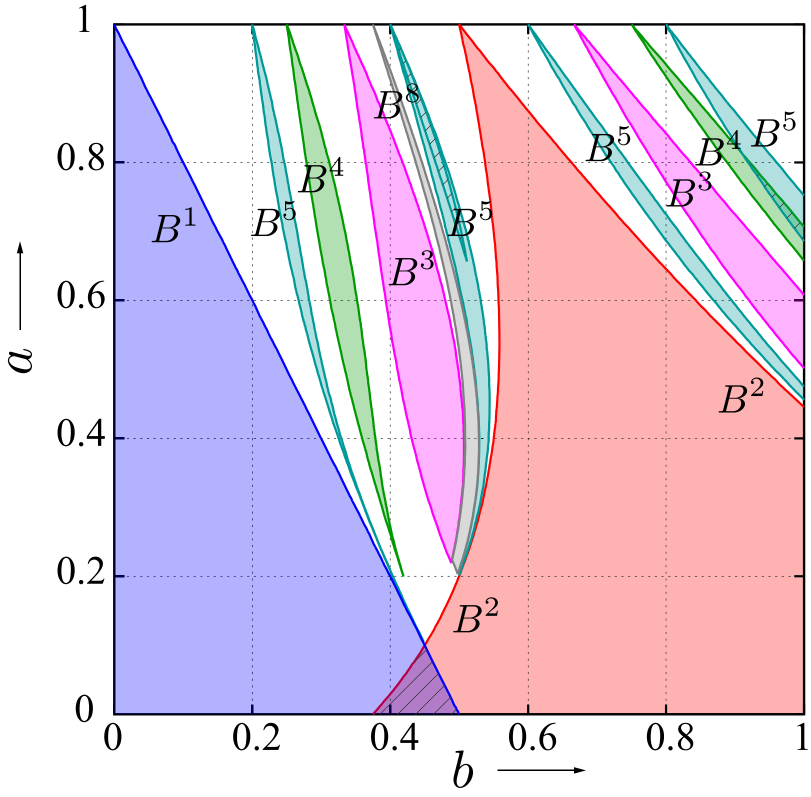


Fig. 6. Bifurcation diagram of the coupled system in the a-b plane ($c = 0.5$ and $\delta = -0.1$).

(iii), and they are shown in Fig. 12. These attractors will emerge dependently on initial conditions. The basins of attraction are shown in Fig. 13.

6. Conclusion

We have investigated bifurcation structures of the NS model and its coupled system in detail. First, definitions of bifurcation phenomena of periodic solutions are formulated as a differentiable two-point boundary value problem, and accurate bifurcation sets are calculated by Newton's method featuring variational equations. As the result, we have shown that BC bifurcations play a dominant role in dynamical behavior of NS models, period-locking regions edged by BC bifurcation sets are computed accurately, and chaotic regions rimmed with tangent bifurcations are found. In addition, multi-stable attractors that do not exist in the single NS model are observed in the coupled system. On the analysis of bifurcation phenomena using Newton's method, degrading the convergence ability due to discontinuity becomes a critical problem. In general, for high-order discrete-time hybrid dynamical systems, the convergence ability of Newton's method becomes worse. To overcome this problem, we have developed the repetitive procedure of Newton's method (Sec. 5.2), i.e., we have introduced an extended border to allow a tolerant error evaluation. Note that our proposed method can be implemented not only in piecewise linear systems but also in piecewise nonlinear systems.

Acknowledgment

This research was funded by ImPACT Program of Council for Science, Technology and Innovation (Cabinet Office, Government of Japan), JSPS KAKENHI Grant Number 25420373, and CREST, JST.

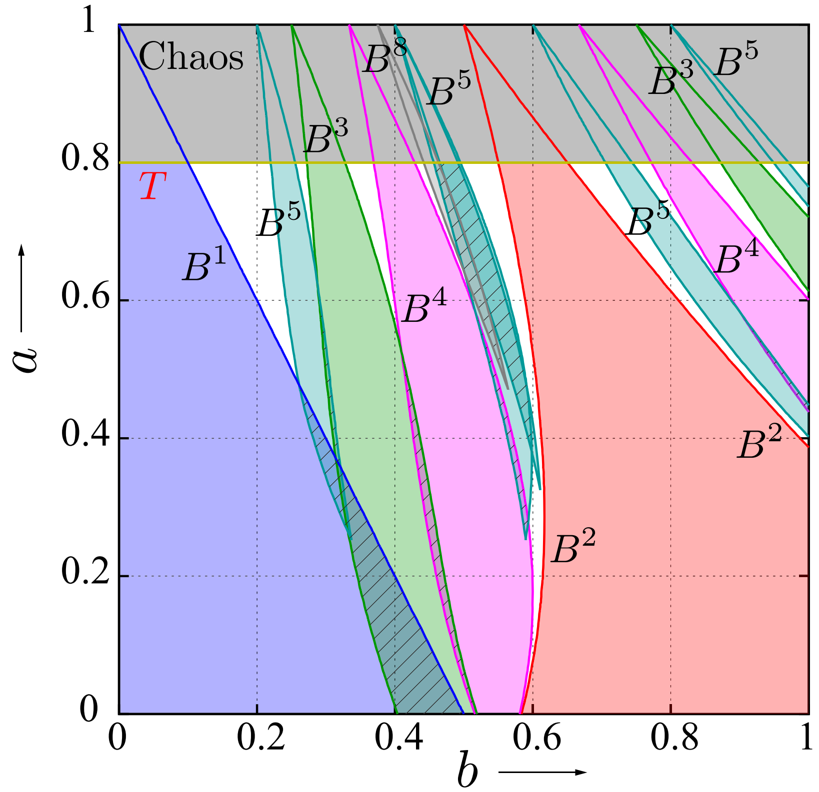


Fig. 7. Bifurcation diagram of the coupled system in the a-b plane ($c = 0.5$ and $\delta = 0.1$).

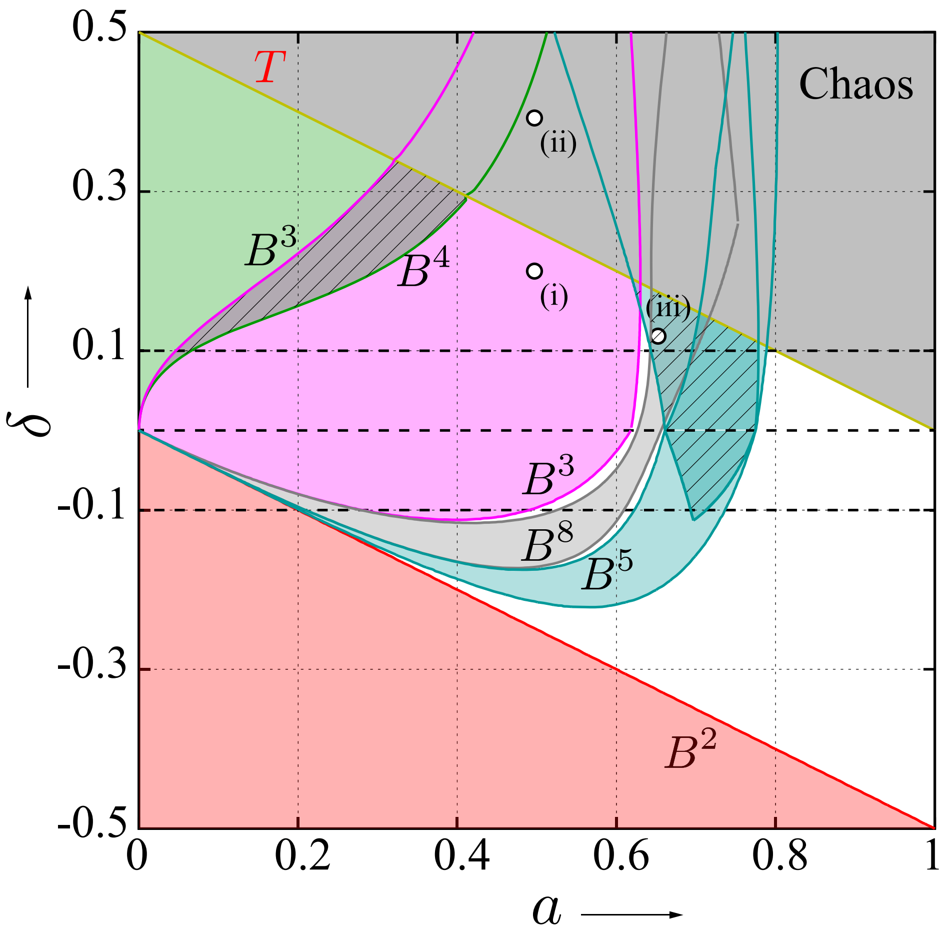


Fig. 8. Bifurcation diagram of the coupled system in the a-δ plane ($b = 0.5$ and $c = 0.5$).

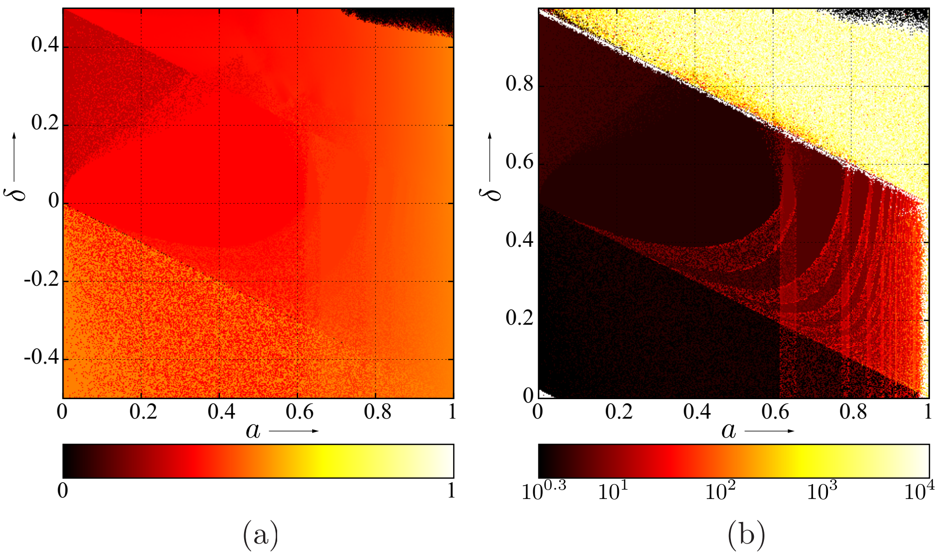


Fig. 9. (a) firing rates and (b) firing periods of the coupled system in the a-δ plane ($b = 0.5$ and $c = 0.5$).

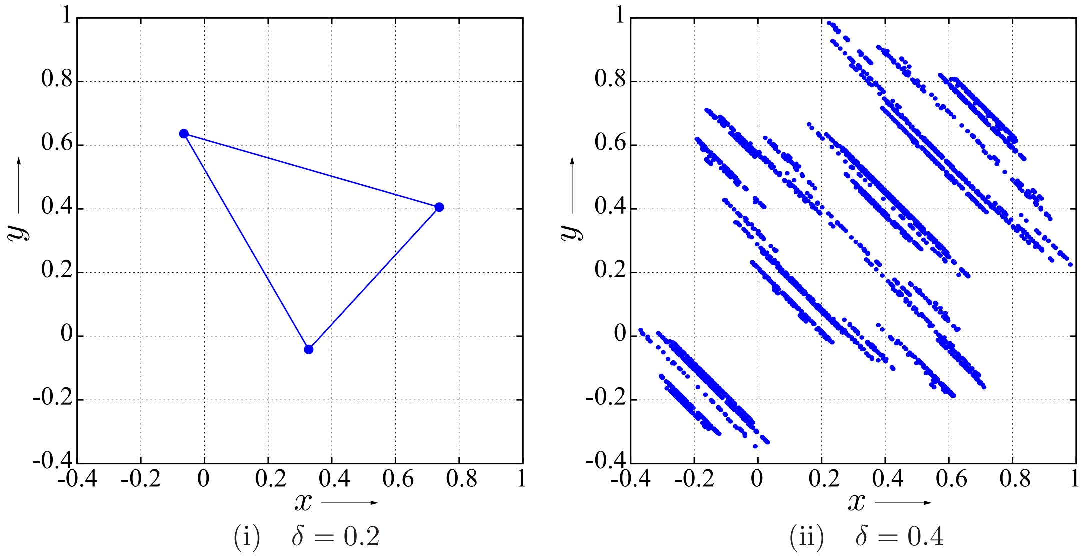


Fig. 10. A 3-periodic attractor and a chaotic attractor ($a = 0.5$, $b = 0.5$ and $c = 0.5$).

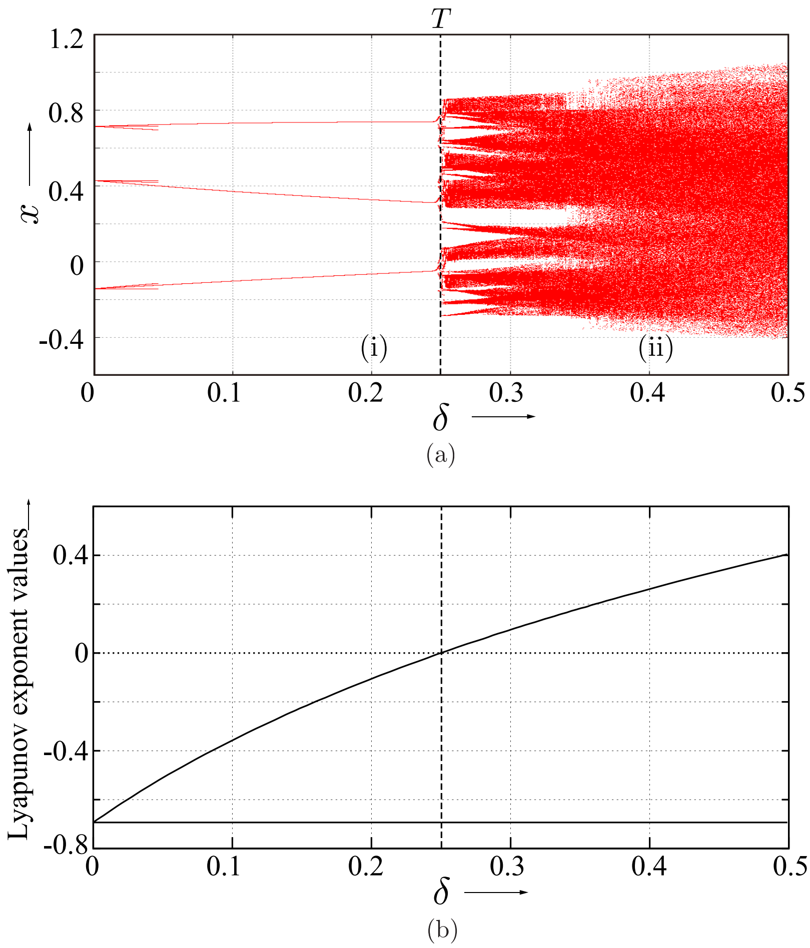


Fig. 11. (a) One-dimensional bifurcation diagram and (b) the Lyapunov exponent value with δ ($a = 0.5$, $b = 0.5$ and $c = 0.5$). The dashed line in (a) shows the parameter values of the tangent bifurcation.

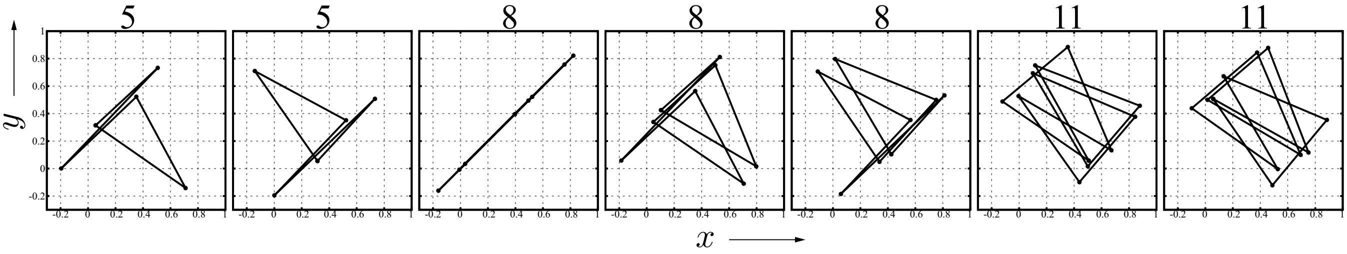


Fig. 12. Coexistence of the attractors in the coupled system ($\delta = 0.11$, $a = 0.65$, $b = 0.5$ and $c = 0.5$).

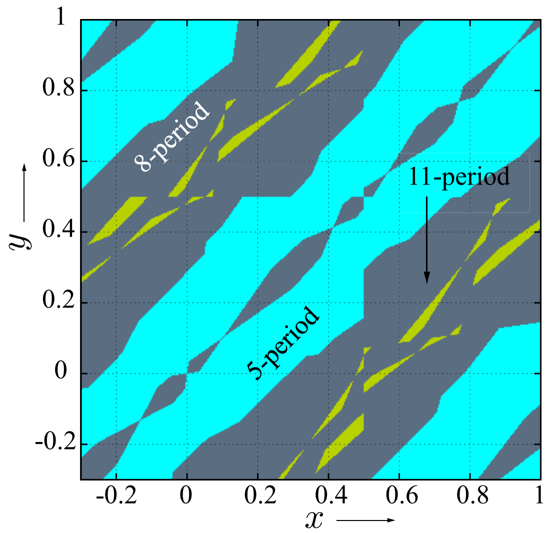


Fig. 13. Basins of attraction of various periodic solutions, $\delta = 0.11$, $a = 0.65$, $b = 0.5$ and $c = 0.5$. Seven different attractors shown in Fig.12 coexist in the state space.

References

Aihara, K., Takabe, T. & Toyoda, M. [1990] "Chaotic neural networks," *Physics Letters A* **144**, 333–340.

Banerjee, S., Karthik, M. S., Yuan, G. & Yorke, J. a. [2000] "Bifurcations in one-dimensional piecewise smooth maps - theory and applications in switching circuits," *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **47**, 389–394, doi:10.1109/81.841921.

Bemporad, A. & Morari, M. [1999] "Control of systems integrating logic, dynamics and constraints," *Automatica* **35**, 407–427.

Bernardo, M., Budd, C. J., Champneys, A. R. & Kowalczyk, P. [2008] *Piecewise-smooth Dynamical Systems: Theory and applications* (London, U. K.: Springer-Verlag, London).

Blondel, V. & Tsitsiklis, J. [1999] "Complexity of stability and controllability of elementary hybrid systems," *Automatica* **35**, 479–489.

Boyland, P. L. [1986] "Bifurcations of circle maps: Arnol'd tongues, bistability and rotation intervals," *Comm. Math. Phys.* **106**, 353–381.

Crutchfield, J. & Kaneko, K. [1988] "Are attractors relevant to turbulence?" *Phys. Rev. Lett.* **60**, 2715–2718, doi:10.1103/PhysRevLett.60.2715.

Fujii, H. & Tsuda, I. [2004] "Neocortical gap junction-coupled interneuron systems may induce chaotic behavior itinerant among quasi-attractors exhibiting transient synchrony," *Neurocomputing* **58–60**, 151–157.

Hata, M. [1982] "Dynamics of caianiello's equation," *J. Math. Kyoto Univ.* **22**, 155–173.

Heemels, W., Schutter, B. D. & Bemporad, A. [2001] "Equivalence of hybrid dynamical models," *Automatica* **37**, 1085–1091.

Ito, D., Ueta, T., Imura, J. & Aihara, K. [2011] "Analysis and controlling of interrupt chaotic systems by a switching threshold," *Proc. NOLTA 2011*, pp. 577–580.

Kinoshita, K. & Ueta, T. [2010] "Bifurcation analysis of coupled nagumo-sato models," *Proc. NOLTA2010*, pp. 488–491.

Kitajima, H., Yoshinaga, T., Aihara, K. & Kawakami, H. [2001] "Chaotic bursts and bifurcation in chaotic neural networks with ring structure," *International Journal of Bifurcation and Chaos* **11**, 1631–1643.

Kousaka, T., Ueta, T. & Kawakami, H. [1999] "Bifurcation of switched nonlinear dynamical systems," *IEEE Trans. CAS-II* **46**, 878–885.

Kuznetsov, Y. A. [2004] *Elements of Applied Bifurcation Theory*, 3rd ed. (New York, U. S.: Springer-Verlag, New York).

Nagumo, J. & Sato, S. [1972] "On a response characteristic of a mathematical neuron model," *Biological Cybernetics* **10**, 155–164.

Nakagawa, M. & Okabe, M. [1992] "On the chaos region of the modified nagumo-sato model," *Journal of the Physical Society of Japan* **61**, 1121–1124.

Oku, M. & Aihara, K. [2012] "Numerical analysis of transient and periodic dynamics in single and coupled nagumo-sato models," *International Journal of Bifurcation and Chaos* **22**.

Politi, A., Livi, R., Oppo, G. & Kapral, R. [1993] "Unpredictable behaviour in stable systems," *Europhys. Lett.* **22**, 571–576, doi:10.1209/0295-5075/22/8/003.

Ueta, T., Miyazaki, H., Kousaka, T. & Kawakami, H. [2004] "Bifurcation and chaos in coupled bvp oscillators," *International Journal of Bifurcation and Chaos* **14**, 1305–1324.