TOPOLOGY OF DEFINABLE ABELIAN GROUPS IN O-MINIMAL STRUCTURES

ELÍAS BARO AND ALESSANDRO BERARDUCCI

Abstract. In this note we prove that every definably connected, definably compact abelian definable group G in an o-minimal expansion of a real closed field with $\dim(G) \neq 4$ is definably homeomorphic to a torus of the same dimension. Moreover, in the semialgebraic case the result holds for all dimensions.

1. Introduction

Let M be an o-minimal expansion of a real closed field. Let H be a definable group in M equipped with Pillay's topology in [Pi:88]. So when M is an expansion of the real line H is a real Lie group, and in general it is an "M-Lie group". An example of definable group is the *n-torus* $\mathbb{T}^n(M)$ over M, defined as the poly-interval $[0,1)^n$ in M with the sum operation modulo 1. When M is the real field $\mathbb{R}_{\text{field}}$ this coincides with the classical torus $(\mathbb{R}\,/\,\mathbb{Z})^n$. Let us recall that $(\mathbb{R}\,/\,\mathbb{Z})^n$ is the only compact abelian connected Lie group of dimension n up to Lie-isomorphisms.

In the rest of the paper we fix a definably connected, definably compact, definable abelian group G in M of dimension n, endowed with Pillay's topology. There are three natural questions:

(1) Is G definably isomorphic to the n-torus $\mathbb{T}^n(M)$?
(2) Is G definably homeomorphic to $\mathbb{T}^n(M)$?
(3) Is G definably homotopy equivalent to $\mathbb{T}^n(M)$?

The answer to question (1) is clearly no. For instance $[0,1) \subset \mathbb{R}$ modulo 1 is Lie isomorphic to $SO(2,\mathbb{R})$ but the isomorphism is not semialgebraic (note, however, that they are semialgebraically homeomorphic). Instead, we could ask if G is definably isomorphic to a product of 1-dimensional definable subgroups. But it turns out that this question still has a negative answer even for $M = \mathbb{R}_{\text{field}}$. Indeed it is possible that $\dim(G) > 1$, but G has no subgroups of dimension one definable in $\mathbb{R}_{\text{field}}$ [PeSte:99, Example 5.2].

On the other hand the homotopy problem (3) has a positive answer by [BeMaOt:08, Theorem 3.4].

In this note we deal with the homeomorphism problem (2), giving a positive solution when $\dim(G) \neq 4$ (see Theorem 3.1). We also show that when G is semialgebraic, namely M is a real closed field without additional structure, then

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the result holds in all dimensions (see Theorem 2.1). We point out that the one dimensional case was already proved in [St:94].

The assumption that G is definably compact and definably connected is not very restrictive. In fact by [PeSta:05, Thm.5.1, Thm.5.7] and [PeSte:99, Thm.1.2], the study of the topology of definable abelian groups can be reduced to the definably compact case. So as a corollary we obtain, with a small proviso, that a definably connected abelian n-dimensional group is definably homeomorphic to a space of the form $\mathbb{T}^m(M) \times M^k$ with $m + k = n$. The proviso is vacuous in the semialgebraic case, while in the general case we require that $m \neq 4$, where m is n minus the dimension of the maximal torsion free definable subgroup of G.

Suitable versions of problems (1),(2),(3) can be posed in the non-abelian case. This can be done as follows. Given a definable group G, there is a canonical real Lie group G/G^{00} associated to G (by [Pi:04] and [BOPP:05]). By [Ba:10] and [BeMa:10] when G is definably compact and definably connected, the isomorphism type of G/G^{00} determines G up to definable homotopy equivalence. One can ask whether the isomorphism type of G/G^{00} determines G up to definable homeomorphism. The results in [Ma:10] reduce the question to the abelian case, which is the one we consider in this paper. Finally let us observe that by [Co:09t, Thm.3.8.8] (see also [Co:09]) the study of the topology of a definable group reduces to the definably compact case.

We shall make use of the "o-minimal Hauptvermutung" proved by M. Shiota in [Sh:97, Chapter III] when M is an expansion of $\mathbb{R}_{\text{field}}$, and extended in [Sh:10, §2] to the case when M is an o-minimal expansion of an arbitrary real closed field.

Fact 1.1. (o-minimal Hauptvermutung) *Let K and L be finite simplicial complexes. Let M be an o-minimal expansion of a real closed field. If there exists a definable, in M, homeomorphism from $|K|$ to $|L|$, then there is a PL-isomorphism from $|K|$ to $|L|$.*

Here $|K|$ denotes a geometrical realization, in M, of the simplicial complex K. In this note all simplicial complexes are closed and finite, so that $|K|(M)$ is always definably compact. By a "PL map" we always mean "finitely PL map", namely the geometrical realization of a simplicial map between finite subdivisions of the relevant complexes. In Section 2 we prove the semialgebraic case of the homeomorphism problem (2) using the Hauptvermutung in the weak form of [Sh:97]. In Section 3 we assume $\dim(G) \neq 4$ and we reduce the general o-minimal case to the semialgebraic case. In this step we need the strong form of the Hauptvermutung (as in [Sh:10]) and the following fact:

Fact 1.2. (Classification of Homotopy tori) *Let X be a closed PL-manifold of dimension $n \neq 4$ homotopy equivalent to the standard torus $\mathbb{T}^n(\mathbb{R})$ (considered as a PL-manifold under a standard triangulation). Then there is a finite covering $f : \widetilde{X} \to X$ such that $\widetilde{X}$ is PL-homeomorphic to $\mathbb{T}^n(\mathbb{R})$.*

When $n \geq 5$ a proof of Fact 1.2 can be found in [HsSi:69, Theorem B] and [Wa:69, Corollary]. (See also [Wa:99, Chapter 15A] for a complete development of homotopy tori.) When $n \leq 3$ it turns out that X is already PL-homeomorphic to a standard torus. Indeed, for $\dim(X) = 1$ or 2 this is well-known and for $\dim(X) = 3$ we can use [KiSi:77, Theorem 5.4, pag 249] together with the positive solution of the three dimensional Poincaré's conjecture. Since our intended readership may not be familiar with the notations in [KiSi:77] we add few lines of explanation. The cited theorem tells us that $S^*(\mathbb{T}^3) = 0$. Unraveling the notations this means that if M_1 and M_2 are PL-manifolds and $f_i\colon M_i \to \mathbb{T}^3$ is a homotopy equivalence for $i = 1, 2$, then there is a PL-homeomorphism $h\colon M_1 \to M_2$ such that $f_2 \circ h$ is homotopic to f_1. In particular, taking $M_2 = \mathbb{T}^3$ and $f_2 = id$, we obtain that any PL-manifold homotopy equivalent to $\mathbb{T}^3$ is PL-homeomorphic to $\mathbb{T}^3$. This statement (= Borel's conjecture for $\mathbb{T}^3$) is known to imply Poincaré's conjecture in dimension 3 [F:96, §1.4], which was not known when [KiSi:77] was written. The solution of the riddle lies in a note hidden inside the proof of Theorem 5.3 in [KiSi:77] whose effect is to modify the definition of S^* in dimension 3: "in dimension 3 we supplement this definition by supposing that M_1 is Poincaré, i.e., contains no fake 3-discs". Granted the positive solution to the 3-dimensional Poincaré's conjecture, the supplement is vacuous.

2. Semialgebraic case

In this section suppose that M is a real closed field without additional structure. So the definable sets in M coincide with the semialgebraic sets. We prove.

Theorem 2.1. *G is semialgebraically homeomorphic to the n-standard torus $\mathbb{T}^n(M)$.*

Proof. By Robson's embedding theorem (see [vdD:98, Theorem 10.1.8]) we can assume that the topology of G (given by [Pi:88]) coincides with the topology induced by the ambient space M^n. By the triangulation theorem we can then assume that the underlying set $\mathrm{dom}(G)$ is the realization of a $\emptyset$-definable finite simplicial complex K. A priori we cannot ensure that the group operation of G is $\emptyset$-definable, but by model completeness of the theory of real closed fields there exist a possibly different group operation on $\mathrm{dom}(G) = |K|$ which is $\emptyset$-definable and continuous with respect to the topology of $|K|$. Since we are only interested in the definable homeomorphism type of G we can assume the group operation is $\emptyset$-definable. We can then consider the group $G(\mathbb{R})$ obtained by interpreting the defining formulas in $\mathbb{R}_{\text{field}}$. By [Pi:88, Remark 2.6], there is a (unique) Nash group structure on $G(\mathbb{R})$. In particular, $G(\mathbb{R})$ is an abelian compact connected real Lie-group and therefore there is a Lie-isomorphism $f\colon G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$. We will show that f is definable in some o-minimal expansion of the real field. In fact it is enough to consider the o-minimal structure $\mathbb{R}_{an}$ studied in [vdD:86]. We need the following:

Fact: Given an analytic function f defined on an open subset V of $\mathbb{R}^n$, its restriction to a definable (i.e. semialgebraic) compact subset $K \subset V$ is definable in $\mathbb{R}_{an}$.

Indeed this is true (almost by definition of $\mathbb{R}_{an}$) when K is a compact poly-interval, and the general case follows by covering K by finitely many poly-intervals contained in V. We then obtain:

Claim: The Lie-isomorphism $f\colon G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$ is definable in $\mathbb{R}_{an}$.

In fact there are semialgebraic charts making $G(\mathbb{R})$ into a Nash group and for each chart V, $f|V$ is analytic. By shrinking the charts we can assume that $f|V$ extends to an analytic map on the closure of V. So by the above fact f is definable in $\mathbb{R}_{an}$.

In particular we have proved that there is a homeomorphism $f\colon G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$ definable in $\mathbb{R}_{an}$. By the semialgebraic triangulation theorem and the o-minimal Hauptvermutung of [Sh:97], there is a semialgebraic homeomorphism $g\colon G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$. Moreover, by model completeness of the theory of real closed field, there is some g as above which is $\emptyset$-definable in $\mathbb{R}_{\text{field}}$. Interpreting the same formulas in M we obtain a semialgebraic homeomorphism from $G(M)$ to $\mathbb{T}^n(M)$ as desired. □

3. General o-minimal case

In this section we assume that M is an arbitrary o-minimal expansion of a real closed field. We will prove:

Theorem 3.1. *If $n = \dim(G) \neq 4$, G is definably homeomorphic to the n-torus $\mathbb{T}^n(M)$.*

As above, we can assume that Pillay's topology on G coincides with the topology induced by the ambient space M^n, and by the triangulation theorem we can then assume that $\text{dom}(G)$ is the geometrical realization $|K|(M)$ of a finite simplicial complex K. We need:

Lemma 3.2. *If $n = \dim(G) \neq 4$, $\text{dom}(G) = |K|(M)$ admits a semialgebraic abelian group operation (possibly unrelated to the original one).*

Theorem 3.1 follows at once from the Lemma and the semialgebraic case (Theorem 2.1). So it remains to prove the lemma.

Proof of Lemma 3.2. Note that $\text{dom}(G) = |K|(M)$ is at the same time a closed definable manifold (with Pillay's topology) and the realization, over M, of a finite simplicial complex. By Shiota's o-minimal Hauptvermutung in [Sh:10], it easily follows (see Fact 3.3 below) that $|K|(M)$ is a closed PL-manifold "over M". This is equivalent to say that the closed star of each vertex of K is PL-homeomorphic to the standard simplex of the correct dimension. By model completeness of the theory of real closed fields, the same holds over $\mathbb{R}$. Namely $|K|(\mathbb{R})$ is a PL-manifold (but we have no way of inheriting the definable group structure of $|K|(M)$). Moreover $|K|(\mathbb{R})$ is homotopy equivalent to the standard torus $\mathbb{T}^n(\mathbb{R})$. Indeed, by [BeMaOt:08, Theorem 3.4] there exists a definable homotopy equivalence from $\text{dom}(G) = |K|(M)$ to $\mathbb{T}^n(M)$ and therefore by [BaOt:10, Theorem 3.1] there is a semialgebraic homotopy equivalence from $|K|(\mathbb{R})$ to $\mathbb{T}^n(\mathbb{R})$. Since $n = \dim(G) \neq 4$, by Fact 1.2, $|K|(\mathbb{R})$ has a finite PL-cover which is PL-homeomorphic to $\mathbb{T}^n(\mathbb{R})$. Namely we have a PL-covering $f : \mathbb{T}^n(\mathbb{R}) \to |K|(\mathbb{R})$ with finite fibers. By model completeness we can assume that f is defined without parameters. So we can go back to M and get a semialgebraic (actually PL) covering

$$f\colon\ \mathbb{T}(M)^n \to |K|(M) = \text{dom}(G).$$

But on $\text{dom}(G)$ we have a definable group operation that can be lifted to $\mathbb{T}^n(M)$ via f (by uniform lifting of paths). So we get a definable group operation $*$ on $\mathbb{T}^n(M)$ making f into a definable covering homomorphism with a finite kernel $\Gamma < (\mathbb{T}^n(M), *)$. Note that $*$ may not coincide with the natural group operation on $\mathbb{T}^n(M)$ (the sum mod 1), so in particular it need not be semialgebraic. In any case however $(\mathbb{T}^n(M), *)$ is an abelian group. Therefore there is k such that Γ is contained in the k-torsion subgroup $(\mathbb{T}^n(M), *)[k]$ of $(\mathbb{T}^n(M), *)$. Our next goal is to obtain a definable group homomorphism

$$h\colon G \to (\mathbb{T}(M)^n, *).$$

Write for simplicity $\mathbb{T}^n$ for $\mathbb{T}^n(M)$. Now G is definably isomorphic to $(\mathbb{T}^n, *)/\Gamma$ and since $\Gamma < (\mathbb{T}^n, *)[k]$ there is a definable covering from $(\mathbb{T}^n, *)/\Gamma$ to $(\mathbb{T}^n, *)/(\mathbb{T}^n, *)[k]$.

The latter group is definably isomorphic to $(\mathbb{T}^n, *)$ because the k-torsion subgroup of a definably compact definably connected abelian group H is the kernel of the surjective homomorphism $H \to H$ sending x to kx (we need the fact that such groups are divisible, as proved in [EdOt:04, Theorem 2.1]). Composing we obtain a finite definable covering homomorphism $h \colon G \to (\mathbb{T}^n, *)$. As already remarked $*$ may not be semialgebraic. However on $\mathbb{T}^n$ we also have a semialgebraic group operation $\cdot$ (the sum mod 1). The idea is to use the covering map h (seen just as a continuous map, not as a group homomorphism) to lift the semialgebraic group operation $\cdot$ to a semialgebraic group operation on $\mathrm{dom}(G)$. The problem however is that h is not semialgebraic. However by [EdJoPe:10, Corollary 2.2], each definable cover of a semialgebraic group, is equivalent to a semialgebraic cover. So there is a semialgebraic covering homomorphism $h' \colon G' \to (\mathbb{T}^n, \cdot)$ and a definable homeomorphism $\psi : \mathrm{dom}(G) \to \mathrm{dom}(G')$ commuting with h and h'. But $\mathrm{dom}(G)$ and $\mathrm{dom}(G')$ are semialgebraic, so by the Hauptvermutung (combined with the triangulation theorem) there is a semialgebraic homeomorphism $\phi : \mathrm{dom}(G) \to \mathrm{dom}(G')$. Now take $h' \circ \phi$. This is a semialgebraic covering from $\mathrm{dom}(G)$ to $(\mathbb{T}^n, \cdot)$, and it can be used to lift $\cdot$ to a semialgebraic group operation on $\mathrm{dom}(G)$. □

Let us prove the missing fact needed in the above proof.

Fact 3.3. *Let K be a finite simplicial complex such that $|K|$ is an n-dimensional closed definable manifold. Then $|K|$ is a PL-manifold, namely the star of each vertex of K is PL-homeomorphic to the standard n-simplex.*

Proof. In this proof simplicial complexes are assumed to be finite but not necessarily closed. We will use without mention the well-known invariance of stars in piecewise linear topology, i.e., the star of a vertex of a closed simplicial complex is PL-isomorphic to the star of that vertex in any simplicial subdivision. Let $\{(U_1, f_1), \dots, (U_s, f_s)\}$ be a definable atlas of $|K|$. That is, each U_i is a definable open subset of $|K|$, each f_i is a definable homeomorphism from U_i to a definable open subset V_i of M^n (with the usual property on transition maps) and $|K| = \bigcup_{i=1}^s U_i$. By shrinking of coverings, we can find definable open subsets $W_1, \dots, W_s$ of $|K|$ such that $|K| = \bigcup_{i=1}^s W_i$ and $W_i \subset \overline{W}_i \subset U_i$ for each $i = 1, \dots, s$. Moreover, by the triangulation theorem we can assume that each V_i is the realization of an open finite simplicial complex and $f_i(\overline{W_i})$ the realization of a subcomplex. Considering a barycentric subdivision if necessary, we can also assume that the star in V_i of each vertex of $f_i(\overline{W_i})$ is a closed finite subcomplex. In particular, since V_i is an open subset of M^n, it follows that the star of each vertex in $f_i(\overline{W_i})$ is PL-isomorphic to a standard n-simplex.

Now, again by the triangulation theorem, there exist a definable homeomorphism $\psi : |L| \to |K|$ compatible with the definable sets U_i, W_i and $\overline{W_i}$. Since $\psi^{-1}(\overline{W_i})$ and $f_i(\overline{W_i})$ are definable homeomorphic, by the o-minimal Hauptvermutung they are PL-isomorphic. Now, given a vertex v of L, the star of v in L is contained in some $\overline{W_i}$ and therefore is PL-isomorphic to the star of some vertex of $f_i(\overline{W_i})$. Hence, the star of each vertex of L is PL-isomorphic to a standard n-simplex.

By the o-minimal Hauptvermutung, there exist a PL-isomorphism of $|L|$ and $|K|$. Hence we deduce that the star of each vertex of K is PL-isomorphic to the star of a vertex of L, which in turn is PL-isomorphic to a standard n-simplex. □

We have thus completed the proof of the Lemma, and Theorem 3.1 follows.

A possible attempt to deal with the case $\dim(G) = 4$ is to replace G with $G\times\mathbb{T}^1$. So by Theorem 3.1 there is definable homeomorphism from $G\times\mathbb{T}^1$ to $\mathbb{T}^5$. However we do not know whether this implies that there is a definable homeomorphism from G to $\mathbb{T}^4$. Finally let us observe that, even in dimension 4, we can always assume $\mathrm{dom}(G) = |K|(M)$ (after a triangulation) and conclude that $|K|(\mathbb{R})$ is homotopy equivalent to $\mathbb{T}^4(\mathbb{R})$ (reasoning as in the proof of Lemma 3.2). Moreover by [FrQu:90, §11.5] a fourth dimensional PL-manifold homotopy equivalent to a standard torus is homeomorphic to it (although not necessarily PL-homeomorphic). So in any case we conclude that $|K|(\mathbb{R})$ is homeomorphic to $\mathbb{T}^4(\mathbb{R})$, but since a priori the homeomorphism could be quite wild, there is no obvious way to obtain from these data a definable homeomorphism from $|K|(M)$ to $\mathbb{T}^4(M)$.

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