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TORIC EMBEDDED RESOLUTIONS OF QUASI-ORDINARY HYPERSURFACE SINGULARITIES

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Introduction.

A germ of complex analytic variety is *quasi-ordinary* if there exists a finite projection, called quasi-ordinary, to the complex affine space $(\mathbb{C}^d, 0)$ with discriminant locus contained in a normal crossing divisor (for instance, the singularities of complex analytic curves are quasi-ordinary). These singularities appear classically in Jung's strategy to obtain the resolution of singularities of surfaces from the embedded resolution of plane curves (see [J], [W] and [L2]). Some properties of complex analytic curve singularities generalize to quasi-ordinary hypersurface singularities: for instance, Jung-Abhyankar's theorem guarantees the existence of fractional power series parametrizations generalizing the classical Newton-Puiseux parametrizations of the plane curve case; by comparing these parametrizations we obtain a finite set of *distinguished* or *characteristic monomials* which generalize the notion of characteristic exponents in the plane branch case.

The results on quasi-ordinary hypersurface singularities concern mainly the analytically irreducible case: Lipman builds a non embedded resolution procedure of a quasi-ordinary surface where only quasi-ordinary singularities occurs and uses it to prove the analytical invariance properties

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of the characteristic monomials (see [L1], and [L3]); another proof of this result was given by Luengo in [Lu]; more generally Gau proved that the characteristic monomials, suitably normalized by an inversion formulae of Lipman [L1], define a complete invariant of the *embedded topological type* of the quasi-ordinary hypersurface singularity (see [Gau]); Gau's proof involves Lipman's description of the divisor class group of the singularity in terms of the characteristic monomials (see [L4]).

An important step to establish the relations between the topological type and the embedded resolutions of a hypersurface singularity, which are well-known in the case of plane curve singularities (see [Z4], [Z3] and [Re]), is to determine if the characteristic monomials of a hypersurface quasi-ordinary singularity determine a procedure of embedded resolution. This is the content of Lipman's open problem 5.1 (see [L5]) which is stated in the context of the generalizations of *equisingularity*, in particular by using Zariski's work on the *dimensionality type* with respect to the classification by *equiresolution*. In the case of an analytically irreducible quasi-ordinary surface germ Ban and McEwan (see [B-M]) have found a such a procedure following the algorithm of resolution of Bierstone and Milman, developed from the work of Hironaka. Villamayor has given a solution to Lipman's problem for any quasi-ordinary hypersurface singularity (see [V2]). Villamayor's approach studies the abelian branch covering of the affine space obtained by taking suitable roots of the regular parameters defining the components of the discriminant locus. By the Jung-Abhankar's Theorem the equation of the quasi-ordinary hypersurface under this extension splits in a product of Weierstrass polynomials of degree one. The singularity obtained is a non transversal intersection of smooth hypersurfaces, whose embedded resolution requires the simplest combinatorial part of Hironaka's method. The important point that he proves is that this resolution procedure is Galois equivariant, in such a way that when taking the quotients by the Galois action the local constructions glue up together defining a modification of the embedded quasi-ordinary hypersurface. The ambient space obtained in this way has only *toric quotient singularities* and a canonical resolution of these singularities (see [V1]) provides an embedded desingularization of the quasi-ordinary hypersurface. The desingularization obtained is not necessarily an isomorphism outside the singular locus of the quasi-ordinary hypersurface.

In this paper we give another solution to Lipman's problem in two different ways.

In the first one we build an embedded resolution of a reduced quasi-ordinary hypersurface germ $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$ as a composition of toric morphisms which depend only on the characteristic monomials (see Theorem 1). The first toric morphism we build is defined by the *dual Newton diagram* of a suitable Weierstrass polynomial $f \in \mathbb{C}\{X_1, \ldots, X_d\}[Y]$ defining the embedding $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$. Suitable here means that Y is a good coordinate: the Newton polyhedron of f, have compact faces of dimension at most one, and it is canonically determined by the characteristic monomials. We study the strict transform S' of S by this modification: we show that the restriction $\pi_1 : S' \to S$ is a finite map. The germ of S' at any of the finitely many points of the fiber $\pi_1^{-1}(0)$ is a *toric quasi-ordinary singularity*, defined as finite branched covering of a normal affine toric variety unramified over its torus (see [GP1]). It follows that it is more natural to build the resolutions for *toric* quasi-ordinary hypersurfaces by generalizing to this case the notions of characteristic monomials and many of their properties in the *classical* quasi-ordinary case. At any point of $\pi_1^{-1}(0)$, the strict transform S' has less characteristic monomials, with respect to a projection canonically determined from the fixed quasi-ordinary projection of S, and we determine them from the given characteristic monomials of $(S, 0)$. By iterating we obtain, in a canonical manner from the fixed quasi-ordinary projection of S, a *partial embedded resolution*: a normal variety of dimension $d + 1$ with only toric singularities (not necessarily quotient singularities) and a modification $\pi = \pi_1 \circ \ldots \circ \pi_k$ such that the strict transform of S is a d-dimensional section transversal to the exceptional fiber $\pi^{-1}(0)$ (which is of dimension one). This implies that any toric resolution of the ambient space is an embedded resolution of the strict transform and provides *a fortiori* an embedded resolution of S. It follows also that the restriction of π to the the strict transform of S is the *normalization map* of S. This implies that the restriction of any of these embedded resolutions to the strict transform of S is an isomorphism outside the singular locus of S. In the case of a plane curve germ we show that our procedure, with respect to a transversal projection, leads to the minimal resolution of the curve and we compare our method with those given by Lê, Oka and A'Campo (see [Le-Ok], [Ok], and [A'C-Ok]).

The second method builds embedded resolutions of an analytically irreducible quasi-ordinary hypersurface germ $(S, 0)$ by generalizing the method of Goldin and Teissier for plane branches (see [G-T]). The approach and results of this part are also inspired those obtained by Lejeune and Reguera in the case of sandwiched surface singularities (see [LJ-R]) and sketched for plane branches in [LJ-R2]. If $g \geqslant 1$ denotes the number of characteristic exponents we re-embed the germ $(S, 0)$ in the affine space $(\mathbb{C}^{d+g}, 0)$, by using certain *approximate roots* of a suitable Weierstrass polynomial defining the embedding $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$. These approximate roots have *maximal contact* in the sense that, at each step of the partial resolution there is one approximate root whose strict transform defines a good coordinate for the strict transform of S. We define a toric modification $p : Z \to \mathbb{C}^{d+g}$ depending only on a rank d semigroup Γ, which is a *partial embedded resolution* of the irreducible germ $(S, 0) \subset (\mathbb{C}^{d+g}, 0)$, and of an affine toric variety $Z^\Gamma \subset \mathbb{C}^{d+g}$ obtained from $(S, 0) \subset (\mathbb{C}^{d+g}, 0)$ by specialization and defined by the semigroup Γ (see Theorem 2). This semigroup, which generalizes the classical semigroup of a plane branch, does not depend on the quasi-ordinary projection and defines a complete invariant of the embedded topological type of S, as characterized by Gau (see [GP2] or [GP3]). As in the first method any toric resolution of singularities of the ambient space Z provides an embedded resolution of S.

We compare the partial resolutions π and p: we prove in Theorem 3 that π is the restriction of p to a $(d + 1)$-dimensional smooth variety of Z containing the strict transform of S.

One of the technical tools common to both methods is the construction of toric embedded resolutions of non necessarily normal affine toric varieties equivariantly embedded, a result obtained in collaboration with Teissier (see Proposition 6, Proposition 6.4 of [T2], and [GP-T]).

One important contribution of our approach is a better understanding of the structure of the exceptional divisor of these resolutions. The ambient space of the partial resolution π, which is canonical and factors any of these embeded resolutions, is built with a toroidal embedding structure such that the associated conic polyhedral complex with integral structure (see [KKMS]) is built explicitly from the characteristic monomials. This description allows us to re-embed this complex as a fan in an affine space of bigger dimension, a technical lemma which is essential to compare the partial resolutions p and π (see Propositions 42 and 45). The toric resolutions of the ambient space are defined by certain regular subdivisions of this fan (which always exists, see [Co] and [KKMS]). These regular subdivisions determine many features of the geometry of the exceptional divisor which are very useful for the applications:

• In collaboration with Némethi and McEwan we have shown that the zeta function of the geometric monodromy of the germ $(S, 0)$ coincides with the zeta function of the plane curve germ obtained from $(S, 0)$ by intersection with $d-1$ coordinate hyperplanes, which are determined by the quasi-ordinary projection (see [M-N] and [GP-M-N]).

• In collaboration with García Barroso we analyse in [GB-GP] the strict transform of the polar hypersurfaces of $(S, 0)$ under the partial resolution of $(S, 0)$ and we obtain a decomposition theorem which provides in the case of plane curve germs a simple algebraic proof of a Theorem of Lê, Michel and Weber ([L-M-W]).

The proofs are written in the analytic case. They provide also two embedded resolutions of quasi-ordinary hypersurface singularities in the algebroid case (over an algebraically closed field of zero characteristic).

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1. Toric maps, Newton polyhedra and partial resolution of singularities.

We introduce the notations and basic definitions of toric geometry and we build embedded resolutions of non necessarily normal affine toric varieties.

1.1. A reminder of toric geometry.

We give some definitions and notations (see [F], [Ew] and [Od] for proofs). If $N \cong \mathbb{Z}^{d+1}$ is a lattice we denote by $N_{\mathbb{R}}$ the real vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by N and by M the dual lattice. A *rational convex polyhedral cone* σ in $N_{\mathbb{R}}$ is the set non negative linear combinations of vectors $a^1, \dots, a^s \in N$. In what follows a *cone* will mean a rational convex polyhedral cone. The cone σ is *strictly convex* if σ contains no linear subspace of dimension > 0; the cone σ is *regular* if the *primitive integral vectors* defining the 1-dimensional faces belong to a basis of the lattice N. We denote by $\overset{\circ}{\sigma}$ the *relative interior* of a cone σ. The *dual* cone $\sigma^\vee$ (resp. *orthogonal* cone $\sigma^\perp$) of σ is the set $\{w \in M_{\mathbb{R}} / \langle w, u \rangle \geqslant 0$, (resp. $\langle w, u \rangle = 0)$

$\forall u \in \sigma\}$. A *fan* Σ is a family of *strictly convex cones* in $N_{\mathbb{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The *support* of the fan Σ is the set $\bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$. The *i*-skeleton $\Sigma^{(i)}$ is the subset of i-dimensional cones of Σ. The fan Σ is *regular* if all its cones are regular.

Any non necessarily normal affine toric variety over the field $\mathbb{C}$ of complex numbers is of the form $Z^{\Lambda} = \operatorname{Spec} \mathbb{C}[\Lambda]$ where Λ is a *monoid*, i.e., a sub-semigroup of finite type of a lattice $-\Lambda + \Lambda$ which generates it as a group. The closed points of Z^{Λ} correspond to homomorphisms of semigroups $\Lambda \to \mathbb{C}$ where $\mathbb{C}$ is considered as a semigroup with respect to multiplication. The torus embedded in Z^{Λ} is the group of homomorphisms of semigroups $\Lambda \to \mathbb{C} - \{0\}$ and acts naturally on the closed points of Z^{Λ}. The *normalization* of Z^{Λ} is obtained from the inclusion $\Lambda \to \mathbb{R}_{\geqslant 0}\Lambda \cap (-\Lambda + \Lambda)$ where $\mathbb{R}_{\geqslant 0}\Lambda$ is the cone spanned by the elements of Λ (see [KKMS]). The action of the torus has a fixed point if and only if the cone $\mathbb{R}_{\geqslant 0}\Lambda$ is strictly convex, then this point is defined by the ideal $(X^u / u \in \Lambda - \{0\})$ of $\mathbb{C}[\Lambda]$ and coincides with the 0-dimensional orbit; the analytic algebra $\mathbb{C}\{\Lambda\}$ of Z^{Λ} at this point can be viewed as a subring of the ring $\mathbb{C}[[\Lambda]]$ of formal complex power series with exponents in the semigroup Λ (see [GP1] lemme 1.1).

In particular, if σ is a cone in the fan Σ the semigroup $\sigma^{\vee} \cap M$ is of finite type, it spans the lattice M and the variety $Z^{\sigma^{\vee} \cap M}$, which we denote also by $Z_{\sigma,N}$ or by Z_{σ} when the lattice is clear from the context, is *normal.*

If $\sigma \subset \sigma'$ are cones in the fan Σ we have an open immersion $Z_{\sigma} \subset Z_{\sigma'}$; the affine varieties Z_{σ} corresponding to cones in a fan Σ glue up to define the *toric variety* Z_{Σ}. The torus, $(\mathbb{C}^*)^{d+1}$, is embedded as an open dense subset $Z_{\{0\}}$ of Z_{Σ}, which acts on each chart Z_{σ}; these actions paste to an action on Z_{Σ} which extends the product on the torus. General toric varieties are defined by this property, the toric varieties which can be defined using fans are precisely the normal ones (see [KKMS]). The toric variety Z_{Σ} is non singular if and only if the fan Σ is regular.

We describe the orbits of the action of the torus on the variety Z_{Σ}. The orbit $\mathbb{O}_{\sigma,N}$ (which we denote also by $\mathbb{O}_{\sigma}$) is the Zariski closed subset of Z_{σ} defined by the ideal $(X^w / w \in (\sigma^{\vee} - \sigma^{\perp}) \cap M)$ of $\mathbb{C}[\sigma^{\vee} \cap M]$. This orbit is a torus for $0 \leqslant \dim \sigma < \operatorname{rk} N$, since the associated coordinate ring is the $\mathbb{C}$-algebra of the sub-lattice $M(\sigma) := M \cap \sigma^{\perp}$ of M of codimension equal to $\dim \sigma$. On the closed orbit $\mathbb{O}_{\sigma}$ we consider the *special point* o_{σ} defined by $X^u(o_{\sigma}) = 1$ for all $u \in M(\sigma)$. If $\dim \sigma = \operatorname{rk} N$ the orbit $\mathbb{O}_{\sigma}$ is reduced to the special point. If $\dim\sigma < \operatorname{rk} N$ we have an exact sequence of lattices:

$$0 \to M(\sigma) \to M \xrightarrow{j} M_\sigma \to 0.$$

If $0 \to N_\sigma \xrightarrow{j^*} N \to N(\sigma) \to 0$ is the dual exact sequence the lattice N_σ spanned by $\sigma \cap N$ is of dimension equal to $\dim\sigma$ and the semigroup $\sigma^\vee_{N_\sigma}$ associated to the cone σ with respect to the lattice N_σ is isomorphic to $j(\sigma^\vee \cap M)$. If we choose a splitting $M \cong M(\sigma) \oplus M_\sigma$ we obtain a semigroup isomorphism $\sigma^\vee \cap M \cong M(\sigma) \oplus (\sigma^\vee_{N_\sigma} \cap M_\sigma)$ inducing an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}[M(\sigma)] \otimes_{\mathbb{C}} \mathbb{C}[\sigma^\vee_{N_\sigma} \cap M_\sigma]$ which defines (non canonically) the product structure

$$Z_{\sigma,N} \cong \mathbb{O}_{\sigma,N} \times Z_{\sigma,N_\sigma}. \tag{1}$$

The map that sends a cone σ in Σ to the orbit $\mathbb{O}_\sigma \subset Z_\Sigma$ is a bijection between the fan Σ and the set of orbits. If σ is a face of τ then Z_σ is an open subset of Z_τ and the orbit $\mathbb{O}_\tau$ is contained in the closure of $\mathbb{O}_\sigma$ in Z_τ since $\tau^\perp \subset \sigma^\perp$, thus the closure of the orbit of σ in Z_Σ is $\overline{\mathbb{O}_\sigma} = \bigcup \mathbb{O}_\tau$ where τ runs through the cones of Σ which have σ as a face.

The orbit closures are normal toric varieties by themselves with respect to the lattice $N(\sigma)$. The cones of the fan associated to $\overline{\mathbb{O}_\sigma}$ are of the form $\tau + (N_\sigma)_\mathbb{R} \subset N_\mathbb{R}/(N_\sigma)_\mathbb{R}$ for $\tau \in \Sigma$ containing σ as a face.

Remark 1.— The singular locus of Z_Σ is the union of those orbits $\mathbb{O}_\sigma$ for σ a non regular cone.

This follows from formula (1) by noticing that the orbit $\mathbb{O}_\sigma$ is contained in the singular locus of Z_σ if and only if o_{σ,N_σ} is a singular point of Z_{σ,N_σ} if and only if the cone σ is not a regular cone.

Definition 1.— *A fan Σ' is a* subdivision *of the fan Σ if both fans have the same support and if any cone of Σ' is contained in a cone of Σ. The fan Σ' is* regular subdivision *if Σ' is a regular fan. A regular subdivision Σ' is a* resolution of the fan Σ *if any regular cone of Σ belongs to Σ'.*

Associated to a subdivision of fans there is a *modification* $\pi_\Sigma : Z_{\Sigma'} \to Z_\Sigma$ inducing an isomorphism between their tori.

Example 1.— Let Σ be a regular subdivision of the cone $\sigma := \mathbb{R}^{d+1}_{\geqslant 0}$ with lattice $N := \mathbb{Z}^{d+1}$. This subdivision defines a modification $\pi_\Sigma : Z_\Sigma \to Z_\sigma = \mathbb{C}^{d+1}$ which we describe in detail:

The variety Z_Σ is non singular, for each cone σ of maximal dimension the variety Z_σ is isomorphic to $\mathbb{C}^{d+1}$ and the restriction $\pi_\sigma : Z_\sigma \to \mathbb{C}^{d+1}$ of the morphism π_Σ is induced by the semigroup inclusion $\mathbb{R}^{d+1}_{\geqslant 0} \cap M \to \sigma^\vee \cap M$. The set of primitive vectors in the 1-skeleton σ is a basis of N and its dual basis is a minimal set of generators of the semigroup $\sigma^\vee \cap M$. These generators give us coordinates to describe the map $\pi_\sigma : Z_\sigma \to \mathbb{C}^{d+1}$ in the form:

$$\begin{aligned} X_1 &= U_1^{a_1^1} U_2^{a_1^2} \cdots U_{d+1}^{a_1^{d+1}} \\ X_2 &= U_1^{a_2^1} U_2^{a_2^2} \cdots U_{d+1}^{a_2^{d+1}} \\ &\cdots \\ X_{d+1} &= U_1^{a_{d+1}^1} U_2^{a_{d+1}^2} \cdots U_{d+1}^{a_{d+1}^{d+1}} \end{aligned} \tag{2}$$

where $(a_1^i, a_2^i, \ldots, a_{d+1}^i)$ is the coordinate of the primitive vector a^i in the 1-skeleton of σ, for $i = 1, \ldots, d+1$. Since the fan Σ is regular, it is easy to see directly from formula (2) that the map π_Σ is an isomorphism over the torus $X_1 \cdots X_{d+1} \neq 0$ of $\mathbb{C}^{d+1}$.

A *resolution of singularities* of a variety Z is a smooth variety Z' with a modification $Z' \to Z$ which is an isomorphism outside the singular locus of Z. The resolution of singularities of normal toric varieties is reduced to a combinatorial property of faces (see [KKMS]). More precisely we have that: Given any fan Σ there is a resolution Σ' of Σ (see definition 1). The associated toric morphism $Z_{\Sigma'} \to Z_\Sigma$ is a resolution of singularities of the variety Z_Σ (see [Co], Theorem 5.1).

We describe now the exceptional locus associated to a subdivision Σ' of a fan Σ. Taking away the cone σ from the fan of the cone σ means geometrically to take away the orbit $\mathbb{O}_\sigma$ from the variety Z_σ. It follows that (see [GS-LJ] Proposition page 199):

$$\pi^{-1}(\mathbb{O}_\sigma) = \bigcup_{\tau \in \Sigma', \circ\tau \subset \circ\sigma} \mathbb{O}_\tau. \tag{3}$$

It follows from (3) that the *exceptional fibers*, i.e., the union of subvarieties of dimension > 1 which are mapped to points, are given by

$$\bigcup_{\dim \sigma = \mathrm{rk} N, \sigma \notin \Sigma'} \pi^{-1}(\mathbb{O}_\sigma)$$

and that the *exceptional locus*, i.e., the subvarieties that are mapped on a variety of smaller dimension, is

$$\bigcup_{\sigma \notin \Sigma'} \pi^{-1}(\mathbb{O}_\sigma) = \bigcup_{\tau \text{ minimal } \in \Sigma' - \Sigma} \overline{\mathbb{O}_\tau}.$$

The *discriminant locus*, i.e., the image of the exceptional locus, is equal to

$$\bigcup_{\sigma \text{ minimal } \in \Sigma - \Sigma'} \overline{\mathbb{O}_\sigma}. \quad (4)$$

1.2. Newton polyhedra and partial resolution of singularities.

The *Newton polyhedron* $\mathcal{N}(\phi)$ of a non zero series $\phi = \sum c_a X^a \in \mathbb{C}\{X\}$ with $X = (X_1, \ldots, X_{d+1})$ is the convex hull of the set $\bigcup_{c_a \neq 0} a + \mathbb{R}^{d+1}_{\geqslant 0}$. More generally the Newton polyhedron of any non-zero germ $\phi = \sum c_a X^a$ of holomorphic function at the special point o_ρ of a normal affine toric variety $Z_\rho = \mathrm{Spec}\mathbb{C}[\rho^\vee \cap M]$ (for a strictly convex cone $\rho^\vee$) is the convex hull of the subset $\bigcup_{c_a \neq 0} a + \rho^\vee$ of $M_\mathbb{R}$. We denote it by $\mathcal{N}_\rho(\phi)$ or by $\mathcal{N}(\phi)$ if the cone ρ is clearly determined by the context. Many of the properties associated with classical Newton polyhedra hold in this case; for instance, if $0 \neq \phi = \phi_1 \cdots \phi_s$ we have that $\mathcal{N}(\phi)$ is the Minkowski sum $\mathcal{N}(\phi_1) + \ldots + \mathcal{N}(\phi_1)$ since the series ϕ_i have coefficients in a domain. It follows from this property that:

Remark 2. — If $0 \neq \phi = \phi_1 \cdots \phi_s$ and $\mathcal{N}(\phi)$ has only one vertex the same holds for each of the Minkowski terms $\mathcal{N}(\phi_i)$, for $i = 1, \ldots, s$.

The *face* $\mathcal{F}_u$ *of the polyhedron* $\mathcal{N}_\rho(\phi)$ *defined by a vector* in $u \in \rho$ is the set of vectors $v \in \mathcal{N}_\rho(\phi)$ such that $\langle u, v \rangle = \inf_{v' \in \mathcal{N}_\rho(\phi)} \langle u, v' \rangle$. All faces of the polyhedron $\mathcal{N}_\rho(\phi)$ can be recovered in this way. The face of $\mathcal{N}_\rho(\phi)$ defined by u is compact if and only if $u \in \mathring{\rho}$.

The *cone* $\sigma(\mathcal{F}) \subset \rho$ *associated to the face* $\mathcal{F}$ of the polyhedron $\mathcal{N}_\rho(\phi)$ is

$$\sigma(\mathcal{F}) := \{u \in \rho \ / \ \forall v \in \mathcal{F}, \text{ we have } \langle u, v \rangle = \inf_{v' \in \mathcal{N}_\rho(\phi)} \langle u, v' \rangle \}.$$

The cones $\sigma(\mathcal{F})$, for $\mathcal{F}$ running through the set of faces of the polyhedron $\mathcal{N}_\rho(\phi)$, define a subdivision $\Sigma(\mathcal{N}_\rho(\phi))$ of the fan of the cone ρ called the *dual Newton diagram*. The relative interiors of the cones in the fan

$\Sigma(\mathcal{N}_\rho(\phi))$ are equal to the equivalence classes of vectors in ρ by the relation: $u \sim u' \Leftrightarrow \mathcal{F}_u = \mathcal{F}_{u'}$. We say that a fan Σ supported on the cone ρ is *compatible* with a set of series $\phi_1, \ldots, \phi_s \in \mathbb{C}\{\rho^\vee \cap M\}$ if it subdivides the fan $\Sigma(\mathcal{N}_\rho(\phi))$ with $\phi = \phi_1 \cdots \phi_s$. A cone in the fan $\Sigma(\mathcal{N}_\rho(\phi_1 \cdots \phi_s))$ is intersection of cones of the fans $\Sigma(\mathcal{N}_\rho(\phi_i))$ therefore Σ is compatible with all the polyhedra $\mathcal{N}_\rho(\phi_i)$. If Σ is compatible with $\mathcal{N}_\rho(\phi)$ all vectors in $\mathring{\sigma}$ define the same face $\mathcal{F}_\sigma$ of $\mathcal{N}_\rho(\phi)$, for $\sigma \in \Sigma$.

Definition 2. — *Let $0 \neq \phi = \sum c_a X^a \in \mathbb{C}\{\rho^\vee \cap M\}$. The* symbolic restriction *$\phi_{|\mathcal{F}}$ of ϕ to the compact face $\mathcal{F}$ of the polyhedron $\mathcal{N}_\rho(\phi)$ is the polynomial $\phi_{|\mathcal{F}} := \sum_{a \in \mathcal{F}} c_a X^a \in \mathbb{C}[\rho^\vee \cap M]$. The* Newton principal part *$\phi_{|\mathcal{N}}$ of ϕ is the sum of those terms of ϕ having exponents lying on the compact faces of the Newton polyhedron $\mathcal{N}_\rho(\phi)$*

We follow here the terminology of [Kou] and [Ok]. The Newton principal part $\phi_{|\mathcal{N}} \in \mathbb{C}[\rho^\vee \cap M]$ does not change if we change the ring $\mathbb{C}[[\rho^\vee \cap M]]$ by extending the lattice M.

Let Σ be any fan supported on ρ defining the modification $\pi_\Sigma : Z_\Sigma \to Z_\rho$. Let $\mathcal{V}$ be a subvariety of Z_ρ such that the intersection of the discriminant locus of π_Σ with each irreducible component $\mathcal{V}_i$ of $\mathcal{V}$ is nowhere dense on $\mathcal{V}_i$. For instance if $\mathcal{V}$ is irreducible this condition holds if the torus is an open dense subset of $\mathcal{V}$. The *strict transform* $\mathcal{V}_\Sigma \subset Z_\Sigma$ is the subvariety of $\pi_\Sigma^{-1}(\mathcal{V})$ such that the restriction $\mathcal{V}_\Sigma \to \mathcal{V}$ is a modification.

If the fan Σ is regular, the toric map $\pi_\Sigma : Z_\Sigma \to Z_\rho$ is a (toric) *embedded pseudo-resolution* of $\mathcal{V}$ if the restriction $\mathcal{V}_\Sigma \to \mathcal{V}$ is a modification such that the strict transform $\mathcal{V}_\Sigma$ is non singular and transversal to the orbit stratification of the exceptional locus of Z_Σ. The modification π_Σ is a (toric) *embedded resolution* of $\mathcal{V}$ if in addition the restriction to the strict transform $\mathcal{V}_\Sigma \to \mathcal{V}$ is an isomorphism outside the singular locus of $\mathcal{V}$ (see [G-T]). If π_Σ is only a pseudo-resolution we can only guarantee that the map $\mathcal{V}_\Sigma \to \mathcal{V}$ is an isomorphism outside the intersection of $\mathcal{V}$ with the discriminant locus of π_Σ. In this case, this set contains the singular locus of $\mathcal{V}$ but it is not necessarily equal to it.

Definition 3. — *If Σ is a (non necessarily regular) subdivision of ρ the toric morphism $\pi_\Sigma : Z_\Sigma \to Z_\rho$ is a* partial (toric) embedded resolution *of $\mathcal{V}$ if for any resolution Σ' of the fan Σ the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is an embedded resolution of $\mathcal{V}$.*

Let $\mathcal{V} \subset Z_\rho$ an irreducible subvariety such that the intersection with the torus is an open dense subset. Let Σ be a subdivision ρ compatible with a set of generators $\phi_1, \dots, \phi_s$ of the ideal of $\mathcal{V} \subset Z_\rho$. We give a combinatorial condition on the Newton polyhedra of $\phi_1, \dots, \phi_s$ for the intersection of the strict transform with the exceptional fiber being non empty.

LEMMA 3.— *Let σ a cone in Σ such that $\overset{\circ}{\sigma} \subset \overset{\circ}{\rho}$. If $\mathbb{O}_\sigma \cap \mathcal{V}_\Sigma \neq \emptyset$ then the face $\mathcal{F}_i$ of the Newton polyhedron $\mathcal{N}(\phi_i)$ of ϕ_i defined by σ is of dimension $\geqslant 1$ for $1 \leqslant i \leqslant s$.*

Proof.— We have that $\phi_i - \phi_{i|\mathcal{F}_i}$ belongs to the ideal generated by $\{X^u / u \in (\mathcal{N}(\phi_i) - \mathcal{F}_i) \cap M\}$. Since Σ is compatible with the ϕ_i the cone $\sigma^\vee$ contains the cone spanned by elements in the polyhedron $-u_0 + \mathcal{N}(\phi_i)$ for any $u_0 \in \mathcal{F}_i$. Let $u_i \in \mathcal{F}_i$ be a vertex then we can factor in the ring $\mathbb{C}[\sigma^\vee \cap M]$:

$$\phi_i \circ \pi_\sigma = X^{u_i} \psi_i \text{ and } \phi_{i|\mathcal{F}_i} \circ \pi_\sigma = X^{u_i} \psi_{i|\mathcal{F}_i} \text{ with } \psi_{i|\mathcal{F}_i} \in \mathbb{C}[\sigma^\perp \cap M]$$

in such a way that the exponent of a term appearing in $X^{-u_i}(\phi_i \circ \pi_\sigma - \phi_{i|\mathcal{F}_i} \circ \pi_\sigma)$ belongs to $(\sigma^\vee - \sigma^\perp) \cap M$ and thus this term vanishes on the orbit $\mathbb{O}_\sigma$. By definition the elements $X^{-u_i} \phi \circ \pi_\sigma$ for $1 \leqslant i \leqslant s$ belong to the ideal defining the strict transform of $\mathcal{V}$. If the face $\mathcal{F}_i$ is a vertex for some i the ideal of $\mathbb{O}_\sigma \cap \mathcal{V}_\Sigma$ in Z_σ is equal to (1) thus $\mathcal{V}_\Sigma \cap \mathbb{O}_\sigma$ is empty. □

The following lemma is an easy consequence of the implicit function theorem.

Let $\rho \subset N_\mathbb{R}$ be a rational strictly convex cone of dimension equal to rk N. We denote by Δ the cone $\rho \oplus \mathbb{R}^g_{\geqslant 0} \subset (N_\Delta)_\mathbb{R}$ where N_Δ is the lattice $N \oplus \mathbb{Z}^g$ with dual lattice M_Δ. The semigroup $\Delta^\vee \cap M_\Delta$ is of the form $(\rho^\vee \cap M) \oplus \mathbb{Z}^g_{\geqslant 0}$. The monomial corresponding to $(\alpha, v) \in \Delta^\vee \cap M_\Delta$ is denoted by $X^\alpha U^v$ or by $X^\alpha U_1^{v_1} \dots U_g^{v_g}$.

LEMMA 4.— *If $\phi_1, \dots, \phi_g \in \mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ verify that $\phi_1(o_\sigma, U) = U_i$, for $i = 1, \dots, g$ then there exist series $\epsilon_i \in \mathbb{C}\{\rho^\vee \cap M\}$ for $i = 1, \dots, g$ such that the ideals of $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ generated by $\phi_1, \dots, \phi_g$ and $U_1 - \epsilon_1, \dots, U_g - \epsilon_g$ coincide.*

Proof.— An homomorphism of semigroups $\mathbb{Z}^s_{\geqslant 0} \xrightarrow{\psi} \rho^\vee \cap M$ extends to an homomorphism $\mathbb{Z}^{s+k}_{\geqslant 0} \xrightarrow{\psi \times \mathrm{Id}} \Delta^\vee \cap M_\Delta$. If ψ is surjective it defines an equivariant embedding $Z_\rho \subset \mathbb{C}^s$ which extends (by using the homomorphism $\psi \times \mathrm{Id}$) to an equivariant embedding $Z_\Delta = Z_\rho \times \mathbb{C}^g \subset \mathbb{C}^{s+g}$. If

$\varphi_1, \dots, \varphi_g$ are power series defining holomorphic functions at $(\mathbb{C}^{s+g}, 0)$ representing $\phi_1, \dots, \phi_g$ the implicit function theorem guarantees the existence of power series ε_i in s variables such that the ideals $(\varphi_1, \dots, \varphi_g)$ and $(U_1 - \varepsilon_1, \dots, U_g - \varepsilon_g)$ coincide. The result follows by passing to the quotient by the binomial ideal defining the embedding $Z_\Delta \subset \mathbb{C}^{s+g}$. □

1.3. Embedded resolution of non necessarily normal toric varieties.

We build an embedded resolution of non necessarily normal affine toric variety Z^Λ equivariantly embedded in a normal affine toric variety Z_ρ (for $\rho^\vee$ a strictly convex cone). We build first a *partial embedded resolution* which is a toric morphism providing an *embedded normalization* inside a normal toric ambient space. Then any toric resolution of the singularities of the ambient space, which always exists, provides an embedded resolution. The advantage of this method is that the partial resolution is completely determined by the embedding $Z^\Gamma \subset Z_\rho$. This result is the fruit of discussions with Professor B. Teissier (see [T2], §6, Proposition 6.4 and [GP-T]).

Let Λ be a monoid. An *equivariant embedding* of Z^Λ in the normal affine toric variety Z_ρ is given by a surjective homomorphism of semigroups $\rho^\vee \cap M \to \Lambda$ which extends to a lattice homomorphism $\varphi : M \to -\Lambda + \Lambda$ and a vector space homomorphism $\varphi_\mathbb{R} : M_\mathbb{R} \to (-\Lambda + \Lambda)_\mathbb{R}$. The torus of Z^Λ is equivariantly embedded in the torus of Z_ρ, the embedding is obtained from the homomorphism φ. The linear subspace $(\mathrm{Ker}(\varphi_\mathbb{R}))^\perp \subset N_\mathbb{R}$, denoted by ℓ in what follows, is of dimension equal to $\mathrm{rk}\, \Lambda$ and the same holds for the cone $\sigma_0 := \ell \cap \rho$. The ideal of the embedding $Z^\Lambda \subset Z_\rho$ is generated by the binomials

$$X^u - X^v \in \mathbb{C}[\rho^\vee \cap M] \text{ such that } \varphi(u) = \varphi(v) \tag{5}$$

(see [St], Chapter 4).

Lemma 5. — *With the above notations suppose that the cone $\rho^\vee$ is strictly convex. Let Σ be any fan compatible with a finite set of binomial equations $X^{u_i} - X^{v_i} = 0$ for $i \in I$ defining the embedding $Z^\Lambda \subset Z_\rho$. Then the fan Σ is compatible with the linear subspace ℓ. If $\sigma \in \Sigma$ and $\mathring{\sigma} \subset \mathring{\rho}$ then $\mathbb{O}_\sigma \cap Z^\Lambda_\Sigma \neq \emptyset$ implies that $\sigma \subset \ell$. Moreover, if $\sigma \subset \ell$ and $\dim \sigma = \dim \ell$ the intersection $Z^\Lambda_\Sigma \cap \mathbb{O}_\sigma$ as schemes is the simple point o_σ and $Z^\Lambda_\Sigma \cap Z_\sigma$ is isomorphic to Z_{σ, N_σ}. If Σ is regular the morphism π_Σ is an embedded pseudo-resolution of singularities of $Z^\Lambda \subset Z_\rho$.*

Proof. — The cone $\sigma_0 = \rho \cap \ell$ is associated to the Minkowski sum of compact edges of $\mathcal{N}(X^{u_i} - X^{v_i})$ for $i \in I$ since $\langle w, u_i \rangle = \langle w, v_i \rangle$, $\forall i \in I$ if and only if $w \in \ell$. Since the fan Σ is compatible with the binomial equations of Z^Λ it follows that a subdivision of σ_0 is contained in Σ, i.e., this fan is compatible with the linear subspace ℓ.

We deduce by duality from the equality $\sigma_0 = \rho \cap \ell$ that

$$\sigma_0^\vee = \rho^\vee + \ell^\vee = \rho^\vee + \ell^\perp = \rho^\vee + \mathrm{Ker}(\varphi_\mathbb{R}). \tag{6}$$

Since the cone $\rho^\vee$ is strictly convex, formula (6) implies that

$$\sigma_0^\perp = \mathrm{Ker}(\varphi_\mathbb{R}) \tag{7}$$

and thus

$$\mathrm{Ker}(\varphi) \subset \sigma_0^\vee \cap M. \tag{8}$$

Let $\sigma \in \Sigma$ with $\overset{\circ}{\sigma} \subset \overset{\circ}{\rho}$, since Σ is compatible with the binomials $X^{u_i} - X^{v_i}$, the ideal generated by $1 - X^{u_i - v_i}$ (up to relabeling) is contained in the ideal defining the strict transform Z^Λ_Σ in the chart Z_σ. Thus the variety Z', defined by $X^{u_i - v_i} - 1 = 0$ for $i \in I$, contains $Z^\Lambda_\Sigma \cap Z_\sigma$. Then we have

$$\begin{aligned} Z' \cap \mathbb{O}_\sigma \neq \emptyset \Leftrightarrow\ & \exists p \in Z_\sigma : X^{u_i - v_i}(p) = 1\ \forall i \in I, X^u(p) = 0 \\ & \forall u \in (\sigma^\vee - \sigma^\perp) \cap M \\ \Leftrightarrow\ & u_i - v_i \in \sigma^\perp \cap M, \forall i \in I \Leftrightarrow \mathrm{Ker}\ (\varphi) \subset \sigma^\vee \Leftrightarrow \sigma \subset \rho \cap \ell. \end{aligned}$$

The chart Z_σ is isomorphic to $\mathbb{O}_\sigma \times Z_{\sigma, N_\sigma}$ by formula (1).

If $\sigma \subset \ell$ and $\dim \sigma = \dim \ell$ we have that $\sigma^\perp = \sigma_0^\perp$ coincides with $\mathrm{Ker}(\varphi_\mathbb{R})$ by (7). We deduce an isomorphism

$$Z' \cong \{o_\sigma\} \times Z_{\sigma, N_\sigma} \subset Z_\sigma \tag{9}$$

from (1) since the lattice $\sigma^\perp \cap M = \mathrm{Ker}(\varphi)$ is generated by $\{u_i - v_i\}_{i \in I}$. Therefore the variety Z' is irreducible and of dimension equal to rk Λ. We deduce from (9) that Z^Λ_Σ intersects the orbit $\mathbb{O}_\sigma$ transversally since the coordinate ring of $Z^\Lambda_\Sigma \cap \mathbb{O}_\sigma$ is $\mathbb{C}$. Since $Z^\Lambda_\Sigma \cap Z_\sigma$ is a subvariety of the irreducible variety Z' and both are of the same dimension they coincide.

If Σ is regular we deduce that Z^Λ_Σ is smooth and intersects transversally the orbit stratification of the exceptional locus of Z_Σ thus π_Σ is an embedded pseudo-resolution of Z^Λ. □

With the notations of Lemma 5 we have:

Proposition 6. — *Suppose that the cone $\rho^\vee$ is strictly convex. Let Σ be a subdivision of ρ containing the cone σ_0.*

1. *The strict transform Z^Λ_Σ of Z^Λ by the morphism π_Σ is isomorphic to $Z_{\sigma_0, N_{\sigma_0}}$ and the restriction $\pi_\Sigma | Z^\Lambda_\Sigma : Z^\Lambda_\Sigma \to Z^\Lambda$ is the normalization map.*
2. *The morphism π_Σ is a partial embedded resolution of $Z^\Lambda \subset Z_\rho$.*

Proof. — We keep notations of Lemma 5. If we choose a splitting $M \cong \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$ we obtain using (8) a semigroup isomorphism

$$\sigma_0^\vee \cap M \cong \mathrm{Ker}(\varphi) \oplus \varphi(\sigma_0^\vee \cap M),$$

which corresponds geometrically to the isomorphism $Z_{\sigma_0} \cong \mathbb{O}_{\sigma_0} \times Z_{\sigma_0, N_{\sigma_0}}$ of (1).

We deduce from (6) that $\sigma_0^\vee = \varphi_{\mathbb{R}}^{-1}(\varphi_{\mathbb{R}}(\rho^\vee))$ and it follows that the semigroup

$$\varphi(\sigma_0^\vee \cap M) = \varphi_{\mathbb{R}}(\rho^\vee) \cap \varphi(M) \tag{10}$$

is the saturated semigroup $\mathbb{R}_{\geqslant 0}\Lambda \cap (-\Lambda + \Lambda)$ of Λ in the lattice it spans; therefore the variety $Z_{\sigma_0, N_{\sigma_0}}$ is isomorphic to the normalization of Z^Λ (see [KKMS]).

Let Σ' be a subdivision of Σ compatible with the equations of Z^Λ. By Lemma 5 if $\sigma \in \Sigma'$, $\overset{\circ}{\sigma} \subset \overset{\circ}{\rho}$ and $\mathbb{O}_\sigma \cap Z^\Lambda_{\Sigma'} \neq \emptyset$ then we have $\sigma \subset \ell$. A fortiori the same property holds replacing Σ' by Σ as a consequence of (3). It follows that the strict transform of the germ (Z^Λ, o_ρ) is contained in the chart corresponding to the cone σ_0. This implies that $Z^\Lambda_\Sigma \subset Z_{\sigma_0}$ since the morphism π_Σ is equivariant and Z^Λ is equivariantly embedded. It follows also from the proof of Lemma 5 that the restriction of π_Σ to $Z^\Lambda_\Sigma \to Z^\Lambda$ corresponds algebraically to the inclusion of $\mathbb{C}[\Lambda]$ in its integral closure thus it is the normalization map.

A resolution Σ' of the fan Σ is subdivided by a regular fan Σ'' which is compatible with the equations of Z^Λ. By Lemma 5 the map $\pi_{\Sigma''} \circ \pi_{\Sigma'} \circ \pi_\Sigma$ is a pseudo-resolution of Z^Λ. A fortiori the same holds for $\pi_{\Sigma'} \circ \pi_\Sigma$ by (3). By definition if $\sigma' \in \Sigma$ is a regular cone then $\sigma' \in \Sigma'$, thus $Z_{\Sigma'} \to Z_\Sigma$ is an isomorphism over the points of the orbit $\mathbb{O}_{\sigma'}$. By Remark 1 the singular locus of Z^Λ_Σ is defined by the intersection of those orbits $\mathbb{O}_{\sigma'}$ for those cones σ' running through the set of non regular faces of σ_0. This shows that $Z^{\Lambda}_{\Sigma'} \to Z^{\Lambda}_{\Sigma}$ is a resolution of singularities of the normalization Z^{Λ}_{Σ} of Z^{Λ}. A fortiori the map $Z^{\Lambda}_{\Sigma} \to Z^{\Lambda}$ is a resolution of singularities. □

1.4. Equivariant branched coverings of normal toric varieties.

Some branched coverings of normal toric varieties are equivariant. Typically, if σ is a rational cone for the lattice N it is also rational for a sub-lattice of the same rank $N' \subset N$ and we have a homomorphism of semigroups $\sigma^\vee \cap M \to \sigma^\vee \cap M'$ where $M \subset M'$ is the inclusion of lattices corresponding to $N' \subset N$ by duality. This homomorphism defines an equivariant morphism

$$Z_{\sigma,N'} \to Z_{\sigma,N} \tag{11}$$

extending the homomorphism of tori $T' \to T$ defined by the lattice extension $M \subset M'$, which has kernel a finite subgroup H of T'. Each $w \in H$ corresponds to a morphism $Z_{\sigma,N'} \to Z_{\sigma,N'}$ given by the homomorphism $\mathbb{C}[\sigma^\vee \cap M'] \to \mathbb{C}[\sigma^\vee \cap M']$ mapping $X^u \mapsto w(u)X^u$. The ring $\mathbb{C}[\sigma^\vee \cap M]$ is the set of invariants of $\mathbb{C}[\sigma^\vee \cap M']$ by the action of the group H and the morphism (11) coincides with canonical projection of the quotient of Z'_σ with respect to the action of the group H by Corollary 1.16 of [Od]. If σ is of maximal dimension the 0-orbit o'_σ of $Z_{\sigma,N'}$ projects to the 0-orbit o_σ of $Z_{\sigma,N}$ and we have that $(Z_{\sigma,N'}, o'_\sigma) \to (Z_{\sigma,N}, o_\sigma)$ is a morphism of analytically irreducible germs. The corresponding homomorphism of analytic algebras $\mathbb{C}\{\sigma^\vee \cap M\} \to \mathbb{C}\{\sigma^\vee \cap M'\}$ extends to a homomorphism $L \to L'$ of their fields of fractions of degree equal to the cardinality of H, i.e., the index of M as a subgroup of M'. This field extension is Galois and the Galois group is obtained from the automorphisms of $\mathbb{C}\{\sigma^\vee \cap M'\}$ defined by the elements of H (see [GP1]).

Let $\nu_1, \dots, \nu_g \in M'$ and define from them a sequence of lattices and integers:

$$\begin{cases} M_0 := M, M_i := M_{i-1} + \nu_i \mathbb{Z}, \text{ for } i = 1, \dots, g \\ n_0 := 1, n_i = \# M_i / M_{i-1} \text{ for } i = 1, \dots, g. \end{cases} \tag{12}$$

The lattices M_i are all sub-lattices of finite index of M'. We have the inclusions of lattices $N' \subset N_g \subset \ldots \subset N_1 \subset N_0 = N$ where N_i denotes the dual lattice of M_i.

LEMMA 7. — *The field of fractions of* $\mathbb{C}\{\rho^\vee \cap M_j\}$ *is* $L[X^{\nu_1}, \dots, X^{\nu_j}]$. *If* $\lambda \in \rho^\vee \cap M'$ *then* $X^\lambda \in \mathrm{Fix}(\mathrm{Gal}(L'/L[X^{\nu_1}, \dots, X^{\nu_j}]))$ *if and only if* $\lambda \in \rho^\vee \cap M_j$.

Proof. — The homomorphism of analytic algebras $\mathbb{C}\{\rho^\vee \cap M\} \to \mathbb{C}\{\rho^\vee \cap M_j\}$ is finite and defines an extension of the corresponding fields of fractions of degree $n_1 \cdots n_j$ equal to the order of the finite group M_j/M. We prove the first assertion by induction on j: for $j = 1$ the roots of the minimal polynomial of X^{ν_1} over L are the different conjugates of X^{ν_1} by the action of the elements of the Galois group of L/L'. We deduce from this that the minimal polynomial of X^{ν_1} is $Y^{n_1} - X^{n_1\nu_1}$ where $n_1 = \#M_1/M_0$ is also the degree of the extension $L[X^{\nu_1}]/L$. Since $L[X^{\nu_1}]$ is contained in the field of fractions of $\mathbb{C}\{\rho^\vee \cap M_1\}$ and both fields define extensions of L of the same degree they are equal. By induction hypothesis the field of fractions of $\mathbb{C}\{\rho^\vee \cap M_{j-1}\}$ is $L[X^{\nu_1}, \dots, X^{\nu_{j-1}}]$ and we can replace L, ν_1 and n_1 in the previous argument by $L[X^{\nu_1}, \dots, X^{\nu_{j-1}}]$, ν_j and n_j respectively to obtain the assertion for j.

If $\nu \in \rho^\vee \cap M_j$ it is clear that ν is fixed by any element of the Galois group of the extension $L'/L[X^{\nu_1}, \dots, X^{\nu_j}]$. The converse follows by the first assertion and Corollary 1.16 of [Od] applied to the inclusion of semigroups $\rho^\vee \cap M_j \subset \rho^\vee \cap M'$.

1.5. A reminder on toroidal embeddings.

Let $\mathcal{X}$ be a normal variety of dimension $d + 1$, and let E_i be a finite set of normal hypersurfaces with complement $\mathcal{U}$ in $\mathcal{X}$. A *toroidal embedding without self intersection* is defined by requiring the triple $(\mathcal{X}, \mathcal{U}, x)$ at any point $x \in \mathcal{X}$ to be formally isomorphic to $(Z_\sigma, T = (\mathbb{C}^*)^{d+1}, z)$ for z a point in some toric variety Z_σ. This means that there is a formal isomorphism between the completions of the local rings at respective points which sends the ideal of $\mathcal{X} - \mathcal{U}$ into the ideal of $Z_\sigma - T$; (see [KKMS]). The variety $\mathcal{X}$ is naturally stratified, with strata $\bigcap_{i \in K} E_i - \bigcup_{i \notin K} E_i$ and the open stratum $\mathcal{U}$.

The *star of a stratum* $\mathfrak{S}$, $\operatorname{star}\mathfrak{S}$, is the union of the strata containing $\mathfrak{S}$ in their closure. We associate to the stratum $\mathfrak{S}$ the set $M^{\mathfrak{S}}$ of Cartier divisors supported on $\operatorname{star}\mathfrak{S} - \mathcal{U}$. We denote by $N^{\mathfrak{S}}$ the dual group $\mathrm{Hom}(M^{\mathfrak{S}}, \mathbb{Z})$. The semigroup of effective divisors defines in the real vector space $M^{\mathfrak{S}}_{\mathbb{R}} := M^{\mathfrak{S}} \otimes \mathbb{R}$ a rational convex polyhedral cone and we denote its dual cone in $N^{\mathfrak{S}}_{\mathbb{R}} := N^{\mathfrak{S}} \otimes \mathbb{R}$ by $\rho^{\mathfrak{S}}$. If $\mathfrak{S}'$ is a stratum in $\operatorname{star}\mathfrak{S}$, we have a group homomorphism defined by restriction of Cartier divisors $M^{\mathfrak{S}} \to M^{\mathfrak{S}'}$

which is onto; by duality we obtain an inclusion $N^{\mathfrak{S}'} \to N^{\mathfrak{S}}$ and the cone $\rho^{\mathfrak{S}'}$ is mapped onto a face of $\rho^{\mathfrak{S}}$ (see [KKMS]). We can associate in this way to a toroidal embedding without self-intersection a *conic polyhedral complex with integral structure* (*c.p.c.* in what follows) see [KKMS]. This generalizes the way of recovering from a normal toric variety the associated fan. This complex is *combinatorially isomorphic* to the cone over the dual graph of intersection of the divisors E_i. We have that the strata of the stratification are in one-to-one correspondence with the faces of the conic polyhedral complex. For instance, the conic polyhedral complex associated to the toroidal embedding defined by Z_Σ and the normal hypersurfaces $\{\overline{\mathbb{O}_\sigma}\}_{\sigma \in \Sigma^{(1)}}$ is isomorphic to the conic polyhedral complex (with integral structure) (Σ, N) defined by the fan Σ and the lattice N.

We can define, in an analogous manner to the case of a fan, a regular subdivision of a conic polyhedral complex. Associated to a subdivision we have an induced *toroidal modification* (see [KKMS] Th. 6* and 8*), i.e., a normal variety $\mathcal{X}'$ with a toroidal embedding $\mathcal{U} \subset \mathcal{X}'$ and a modification $\mathcal{X}' \to \mathcal{X}$ provided with a commutative diagram:

$$\begin{array}{ccc} \mathcal{U} & \to & \mathcal{X} \\ \downarrow & \nearrow & \\ \mathcal{X}' & & \end{array}$$

The notion of toric partial embedded resolution generalize easily in the toroidal case.

2. Toric quasi-ordinary singularities.

We introduce toric quasi-ordinary singularities and we extend to this case many notions and properties of quasi-ordinary singularities.

Let (S, o) be a germ of analytically irreducible complex variety of dimension d. We denote by R the associated analytic algebra. A sufficiently small representative $S \to S'$ of a finite map germ $(S, o) \to (S', o')$ has finite fibers, its image is an open neighborhood of o' and the maximal cardinality of the fibers is equal to to the *degree* of the map. The *discriminant locus*, i.e., the set of points of having fibers of cardinality less than the degree, is an analytical subvariety of S'. Outside the discriminant locus, the map is an *unramified* covering. We can think of the discriminant locus as an analytic space or as a germ at o'.

Definition 4. — *A germ of complex analytic variety* (S, o) *is a* quasi-ordinary singularity *if there exist a finite morphism* $(S, o) \to (\mathbb{C}^d, 0)$ *(called a* quasi-ordinary projection) *and some analytical coordinates* $(X_1, \dots, X_d)$ *at* 0*, such that the morphism is unramified over the torus* $X_1 \dots X_d \neq 0$ *in a neighborhood of the origin.*

The class of quasi-ordinary singularities contains all curve singularities. The Jung-Abhyankar Theorem guarantees that R can be viewed as a subring of $\mathbb{C}\{X_1^{1/m}, \dots, X_d^{1/m}\}$ for some suitable integer m (see [J] for a topological proof in the surface case and [A1], Th. 3 for an algebraic proof).

The finite map germ $(S, o) \to (S', o')$ corresponds algebraically to a local homomorphism $R' \to R$ of their analytic algebras which gives R the structure of finite module over R'. In particular if R is generated over R' by one element there is a surjection $R'[Y] \to R$ which corresponds geometrically to an embedding $(S, o) \subset (S' \times \mathbb{C}, (o', 0))$. We say that (S, o) is an *hypersurface relative to the base* (S', o'). We define toric quasi-ordinary singularities by replacing the base $(\mathbb{C}^d, 0)$ by the germ (Z_ρ, o_ρ) of an affine toric variety at its zero orbit (for a strictly convex cone $\rho^\vee$).

Definition 5 (see [GP1]). — *The germ* (S, o) *is a* toric quasi-ordinary singularity *if there exists a finite morphism* $(S, o) \to (Z_\rho, o_\rho)$ *unramified over the torus in a neighborhood of the zero-orbit* o_ρ *of a suitable normal affine toric variety* Z_ρ.

Remark 8. — The classical quasi-ordinary singularities are obtained when $(\rho, M) = (\mathbb{R}^d_{\geqslant 0}, \mathbb{Z}^d)$.

By definition the analytic algebra R of a toric quasi-ordinary singularity is a $\mathbb{C}\{\rho^\vee \cap M\}$-algebra of finite type. The germ (S, o) is an hypersurface relative to the toric base if there exists $x \in R$ such that $R = \mathbb{C}\{\rho^\vee \cap M\}[x]$. Then the $\mathbb{C}\{\rho^\vee \cap M\}$-algebra homomorphism $\mathbb{C}\{\rho^\vee \cap M\}[Y] \to R$ that maps $Y \mapsto x$ is surjective. Its kernel is a principal ideal generated by a monic polynomial f such that $f(o_\rho, Y) = Y^{\deg f}$ and $\deg f$ is equal to the degree of the map $(S, o) \to (Z_\rho, o_\rho)$. The polynomial f is a *quasi-ordinary polynomial*, i.e., the discriminant $\Delta_Y f$ of f with respect to Y is of the form

$$\Delta_Y f = X^\eta H \text{ with } H(o_\rho) \neq 0.$$

Conversely each monic quasi-ordinary polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ such that $f(o_\rho, Y) = Y^{\deg f}$ defines a germ of toric quasi-ordinary hypersurface. The $\mathbb{C}\{\rho^\vee \cap M\}$-algebra homomorphism $\mathbb{C}\{\rho^\vee \cap M\}[Y] \to R$ defines an embedding $S \subset Z_\rho \times \mathbb{C}$ that maps $o \mapsto (o_\rho, 0)$. The quasi-ordinary projection of (S, o) is induced by the first projection of the product $Z_\rho \times \mathbb{C}$.

The product $Z_\rho \times \mathbb{C}$ is the toric variety Z_ϱ defined by the cone $\varrho = \rho \times \mathbb{R}_{\geqslant 0}$ with respect to the lattice N' dual to the lattice $M' := M \oplus y\mathbb{Z}$. Then we have $\varrho^\vee \cap M' \cong (\rho^\vee \cap M) \oplus y\mathbb{Z}_{\geqslant 0}$. We denote the monomial corresponding to $u + sy \in (\rho^\vee \cap M) \oplus y\mathbb{Z}_{\geqslant 0}$ by $X^u Y^s$.

If f is an irreducible quasi-ordinary polynomial the associated analytic algebra R is the domain $R = \mathbb{C}\{\rho^\vee \cap M\}[Y]/(f)$. There exists a fractional power series $\zeta \in \mathbb{C}\{\rho^\vee \cap \frac{1}{n} M\}$ which is a root of f where n is the degree of f (see Théorème 1.1 and Remarque 1 of [GP1]). The inclusion $\mathbb{C}\{\rho^\vee \cap M\} \subset \mathbb{C}\{\rho^\vee \cap \frac{1}{n} M\}$ corresponds to a branched covering of a normal affine toric variety and defines a Galois extension $L \subset L_n$ of their corresponding fields of fractions (see subsection 1.4).The minimal polynomial of the root ζ over the field L is equal to f, we have $R \cong \mathbb{C}\{\rho^\vee \cap M\}[\zeta]$ and the field of fractions of R is $L[\zeta]$ since ζ is finite over L. The conjugates $\zeta^{(i)}$ of ζ by the action of the Galois group of $L \subset L_n$ define all the roots of f since the extension $L[\zeta] \subset L_n$ is Galois.

We call *(toric) quasi-ordinary branches* the roots of (toric) quasi-ordinary polynomials.

If f is a reduced quasi-ordinary polynomial of degree n then it splits on $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!} M\}$. The difference $\zeta^{(s)} - \zeta^{(t)}$ of two different roots of f divides the discriminant of f on the ring $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!} M\}$. By Remark 2, the Newton polyhedron of $\zeta^{(s)} - \zeta^{(t)}$ has only one vertex therefore $\zeta^{(s)} - \zeta^{(t)}$ is of the form $X^{\lambda_{st}} H_{st}$ where H_{st} is a unit in $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!} M\}$. It follows that the irreducible factors of f are quasi ordinary polynomials. The monomials $X^{\lambda_{st}}$ so obtained are called *characteristic monomials* and the exponents $\lambda_{st} \in \rho^\vee \cap \frac{1}{n} M$ are called the *characteristic exponents.* If $\mathrm{rk} M = 1$ and if f is irreducible the characteristic exponents correspond to the classical Puiseux characteristic exponents in arbitrary coordinates. We do not need the classical argument to define the characteristic monomials which uses the factoriality of the ring $\mathbb{C}\{X_1, \dots, X_d\}$ (see [L3]), a property which does not hold for the rings of the form $\mathbb{C}\{\rho^\vee \cap M\}$ in general. The notion of characteristic monomials in the classical quasi-ordinary case is already present in Zariski's work (see [Z5]); in the analytically irreducible hypersurface case many geometrical and topological features of these singularities are determined in terms of the characteristic monomials by Lipman, Luengo, Gau and others (see [L1], [L3], [L4], [Lu] and [Gau]).

We define a partial order $\leqslant_\rho$ (or $\leqslant$ for short) on the cone $\rho^\vee$:

$$u \leqslant_\rho u' \Leftrightarrow u' \in u + \rho^\vee \Leftrightarrow \forall w \in \rho : \langle u' - u, w \rangle \geqslant 0.$$

We can extend this partial ordering to a total one on the subset $\rho^\vee \cap M$ by taking an *irrational* vector $\eta \in \rho$, i.e., the coordinates of η with respect to any base of the lattice N are linearly independent over $\mathbb{Q}$, and defining then $\leqslant_\eta$ by $u \leqslant_\eta u' \Leftrightarrow \langle \eta, u - u' \rangle \leqslant 0$.

Lemma 9 (see [Z5] and [L4] in the classical case) . — *Let f_1 be an irreducible factor of the reduced toric quasi-ordinary polynomial f. If $f_1(\zeta^{(s_0)}) = 0$ then we have*

$$\left\{\lambda_{s_0 t} / \zeta^{(s_0)} \neq \zeta^{(t)},\ f(\zeta^{(t)}) = 0 \right\} = \left\{\lambda_{st} / \zeta^{(s)} \neq \zeta^{(t)},\ f(\zeta^{(t)}) = 0 \text{ and } f_1(\zeta^{(s)}) = 0\right\}$$

and this set is totally ordered by $\leqslant_\rho$.

Proof. — The equality above follows since the extension $L[\zeta^{(s_0)}] \subset L_{n!}$ is Galois and the elements of the Galois group act on a series in $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!} M\}$ by changing the coefficients of its terms. Then, if $\zeta^{(t)} \neq \zeta^{(t')}$ are roots of f different to $\zeta^{(s_0)}$ we have that

$$X^{\lambda_{t't}} H_{t't} = \zeta^{(t')} - \zeta^{(t)} = \zeta^{(t')} - \zeta^{(s_0)} - (\zeta^{(t)} - \zeta^{(s_0)}) = X^{\lambda_{t's_0}} H_{t's_0} - X^{\lambda_{ts_0}} H_{ts_0}.$$

Therefore $\lambda_{t't} = \min_\rho \{\lambda_{t's_0}, \lambda_{ts_0}\}$ and the assertion follows. □

Definition 6 (see [GP2]) . — *Two irreducible quasi-ordinary polynomials $f^{(i)}$ and $f^{(j)}$ have order of coincidence $\lambda_{(i,j)}$ if their product $f^{(i)} f^{(j)}$ is a quasi-ordinary polynomial and $\lambda_{(i,j)}$ is the largest exponent of the set $\{\lambda_{st} / f^{(i)}(\zeta^{(s)}) = 0, f^{(j)}(\zeta^{(t)}) = 0\}$.*

We say that the order of coincidence of $f^{(i)}$ with itself is $\lambda_{(i,i)} := +\infty$. We deduce from the proof of Lemma 9 and Definition 6 the following property:

Lemma 10. — *If $f = f^{(1)} \cdots f^{(r)}$ is the factorization of a quasi-ordinary polynomial with monic irreducible factors we have that: $\min\{\lambda_{(i,j)}, \lambda_{(j,l)}\} \geqslant \lambda_{(i,l)}$ with equality if $\lambda_{(i,j)} \neq \lambda_{(j,l)}$ for $i, j, l \in \{1, \ldots, r\}$.*

In particular when f is irreducible it follows that the set of characteristic exponents is totally ordered by $<_\rho$ (see [L3]). In this case we relabel the characteristic exponents by $\lambda_1 <_\rho \lambda_2 <_\rho \ldots <_\rho \lambda_g$ and we denote $\lambda_{g+1} = +\infty$. Following Lipman (see [L4], page 61) we associate to the characteristic exponents sequences of lattices and integers. In the plane branch case the sequence of integers coincide with the first component of the characteristic pairs in arbitrary coordinates.

DEFINITION 7. — *The lattices M_i and the integers n_i associated to the sequence of characteristic exponents $\lambda_1, \ldots, \lambda_g$ for $i = 0, \ldots, g$ by formulae (12) are called characteristic.*

We denote by $e_{i-1} = n_i \cdots n_g$, for $i = 1, \ldots, g$ and we set $n_0 := 1$. We denote by $N_g \subset \cdots \subset N_1 \subset N_0 = N$ the sequence of dual lattices of $M = M_0 \subset \cdots \subset M_g$.

If f is reduced the set of characteristic exponents is not totally ordered by $\leqslant_\rho$, for example the characteristic exponents $(1, 0), (\frac{3}{2}, 0), (1, \frac{3}{2})$ of $f = ((Y - X_1)^2 - X_1^3))((Y + X_1)^2 - X_1^2 X_2^3))$ are not totally ordered for $\leqslant_{\mathbb{R}^2_{\geqslant 0}}$.

LEMMA 11 (see [L3]). — *If f is an irreducible toric quasi-ordinary polynomial and if ζ is a root of f we have*

1. *The characteristic integers n_i verify that $n_i > 1$ for $i = 1, \ldots, g$ and $n_1 \cdots n_g = \deg f$.*

2. *The field of fractions of R is equal to $L[\zeta] = L[X^{\lambda_1}, \ldots, X^{\lambda_g}]$.*

Proof. — Let ζ' be a conjugate of ζ by an element of the Galois group of the field extension $L_n \supset L[X^{\lambda_1}, \ldots, X^{\lambda_j}]$. If $\zeta' \neq \zeta$ we have $\zeta' - \zeta = X^{\lambda_k} H_k$ for a unit H_k and $k > j$ (since $X^{\lambda_1}, \ldots, X^{\lambda_j}$ are fixed for this Galois group). In particular for $j = g$ the only possibility is $\zeta' = \zeta$ thus $\zeta \in L[X^{\lambda_1}, \ldots, X^{\lambda_g}]$ since the extension $L_n \supset L[X^{\lambda_1}, \ldots, X^{\lambda_j}]$ is Galois. Conversely any element of the Galois group of the extension $L_n \supset L[\zeta]$ fix ζ and therefore all the terms appearing in ζ, in particular $X^{\lambda_1}, \ldots, X^{\lambda_g}$, belong to $L[\zeta]$ since the extension $L_n \supset L[\zeta]$ is Galois. It follows that $n_i > 1$ for $i = 1, \ldots, g$, and that the degree n of the extension $L[\zeta] \supset L$ is equal to $n_1 \cdots n_g$. □

We have the following conditions for a power series $\zeta \in \mathbb{C}\{\rho^\vee \cap \frac{1}{n} M\}$ to be a quasi-ordinary branch (see [L3], prop. 1.5 or [Gau], prop 1.3 in the classical case).

LEMMA 12. — *Let* $\zeta = \sum c_\lambda X^\lambda$ *be a non unit in* $\mathbb{C}\{\rho^\vee \cap \frac{1}{m} M\}$. *Then the minimal polynomial of* ζ *over* $\mathbb{C}\{\rho^\vee \cap M\}$ *is quasi-ordinary if and only if there exist elements* $\lambda_i \in \rho^\vee \cap \frac{1}{m} M$, *for* $1 \leqslant i \leqslant g$ *such that*

1. $\lambda_1 <_\rho \lambda_2 <_\rho \ldots <_\rho \lambda_g$, *and* $c_{\lambda_i} \neq 0$ *for* $1 \leqslant i \leqslant g$.
2. *If* $c_\lambda \neq 0$ *then* λ *is the sub-lattice* $M + \sum_{\lambda_i \leqslant_\rho \lambda} \mathbb{Z}\lambda_i$ *of* $M_\mathbb{Q}$.
3. λ_j *is not in the sub-lattice* $M + \sum_{\lambda_i <_\rho \lambda_j} \mathbb{Z}\lambda_i$, *of* $M_\mathbb{Q}$ *for* $j = 1, \ldots, g$.

If such elements exist they are uniquely determined by ζ *and they are the characteristic exponents of* ζ.

Proof. — If the minimal polynomial of ζ over $\mathbb{C}\{\rho^\vee \cap M\}$ is quasi-ordinary then the result follows from Lemmas 9, 11 and 7 applied to sequence of characteristic exponents. Conversely, if ζ' is the conjugate of ζ by an element of the Galois group of $L_n \supset L$ and if $\zeta \neq \zeta'$ let us consider the sequence of lattices M_i and integers n_i associated to $\lambda_1, \ldots, \lambda_g$ by (12). There is some $j \geqslant 1$ such that the monomials X^ν are fixed for $\nu \in M_{j-1}$ and X^{λ_j} is not fixed by this element by Lemma 7 and Hypothesis 3. Then Hypothesis 1 and 2 imply that the difference $\zeta' - \zeta$ is of the form $\zeta' - \zeta = X^{\lambda_j} H_j$ for a unit H_j. □

Remark 13. — The characteristic lattices associated to f provide a canonical way of writing the terms of its roots:

$$\zeta = p_0 + p_1 + \ldots + p_g,$$

where $p_0 \in \mathbb{C}\{\rho^\vee \cap M\}$ and the monomial X^λ appears in the summand p_j implies that $\lambda_j \leqslant_\rho \lambda$ and $\lambda_{j+1} \nleqslant_\rho \lambda$ for $j = 1, \ldots, g$.

It is shown by Lipman (see [L4], remark 7.3.2) that an analytically irreducible quasi-ordinary hypersurface germ of dimension d is *normal* if and only if it is isomorphic to a germ of the form $Y^n - X_1 \ldots X_c = 0$ for some $1 \leqslant c \leqslant d$; otherwise it is well-known that its normalization is a *quotient singularity* (see [L4]); in the two dimensional case it is the germ of an affine toric surface (see [B-P-V], Chapter III, Theorem 5.2). In [GP2] is proved that the normalization of an irreducible quasi-ordinary hypersurface germ is isomorphic to the germ of an affine normal toric variety at its zero orbit and that this singularity is determined from the set of characteristic exponents. The following proposition generalizes this fact for toric quasi-ordinary hypersurface germs.

PROPOSITION 14. — *The integral closure of the ring R in its field of fractions is equal to* $\mathbb{C}\{\rho^\vee \cap M_g\}$.

Proof. — The analytic algebra of the quasi-ordinary hypersurface is of the form $R = \mathbb{C}\{\rho^\vee \cap M\}[\zeta]$. By Lemma 12 we have a ring extension $R \subset \mathbb{C}\{\rho^\vee \cap M_g\}$ which is integral since $\mathbb{C}\{\rho^\vee \cap M_g\}$ is integral over $\mathbb{C}\{\rho^\vee \cap M\}$. By Lemmas 7 and 11 the rings R and $\mathbb{C}\{\rho^\vee \cap M_g\}$ have the same field of fractions. These two conditions imply that both rings have the same integral closure over their field of fractions. The ring $\mathbb{C}\{\rho^\vee \cap M_g\}$ is integrally closed since it is the analytic algebra of the normal variety Z_{ρ,N_g} at the point o_ρ. □

2.1. The Eggers-Wall tree of a reduced quasi-ordinary polynomial.

We structure the partially ordered set of characteristic monomials of a reduced toric quasi-ordinary polynomial in a labeled tree. When $\mathrm{rk} M = 1$ the germ S defined by a reduced quasi-ordinary polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ at the origin is just a germ of complex plane curve. It is well-known that the intersection multiplicities of the different branches of the curve at the origin and the semigroups associated to each of them define a complete invariant of the *embedded topological type* of the plane curve germ $(S, 0)$ (see [Re]). Eggers shows that this information can be encoded by structuring in a labeled tree the characteristic exponents of each irreducible factor and the orders of coincidence between any two of them (see [Eg]). Wall (see [Wa]) gives a different definition of Egger's tree to give a new proof of theorem of García Barroso in [GB1] on the structure of polar curves (see [GB2]). Wall's definition encodes the same amount of information as Egger's definition does and involves the use of a simplicial 1-chain on the tree which is defined from the sequence of characteristic integers of the irreducible factors (see Definition 7). In the case of a classical quasi-ordinary hypersurface, Zariski's result stated in Lemma 9 can be reformulated as follows: If $f = 0$ defines a classical quasi-ordinary hypersurface and if f_1 is an irreducible factor of f the set of characteristic exponents of f_1 union the set of orders of coincidence of f_1 with the factors of f is totally ordered with respect to the partial order defined by the divisibility of the corresponding monomials. Zariski's observation and the sequences of characteristic integers are exactly what is necessary to extend Wall's definition to the quasi-ordinary case in terms of a fixed quasi-ordinary projection $(X, Y) \mapsto X$. This is done more generally by Popescu-Pampu (see [PP2]) for a *Laurent quasi-ordinary polynomial* f, obtaining a result on the structure of $\frac{\partial f}{\partial Y}$ in terms of the tree of f when $\frac{\partial f}{\partial Y}$ is quasi-ordinary.

The definition of the tree in our case runs as follows: Let $f = f^{(1)} \cdots f^{(r)}$ be the factorization in monic irreducible polynomials of f. Each factor $f^{(i)}$ of f is quasi-ordinary and the subset $\theta(f^{(i)})^{(0)}$ of $\rho^\vee \cap M_g \cup \{+\infty\}$ whose elements are $0, +\infty$, the characteristic exponents $\lambda_1^{(i)} <_\rho \cdots <_\rho \lambda_{g(i)}^{(i)}$ of $f^{(i)}$ (if they exist) and the orders of coincidence of $f^{(i)}$ with the irreducible factors of f is totally ordered by Lemma 9; we denote by $n_k^{(i)}$ and $e_k^{(i)}$ for $k = 1, \ldots, g(i)$, the sequences of integers associated to $f^{(i)}$ by Definition 7 for $i = 1, \ldots, r$.

The *elementary branch* $\theta(f^{(i)})$ associated to $f^{(i)}$ is the abstract simplicial complex of dimension one with vertices running through the elements of the totally ordered subset $\theta(f^{(i)})^{(0)}$ and edges running through the segments joining consecutive vertices for the partial order $\leqslant$. The underlying topological space is homeomorphic to the segment $[0, +\infty]$. We denote the vertex of $\theta(f^{(i)})$ corresponding to $\lambda \in \theta(f^{(i)})^{(0)}$ by $P_\lambda^{(i)}$. The simplicial complex $\theta(f)$ obtained from the disjoint union $\bigsqcup_{i=1}^s \theta(f^{(i)})$ by identifying in $\theta(f^{(i)})$ and $\theta(f^{(j)})$ the sub-simplicial complex corresponding to $\overline{P_0^{(i)} P_{\lambda_{(i,j)}}^{(i)}}$ and $\overline{P_0^{(j)} P_{\lambda_{(i,j)}}^{(j)}}$ for $1 \leqslant i < j \leqslant r$ is a tree. We give to a vertex $P_\lambda^{(i)}$ of $\theta_f(f)$ the valuation λ. This defines a 0-chain $C_0(f)$ on $\theta_f(f)$ which attaches the value λ to each vertex $P_\lambda^{(i)}$ in the Eggers tree (counting each vertex only once).

For $i = 1, \ldots, r$ we define an integral 1-chain whose segments are obtained by by subdividing the segments of the chain

$$\overline{P_0^{(i)} P_{\lambda_1^{(i)}}} + n_1^{(i)} \overline{P_{\lambda_1^{(i)}}^{(i)} P_{\lambda_2^{(i)}}^{(i)}} + \cdots + n_1^{(i)} \cdots n_{g(i)}^{(i)} \overline{P_{\lambda_g^{(i)}}^{(i)} P_{+\infty}^{(i)}} \tag{13}$$

with the points corresponding to the orders of coincidence of $f^{(i)}$, the coefficient of an oriented segment in the subdivision is the same as the coefficient of the oriented segment of (13) containing it. It follows from Definition 7 that these 1-chains paste on $\theta_f(f)$ and define a 1-chain $C_1(f)$ with coefficients in $\mathbb{Z}$.

Definition 8. — *The Eggers-Wall tree is the simplicial complex $\theta(f)$ with the chains $C_1(f)$ and $C_0(f)$.*

The chains $C_1(f)$ and $C_0(f)$ determine the number of factors of f, the characteristic exponents of each factor and the orders of coincidence. The vertex $P_\lambda^{(i)}$, if $\lambda \neq 0, +\infty$ is not a characteristic exponents of $f^{(i)}$ if and only if the coefficients of the two segments of $\theta(f^{(i)})$ containing $P_\lambda^{(i)}$ coincide.

3. Embedded resolution procedure.

In this section we build an embedded resolution of a reduced quasi-ordinary polynomial which is a composition of toric morphism determined by the characteristic monomials.

3.1. Definition of good coordinates.

We introduce the notion of Y being a *good coordinate* in terms of the coincidence of the parametrizations of f. In the following section we build the toric morphisms of the resolution using this notion. Different choices of good coordinates provide isomorphic morphisms.

We keep the notations of Section 2.1. We suppose that f is a quasi-ordinary polynomial with r irreducible factors $f^{(1)}, \dots, f^{(r)}$. Define $\mathcal{A}(i) := (M \cap \{\lambda_{(i,j)}\})_j \cup \{\lambda_1^{(i)}\}$ for $1 \leqslant i \leqslant r$. By Lemma 9, if the set $\mathcal{A}(i)$ is non empty it is totally ordered by $<_\rho$.

Then we can define

$$\lambda_{\kappa(i)} := \left\{ \begin{array}{l} \min \mathcal{A}(i) \text{ if } \mathcal{A}(i) \neq \emptyset \\ +\infty \text{ otherwise} \end{array} \right\} \text{ for } i = 1, \dots, r. \tag{14}$$

LEMMA 15.

1. *If $\lambda_{\kappa(i)} \nleq \lambda \notin M$ the term X^λ does not appear in the expansions of the roots of $f^{(i)}$. In particular if X^λ appears in the expansions of the roots of $f^{(j)}$ then λ is $\geqslant \lambda_{(i,j)}$ and the equality $\lambda = \lambda_{(i,j)}$ implies that $\lambda_{(i,j)} = \lambda_1^{(j)}$.*

2. *The case $\lambda_{\kappa(i)} = +\infty$ happens if and only if $f^{(i)}$ is the only factor of f without characteristic exponents and $\lambda_{(i,j)} = \lambda_1^{(j)}$ for all $j \neq i$.*

3. *If $\lambda_{\kappa(i_0)} \in M$ then $\lambda_{\kappa(i_0)}$ is $\geqslant \lambda_{\kappa(j)}$ for all $j \neq i_0$.*

4. *The set $\{\lambda_{\kappa(1)}, \dots, \lambda_{\kappa(r)}\}$ is totally ordered by $<$.*

Proof.— If $f^{(i)}$ has no characteristic exponent the terms in the expansion of its root have exponents in $\rho^\vee \cap M$. Otherwise, $\lambda_{\kappa(i)} \nleq \lambda \notin M$

implies that $\lambda_1^{(i)} \not\leqslant \lambda \notin M$ thus the term X^λ does not appear in the expansion of the roots of $f^{(i)}$ by Lemma 12. If X^λ appears in the expansion of the roots of $f^{(j)}$ then it appears in any difference of roots of $f^{(i)}$ and $f^{(j)}$ thus $\lambda \geqslant \lambda_{(i,j)}$. Moreover, if $\lambda = \lambda_{(i,j)}$ then $\lambda \notin M$ implies that $\lambda_{(i,j)} \geqslant \lambda_1^{(j)}$ by Lemma 12. Since $\lambda_{\kappa(i)} \not\leqslant \lambda_{(i,j)}$ we have that $\lambda_{\kappa(i)} \not\leqslant \lambda_1^{(j)} \notin M$ and therefore $\lambda_1^{(j)} \geqslant \lambda_{(i,j)}$, and the equality $\lambda_{(i,j)} = \lambda_1^{(j)}$ follows.

For the second assertion notice that if $f^{(i)}$ and $f^{(j)}$ are two different factors without characteristic exponents then $\lambda_{(i,j)}$ belongs to M thus $\lambda_{\kappa(i)}, \lambda_{\kappa(j)} \neq +\infty$. If $\lambda_{\kappa(i)} = +\infty$ then $\lambda_{(i,j)}$ is not in M for all $j \neq i$; thus the exponent $\lambda_{(i,j)}$ appears on a term of the parametrization of $f^{(j)}$ and therefore we have $\lambda_{(i,j)} \geqslant \lambda_1^{(j)}$ by Lemma 12. The first assertion for $\lambda = \lambda_1^{(j)}$ implies that $\lambda_{(i,j)} \leqslant \lambda_1^{(j)}$ and equality follows.

Now suppose that $\lambda_{\kappa(i_0)} \in M$. If $j \neq i$ the exponents $\lambda_{\kappa(i_0)}$ and $\lambda_{(i_0,j)}$ are comparable by Lemma 9. We distinguish two cases:

(a) $\lambda_{\kappa(i_0)} \leqslant \lambda_{(i_0,j)}$. Notice that assertion 1 implies that if $f^{(j)}$ has some characteristic exponent then $\lambda_1^{(j)} > \lambda_{\kappa(i_0)}$. If $\lambda_{\kappa(i_0)} < \lambda_{(i_0,j)}$ there is $j \neq l_0 \neq i_0$ such that $\lambda_{\kappa(i_0)} = \lambda_{(i_0,l_0)} = \min\{\lambda_{(i_0,l_0)}, \lambda_{(i_0,j)}\} = \lambda_{(j,l_0)}$ by Lemma 10; hence the exponents $\lambda_{(j,l)}$ and $\lambda_{\kappa(i_0)}$ are comparable by Lemma 9. If $\lambda_{\kappa(i_0)} = \lambda_{(i_0,j)}$ set $l_0 = j$.

If $\lambda_{(j,l)} < \lambda_{\kappa(i_0)}$ we deduce from Lemma 10 that

$$\lambda_{(j,l)} = \min\{\lambda_{(j,l_0)}, \lambda_{(j,l)}\} = \lambda_{(l,l_0)} = \min\{\lambda_{(i_0,l_0)}, \lambda_{(l,l_0)}\} = \lambda_{(i_0,l)},$$

and $\lambda_{(j,l)}$ does not belong to M by definition of $\lambda_{\kappa(i_0)}$. This shows that $\lambda_{\kappa(j)} = \lambda_{\kappa(i_0)}$.

(b) $\lambda_{(i_0,j)} < \lambda_{\kappa(i_0)}$. By definition of $\lambda_{\kappa(i_0)}$ we have that $\lambda_{(i_0,j)} \notin M$ and then assertion *1* implies that $\lambda_{(i_0,j)} = \lambda_1^{(j)}$. If $\lambda_{(j,l)} < \lambda_1^{(j)}$ we deduce using Lemma 10 that $\lambda_{(j,l)} = \min\{\lambda_{(j,l)}, \lambda_{(i_0,j)}\}$ is equal to $\lambda_{(i_0,l)}$ and $< \lambda_{\kappa(i_0)}$. It follows that $\lambda_{(i_0,l)} \notin M$, thus $\lambda_{\kappa(j)} = \lambda_1^{(j)} < \lambda_{\kappa(i_0)}$.

For the last assertion we only have to prove that if $\lambda_{\kappa(i)} = \lambda_1^{(i)}$ and $\lambda_{\kappa(j)} = \lambda_1^{(j)}$ they are comparable by $<$. By Lemma 9, $\lambda_{(i,j)}$ is comparable with $\lambda_1^{(i)}$ and $\lambda_1^{(j)}$. The case $\lambda_{(i,j)} < \lambda_1^{(i)}, \lambda_1^{(j)}$ implies that $\lambda_{(i,j)} \in M$ by Lemma 12, thus $\lambda_{\kappa(i)} \leqslant \lambda_{(i,j)}$ a contradiction. Therefore we can assume that $\lambda_1^{(i)} \leqslant \lambda_{(i,j)}$, replacing i by j if necessary. It follows from the definition of order of coincidence that if $\lambda_1^{(i)} < \lambda_{(i,j)}$ then $\lambda_1^{(i)} = \lambda_1^{(j)}$. If $\lambda_1^{(i)} = \lambda_{(i,j)}$ then the result follows from Lemma 9. $\square$

We relabel the factors $f^{(i)}$ of f in order to have: $\lambda_{\kappa(1)} \leqslant \lambda_{\kappa(2)} \leqslant \cdots \leqslant \lambda_{\kappa(r)}$. If $\lambda \in \rho \cap M$, the monomial X^{λ} appears in all the roots of $f^{(r)}$ with the same coefficient $c_{\lambda}^{(r)}$. Then we define

$$\phi_0 := \sum_{\lambda_{\kappa(r)} \not\leq \lambda \in \rho^{\vee} \cap M} c_{\lambda}^{(r)} X^{\lambda},$$

$$Y' := \begin{cases} Y + \phi_0 \text{ if } \lambda_{\kappa(r)} \notin M \\ Y + \phi_0 + cX^{\lambda_{\kappa(r)}}, \text{ for } c \in \mathbb{C}^* \text{ generic, if } \lambda_{\kappa(r)} \in M. \end{cases} \tag{15}$$

Generic here means that if $\lambda_{\kappa(r)} = \lambda_{\kappa(l)} \in M$ then $c - c_{\lambda_{\kappa(l)}}^{(l)} \neq 0$.

LEMMA 16. — *The polynomial Y' has order of coincidence equal to $\lambda_{\kappa(i)}$ with $f^{(i)}$ for $i = 1, \ldots, r$.*

Proof. — It follows from Lemma 15 that if $\lambda_{\kappa(i)} < \lambda_{\kappa(r)}$ then $\lambda_{\kappa(i)}$ is the order of coincidence of $f^{(i)}$ and $f^{(r)}$ (remark that $\lambda_{\kappa(i)} \notin M$ by assertion 3 of Lemma 15, thus $\lambda_{\kappa(i)} = \lambda_1^{(i)}$ is $\geqslant \lambda_{(i,r)}$ by assertion 1 of Lemma 15; it follows from this fact that $\lambda_{(i,r)} \notin M$ thus $\lambda_1^{(i)} \leqslant \lambda_{(i,r)}$ by Lemma 15). This implies that the order of coincidence of Y' with $f^{(i)}$ is well defined and equal to $\lambda_{\kappa(i)}$. The generic choice of c guarantees in the case $\lambda_{\kappa(r)} \in M$ that the order of coincidence of Y' with those factors $f^{(i)}$ of f with $\lambda_{\kappa(i)} = \lambda_{\kappa(r)}$ is $\lambda_{\kappa(r)}$. □

DEFINITION 9. — *We say that Y is a* good coordinate *for the reduced quasi-ordinary polynomial $f \in \mathbb{C}\{\rho^{\vee} \cap M\}[Y]$ if the order of coincidence of Y with $f^{(i)}$ is well defined and equal to $\lambda_{\kappa(i)}$, for $i = 1, \ldots, r$.*

If Y is not a good coordinate for f then the $\mathbb{C}\{\rho^{\vee} \cap M\}$-automorphism of $\mathbb{C}\{\rho^{\vee} \cap M\}[Y]$ that maps $Y \mapsto Y'$, for Y' defined in Lemma 16, transforms $f \mapsto f' \in \mathbb{C}\{\rho^{\vee} \cap M\}[Y']$. The polynomial f' is quasi-ordinary, f' and f have the same Eggers-Wall tree and Y' is a good coordinate for f'.

In Section 3.2.2 we show that if Y is a good coordinate for f the characteristic monomials determine its Newton polyhedron.

3.2. The first toric morphism of the embedded resolution.

We build the first toric morphism of the embedded resolution and we prove that it simplifies the singularity preserving at the same time the quasi-ordinary structure.

3.2.1. The case of a Newton polyhedron with only one compact edge.

We deal first with the case when all the irreducible factors of f are parametrized by series of the form $X^\lambda \varepsilon$ with $\varepsilon(o_\rho) = c$.

We denote by M_λ the lattice $M + \lambda\mathbb{Z}$ for $\lambda \in \frac{1}{n}M$ (resp. N_λ for the dual lattice), by M'_λ the lattice $M_\lambda \oplus y\mathbb{Z}$ (resp. N'_λ for the dual lattice) and by n_λ the integer $|M_\lambda/M|$. Let Σ be a subdivision of ϱ containing cone $\sigma := \varrho \cap \ell$ where ℓ is the linear subspace of $N'_\mathbb{R}$ orthogonal to the compact face $[n\lambda, ny]$ of the polyhedron $\mathcal{N}(f)$ (where $n = \deg f$). The subdivision Σ of ϱ is rational for the lattices N'_λ and N'. We have the following commutative diagram of equivariant maps:

$$\begin{array}{ccc} Z_{\Sigma,N'_\lambda} & \xrightarrow{\Pi_\Sigma} & Z_{\varrho,N'_\lambda} \\ \downarrow & & \downarrow \\ Z_{\Sigma,N'} & \xrightarrow{\pi_\Sigma} & Z_{\varrho,N'} \end{array} \tag{16}$$

where the vertical arrows are defined by lattice extension and the horizontal arrows are defined by the subdivision Σ. Often we do not precise the lattice if it is corresponds to the below line of the diagram 16.

LEMMA 17. — *The lattice homomorphism $\varphi : M' \to M_\lambda$ that maps $y \mapsto \lambda$ and fixes $u \in M$ induces an isomorphism*

$$M_\sigma \cong M_\lambda. \tag{17}$$

If we choose an splitting $M' \cong M_\sigma \oplus \mathrm{Ker}(\varphi)$ we have a semigroup isomorphism

$$\sigma^\vee \cap M' \cong n_\lambda(y - \lambda_1)\mathbb{Z} \oplus (\rho^\vee \cap M_\lambda) \tag{18}$$

which corresponds to an isomorphism $Z_{\sigma,N'} \cong \mathbb{O}_{\sigma,N'} \times Z_{\rho,N_\lambda}$.

Proof. — We use the combinatorial arguments in the proofs of Lemma 5 and Proposition 6 to prove (17) using that $\sigma^\perp = \mathrm{Ker}(\varphi_\mathbb{R})$ by (7); then (18) holds by (10). □

We denote by $S^{(i)}_\Sigma$ the strict transform of the germ $S^{(i)}$ defined by the irreducible factor $f^{(i)}$ of f for $i = 1, \dots, r$.

LEMMA 18. — *The intersection $S^{(i)}_\Sigma \cap \pi_\Sigma^{-1}(o_\rho)$ is the point $o_1 = (c', o_\rho) \in \mathbb{O}_\sigma$ counted $e^{(i)}_\lambda := (\deg f^{(i)})/n_\lambda$ times, where $c' = c^{n_\lambda}$ and c is the coefficient of X^{λ_1} in any root of the polynomial $f^{(i)}$ defining $S^{(i)}$. In* *particular, the intersection* $S_\Sigma^{(i)} \cap \mathbb{O}_\sigma$ *is transversal if and only if* $e_\lambda^{(i)} = 1$. *The strict transform* S_Σ *of* S *is a germ at the point* o_1.

Proof.— To simplify the proof we drop the super-index (i). If $\tau \in \Sigma$ with $\overset{\circ}{\tau} \subset \overset{\circ}{\varrho}$ then $S_\Sigma \cap \mathbb{O}_\tau \neq \emptyset$ implies that $\tau = \sigma$ since the face of $\mathcal{N}_\varrho(f)$ defined by σ is of dimension $\geqslant 1$ (by Lemma 3). The strict transform S_Σ is defined on Z_σ by $X^{-n\lambda} f = 0$ and it follows that the ideal of $\mathbb{O}_\sigma \cap S_\Sigma$ is generated by $(X^{n_\lambda (y-\lambda)} - c^{n_\lambda})^{e_\lambda}$ where c is the coefficient of X^λ in any root of f. This implies that the intersection of the strict transform S_Σ with $\pi_\Sigma^{-1}(o_\varrho)$ is reduced to the point $o_1 = (c', o_\rho)$ counted e_λ times. In particular, the intersection is transversal if and only if $e_\lambda = 1$. This shows also that the strict transform S_Σ is a germ at the point o_1 since this is the only point of intersection with the exceptional fiber. □

PROPOSITION 19.— *The restriction of the projection* $\mathbb{O}_\sigma \times Z_{\rho, N_\lambda} \cong Z_{\sigma, N'} \to Z_{\rho, N_\lambda}$ *to* (S_Σ, o_1) *is quasi-ordinary. The germ* (S_Σ, o_1) *is defined by a quasi-ordinary polynomial* $f_\Sigma \in \mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$ *(where* $W = Y^{n_\lambda} X^{-n_\lambda \lambda} - c^{n_\lambda}$*) with characteristic exponents* $\lambda' - \lambda$ *for those characteristic exponents* $\lambda' > \lambda$ *of* f. *If* $\lambda_{(i,j)}$ *is the order of coincidence between the irreducible components* $f^{(i)}$ *and* $f^{(j)}$ *of* f *then the order of coincidence of* $f_\Sigma^{(i)}$ *and* $f_\Sigma^{(j)}$ *is* $\lambda_{(i,j)} - \lambda$. *If* (S, o) *is irreducible the same holds for* (S_Σ, o_1).

Proof.— We deal first with the case $\lambda \in M$, i.e., $n_\lambda = 1$ and $N_\lambda = N$. By Lemma 18 the chart Z_σ contains the strict transform S_Σ. By hypothesis the roots $\zeta^{(i)}$ of f are of the form $\zeta^{(i)} = cX^\lambda + \sum_{\lambda' > \lambda} c_{\lambda'}^{(i)} X^{\lambda'}$, i.e., the coefficient of the monomial X^λ is the same for all of them. By Lemma 18 the strict transform of $Y - \zeta^{(i)} = 0$ by the morphism $Z_{\sigma, N'} \to Z_{\varrho, N'}$ is defined by

$$0 = X^{y-\lambda} - c + \sum_{\lambda' > \lambda} c_{\lambda'}^{(i)} X^{\lambda' - \lambda} \tag{19}$$

where the terms $X^{\lambda' - \lambda}$ vanish on the orbit $\mathbb{O}_\sigma$ for all $\lambda' > \lambda$. By Lemma 17 the chart Z_{σ, N'_λ} (resp. $Z_{\sigma, N}$) is isomorphic to $\mathbb{O}_{\sigma, N'_\lambda} \times Z_{\rho, N_\lambda}$ (resp. to $\mathbb{O}_{\sigma, N'} \times Z_{\rho, N}$). Since $n_\lambda = 1$ the toric morphism $Z_{\sigma, N'_\lambda} \to Z_{\sigma, N'}$ restricts to an isomorphism of the orbits $\mathbb{O}_{\sigma, N'_\lambda} \cong \mathbb{O}_{\sigma, N'} = \mathbb{O}_\sigma$ by (18), the coordinate ring of the orbit $\mathbb{O}_\sigma$ being equal to $\mathbb{C}[YX^{-\lambda}]$. We study the strict transform of $Y - \zeta^{(i)} = 0$ (resp. of S) at the point of intersection with the orbit $\mathbb{O}_\sigma$ by replacing the invertible term $X^{y-\lambda}$ by the unit $c + W$ on (19) (resp. on $X^{-n\lambda} f = 0$). We obtain a polynomial $f_\Sigma \in \mathbb{C}\{\rho^\vee \cap M\}[W]$ from $X^{-n\lambda} f$ which splits in $\mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$: $f_\Sigma = \prod_{i=1}^n (W - \tau^{(i)})$; where

$\tau^{(i)} = \sum_{\lambda' > \lambda} c_{\lambda'}^{(i)} X^{\lambda' - \lambda}$. It follows from Lemma 12 that the series $\tau^{(i)}$ are quasi-ordinary branches and that their characteristic exponents are obtained from those of $\zeta^{(i)}$ by subtracting λ. If f is irreducible the same thing happens for f_Σ. Otherwise, we have $\tau^{(i)} - \tau^{(j)} = X^{-\lambda}(\zeta^{(i)} - \zeta^{(j)})$ and this implies the assertion about the orders of coincidence.

If $n_\lambda > 1$ we reduce to the previous case by passing through the diagram (16):

Each irreducible factor of f splits into n_λ irreducible factors in $\mathbb{C}\{\rho^\vee \cap M_\lambda\}[Y]$ having order of coincidence equal to λ. We factor f as a product $F_1 \cdots F_{n_\lambda}$ in $\mathbb{C}\{\rho^\vee \cap M_\lambda\}[Y]$, the F_i being defined by the properties: the order of coincidence of $F_i \neq F_j$ (resp. of any two factors of F_i) is $= \lambda$ (resp. is $> \lambda$). The Eggers-Wall tree of F_i is obtained from the Eggers-Wall tree of f by deleting the vertex P_λ and dividing by n_λ the coefficients of the chain $C_1(f)$ between P_λ and the extreme points $P_{+\infty}^{(j)}$ of the tree (this follows from Lemma 11 and Definition 7). Then the strict transforms of $F_i = 0$ by Π_Σ are disjoint germs at the n_λ points of intersection with $\mathbb{O}_{\sigma, N'_\lambda}$ by Lemma 18.

By Lemma 17 the toric morphism $Z_{\sigma, N'_\lambda} \to Z_{\sigma, N}$ corresponds to the semigroup inclusion

$$n_\lambda (y - \lambda)\mathbb{Z} \oplus (\rho^\vee \cap M_\lambda) \to (y - \lambda)\mathbb{Z} \oplus (\rho^\vee \cap M_\lambda).$$

This map is an unramified covering of degree n_λ and it commutes with the projections onto the factor Z_{ρ, N_λ} of Z_{σ, N'_λ} and $Z_{\sigma, N}$. This provides an isomorphism between the strict transform of F_i by Π_Σ and S_Σ which commutes with the projection onto factor Z_{ρ, N_λ} for $i = 1, \ldots, n_\lambda$. A fortiori the restriction of the projection $Z_{\sigma, N'} \to Z_{\rho, N_\lambda}$ to S_Σ is quasi-ordinary and the result follows. □

With the same hypothesis of Proposition 19 we have:

Corollary 20. — *If (S, o) is analytically irreducible and if $\lambda = \lambda_1$ is the only characteristic exponent of ζ the strict transform S_Σ of S is isomorphic to the germ Z_{ρ, N_1} and the restriction of π_Σ to $S_\Sigma \to S$ is the normalization map. The morphism π_Σ is a partial embedded resolution of $S \subset Z_\varrho$. If Σ' is a resolution of the fan Σ the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is an embedded resolution of $S \subset Z_\varrho$.*

Proof. — It follows from Lemma 18 that S_Σ is isomorphic to the germ (Z_{ρ, N_1}, o_ρ) and to the normalization of (S, o) by Proposition 14. We argue as in Proposition 6 and Lemma 5 to extend the result in this case. □

The following remark is a consequence of the proof of Proposition 19.

Remark 21.— If f is irreducible, $\lambda = \lambda_1$ and if $f_1 \in \mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$ defines a good coordinate for f_Σ then the image of $f_1 = 0$ by π_Σ is defined by an irreducible quasi-ordinary polynomial in $\mathbb{C}\{\rho^\vee \cap M\}[Y]$ with only one characteristic exponent λ_1 and with maximal order of coincidence with f.

The following result has been suggested by Némethi and McEwan see ([M-N] and [GP-M-N]).

LEMMA 22.— *The morphism π_Σ of Proposition 19 is an isomorphism over $Z_\varrho - S$.*

The discriminant of the morphism Π_Σ is described by (4). It follows from this formula that the functions X^λ and Y vanishes on those orbits of Z_{ϱ,N'_λ} which are contained in the discriminant locus of Π_Σ. The image of these orbits by the map $Z_{\varrho,N'_\lambda} \to Z_\varrho$ is the discriminant of π_Σ and it is contained in S since all the roots of f are of the form $Y = X^\lambda$ up to multiplication by a unit. □

3.2.2. The general case.

We build the first toric morphism of the embedded resolution in the general case.

We suppose from now on that Y is a good coordinate for f . The Newton polyhedron of each irreducible factor $f^{(i)}$ with $\lambda_{\kappa(i)} \neq +\infty$ has only one compact edge vertices $(\deg f^{(i)}, 0)$ and $(0, \deg f^{(i)} \lambda_{\kappa(i)})$ where $X^{\lambda_{\kappa(i)}}$ is the initial monomial of any root of $f^{(i)}$. Since the set of $\{\lambda_{\kappa(i)}\}$ is completely ordered by $<_\rho$ the set of compact faces of $\mathcal{N}_\rho(f)$ defines a *monotone polygonal path* with *inclinations* running through $\{\lambda_{\kappa(1)}, \dots, \lambda_{\kappa(r)}\} - \{+\infty\}$ independently of the choice of good coordinate (see [GP1] for the terminology). This fact is a special feature of quasi-ordinary singularities and it is a generalization of the plane curve case.

The dual fan Σ_1 of the polyhedron $\mathcal{N}(f)$ is obtained by intersecting ϱ with the linear hyperplanes $\ell_{\kappa(j)} := \langle y - \lambda_{\kappa(j)}, u\rangle = 0$ for those $\lambda_{\kappa(j)} \neq +\infty$. Since we have that $\{\lambda_{\kappa(j)}\}$ is totally ordered by $<_\rho$ we find that the cones $\varrho \cap \ell_{\kappa(j)}$ belong to Σ_1 since they cannot intersect in the interior of ϱ. Geometrically, this implies that the exceptional locus of π_{Σ_1} is a *bamboo* of $\mathbf{P}^1_{\mathbb{C}}$, each one of them being the closure of the orbit $\mathbb{O}_{\varrho \cap \ell_{\kappa(j)}}$ (we say that a curve is a bamboo if the dual graph of intersection of its irreducible components is isomorphic to the subdivision of a segment).

Proposition 23. — *If $\lambda_{\kappa(i)} \neq +\infty$ then we have:*

1. *The strict transform of $S^{(i)}$ by π_{Σ_1} is a germ $(S^{(i)}_{\Sigma_1}, o^{(i)}_1)$ at the point of intersection with the exceptional curve $\pi^{-1}_{\Sigma_1}(o_\varrho)$.*

2. *The Eggers-Wall tree of a polynomial defining the strict transform S_{Σ_1} at the point $o_1^{(i)}$ is obtained from $\theta(f)$ by removing the segment $[P^{(j)}_0, P^{(j)}_{\lambda_{\kappa(i)}}[$ from the sub-tree of $\theta(f)$ given by $\bigcup \theta(f^{(j)})$, for $f^{(j)}$ with order of coincidence $> \lambda_{\kappa(i)}$ with $f^{(i)}$. The coefficients of the vertices of the resulting tree are obtained by subtracting $\lambda_{\kappa(i)}$. The coefficients of the associated 1-chain are obtained by dividing by $n_{\lambda_{\kappa(i)}}$.*

Proof. — The first assertion follows from Lemma 18. It follows from Proposition 19 that $o_1^{(i)} = o_1^{(j)}$ if and only if the irreducible factors of the symbolic restrictions of $f^{(i)}$ and $f^{(j)}$ to the compact edges of their Newton polyhedra coincide

This is equivalent $f^{(i)}$ and $f^{(j)}$ have order of coincidence $> \lambda_{\kappa(i)} = \lambda_{\kappa(j)}$. The second assertion follows from Proposition 19 since the characteristic exponents and the order of coincidence of $f^{(i)}_\Sigma$ and $f^{(j)}_\Sigma$ are obtained from those corresponding to $f^{(i)}$ and $f^{(j)}$ by subtracting λ_1. The strict transform $S^{(i)}_\Sigma$ is a toric quasi-ordinary hypersurface relative to the base $Z_{\rho, N_{\lambda_{\kappa(i)}}}$ by Proposition 19 and the statement about the coefficients of the associated 1-chain follows from this change of lattice by Lemma 7. □

Remark 24. — If $\lambda_{\kappa(i)} = +\infty$ the strict transform of $S^{(i)}$ is the germ of the closure of the orbit associated to the edge $y\mathbb{R}_{\geqslant 0}$ at the point of intersection with the exceptional curve $\pi^{-1}_{\Sigma_1}(o_\varrho)$.

The assertion follows from the description of the exceptional locus and the discriminant locus of a toric modification given in Section 1.1 once it is noticed that the point of intersection $o_1^{(i)}$ of $S^{(i)}$ with $\pi^{-1}_{\Sigma_1}(o_\varrho)$ is the orbit associated to the $(d+1)$-dimensional cone of Σ_1 which contains the cone $y\mathbb{R}_{\geqslant 0}$.

3.3. The toric embedded resolution.

We show the way to iterate the procedure of the previous section to build an embedded resolution of $S \subset Z_\varrho$ by first eliminating the characteristic exponents and then by resolving the toric singularities of the ambient space.

By Proposition 23, the germs defined by the strict transform at each of the points $o_1^{(i)}$ of intersection with the exceptional curve are simpler toric quasi-ordinary hypersurface singularities. In a finite number of iterations of this procedure the strict transform becomes a union of r toric quasi-ordinary hypersurface germs with no characteristic exponents at all, i.e., is collection of r germs of affine toric varieties at the special points. It follows from Propositions 19 and 14 that the strict transform of (S, o) is its normalization. Thus this method provides an *embedded normalization* (in a normal environment) of the germ $(S, o) \subset (Z_\varrho, o_\varrho)$. We keep the information of the toric singularities of the ambient space by defining at each stage a toroidal embedding without self-intersection:

First, we associate to the toric quasi-ordinary hypersurface $(S, o) \subset (Z_\varrho, o_\varrho)$ embedded with a good coordinate the toroidal embedding defined by (Z_ϱ, N'_0). Its conic polyhedral complex Θ_0 is equal to (ϱ, N'_0). Then, we associate to each point of intersection $o_1^{(i)}$ of the strict transform S_{Σ_1} with the exceptional fiber a normal hypersurface $S_1^{(i)}$ defined by taking a good coordinate for the quasi-ordinary projection of S_{Σ_1} of Proposition 19. Obviously, if $o_1^{(i)} = o_1^{(j)}$ we have $S_1^{(i)} = S_1^{(j)}$ (see Remark 21).

LEMMA 25. — *The c.p.c. Θ_1 associated to the toroidal embedding defined by the variety Z_{Σ_1} and the set normal hypersurfaces $\{\overline{\mathbb{O}_\sigma}\}_{\sigma \in \Sigma_1^{(1)}} \cup \{(S_1^{(i)}, o_1^{(i)})\}_{\lambda\kappa(i) \neq +\infty}$ is obtained from the c.p.c. Σ_1 by adding for each point in the set $\{o_1^{(i)}\}_{i=1,\dots,r}^{\lambda_{\kappa(i)} \neq +\infty}$ the c.p.c. $(\varrho, N'_{\lambda_{\kappa(i)}})$ and pasting it to Σ_1 by identifying $(\rho \times \{0\}, N_{\lambda_{\kappa(i)}} \times \{0\})$ with $(\varrho \cap \ell_{\kappa(i)}, N_{\varrho \cap \ell_{\kappa(i)}})$ by the lattice isomorphism corresponding to (17) by duality. The c.p.c. Θ_1 is independent of the choice of good coordinates.*

Proof. — To simplify the proof we drop the index i, we denote $\lambda_{\kappa(i)}$ by λ and we keep notations of Proposition 19 and Lemmas 17 and 18. The germ (S_1, o_1) is defined by the vanishing of a monic polynomial $f_1 \in \mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$ of degree one where $W = X^{n_\lambda(y-\lambda)} - c$. We deduce from Lemma 17 that the analytic algebra of the germ (Z_Σ, o_1) is isomorphic to $\mathbb{C}\{\varrho^\vee \cap M'_\lambda\}$ by the isomorphism that maps $f_1 \mapsto X^{y_1}$ and $X^u \mapsto X^u$ for all $u \in \rho \cap M_\lambda$. Since the c.p.c. associated to the torus embedding of Z_{ϱ, N_1} is (ϱ, N_1) the same holds for the toroidal embedding corresponding to $\Sigma_1^{(1)}$ and the set of normal hypersurfaces $\mathcal{H} = \{\overline{\mathbb{O}_\sigma}\}_{\sigma \in (\varrho \cap \ell)^{(1)}} \cup \{S_1\}$. The sub-c.p.c. associated to the toroidal embedding corresponding to $\mathcal{H} - \{S_1\}$ is $(\varrho \cap \ell, N_{\varrho \cap \ell})$; it is isomorphic to (ρ, N_1), the pasting isomorphism being obtained from (17) by duality. □

Then we continue as follows:

If the quasi-ordinary polynomial defining the germ of the strict transform $(S_{\Sigma_1}, o_1^{(i)})$ has some characteristic exponent we put it in good coordinates; then its Newton polyhedron defines a subdivision of $(\varrho, N'_{\lambda_{\kappa(i)}})$, for $1 \leqslant i \leqslant r$. These subdivisions glue up to define a subdivision Σ_2 of the c.p.c. Θ_1 since the pasting cones $(\rho \times \{0\}, N_{\lambda_{\kappa(i)}})$ are not subdivided, for $1 \leqslant i \leqslant r$.

The corresponding toric modifications, defined locally, paste into a toroidal modification $\pi_2 : Z_2 \to Z_1$; (we denote the variety Z_{Σ_1} by Z_1, the morphism π_{Σ_1} by π_1, and S_{Σ_1} by S'_1). By iterating this procedure we obtain: A modification $\pi_k : Z_k \to Z_{k-1}$, where the variety Z_k is given with the structure of toroidal embedding (Σ_k denoting its associated c.p.c.). The strict transform S'_k of S by $\pi_k \circ \ldots \circ \pi_1$ at the points of intersection with the exceptional fiber is given with a quasi-ordinary projection and the associated Eggers-Wall tree is obtained from the eventually non connected tree of S'_{k-1} as indicated by Proposition 23. If the quasi-ordinary polynomial defining the germ S'_k at any of these points has some characteristic monomial we define a finer toroidal embedding for Z_k (with c.p.c. Θ_k defined by using Lemma 25) and a subdivision Σ_{k+1} of Θ_k with associated modification $\pi_{k+1} : Z_{k+1} \to Z_k$. In a finite number k_0 of steps the quasi-ordinary polynomials defining the germ S'_{k_0} at the points of intersection with the exceptional fiber have no characteristic monomials. Then it follows from Corollary 20 that

Theorem 1. — *The proper morphism $\pi = \pi_{k_0} \circ \ldots \circ \pi_1$ is a partial embedded resolution of the quasi-ordinary hypersurface germ $(S, o) \subset (Z_\rho, o_\rho)$. The restriction $S' \to S$ of π to the strict transform S' of S is the normalization map.*

An embedded resolution of $S \subset Z_\varrho$ is obtained by composing π with any toric resolution of the toroidal embedding Z_{k_0} with the c.p.c. Σ_{k_0} (or also with the c.p.c. Θ_{k_0}).

Remark 26. — The irreducible components of the exceptional fiber $\pi^{-1}(o_\varrho)$ of the partial resolution are projective lines $P^1_{\mathbb{C}}$. The dual intersection graph of the components of $\pi^{-1}(o_\varrho)$ is obtained from the Eggers-Wall tree $\theta(f)$ by deleting the extremal segments.

One of this segment joins the base vertex P_0 to one defined by the first characteristic exponent of the reduced f and the others corresponds to the segment containing the point $P^{(i)}_{+\infty}$ for $i = 1, \ldots, r$.

3.4. The case of plane curve germs.

The case of plane curve germs corresponds to rk $N = 1$. We keep the same notations. The partial resolution procedure depends only on the Eggers-Wall tree of $f \in \mathbb{C}\{X\}[Y]$ with respect to the projection $(X, Y) \mapsto X$ or more precisely on the choice of the curve $X = 0$. If f is irreducible then our construction is closely related to the construction of the "*Tschirnhausen good resolution tower*" of A'Campo and Oka (see [A'C-Ok], Theorem 4.5). In particular if the curve $X = 0$ is not contained in the tangent cone of S we show that this procedure leads to a minimal embedded resolution of the curve.

Let $f \in \mathbb{C}\{X\}[Y]$ be a reduced polynomial with Y a good coordinate for f. We keep notations of Theorem 1 and we give some more definitions and notations. We denote by $\Theta_{k_0}^{\text{reg}}$ the minimal regular subdivision of the c.p.c. Θ_{k_0} (for the minimal regular subdivision in the toric two dimensional case see Proposition 1.19 of [Od]). This provides a resolution $p : Z_{\Theta_{k_0}^{\text{reg}}} \to \mathbb{C}^2$ where $p := \pi \circ \pi_{\Theta_{k_0}^{\text{reg}}}$ which is canonically determined from the projection $(X, Y) \mapsto X$.

Denote by $\mathcal{G}(p, 0)$ (resp. $\mathcal{G}(p, f)$) the subset of $\Theta_{k_0}^{\text{reg}}$ whose elements are the cones corresponding to non empty intersections of pairs of components of the exceptional divisor of the resolution p, (resp. of the total transform of S by p). Denote by $\mathcal{G}(\pi, 0)$ (resp. by $\mathcal{G}(\pi, f)$) the subset of Θ_{k_0} of those cones corresponding to non empty intersections of pairs of components of the exceptional divisor of the partial resolution π (resp. of the total transform of S by π).

Recall that each edge of $\Theta_{k_0}^{\text{reg}}$ corresponds to an irreducible divisor in the toroidal embedding and any pair of these divisors intersect if and only if the corresponding edges belong to the same cone. It follows that $\mathcal{G}(p, 0)$ (resp. $\mathcal{G}(p, f)$) is combinatorially isomorphic to the *resolution graph of the resolution* (resp. to the *total resolution graph of the resolution*), we just drop the dimension of the faces by one. We deduce from Proposition 23, Remark 24 and an easy induction that the Eggers-Wall tree $\theta(f)$ is combinatorially isomorphic to $\mathcal{G}(\pi, Xf)$.

The valency of a cone e in a conic polyhedral complex is the number of cones of the complex containing e as a facet. We denote by $\#1$ the edge of $\mathcal{G}(p, 0)$ (resp. $\mathcal{G}(p, f)$) which corresponds to the first blow up and we define

$$\delta(e) := \begin{cases} \text{valency of } e & \text{if} \quad e \neq \#1 \\ 1 + \text{ valency of } e & \text{if} \quad e = \#1. \end{cases}$$

The valency of e and the integer $\delta(e)$ depends on the complex containing e. The following lemma implies that the set of non extremal vertices of $\theta(f)$ correspond bijectively with the rupture vertices of $\mathcal{G}(p,f)$ (which are defined by those e with $\delta(e) \geqslant 3$).

Lemma 27. — *Let $f \in \mathbb{C}\{X\}[Y]$ be a reduced polynomial of degree > 1, such that Y is a good coordinate for f. For any edge e in $\mathcal{G}(\pi,0)$ we have*

1. *The integer $\delta(e)$ in $\mathcal{G}(p,Xf)$ is $\geqslant 3$.*
2. *If $\lambda_{\kappa(1)}^{-1} \notin \mathbb{Z}_{>1}$ then $\delta(e)$ in $\mathcal{G}(p,f)$ is $\geqslant 3$.*

Proof. — Recall that we have relabeled the factors of f in order to have $\lambda_{\kappa(1)} \leqslant \ldots \leqslant \lambda_{\kappa(r)}$. We show first the assertion for the exceptional divisors appearing in the first toric modification π_{Σ_1}. The extremal edges of the fan Σ_1, which are defined by the vectors u_1, u_2 of the canonical basis, correspond to the divisors $X = 0$ and $Y = 0$ respectively. If $\lambda_{\kappa(j)} \neq +\infty$, there is an exceptional divisor $D_{\lambda_{\kappa(j)}}$ of π_{Σ_1} corresponding to $d_{\lambda_{\kappa(j)}} \in \mathcal{G}(\pi,f)$. We denote by the same letter the edge $d_{\lambda_{\kappa(j)}}$ of Σ_1 and the primitive vector $(n_{\lambda_{\kappa(j)}}, n_{\lambda_{\kappa(j)}}\lambda_{\kappa(j)})$ on this edge for the lattice N_0'. We say that a two dimensional cone σ is on the *left* (resp. on the *right*) of the vector $d_{\lambda_{\kappa(j)}} \in \sigma$ if $\sigma \subset \langle d_{\lambda_{\kappa(j)}}, u_2\rangle$ (resp. $\sigma \subset \langle u_1, d_{\lambda_{\kappa(j)}}\rangle$).

By Proposition 23, the divisor $D_{\lambda_{\kappa(j)}}$ meets the strict transform of S by π_{Σ_1}.

If $\lambda_{\kappa(r)} > \lambda_{\kappa(j)}$ (resp. if $\lambda_{\kappa(j)} > \lambda_{\kappa(1)}$) then there exists a two dimensional cone on the left (resp. right) of $d_{\lambda_{\kappa(j)}}$ in $\mathcal{G}(p,f)$, obtained from the minimal regular subdivision of the cone $\sigma \in \mathcal{G}(\pi,f)$, on the right (resp. on the left) of $d_{\lambda_{\kappa(j)}}$. Therefore if $\lambda_{\kappa(r)} > \lambda_{\kappa(j)} > \lambda_{\kappa(1)}$ we have $\delta(d_{\lambda_{\kappa(j)}}) \geqslant 3$.

If $\lambda_{\kappa(r)} = +\infty$ then Y divides f and $Y = 0$ is a component of the strict transform of S by π_{Σ_1}. If $\lambda_{\kappa(r)} \neq +\infty$ two cases may occur: a) if the cone $\sigma = \langle d_{\lambda_{\kappa(r)}}, u_2\rangle$ is not regular we have a two dimensional cone in $\mathcal{G}(\pi,f)$ on the left of $d_{\lambda_{\kappa(r)}}$; b) the cone σ is regular thus $\lambda_{\kappa(r)} \in M$. By the proof of Lemma 15 there exists $i \neq r$ such that $\lambda_{\kappa(i)} = \lambda_{(i,r)} = \lambda_{\kappa(r)}$. By Proposition 23 this implies that the strict transforms of $f^{(i)} = 0$ and $f^{(r)} = 0$ meet the divisor $D_{\lambda_{\kappa(r)}}$ in two different points so that we have $\delta(d_{\lambda_{\kappa(r)}}) \geqslant 3$.

Now we deal with the divisor $D_{\lambda_{\kappa(1)}}$. The cone $\langle u_1, d_{\lambda_{\kappa(1)}}\rangle$ belongs to $\mathcal{G}(\pi,Xf)$ and we deduce from this that $\delta(d_{\lambda_{\kappa(1)}}) \geqslant 3$ in $\mathcal{G}(\pi,Xf)$. If the cone $\langle u_1, d_{\lambda_{\kappa(1)}}\rangle$ is not regular we can argue as before to show the existence of a two dimensional cone of $\mathcal{G}(\pi, f)$ on the right of $d_{\lambda_{\kappa(1)}}$. Otherwise we have $n_{\lambda_{\kappa(1)}}\lambda_{\kappa(1)} = 1$ and if $\lambda_{\kappa(1)}^{-1} \notin \mathbb{Z}_{>1}$ the only possibility is $d_{\lambda_{\kappa(1)}} = (1, 1)$. Then we have $\lambda_{\kappa(1)} \in M$ and thus $\lambda_{\kappa(r)} = \lambda_{\kappa(1)}$ by Lemma 15. This case has already been solved.

These facts give the assertion for e corresponding to an exceptional divisor of π_{Σ_1}. When we iterate, the curve $X = 0$ corresponds to the equation of the exceptional divisor meeting the strict transform, thus after the first step we are always in the case *1* and proposition follows. □

An exceptional divisor D of the resolution p is *collapsible* if it has *self-intersection number* equal to -1 and the corresponding edge $d \in \mathcal{G}(p, 0)$ has $\delta(d) \leqslant 2$ in $\mathcal{G}(p, f)$. If the divisor D is collapsible, the modification obtained by blowing down D is still a resolution and the corresponding resolution graph is obtained from $\mathcal{G}(p, 0)$ by deleting the point corresponding to D. The self intersection of the divisors which are images of compact divisors meeting D is increased by one. In a finite number of steps we obtain a *minimal resolution*, i.e., a resolution in which no exceptional divisor is collapsible. The minimal resolution is unique up to isomorphism (see [Lau]).

COROLLARY 28.— *If $\lambda_{\kappa(1)}^{-1} \notin \mathbb{Z}_{>1}$, in particular if the projection $(X, Y) \mapsto X$ is transversal for all the components of f then the morphism p is the minimal resolution.*

Proof.— The self intersection numbers of the exceptional divisors of the minimal resolution of a toric surface singularity are $\leqslant -2$ (see Proposition 1.19 of [Od]). This implies that the exceptional divisors corresponding to edges in $\mathcal{G}(p, f) - \mathcal{G}(\pi, f)$ are not collapsible. Then the corollary follows from Lemma 27. □

Remark 29.— The number of *local toroidal morphisms* used in the partial resolution π is not necessarily equal to the complexity of the resolution (as defined by [Le-Ok]).

For instance $f = ((Y-X)^2 - X^3))((Y+X)^2 - X^5))$ has characteristic exponents $\{1, \frac{3}{2}, \frac{5}{2}\}$. The projection $(X, Y) \mapsto X$ is transversal for the two irreducible components. It follows easily that the number of local toroidal morphisms used to define our partial resolution is three; our good coordinates (15) are generic. On the other hand, the resolution graph is a bamboo so that the resolution complexity is equal to ones the curve can be resolved with one toric morphism, with respect to a special choice of coordinates.

4. The semigroup associated to a toric quasi-ordinary branch.

We associate to the quasi-ordinary branch ζ a semigroup Γ which is determined from the characteristic exponents; the construction of Γ involves also a generalization of the notion of the plane curves with maximal contact with a given branch given by Lejeune [LJ] and this relation can be described by using the *approximate roots* of the polynomial f. The main part of the results and the proofs of this section is given in [GP3].

4.1. Definition of the semigroup.

In the following sections we study a fixed toric quasi-ordinary singularity S parametrized by a toric quasi-ordinary branch $\zeta \in \mathbb{C}\{\rho^\vee \cap \frac{1}{n} M\}$ with $g \geqslant 1$ characteristic exponents $\{\lambda_1, \dots, \lambda_g\}$ and with minimal polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$. If rk $M = 1$ then the singularity S is a plane branch and the set of intersection multiplicities $(S, S')_0$ of S, such as plane curve germs S' do not contain S as a component, forms a sub-semigroup of $(\mathbb{Z}_{\geqslant 0}, +)$ which is an invariant of the germ S and which is generated by the following elements (see [Z6]):

$$\bar{\gamma}_1 = n\lambda_1, \quad \bar{\gamma}_{j+1} = n_j \bar{\gamma}_j + n\lambda_{j+1} - n\lambda_j, \ \text{ for } j = 1, \dots, g-1. \tag{20}$$

For $j = 0, \dots, g-1$, we expand

$$\begin{aligned} \bar{\gamma}_{j+1} = {} & n\left((n_1 - 1)n_2 \cdots n_j \lambda_1 + (n_2 - 1)n_3 \cdots n_j \lambda_2 \right. \\ & \left. + \cdots + (n_j - 1)\lambda_j + \lambda_{j+1}\right. \\ & \stackrel{\text{Def.7}}{=} n_1 \dots n_j \left((e_0 - e_1)\lambda_1 + (e_1 - e_2)\lambda_2 \right. \\ & \left. + \dots + (e_{j-1} - e_j)\lambda_j + e_j \lambda_{j+1}\right. . \end{aligned} \tag{21}$$

We denote $\frac{1}{n}\bar{\gamma}_i$ by γ_i for $i = 1, \dots, g$ and we have

$$\gamma_1 = \lambda_1, \gamma_{j+1} = n_j \gamma_j + \lambda_{j+1} - \lambda_j, \ \text{ for } j = 1, \dots, g-1. \tag{22}$$

Definition 10. — *We associate to the quasi-ordinary branch ζ the sequence of semigroups $\Gamma_j = \rho^\vee \cap M + \gamma_1 \mathbb{Z}_{\geqslant 0} + \cdots + \gamma_g \mathbb{Z}_{\geqslant 0}$ for $j = 0, \dots, g$.*

We denote Γ_g by Γ and $n\Gamma_j$ by $\bar{\Gamma}_j$ for $j = 0, \dots, g$. The classical semigroup of a plane branch is $\bar{\Gamma}_g$.

If ζ is a *classical* quasi-ordinary branch suitably *normalized*(1). Lipman proved that the sequence of characteristic exponents is an analytical invariant of the germ it parametrizes when $\dim S = 2$, by building a (non embedded) resolution of the germ (see [L1], [L3]) which determines the characteristic exponents. Luengo gives another proof also using resolutions (see [Lu]). If the germ is analytically irreducible the characteristic exponents define a complete invariant of the embedded topological type of the hypersurface $S \subset \mathbb{C}^{d+1}$ it parametrizes (see [Gau] and [L4]). We proved in [GP2] that if τ and ζ are quasi-ordinary branches parametrizing S then the semigroups associated to them are isomorphic and moreover that the minimal set of generators of this semigroup defines the sequence of characteristic exponents of any normalized quasi-ordinary branch parametrizing S. By Gau's characterization it follows that the semigroup Γ defined above is a complete topological invariant of the embedded topological type of germ $(S, 0)$.

The following lemma generalizes the properties of the semigroups of plane branches (see [T1], Chapitre I, Lemma 2.2.1) to the quasi-ordinary hypersurface case (see [GP2]).

Lemma 30 (See [GP3]).

1. *The sub-lattice of M generated by Γ_j is equal to M_j, for $0 \leqslant j \leqslant g$.*

2. *The order of the image of γ_j in the group M_j/M_{j-1} is equal to n_j for $j = 1, \dots, g$.*

3. *We have that $\gamma_j > n_{j-1}\gamma_{j-1}$ for $j = 2, \dots, g$.*

4. *If a vector $u_j \in \rho^\vee \cap M_j$ then we have $u_j + n_j\gamma_j \in \Gamma_j$.*

5. *The vector $n_j\gamma_j$ belongs to the semigroup Γ_{j-1} for $j = 1, \dots, g$, moreover we have a unique relation:*

$$n_j\gamma_j = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \cdots + l_{j-1}^{(j)}\gamma_{j-1} \tag{23}$$

such that $0 \leqslant l_i^{(j)} \leqslant n_i - 1$ and $\alpha^{(j)} \in M_0$, for $j = 1, \dots, g$.

In the plane branch case several authors have studied the properties of those curves S' such that the intersection multiplicity with S at the

(1) In the case of a plane branch this condition means that $X = 0$ is not contained in the tangent cone of the curve.

origin belongs to the unique minimal set of generators of the semigroup of the branch (see [Z6]). Lejeune introduced the notion of curves of maximal contact with a given plane curve germ for curves defined over a field of arbitrary characteristic in terms of the resolution (see [LJ]). If the characteristic is zero it turns out that both notions are equivalent (see [Ca]). If the projection (X, Y) is transversal we can study these curves by means of the minimal polynomials of suitable truncations of the roots of f. When we do this with respect to an arbitrary projection, the curves we obtain provide a non necessarily minimal set of generators of the semigroup of the branch S. These curves can be represented by some of the *approximate roots* of the polynomial f (see [A-M]) and we call them *semi-roots.* following the terminology of [A3]. See Popescu-Pampu's survey [PP1] for more on the notion of semi-root.

Definition 11. — *A j^{th}-semi-root of f is an irreducible quasi-ordinary polynomial in $\mathbb{C}\{\rho^\vee \cap M\}[Y]$ of degree $n_0 \dots n_j$ which has order of coincidence equal to λ_{j+1} with f, for $j = 0, \dots, g$.*

The minimal polynomials of the quasi-ordinary branches $p_0 + \dots + p_j$ obtained by truncating ζ in Remark 13 are j^{th}-semi-roots of f for $j = 0, \dots, g$.

Proposition 31 (see [GP2] and [GP3]). — *Let $q \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ a monic polynomial of degree $n_0 \dots n_j$. Then q is a j-semi-root of f if and only if $q(\zeta) = X^{\gamma_{j+1}} \varepsilon_j$ for a unit ε_j.*

The notion of semi-root extends the properties of maximal contact with respect to the resolution to the quasi-ordinary case (see Proof of Theorem 1 and Remark 21).

Remark 32. — The polynomial q_j is a j-semi-root of f is and only if the strict transform of $q_j = 0$ by the morphism $\pi_j \circ \dots \circ \pi_1$ is a germ defined by a good coordinate and conversely.

This follows from Proof of Theorem 1 and Remark 21.

Let A a ring containing $\mathbb{Q}$ as a subring. *Approximate roots* are defined by Abhyankar and Moh, (see [A-M], [G-P], and [PP1]). If p is any monic polynomial and k divides the degree of p there is a unique monic polynomial r in $A[Y]$ of degree $\frac{\deg(p)}{k}$ such that $\deg(p - r^k) < \deg(p) - \frac{\deg(p)}{k}$. We say that r is a k-semi-root of p. We can use Proposition 31 to prove that the e_j-approximate roots of a quasi-ordinary polynomial f are semi-roots, and therefore are irreducible quasi-ordinary polynomials with a prescribed order of coincidence with the polynomial f (see [GP2] and [GP3]).

4.2. Expansion in terms of semi-roots.

The expansions in terms of semi-roots are introduced by Abhyankar in the plane curve case (see [A3]) and used by Popescu-Pampu in the case of a quasi-ordinary hypersurface singularity (see [PP2]).

We fix from now on a *complete set* $q_0, \dots, q_g$ *of semi-roots of* f ($\deg q_i = n_0 \cdots n_i$ for $i = 0, \dots, g$). We assume that the coefficient of the term $X^{\gamma_{j+1}}$ appearing in $q_j(\zeta)$ by Proposition 31 is equal to one for $j = 0, \dots, g-1$ in order to simplify some computations.

We recall now the classical q-adic expansion of a polynomial $p_0 \in A[Y]$ with coefficients on a domain A in terms of a polynomial $q \in A[Y]$ having invertible leading term (see [Z6]). The sequence of Euclidean divisions:

$$p_0 = p_1 q + a_0, \;\; p_1 = p_2 q + a_1, \;\; \dots, \;\; p_s = p_{s+1} q + a_s,$$

(where s is the first integer for which $p_{s+1} = 0$) provides a unique decomposition of the form

$$H = a_0 + a_1 q + a_2 q^2 + \cdots + a_s q_s^s, \text{ for } 0 \leqslant \deg a_i \leqslant \deg q - 1.$$

LEMMA 33 (see [PP2]). — *Any polynomial* $h \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ *can be written in a unique way as*

$$h = \sum c_{l_1,\dots,l_{g+1}} q_0^{l_1} q_1^{l_2} \cdots q_g^{l_{g+1}} \tag{24}$$

with $c_{l_1,\cdots,l_{g+1}} \in \mathbb{C}\{\rho^\vee \cap M\}$, $0 \leqslant l_k \leqslant n_k - 1$ *for* $k = 1, \dots, g$ *and* $l_{g+1} \in \mathbb{Z}_{\geqslant 0}$.

If $c_{l_1,\dots,l_g,0}$ *and* $c_{l'_1,\dots,l'_g,0}$ *are two different coefficients of the expansion the Newton principal parts of* $c_{l_1,\cdots,l_g,0} q_0^{l_1}(\zeta) \cdots q_{g-1}^{l_g}(\zeta)$ *and* $c_{l'_1,\cdots,l'_g,0} q_0^{l'_1}(\zeta) \cdots q_{g-1}^{l'_g}(\zeta)$ *(viewed in the ring* $\mathbb{C}\{\rho^\vee \cap M_g\}$*) have no term in common.*

Proof. — The q_g-adic expansion of h is of the form: $h = a_0^{(g)} + a_1^{(g)} q_g + \cdots + a_{s_g}^{(g)} q_g^{s_g}$. We build the q_{g-1}-adic expansions of the coefficients:

$$a_j^{(g)} = a_{0,j}^{(g-1)} + a_{1,j}^{(g-1)} q_{g-1} + \cdots + a_{s_{g-1},j}^{(g-1)} q_{g-1}^{s_{g-1}}$$

where $0 \leqslant \deg a_{l,j}^{(g-1)} \leqslant n_0 \cdots n_{g-1} - 1$ for $0 \leqslant l \leqslant s_{g-1}$ and $0 \leqslant s_{g-1} \leqslant n_g - 1$ since $a_j^{(g)}$ is of degree $< n_0 \cdots n_g = n$. An expansion satisfying the required properties is obtained by iterating this procedure. The unicity follows from the unicity of Euclidean division. For the last assertion, remark that by Lemma 31 the Newton principal part of $q_{k-1}(\zeta)$ (viewed in $\mathbb{C}\{\rho^\vee \cap M_g\}$) is equal to X^{γ_k} for $k = 1, \dots, g$. It follows from *2* in Lemma 30 that the Newton principal parts of $c_{l_1,\dots,l_g,0} q_0^{l_1}(\zeta) \cdots q_{g-1}^{l_g}(\zeta)$ and of $c_{l'_1,\dots,l'_g,0} q_0^{l'_1}(\zeta) \cdots q_{g-1}^{l'_g}(\zeta)$ do not have any term in common if $(l_1, \dots, l_g) \neq (l'_1, \dots, l'_g)$. □

The following proposition (see [GP2]) generalizes [Z6], Chapitre II, Th. 3.9. in the plane branch case.

Proposition 34. — *If $h \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ is of degree $< n_0 n_1 \dots n_j$ then the Newton principal part of $h(\zeta)$ belongs to $\mathbb{C}[\Gamma_j]$, for $j = 1, \dots, g$.*

Proof. — The result is trivial if $\deg h = 0$. If $\deg h < n_1 \dots n_j$ then the $(q_0, \dots, q_g)$-expansion of h is of the form: $h = \sum c_{l_1,\dots,l_j} q_0^{l_1} q_1^{l_2} \cdots q_{j-1}^{l_j}$. By Lemma 33 the Newton principal parts of $c_{l_1,\dots,l_j} q_0^{l_1}(\zeta) \cdots q_{j-1}^{l_j}(\zeta)$ and of $c_{l'_1,\dots,l'_j} q_0^{l'_1}(\zeta) \cdots q_{j-1}^{l'_j}(\zeta)$ do not have terms in common, thus the polynomial $h(\zeta)_{|\mathcal{N}}$ is a sum of some of the terms in the Newton principal parts of the summands $c_{l_1,\dots,l_j} q_0^{l_1}(\zeta) \cdots q_{j-1}^{l_j}(\zeta)$ and therefore it belongs to $\mathbb{C}[\Gamma_j]$ by Proposition 31, for $j = 1, \dots, g$. □

We call the expansion (24) above the $(q_0, \dots, q_g)$-*expansion of* h.

Lemma 35. — *The $(q_0, \dots, q_g)$-expansion of $q_{j-1}^{n_j}$ is of the following form, for $1 \leqslant j \leqslant g$:*

$$q_{j-1}^{n_j} = c_j^* q_j + \sum c^{(j)}_{l_1,\dots,l_j} q_0^{l_1} q_1^{l_2} \cdots q_{j-1}^{l_j} \tag{25}$$

where $c_j^* \in \mathbb{C}^*$, the other coefficients belong to $\mathbb{C}\{\rho^\vee \cap M\}$, we have $0 \leqslant l_k \leqslant n_{k+1} - 1$ for $k = 0, \dots, j-1$. The coefficient $c^{(j)}_{l_1^{(j)},\dots,l_{j-2}^{(j)},0}$ appears and it is of the form $X^{\alpha^{(j)}} \cdot$ unit, where the integers $l_1^{(j)}, \dots, l_{j-2}^{(j)}$ and the exponent $\alpha^{(j)}$ are given by formula (23). Moreover, if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_1,\dots,l_j}$ then

$$n_j \gamma_j \leqslant \alpha' + l_1 \gamma_1 + \cdots + l_j \gamma_j \tag{26}$$

and equality holds if and only if $(l_1, \dots, l_j) = (l_1^{(j)}, \dots, l_{j-2}^{(j)}, 0)$ and $\alpha' = \alpha^{(j)}$.

Proof. — Since $\deg q_{j-1}^{n_j} = n_1 \cdots n_j$ the algorithm to calculate the $(q_0, \ldots, q_g)$-expansion begins by dividing $q_{j-1}^{n_j}$ by q_j. This gives $q_{j-1}^{n_j} = c_{j+1}^* q_{j+1} + r_j$, where $c_{j+1}^* \in \mathbb{C}^*$ since both polynomials have the same degree. The q_k that may appear in the expansion of r_j are those of degree $\leqslant \deg r_j < n_1 \cdots n_j$. We deduce from the second assertion of Lemma 33 that

$$\mathcal{N}(c^{(j)}_{l_1,\ldots,l_j} q_0^{l_1}(\zeta) \cdots q_{j-1}^{l_{j-1}}(\zeta)) \subset n_j\gamma_j + \rho^\vee = \mathcal{N}(q_{j-1}^{n_j}(\zeta)).$$

This implies that if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_1,\ldots,l_j}$ then formula (26) holds. If equality in (26) holds for a term the term $X^{\alpha'}$ appearing on the series $c^{(j)}_{s_1^{(j)},\ldots,s_j^{(j)}}(\zeta)$ it follows that the series is the form $X^{\alpha'} \cdot$ unit. Assertion 2 of Lemma 30 implies that $s_j^{(j)} = 0$ in the relation $n_j\gamma_j = \alpha' + s_1^{(j)}\gamma_1 + \ldots + s_j^{(j)}\gamma_j$. Then it follows that $(l_1, \ldots, l_{j-2}) = (l_1^{(j)}, \ldots, l_{j-2}^{(j)})$ and that $\alpha' = \alpha^{(j)}$ by unicity in (23). □

5. Partial embedded resolution with one toric morphism.

In this section we build a partial embedded resolution of the toric quasi-ordinary germ embedded in an affine toric variety by using the semi-roots. We follow the approach of [G-T] for irreducible germs of plane curves.

We denote by Δ the cone $\rho \oplus \mathbb{R}^g_{\geqslant 0} \subset (N_\Delta)_\mathbb{R}$ where N_Δ is the lattice $N \oplus \mathbb{Z}^g$ with dual lattice M_Δ. We denote by $u_1, \ldots, u_g$ the canonical basis of $\{0\} \oplus \mathbb{Z}^g$. An element of $\Delta^\vee \cap M_\Delta$ is of the form (α, v) where $\alpha \in \rho^\vee \cap M$ and $v = v_1 u_1^* + \cdots v_g u_g^*$ where $u_1^*, \ldots, u_g^*$ is the dual basis of $u_1, \ldots, u_g$ and $v_i \in \mathbb{Z}_{\geqslant 0}$. We denote the monomial corresponding to (α, v) by $X^\alpha U^v$, $X^\alpha U_1^{v_1} \ldots U_g^{v_g}$ or $X^{\alpha + \sum v_i u_i^*}$ depending on the context.

The embedding $S \subset Z_\Delta$ which is studied in this section corresponds algebraically to the homomorphism of $\mathbb{C}\{\rho^\vee \cap M\}$-algebras:

$$\left\{ \begin{array}{l} \Psi_0 : \mathbb{C}\{\rho^\vee \cap M\}[U_1, \ldots, U_g] \to R \\ U_j \mapsto q_{j-1}(\zeta), \ \text{for } j = 1, \ldots, g \end{array} \right. \tag{27}$$

(which is surjective since in particular $R = \mathbb{C}\{\rho^\vee \cap M\}[q_0(\zeta)]$).

In the plane branch case Teissier shows that this embedding specializes to the *monomial curve*, an affine curve monomially embedded with the same semigroup (see [T1]). In the general case the generalization of monomial curve is given by an equivariant embedding $Z^\Gamma \subset Z_\Delta$ which is defined from the restriction of the lattice homomorphism

(28) $\varphi : M_\Delta \to M_g$ that maps $\alpha + v \mapsto \alpha + v_1\gamma_1 + \ldots + v_g\gamma_g$

to the semigroup $\Delta^\vee \cap M_\Delta$ and its image Γ.

5.1. Specialization through graded rings.

In the plane branch case the embedding of the monomial curve is determined by a system of generators the *graded ring*([2]) associated to the *filtration* of R induced by the powers of the maximal ideal of its integral closure (see [T1]). In our case we show that the homomorphism Ψ_0 can be filtered in such a way that the homomorphism of the associated graded rings, forgetting the graded structure, defines the embedding $Z^\Gamma \subset Z_\Delta$ above.

The filtration of the ring $\mathbb{C}\{\rho^\vee \cap M\}$ (resp. of $\mathbb{C}[[\rho^\vee \cap M]]$) defined by a vector $\eta \in \rho$ is given by the ideals:

$$\mathcal{I}_j = \left\{ \sum_{u \in \rho^\vee \cap M} c_u X^u \;/\; \min_{c_u \neq 0} \langle \eta, u \rangle \geqslant j \right\} \text{ for } j \in \eta(\rho^\vee \cap M).$$

Since the ring $\mathbb{C}\{\rho^\vee \cap M\}$ is Noetherian the ordered sub-semigroup $\eta(\rho^\vee \cap M)$ of $\mathbb{R}_{\geqslant 0}$ is isomorphic to $\mathbb{Z}_{\geqslant 0}$ (see the proof of Lemma 1.4 of [GP1]). The vector η defines a *weighted filtration* of $\mathbb{C}\{\rho^\vee \cap M\}[U_1, \ldots, U_g]$ (resp. of $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ or $\mathbb{C}[[\Delta^\vee \cap M_\Delta]]$) given by the ideals $\mathcal{J}_j$ generated by those series having only terms $X^\alpha U^v$ of *weights* $w := \varphi(\alpha, v)$ such that $\langle \eta, w \rangle \geqslant j$, for j running through the semigroup $\eta(\rho^\vee \cap M_g)$. The homomorphism Ψ_0 is *filtered* since $\Psi_0(\mathcal{J}_k) \subset \mathcal{I}_k$ for all $k \in \eta(\rho^\vee \cap M_g)$, and then it defines an homomorphism of the associated graded rings.

Proposition 36. — *The sequence of graded ring homomorphisms associated to the filtered sequence of homomorphisms (with the filtrations defined by* $\eta \in \mathring{\rho}$*)*

(29) $$\mathbb{C}\{\rho^\vee \cap M\}[U_1, \ldots, U_g] \Psi_0 \longrightarrow R \hookrightarrow \mathbb{C}\{\rho^\vee \cap M_g\}$$

is isomorphic to

$$\mathbb{C}[\rho^\vee \cap M][U_1, \ldots, U_g] \longrightarrow \mathbb{C}[\Gamma] \hookrightarrow \mathbb{C}[\rho^\vee \cap M_g]$$

([2]) See [Bbk] for the definitions and properties of commutative algebra used in the following sections.

where the first homomorphism is defined by $X^\alpha U^v \mapsto X^{\varphi(\alpha,v)}$*, and the graduations are defined by* η*. If the vector* η *is irrational the semigroup* Γ *is determined by the graduation.*

Proof. — If $\eta \in \overset{\circ}{\rho}$ the symbolic restriction $\phi_{|_\eta}$ of $\phi \in \mathbb{C}\{\rho^\vee \cap M_g\}$ to the face defined by η on the polyhedron $\mathcal{N}_\rho(\phi)$ belongs to $\mathbb{C}[\rho^\vee \cap M_g]$ since this face is compact. If $\phi \in \mathbb{C}\{\rho^\vee \cap M_g\}$ there exists a unique integer k such that $\phi \in \mathcal{I}_k - \mathcal{I}_{k+1}$ and then we have $\phi = \phi_{|_\eta} \bmod \mathcal{I}_{k+1}$. It follows from the property: $\phi_{|_\eta} \phi'_{|_\eta} = (\phi\phi')_{|_\eta}$ for $0 \neq \phi, \phi' \in \mathbb{C}\{\rho^\vee \cap M_g\}$, that the graded ring associated to this filtration is isomorphic to the graded ring $\mathbb{C}[\rho^\vee \cap M_g]$ where the j-homogeneous term of the graduation is $\bigoplus_{\langle \eta, u\rangle = j} \mathbb{C}X^u$ for $j \in \eta(\rho^\vee \cap M_g)$. We deduce analogously that the graded ring associated to the weighted filtration is isomorphic to $\mathbb{C}[\Delta^\vee \cap M_\Delta]$ where the non zero elements in the j-homogeneous term are those polynomials such that $\langle \eta, w\rangle = j$ for w running through the weights of the monomials appearing on them.

Under these identifications we have that

– The graded ring associated to R with the induced filtration is isomorphic to the graded subring of $\mathbb{C}[\rho^\vee \cap M_g]$ generated as a $\mathbb{C}$-algebra by the symbolic restrictions $\phi_{|\eta}$ of $0 \neq \phi \in R$ to the face defined by η on the polyhedron $\mathcal{N}(\phi)$. We deduce from Proposition 34 and Proposition 31 that this graded subring is isomorphic to $\mathbb{C}[\Gamma]$.

– The initial term of $\Psi_0(U_i) = q_{i-1}(\zeta)$ is equal to X^{γ_i} (the coefficient has been normalized to be one) thus the homomorphism $\mathrm{gr}(\Psi_0)$ corresponds to the $\mathbb{C}[\rho^\vee \cap M]$-homomorphism $\mathbb{C}[\rho^\vee \cap M][U_1, \ldots, U_g] \to \mathbb{C}[\Gamma]$ that maps $U_i \mapsto X^{\gamma_i}$ for $i = 1, \ldots, g$.

If the vector η is *irrational* we can recover the semigroup $\rho^\vee \cap M_g$ (resp. Γ) from the graduation of $\mathbb{C}[\rho^\vee \cap M]$ (resp. of $\mathbb{C}[\Gamma]$) since each term of the graduation is of dimension one (resp. zero or one) over $\mathbb{C}$, the vector η defining a total ordering on $\rho^\vee \cap M_g$. □

Remark 37. — The sequence of homomorphisms (29) extends to the sequences:

$$
\begin{array}{ccccc}
\mathbb{C}[[\Delta^\vee \cap M_\Delta]] & \xrightarrow{\hat{\Psi}} & \hat{R} & \hookrightarrow & \mathbb{C}[[\rho^\vee \cap M_g]] \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{C}\{\Delta^\vee \cap M_\Delta\} & \xrightarrow{\Psi} & R & \hookrightarrow & \mathbb{C}\{\rho^\vee \cap M_g\}
\end{array}
\tag{30}
$$

where $\hat{R}$ denotes the completion of the ring R with respect to the maximal ideal $\mathfrak{M}_R$. The assertion of Proposition 36 remains true for each line of the above diagram.

We notice that $\hat{R}$ coincides with the completion with respect to the filtration defined by η: we have that $\mathfrak{M}_R^{s_j} \subset \mathcal{I}_j$ where s_j is the minimal power of $\mathfrak{M}_R$ containing the set of monomials in $\mathcal{I}_j - \mathcal{I}_{j+1}$ which is finite since $\eta \in \overset{\circ}{\rho}$.

5.2. Equations for the embeddings.

We build equations of the embeddings of $Z^\Gamma \subset Z_\Delta$ and $S \subset Z_\Delta$.

Proposition 38. — *The ideal of the embedding $Z^\Gamma \subset Z_\Delta$ is generated by the binomials*

$$\left\{\begin{array}{lllll} h_1 & := & U_1^{n_1} & - & X^{\alpha^{(1)}} \\ h_2 & := & U_2^{n_2} & - & X^{\alpha^{(2)}} U_1^{l_1^{(2)}}, \\ \cdots & \cdots & \cdots & \cdots & \cdots, \\ h_g & := & U_g^{n_g} & - & X^{\alpha^{(g)}} U_1^{l_1^{(g)}} \dots U_{g-1}^{l_{g-1}^{(g)}}, \end{array}\right. \tag{31}$$

which correspond to relations (23).

Preuve. — The ideal I of the embedding $Z^\Gamma \subset Z_\Delta$ is generated by the binomials $X^\alpha U^\omega - X^{\alpha'} U^{\omega'}$ of $\mathbb{C}[\Delta^\vee \cap M_\Delta]$ verifying (see (5)):

$$\varphi(\alpha, \omega) = \varphi(\alpha', \omega'). \tag{32}$$

The binomials $h_1, \dots, h_g$ above verify this condition by Lemma 30. If B is a binomial in I, we can factor the common term in U_g to obtain a binomial in I of the form $X^\alpha U^\omega - X^{\alpha'} U^{\omega'}$ with $w'_g = 0$. Then the integer ω_g is a multiple of n_g (since $n_g \gamma_g \in M_{g-1}$ by Lemma 30 we obtain from the equality (32) a relation $r\gamma_g \in M_{g-1}$ where r is the reminder of the Euclidean division of ω_g by n_g and then Lemma 30 implies that $r = 0$). We can show by induction on ω_g / n_g that the reminder of the Euclidean division of $X^\alpha U^\omega - X^{\alpha'} U^{\omega'}$ by h_g as polynomials in U_g is a binomial B_1 in $\mathbb{C}[\rho^\vee \cap M][U_1, \dots, U_{g-1}]$. The binomial B_g obtained by iterating this procedure belongs to $\mathbb{C}[\rho^\vee \cap M]$ and to the ideal I. The relation (32) corresponding to B_g is trivial since the homomorphism φ is injective on M thus $B_g = 0$. This implies that the ideal I is generated by $h_1, \dots, h_g$. □

Proposition 39. — *The ideal of the embedding $S \subset Z_\Delta$ defined by (27) is generated by elements of the ring $\mathbb{C}\{\rho^\vee \cap M\}[U_1, \ldots, U_g]$ which are of the form:*

(33)

$$\begin{cases} H_1 := U_1^{n_1} - X^{\alpha^{(1)}} + c_1^* U_2 + r_1(U_1), \\ H_2 := U_2^{n_2} - X^{\alpha^{(2)}} U_1^{l_1^{(2)}} + c_2^* U_3 + r_1(U_1, U_2), \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ H_{g-1} := U_{g-1}^{n_{g-1}} - X^{\alpha^{(g-1)}} U_1^{l_1^{(g-1)}} \ldots U_{g-2}^{l_{g-2}^{(g-1)}} + c_{g-1}^* U_g + r_{g-1}(U_1, U_2, \ldots, U_{g-1}), \\ H_g := U_g^{n_g} - X^{\alpha^{(g)}} U_1^{l_1^{(g)}} \ldots U_{g-1}^{l_{g-1}^{(g)}} + r_g(U_1, U_2, \ldots, U_g). \end{cases}$$

The weight of a term $X^\alpha U_1^{l_1} U_2^{l_2} \cdots U_j^{l_j}$ appearing in the expansion of $r_j(U_1, U_2, \ldots, U_j)$ is $\geqslant n_j \gamma_j$ and equality never holds, for $j = 1, \ldots, g$. The terms appearing in the expansion of $r_j(U_1, U_2, \ldots, U_j)$ are determined explicitly by formula (25).

Proof. — It follows from the definition of the homomorphism Ψ_0 and formula (25) that the polynomials H_i above belong to the kernel of Ψ_0 (and then to the kernels of Ψ and $\hat{\Psi}$). By Proposition 38 and Lemma 35 their initial forms with respect to the filtration defined by $\eta \in \overset{\circ}{\rho}$ generate $\mathrm{Ker}(\mathrm{gr}(\hat{\Psi}))$. Then we have that $\mathrm{gr}(\mathrm{Ker}(\hat{\Psi})) = \mathrm{Ker}(\mathrm{gr}(\hat{\Psi}))$. We deduce using that the ideal $\mathrm{Ker}(\hat{\Psi})$ is complete for the induced filtration, that the polynomials $H_1, \ldots, H_g$ generate $\mathrm{Ker}(\hat{\Psi})$(3).

Since the inclusion $\mathbb{C}\{\Delta^\vee \cap M_\Delta\} \to \mathbb{C}[[\Delta^\vee \cap M_\Delta]]$ is an homomorphism of local rings continuous for the $\mathfrak{M}$-adic topologies which extends to the identity homomorphism between the respective completions, we have that the ring $\mathbb{C}[[\Delta^\vee \cap M_\Delta]]$ is a *faithfully flat* $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$-module(4). The ideal J generated by $(H_1, \ldots, H_g)$ on $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ is contained in $\mathrm{Ker}(\Psi)$ and we have shown that $J\mathbb{C}[[\Delta^\vee \cap M_\Delta]] = \mathrm{Ker}(\hat{\Psi})$. The faithfully flat property implies that J coincides with the contraction of $\mathrm{Ker}(\hat{\Psi})$ in $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$(5). Therefore we obtain that $J = \mathrm{Ker}(\Psi)$.

Let $\mathcal{U}$ be the subset of those elements in $\mathbb{C}\{\rho^\vee \cap M\}[U]$ with non zero constant term as power series. The image by Ψ_0 of a series in $\mathcal{U}$ is a unit. This implies that the localization $\mathcal{U}^{-1}\Psi_0 : \mathcal{U}^{-1}\mathbb{C}\{\rho^\vee \cap M\}[U] \to$

(3) See Proposition 12 No 9, §2, Chapitre III, of [Bbk].

(4) See Proposition 10 No 5, §3, Chapitre III of [Bbk].

(5) Proposition 9 No 5, §3, Chapitre I of [Bbk].

$\mathbb{C}\{\rho^\vee \cap M_g\}$ is a well-defined homomorphism. The same argument shows that $\mathrm{Ker}(\mathcal{U}^{-1}\Psi_0)$ is generated by $H_1, \ldots, H_g$. Since $\mathcal{U} \cap \mathrm{Ker}\Psi_0 = \emptyset$ we deduce from the standard properties of the localization that $H_1, \ldots, H_g$ generate $\mathrm{Ker}(\Psi_0)$. □

5.3. Simultaneous partial embedded resolution.

We show that the partial embedded resolution of $Z^\Gamma \subset Z_\Delta$ built in Proposition 6 is also a partial embedded resolution of $S \subset Z_\Delta$.

The linear subspace $\ell \subset (N_\Delta)_\mathbb{R}$ orthogonal to $\mathrm{Ker}(\varphi)$ is of dimension d and is also orthogonal to the Minkowski sum of compact edges of $\mathcal{N}(h_i)$ for $i = 1, \ldots, g$.

Lemma 40. — *Let Σ_0 be the smallest subdivision of Δ compatible with the Newton polyhedron of $H_1 \cdots H_g$. The cone $\sigma_0 = \Delta \cap \ell$ belongs to Σ_0. The strict transform S_{Σ_0} of S is defined on the chart Z_{σ_0} by the equations: $U_i^{-n_i} H_i = 0$ for $i = 1, \ldots, g$. The intersection $S_{\Sigma_0} \cap \mathbb{O}_{\sigma_0}$ as schemes is reduced to the simple point o_{σ_0}. The germ $(S_{\Sigma_0}, o_{\sigma_0})$ is isomorphic to the germ of toric variety $Z_{\sigma_0, N_{\sigma_0}}$ at the distinguished point. If Σ is any subdivision of Δ containing the cone σ_0 and if $\sigma \in \Sigma$ with $\mathring{\sigma} \subset \mathring{\Delta}$ then $S_\Sigma \cap \mathbb{O}_\sigma \neq \emptyset$ implies that $\sigma = \sigma_0$. Moreover, if Σ' is a regular subdivision of Σ then the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is an embedded pseudo-resolution of S.*

Proof. — A vector $v \in \mathring{\sigma}_0$ vanish on $\mathrm{Ker}(\varphi_\mathbb{R})$ thus it is of the form $v = \tilde{v} \circ \varphi$ for $\tilde{v} \in N_g$ belonging to $\mathring{\rho}$ since $\tilde{v}$ vanishes only at the vertex of the cone $\rho^\vee$ (this follows from $\varphi_\mathbb{R}^{-1}(\rho^\vee) = \Delta^\vee + \mathrm{Ker}(\varphi_\mathbb{R})$ and $\mathring{\sigma}_0 \subset \mathring{\Delta}$). We deduce from this that the face defined by v on the polyhedron $\mathcal{N}(H_i)$ corresponds to the monomials of weight w such that $\langle \tilde{v}, w \rangle$ is minimal. By Proposition 39 the symbolic restriction of H_i to this face is equal to h_i. Conversely, if h_i is the symbolic restriction of H_i to the face defined by v it follows that $v \in \ell$ and since these are compact faces we have that $v \in \mathring{\Delta}$ thus $v \in \mathring{\sigma}_0$.

The common zero locus S' of the functions $U_i^{-n_i} H_i$ for $i = 1, \ldots, g$ on the chart Z_{σ_0} contains $S_{\Sigma_0} \cap Z_{\sigma_0}$. Then we deduce from the proof of Lemma 3 that

$$U_i^{-n_i} H_i = U_i^{-n_i} h_i + \text{ terms vanishing on the orbit } \mathbb{O}_{\sigma_0}, \quad \text{for } i = 1, \ldots, g. \tag{34}$$

Since the equations $U_i^{-n_i} h_i = 0$ for $i = 1, \dots, g$, define on the chart Z_{σ_0} the strict transform $Z^{\Gamma}_{\Sigma_0}$ we deduce from (34) above that $S' \cap \mathbb{O}_{\sigma_0}$ coincides as schemes intersection with $Z^{\Gamma}_{\Sigma_0} \cap \mathbb{O}_{\sigma_0}$, thus it is equal to the simple point o_{σ_0} by Lemma 5. If the germ (S', o_{σ_0}) is analytically irreducible it must coincide with the sub-germ $(S_{\Sigma_0}, o_{\sigma_0})$ since both are of the same dimension. We show this fact by proving that (S', o_{σ_0}) is isomorphic to $(Z_{\sigma_0, N_{\sigma_0}}, o_{\sigma_0})$:

We notice that the chart Z_{σ_0} is isomorphic to $\mathbb{O}_{\sigma_0} \times Z_{\sigma_0, N_{\sigma_0}}$ by (31). The binomials $W_i := U_i^{-n_i} h_i$ for $i = 1, \dots, g$, define a regular system of parameters at the point o_{σ_0} of the orbit $\mathbb{O}_{\sigma_0}$ therefore we can apply Lemma 4 to the equations (34) to show the existence of $\phi_1, \dots, \phi_g \in \mathbb{C}\{\rho^\vee \cap M_g\}$ such that the germ (S', o_{σ_0}) is given by $W_i = \phi_i$ for $i = 1, \dots, g$.

Let Σ be any subdivision of Σ_0 containing the cone σ_0. The restriction $\pi : S_\Sigma \to S$ of π_Σ is a modification and since (S, o_Δ) is analytically irreducible the exceptional fiber is connected by the Main Theorem of Zariski (see [Mu] and [Z2]). On the other hand we have that

$$\pi^{-1}(o_\Delta) = \bigcup_{\sigma \in \Sigma, \overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}} (S_\Sigma \cap \mathbb{O}_\sigma) \text{ (by (3))};$$

and we have shown that on the open set $S_\Sigma \cap Z_{\sigma_0}$ of S_Σ the exceptional fiber is reduced to the point o_{σ_0}, therefore the exceptional fiber $\pi^{-1}(o_\Delta)$ contains no other points (otherwise would not be a connected set).

If Σ' is a regular subdivision of Σ it follows that $\mathbb{O}_\sigma \cap S_{\Sigma'} \neq \emptyset$ if and only if $\sigma \subset \sigma_0$. Thus we can cover the strict transform with those charts Z_σ for $\sigma \subset \sigma_0$ and $\dim \sigma = \dim \sigma_0$. It follows as in the case when σ_0 is a regular cone, that the strict transform is smooth and transversal to the canonical stratification of the exceptional divisor therefore $\pi_{\Sigma'} \circ \pi_\Sigma$ is a pseudo-resolution. □

THEOREM 2. — *Let Σ be any subdivision of Δ containing the cone σ_0.*

1. *The strict transform S_Σ is a germ at the point o_{σ_0} isomorphic to $(Z_{\sigma_0, N_{\sigma_0}}, o_{\sigma_0})$ and the restriction $\pi_\Sigma | S_\Sigma : S_\Sigma \to S$ is the normalization map.*

2. *The morphism π_Σ is a partial embedded resolution of $S \subset Z_\Delta$.*

Proof. — The first assertion follows from Lemma 40 taking in account (10) (which implies that $(Z_{\sigma_0, N_{\sigma_0}}, o_{\sigma_0})$ is isomorphic to (Z_{ρ, N_g}, o_ρ)) and Proposition 14 which implies that the integral closure of R is $\mathbb{C}\{\rho^\vee \cap M_g\}$.

By Lemma 40 if Σ' is a regular subdivision of Σ the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is a embedded pseudo-resolution of S; we show that if Σ' is a resolution of the fan Σ then the restriction $S_{\Sigma'} \to S$ is a resolution of singularities.

The germ S_Σ is parametrized by $W_i = \phi_i$ for $i = 1, \dots, g$, on the chart $Z_{\sigma_0} \cong \mathbb{O}_{\sigma_0} \times Z_{\sigma_0, N_{\sigma_0}}$ thus the restriction $S_\Sigma \to Z_{\sigma_0, N_{\sigma_0}}$ of the second projection is an isomorphism of germs. It follows that the singular locus of S_Σ lies over the singular locus of the toric variety $Z_{\sigma_0, N_{\sigma_0}}$. It is easy to see that the orbit $\mathbb{O}_\tau$ of Z_{σ_0} is the set lying over the orbit $\mathbb{O}_{\tau, N_{\sigma_0}}$ of $Z_{\sigma_0, N_{\sigma_0}}$ thus the singular locus of S_Σ is equal to $\bigcup (S_\Sigma \cap \mathbb{O}_\tau)$ for τ running through the set of non regular faces of σ_0. If Σ' is a resolution of the fan Σ and if $\sigma' \in \Sigma$ is a regular cone then $\sigma' \in \Sigma'$, thus $Z_\Sigma \to Z_{\Sigma_0}$ is an isomorphism over the points of the orbit $\mathbb{O}_{\sigma'}$ by (4). Therefore the restriction $S_{\Sigma'} \to S_\Sigma$ is an isomorphism outside the singular locus of S_Σ and since $S_{\Sigma'}$ is smooth this modification is a resolution of singularities of the normalization S_Σ. A fortiori the composed map $S_{\Sigma'} \to S_\Sigma$ is resolution of singularities of S. □

5.4. Relation between the partial embedded resolution procedures.

We show that the partial embedded resolutions of an analytically irreducible toric quasi-ordinary germ S defined in Sections 3 and 5 coincide when the second is suitably chosen.

In Section 3 we have built a partial embedded resolution π of a toric quasi-ordinary hypersurface $S \subset Z_\varrho$ which depends only on the characteristic exponents of a toric quasi-ordinary polynomial f defining the embedding. Since the germ S is analytically irreducible, the morphism $\pi : Z' \longrightarrow Z_\varrho = Z_0$ is the composition of g toroidal modifications $\pi_i : Z_i \to Z_{i-1}$ for $i = 1, \dots, g$ and g the number of characteristic exponents. In Section 5 we have built an embedding of (S, o) as a codimension g sub-germ of the toric variety (Z_Δ, o_Δ) and we have proved that if Σ is a subdivision of Δ compatible with certain linear subspace, the toric morphism $\pi_\Sigma : Z_\Sigma \to Z_\Delta$ is partial embedded resolution of $S \subset Z_\Delta$. Furthermore, the restriction of π (resp. of π_Σ) to the strict transform S' (resp. S_Σ) of S is the normalization map (see Theorems 1 and 2).

The embedding $S \subset Z_\Delta$ defined by (27) extends to an embedding of the pair (S, Z_ϱ): the image of (Z_ϱ, o_ϱ) under this embedding is the sub-germ $(\mathcal{Z}, o_\Delta)$ of (Z_Δ, o_Δ) defined by the equations (see (33)):

(35)

$$\begin{cases} -c_1^* U_2 = U_1^{n_1} - X^{\alpha^{(1)}} + r_1(U_1), \\ -c_2^* U_3 = U_2^{n_2} - X^{\alpha^{(2)}} U_1^{l_1^{(2)}} + r_1(U_1, U_2), \\ \cdots \\ -c_{g-1}^* U_g = U_{g-1}^{n_{g-1}} - X^{\alpha^{(g-1)}} U_1^{l_1^{(g-1)}} \cdots U_{g-2}^{l_{g-2}^{(g-1)}} + r_{g-1}(U_1, U_2, \ldots, U_{g-1}). \end{cases}$$

Since $c_i^* \in \mathbb{C}^*$ we can eliminate the variables $U_2, \ldots, U_g$ in the equation:

$$U_g^{n_g} - X^{\alpha^{(g)}} U_1^{l_1^{(g)}} \cdots U_{g-1}^{l_{g-1}^{(g)}} + r_g(U_1, U_2, \ldots, U_g) = 0 \tag{36}$$

by using (35), and we obtain in this way a quasi-ordinary polynomial defining the embedding $S \subset Z_\varrho$.

Remark 41. — If we vanish the $r_1, \ldots, r_g$ in (33) we obtain:

$$\begin{cases} \tilde{H}_1 := U_1^{n_1} - X^{\alpha^{(1)}} + c_1^* U_2, \\ \tilde{H}_2 := U_2^{n_2} - X^{\alpha^{(2)}} U_1^{l_1^{(2)}} + c_2^* U_3, \\ \cdots \\ \tilde{H}_{g-1} := U_{g-1}^{n_{g-1}} - X^{\alpha^{(g-1)}} U_1^{l_1^{(g-1)}} \cdots U_{g-2}^{l_{g-2}^{(g-1)}} + c_{g-1}^* U_g, \\ \tilde{H}_g := U_g^{n_g} - X^{\alpha^{(g)}} U_1^{l_1^{(g)}} \cdots U_{g-1}^{l_{g-1}^{(g)}}. \end{cases} \tag{37}$$

We can eliminate recursively from the equations $\tilde{H}_i = 0$, for $i = 1, \ldots, g-1$ the variables $U_2, \ldots, U_g$ in the equation $\tilde{H}_g = 0$ obtaining in this way a canonical equation of a quasi-ordinary hypersurface with the same characteristic monomials. The exponents appearing in these polynomials are completely determined by the characteristic monomials of $(S, 0)$. See [T1] for Teissier's analogous statement in the case of plane branches.

DEFINITION 12. — *A subdivision Σ of Δ is suitable with respect to the embedding of the pair $(\mathcal{Z}, S)$ in Z_Δ, if it is the dual Newton diagram of $\tilde{H}_1 \cdots \tilde{H}_g$.*

It follows from Remark 41 that the suitable subdivision Σ of Δ is uniquely determined from the given characteristic monomials of $(S, 0)$. We prove that the strict transform $\mathcal{Z}_\Sigma$ of $\mathcal{Z}$ by the toric modification π_Σ is a *section* of the toric variety $\mathcal{Z}_\Sigma$, transversal to the exceptional fiber of the modification π_Σ. More generally it is transversal to the orbit stratification of $\mathcal{Z}_\Sigma$ and the set of non empty intersections $\mathcal{Z}_\Sigma \cap \mathbb{O}_\sigma$ define the stratification corresponding to a natural toroidal embedding structure which is determined by Σ. In particular we obtain that the restriction

$p : \mathcal{Z}_\Sigma \to \mathcal{Z}$ of π_Σ to $\mathcal{Z}_\Sigma$ is a partial embedded resolution of $S \subset \mathcal{Z}$. The main result of this section is that the partial embedded resolutions defined by π and by p are isomorphic:

Theorem 3. — *If Σ is the suitable subdivision of Δ with respect to the embedding of the pair $(\mathcal{Z}, S)$ in Z_Δ, then the strict transform $(\mathcal{Z}_\Sigma, S_\Sigma)$ of the pair $(\mathcal{Z}, S)$ by the toric modification π_Σ is equal to an embedding of the pair (Z', S') in Z_Σ such that the following diagram commutes:*

$$\begin{array}{ccc} (Z', S) & \longrightarrow & (\mathcal{Z}_\Sigma, S_\Sigma) \\ \pi \downarrow & & p \downarrow \\ (Z_\varrho, S) & \longrightarrow & (\mathcal{Z}, S). \end{array}$$

Therefore the morphism $p : \mathcal{Z}_\Sigma \to \mathcal{Z}$ is the composition of g toroidal modifications.

In the plane branch case an analogous statement (using resolution instead of partial resolution) has been announced by Goldin and Teissier without proof in [G-T]; Lejeune and Reguera have sketched in that case toric resolutions of the monomial curve such that the restrictions to the strict transform of the smooth surface, which contains the re-embeded plane branch, are equal to the minimal resolution of the branch (see [LJ-R2]).

We introduce first some notations in order to describe the *suitable* subdivision Σ of Δ. The following subsets of Δ defined for $0 \leqslant j < j+k \leqslant g$:

$$(38) \qquad \begin{aligned} \rho_j^{j+k} = \{ & a + \langle a, \gamma_1 \rangle u_1 + \cdots + \langle a, \gamma_j \rangle u_j + n_j \langle a, \gamma_j \rangle u_{j+1} + \cdots \\ & + n_j \cdots n_{j+k-1} \langle a, \gamma_j \rangle u_{j+k} / a \in \rho \} \end{aligned}$$

are the cones which correspond by duality to certain Minkowski sums of edges of $\mathcal{N}(\tilde{H}_i)$ for $i = 1, \ldots, k$. The cone ρ_0^k coincides with $\rho \times \{0\} \subset \Delta$ for $1 \leqslant k \leqslant g$. We denote by Ξ the $(d+1)$-dimensional fan whose elements are the faces of the $2g$ cones of dimension $d+1$:

$$(39) \qquad \rho_j^g + \mathbb{R}_{\geqslant 0} u_j, \quad \rho_j^g + \rho_{j-1}^g \quad \text{for } j = 1, \ldots, g.$$

We will show below that Ξ is a subfan of the suitable subdivision Σ (see Remark 44).

Proposition 42. — *Let Σ a suitable subdivision of Δ. If $\sigma \in \Sigma$ and if $\overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}$ then $\mathcal{Z}_\Sigma \cap \mathbb{O}_\sigma \neq \emptyset$ implies that $\sigma \in \Xi$. If $\sigma \in \Xi^{(d+1)}$ then $\mathcal{Z}_\Sigma \cap \mathbb{O}_\sigma$ is reduced to a simple point x_σ and the germ $(\mathcal{Z}_\Sigma, x_\sigma)$ is isomorphic to $(Z_{\sigma,(N_\Delta)_\sigma}, o_\sigma)$. The set $\{\mathcal{Z}_\Sigma \cap \mathbb{O}_\sigma\}_{\sigma \in \Xi}$ is the stratification associated to* *a toroidal embedding structure on $\mathcal{Z}_\Sigma$ which has Ξ as associated conic polyhedral complex.*

In order to prove Proposition 42 we characterize in the lemma below some convexity properties of the the Newton polyhedra of the polynomials $H_1, \ldots, H_{g-1}$ defining the embedding $\mathcal{Z} \subset Z_\Delta$. Lemma 43 below is inspired by a result of Lejeune and Reguera in the case of sandwiched surface singularities (see Proposition 1.3 of [LJ-R]). We need some useful notations. The exponents

$$u_{j+1}^*,\, n_j u_j^*,\, \varpi_j := \alpha^{(j)} + l_1^{(j)} u_1^* + \cdots + l_{j-1}^{(j)} u_{j-1}^*$$

are the vertices of the two dimensional face $\mathcal{T}^j$ of the polyhedron $\mathcal{N}(H_j)$ by Proposition 39. This face and its edges

$$\mathcal{T}_1^j := [u_{j+1}^*, n_j u_j^*],\, \mathcal{T}_2^j := [u_{j+1}^*, \varpi_j],\, \mathcal{T}_3^j := [n_j u_j^*, \varpi_j],$$

play a significant role in what follows. Any other vertex ϖ_j' of the Newton polyhedron of H_j corresponds to a monomial of weight $> n_j\gamma_j$, i.e., we have $\varpi_j' = \alpha' + l_1 u_1^* + \cdots + l_j u_j^*$ and $\alpha' + l_1\gamma_1 + \cdots + l_j\gamma_j > n_j\gamma_j$.

We prove the following lemma by using the properties of the semi-group Γ (see lemma 30).

LEMMA 43. — *Let $\mathcal{E}_i$ be a compact edge of $\mathcal{N}(H_i)$ for $i = 1, \ldots, g$. If $\cap_{i=1}^j \overset{\circ}{\sigma}(\mathcal{E}_i) \neq \emptyset$ for $1 \leqslant j \leqslant g$, then we have $\mathcal{E}_i$ is an edge of $\mathcal{T}^i$, of the form $\mathcal{E}_i = \mathcal{T}^i_{s(i)}$, for certain $s(i) \in \{1, 2, 3\}$. The intersection $\cap_{i=1}^j \sigma(\mathcal{E}_i)$ and the possibilities for $s(i)$ are described in the following four cases:*

$$\textit{Case } (A) \begin{cases} \cap_{i=1}^j \sigma(\mathcal{E}_i) = \rho_j^{j+1} + \mathbb{R}_{\geqslant 0} u_{j+1} + \mathbb{R}_{\geqslant 0} u_{j+2} + \ldots + \mathbb{R}_{\geqslant 0} u_g, \\ s(i) = 3, \text{ for } 1 \leqslant i \leqslant j \leqslant g-1. \end{cases}$$

$$\textit{Case } (B) \begin{cases} \cap_{i=1}^j \sigma(\mathcal{E}_i) = \rho_{j_0}^{j+1} + \mathbb{R}_{\geqslant 0} u_{j_0} + \mathbb{R}_{\geqslant 0} u_{j+2} + \ldots + \mathbb{R}_{\geqslant 0} u_g, \\ s(i) = 3, \text{ for } 1 \leqslant i \leqslant j_0 - 1, \\ s(i) = 1, \text{ for } j_0 \leqslant i \leqslant j \leqslant g-1; \\ \text{where } 1 \leqslant j_0 \leqslant g-1. \end{cases}$$

$$\textit{Case } (C) \begin{cases} \cap_{i=1}^j \sigma(\mathcal{E}_i) = \rho_{j_0}^{j+1} + \rho_{j_0-1}^{j+1} + \mathbb{R}_{\geqslant 0} u_{j+2} + \ldots + \mathbb{R}_{\geqslant 0} u_g, \\ s(i) = 3, \ \textit{for } 1 \leqslant i \leqslant j_0, \\ s(j_0) = 2, \\ s(i) = 1, \text{ for } j_0 + 1 \leqslant i \leqslant j \leqslant g-1; \\ \text{where } 1 \leqslant j_0 \leqslant g-1. \end{cases}$$

$$\textit{Case } (D) \begin{cases} \cap_{i=1}^j \sigma(\mathcal{E}_i) = \rho_g^g, \\ s(i) = 3, \text{ for } 1 \leqslant i \leqslant j = g. \end{cases}$$

Proof. — The compact faces of Newton polyhedra are determined by elements $a+v \in \Delta$ which belong to the relative interior of Δ; i.e., $a+v$ is of the form $a \in \overset{\circ}{\rho}$ and $v = \sum_{i=1}^{g} v_i u_i$ with $v_i > 0$. We calculate the values of $a+v$ on the vertices of the Newton polyhedron of H_j in terms of the weight of the corresponding monomial. We prove the lemma by induction on j, for $j = 1$ we show first that the compact edges of $\mathcal{N}(H_1)$ are exactly $\mathcal{T}_i^1$ for $i = 1, 2, 3$. We have the following:

$$\begin{array}{rclcl}
\text{(i)} & = & \langle a+v, \varpi_1 & = & \langle a, \alpha^{(1)} \rangle = n_1 \langle a, \gamma_1 \rangle \\
\text{(ii)} & = & \langle a+v, n_1 u_1^* \rangle & = & n_1 v_1 \\
\text{(iii)} & = & \langle a+v, u_2^* \rangle & = & v_2 \\
\text{(iv)} & = & \langle a+v, \varpi_1' \rangle & = & \langle a+v, \alpha_1 + l_1 u_1^* \rangle \\
 & & & = & \langle a, \alpha' + l_1 \gamma_1 \rangle + l_1 (v_1 - \langle a, \gamma_1 \rangle) > (n_1 - l_1) \langle a, \gamma_1 \rangle + l_1 v_1
\end{array}$$

where the inequality on (iv) follows from (26) since $a \in \overset{\circ}{\rho}$. We suppose that $a+v$ determines a compact edge e_1 of $\mathcal{N}(H_1)$. Three cases appear:

– If $v_1 = \langle a, \gamma_1 \rangle$ then (iv) > (i) = (ii) thus $v_2 > n_1 \langle a, \gamma_1 \rangle$ and $\mathcal{E}_1 = \mathcal{T}_3^1$.

– If $v_1 > \langle a, \gamma_1 \rangle$ then (ii), (iv) > (i) thus $v_2 = n_1 \langle a, \gamma_1 \rangle$ and $\mathcal{E}_1 = \mathcal{T}_2^1$.

– If $v_1 < \langle a, \gamma_1 \rangle$ then (i), (iv) > (ii) thus $v_2 = n_1 v_1$ and $\mathcal{E}_1 = \mathcal{T}_1^1$.

The equality (i) = (ii) = (iii) corresponds to the two dimensional face $\mathcal{T}^1$. It follows that

$$\sigma(\mathcal{E}_1) = \begin{cases} \rho_1^2 + \mathbb{R}_{\geqslant 0} u_2 + \mathbb{R}_{\geqslant 0} u_3 + \cdots + \mathbb{R}_{\geqslant 0} u_g, & \text{if } \mathcal{E}_1 = \mathcal{T}_3^1 \\ \rho_1^2 + \mathbb{R}_{\geqslant 0} u_1 + \mathbb{R}_{\geqslant 0} u_3 + \cdots + \mathbb{R}_{\geqslant 0} u_g, & \text{if } \mathcal{E}_1 = \mathcal{T}_2^1 \\ \rho_1^2 + \rho_0^2 + \mathbb{R}_{\geqslant 0} u_3 + \cdots + \mathbb{R}_{\geqslant 0} u_g, & \text{if } \mathcal{E}_1 = \mathcal{T}_1^1. \end{cases}$$

We suppose the result true for $j-1$. We consider a vector $a+v \in \bigcap_{i=1}^{j-1} \overset{\circ}{\sigma}(\mathcal{T}_{s(i)}^i)$ determining an edge $\mathcal{E}_j$ of $\mathcal{N}(H_j)$, i.e., $a+v \in \overset{\circ}{\sigma}(\mathcal{E}_j)$. The values of $a+v$ on the vertices of $\mathcal{T}_1^j$ are

$$\begin{array}{rcl}
\text{(ii)} & = & \langle a+v, n_j u_j^* \rangle = n_j v_j \\
\text{(iii)} & = & \langle a+v, u_{j+1}^* \rangle = v_{j+1}.
\end{array}$$

We deal first with the case (A) where $s(i) = 3$ for $1 \leqslant i \leqslant j-1$. Then $a+v \in \rho_{j-1}^j$ and it follows as before that

$$\begin{array}{rcl}
\text{(i)} & = & \langle a+v, \varpi_j \rangle = n_j \langle a, \gamma_j \rangle \\
\text{(iv)} & = & \langle a+v, \varpi_j' \rangle = \langle a+v, \alpha_j + \sum_{i=1}^{j} l_i u_i^* \rangle > n_j \langle a, \gamma_j \rangle + l_j (v_j - \langle a, \gamma_j \rangle)
\end{array}$$

where the inequality is obtained from (26) by adding and subtracting the term $l_j\langle a,\gamma_j\rangle$. Three cases appear if v_j is $=$ (resp. $>$ or $<$) to (resp. than) $\langle a,\gamma_j\rangle$ and we obtain the result by arguing as in the case $j=1$ by replacing appropriately the index 1 by j.

In any other case by induction hypothesis there exists $1\leqslant j_0\leqslant j-1$ such that $s(i)=3$ for $1\leqslant i<j_0$ and $s(j_0)\in\{1,2\}$. It follows that the vector $a+v$ is of the form

$$a+v=a+\sum_{i=1}^{j_0}\langle a,\gamma_i\rangle u_i+\sum_{i>j_0}^{g} v_iu_i.$$

We bound the value of $a+v$ on a vertex of the polyhedron $\mathcal{N}(H_j)$ not lying on $\mathcal{T}_1^j$.

$$\begin{aligned}
\text{(iv)} &= \langle a+v,\varpi'_j\rangle = \langle a+v,\alpha_j+\sum_{i=1}^{j} l_iu_i^*\rangle\\
&= \langle a,\alpha_j+\sum_{i=1}^{j} l_i\gamma_i\rangle-\sum_{i=j_0}^{j} l_i\langle a,\gamma_i\rangle+\sum_{i=j_0}^{j} l_iv_i\\
&> (n_j-l_j)\langle a,\gamma_j\rangle-\sum_{i=j_0}^{j-1} l_i\langle a,\gamma_i\rangle+\sum_{i=j_0}^{j} l_iv_i\ >\cdots\\
&> ((\cdots((n_j-l_j)n_{j-1}-l_{j-1})\cdots)n_{j_0}-l_{j_0})\,\langle a,\gamma_{j_0}\rangle+\sum_{i=j_0}^{j} l_iv_i.
\end{aligned}\tag{40}$$

The first inequality is given by (26) and the others are deduced from the inequality $n_i\gamma_i<\gamma_{i+1}$ in Lemma 30.

In case (B) by induction hypothesis we have that $v_{j_0}=\langle a,\gamma_{j_0}\rangle+c$, for some $c>0$ and $v_i=n_{j_0}\cdots n_{i-1}\langle a,\gamma_{j_0}\rangle$ for $j_0<i\leqslant j$. In case (C) we have that

$$\begin{cases} n_{j_0-1}\langle a,\gamma_{j_0-1}\rangle < v_{j_0} < \langle a,\gamma_{j_0}\rangle & \text{if } j_0>1\\ 0 < v_{j_0} < \langle a,\gamma_{j_0}\rangle & \text{if } j_0=1\end{cases}$$

and that $v_j=n_{j_0}\cdots n_{i-1}v_{j_0}$ for $j_0<i\leqslant j$. In both cases (B) and (C) when we substitute the v_i on (40) we deduce that (iv), (i) $>$ (ii) therefore $v_{j+1}=n_jv_j$ and $\mathcal{E}_j=\mathcal{T}_1^j$.

Finally, when $j=g$ the polynomial H_g has no term in U_{g+1}. In particular a vector $a+v\in\bigcap_{i=1}^{g-1}\circ\sigma(\mathcal{T}^i_{s(i)})$ for $s(i)$ in case (B) or (C), determines the vertex $n_gu_g^*$ of the polyhedron $\mathcal{N}(H_g)$. The only remaining case is (A) and then the condition on $a+v$ to determine a compact edge of $\mathcal{N}(H_g)$ is $v_g = \langle a, \gamma_g \rangle$; the edge is equal to $\mathcal{T}_3^g = [\varpi, n_g u_g^*]$ and $\bigcap_{i=1}^g = \rho_g^g$. □

Remark 44. — The cones of the form $\cap_{i=1}^{g-1} \sigma(\mathcal{T}_{s(i)}^i)$ defined by Lemma 43 when $j = g - 1$ are

$$\begin{cases} \rho_k^g + \mathbb{R}_{\geqslant 0} u_k, \ \rho_k^g + \rho_{k-1}^g & \text{for } k = 1, \dots, g-1. \\ \rho_{g-1}^g + \mathbb{R}_{\geqslant 0} u_g. \end{cases}$$

If we subdivide $\rho_{g-1}^g + \mathbb{R}_{\geqslant 0} u_g$ with ρ_g^g we obtain the fan Ξ. It follows that Ξ is a subfan of the dual Newton diagram of $\tilde{H}_1 \dots \tilde{H}_g$. Theorem 3 holds more generally for any subdivision of Δ containing Ξ.

Proof of Proposition 42. — Let Σ' be any subdivision of Σ which is compatible with the Newton polyhedra of $H_1, \dots, H_{g-1}$ and $\sigma \in \Sigma'$ with $\mathring{\sigma} \subset \mathring{\Delta}$. By Lemma 3 a necessary condition to have $\mathcal{Z}_{\Sigma'} \cap \mathbb{O}_\sigma \neq \emptyset$ is that σ determines a face $\mathcal{F}_i$ of dimension $\geqslant 1$ of each polyhedron $\mathcal{N}(H_i)$ for $i = 1, \dots, g-1$. Then we have $\sigma \subset \bigcap_{i=1}^{g-1} \sigma(\mathcal{F}_i) \subset \bigcap_{i=1}^{g-1} \sigma(\mathcal{E}_i)$ for $\mathcal{E}_i$ any fixed edge of the face $\mathcal{F}_i$. The possible edges $\mathcal{E}_i$ that may appear are determined by Lemma 43 and by duality σ is contained in the support of Ξ. By using (3) we deduce that if $\sigma \in \Sigma - \Xi$ then $\mathbb{O}_\sigma \cap \mathcal{Z}_\Sigma = \emptyset$.

The proof of the second assertion is analogous to the proof of Lemma 40. Let $\sigma \in \Xi^{(d+1)}$, for instance $\sigma = \rho_j^g + \mathbb{R}_{\geqslant 0} u_j$ (the proof in the case $\sigma = \rho_{j-1}^g + \rho_j^g$ is analogous) for $j = 1, \dots, g$. The common zero locus $\mathcal{Z}' \subset Z_\sigma$ of the set functions $X^{-m_1} H_1, \dots, X^{-m_{g-1}} H_{g-1}$ for

$$m_i = \begin{cases} \varpi_i & \text{if } i = 1, \dots, j-1 \\ u_{i+1}^* & \text{if } i = j, \dots, g-1 \end{cases}$$

contains $\mathcal{Z}_\Sigma \cap Z_\sigma$. Each series $X^{-m_i} H_i$ is of the form

$$X^{-m_i} H_i = B_i + \text{ terms vanishing on } \mathbb{O}_\sigma$$

where

$$B_i = \begin{cases} 1 - X^{n_i u_i^* - m_i} & \text{if } i = 1, \dots, j-1 \\ c_i^* + X^{n_i u_i^* - m_i} & \text{if } i = j, \dots, g-1. \end{cases}$$

The edge $\mathcal{E}_i := [n_i u_i^*, m_i]$ is a face of the polyhedron $\mathcal{N}(H_i)$ and by Lemma 43 we have $\sigma = \bigcap_{j=1}^{g-1} \sigma(\mathcal{E}_i)$ thus $\sigma^\perp = \oplus_{j=1}^{g-1} (\sigma(\mathcal{E}_i))^\perp$ since the edges $\mathcal{E}_i$ are affinely independent. Moreover, the vector $n_i u_i^* - m_i$ is primitive for the lattice M_Δ and it follows that $\sigma^\perp \cap M_\Delta = \oplus_{i=1}^{g-1} (n_i u_i^* - m_i)\mathbb{Z}$. It follows that the intersection $\mathcal{Z}' \cap \mathbb{O}_\sigma$ as schemes, defined by the equations $B_1 = \cdots = B_{g-1} = 0$, is a simple point x_σ and that $B_1, \ldots, B_{g-1}$ define a regular system of parameters at the point x_σ of $\mathbb{O}_\sigma$. The germ $(\mathcal{Z}', x_\sigma)$ is analytically irreducible since it is isomorphic to $(Z_{\sigma,(N_\Delta)_\sigma}, o_\sigma)$ by Lemma 4. It follows that it coincides with $(\mathcal{Z}_\Sigma, x_\sigma)$ since this germ is contained in $(\mathcal{Z}', x_\sigma)$ and both have the same dimension. Moreover, if τ is a face of σ then the isomorphism above induces an isomorphism between $\mathcal{Z}_\Sigma \cap \mathbb{O}_\tau$ and the orbit corresponding to τ in $Z_{\sigma,(N_\Delta)_\sigma}$. We conclude from this that $\mathcal{Z}_\Sigma$ has a toroidal embedding structure with associated c.p.c. Ξ. □

We recall some facts and notations about the partial embedded resolution of as an hypersurface (see Theorem 1). Denote by (ϱ_i, N'_i) the dual of the pair $(\rho \times \mathbb{R}_{\geqslant 0} y_i, M'_i)$ where M'_i denotes the lattice $M_i \oplus y_i\mathbb{Z}$; each ϱ_i is of the form $\rho \times \mathbb{R}_{\geqslant 0}$, for $i = 0, \ldots, g-1$. The partial embedded resolution is a composition of g toroidal modifications $\pi_i : Z_i \to Z_{i-1}$ for $Z_0 = Z_\varrho$ and $i = 1, \ldots, g$. Each variety Z_i is given with a toroidal embedding structure having c.p.c. Σ_i. The c.p.c. Σ_1 is isomorphic to the subdivision of (ϱ, N'_0) by the linear form $n_1(y_0 - \lambda_1) \in M'_0$. The c.p.c. Σ_j is obtained from Σ_{j-1} by adding the subdivision of the cone ϱ_{j-1} defined by $n_j(y_{j-1} - \lambda_j + \lambda_{j-1}) \in M'_{j-1}$; this subdivision has $(d+1)$-dimensional cones:

$$\sigma_j^- = \{(a,v) \in \varrho_{j-1}/0 \leqslant v \leqslant \langle a, \lambda_j - \lambda_{j-1}\rangle\}$$
$$\text{and} \quad \sigma_j^+ = \{(a,v) \in \varrho_{j-1}/v \geqslant \langle a, \lambda_j - \lambda_{j-1}\rangle\}. \tag{41}$$

It is glued to Σ_{j-1} by identifying the face $\rho \times \{0\}$ of σ_j^- with $\sigma_{j-1}^+ \cap \sigma_{j-1}^-$ (see Lemma 25).

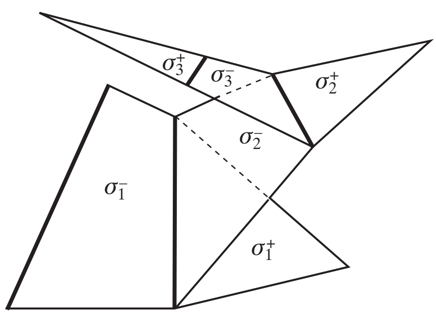

Figure 1. A transversal section of the convex polyhedral complex associated to a quasi-ordinary surface with three characteristic exponents.

Proposition 45. — *There is an isomorphism $\Sigma_g \cong \Xi$ of conic polyhedral complex with integral structure.*

Proof. — It is sufficient to prove that the pair $(\sigma, (N'_{j-1})_\sigma)$ is isomorphic to $(\tau, (N_\Delta)_\tau)$ when (σ, τ) is equal to $(\sigma_j^-, \rho_j^g + \rho_{j-1}^g)$ or to $(\sigma_j^+, \rho_j^g + \mathbb{R}_{\geqslant 0} u_j)$.

In the first case we define an homomorphism $\xi : N'_{j-1} \to N_\Delta$ by

$$(a, v) \mapsto a + \sum_{i=1}^{j-1} \langle a, \gamma_{j-1} \rangle u_i + \sum_{i \geqslant j} n_j \cdots n_{i-1} w(a, v) u_i \tag{42}$$

where $w(a, v) = v + n_{j-1} \langle a, \gamma_{j-1} \rangle$. It follows that $\xi_{\mathbb{R}}(\sigma_j^-) = \rho_j^g + \rho_{j-1}^g$ since $(a, v) \in \sigma_j^-$ implies that $n_{j-1} \langle a, \gamma_{j-1} \rangle \leqslant w(a, v) \leqslant \langle a, \gamma_j \rangle$ by (22).

In the second case we have that $N'_{j-1} = N_j \oplus y_{j-1}^* \mathbb{Z}$ (this follows from Lemma 17: the inclusion $N_j \hookrightarrow N'_{j-1}$ is dual to the homomorphism $M'_{j-1} \to M_j$ that maps $y_{j-1} \mapsto \lambda_j - \lambda_{j-1}$ and fixes M_{j-1}). Thus we have $(N'_{j-1})_{\mathbb{R}} = (N_j)_{\mathbb{R}} \oplus y_{j-1}^* \mathbb{R}$ and with this decomposition the cone σ_j^+ is also defined by the formula (41) above. Then we argue analogously, the corresponding lattice homomorphism ξ is defined by (42) when $w(a, v) = v + \langle a, \gamma_j \rangle$. It follows from (22) that $\xi_{\mathbb{R}}(\sigma_j^+) = \rho_j^g + \mathbb{R}_{\geqslant 0} u_j$. □

We have all the ingredients to prove Theorem 3.

Proof of Theorem 3. — The intersection of $\mathcal{Z}$ with the torus of Z_Δ is isomorphic with Z_ϱ minus the hypersurface defined by $q_0 \dots q_{g-1} = 0$. It follows from their definitions that the morphisms π and p are isomorphic over this set which is the open stratum of the stratification of $\mathcal{Z}'$ and $\mathcal{Z}_\Sigma$.

Let o_τ (resp. $o_{\tau'}$) be 0-dimensional stratum of $\mathcal{Z}'$ (resp. of $\mathcal{Z}_\Sigma$) associated to the cones $\tau \in \Sigma^{(d+1)}$ (resp. $\tau' \in \Xi^{(d+1)}$). If τ corresponds to τ' by the bijection established in Proposition 45 we can extend the isomorphism from the open strata to an isomorphism $(Z', o_\tau) \to (\mathcal{Z}_\Sigma, o_{\tau'})$ by means of Proposition 45 and inducing isomorphisms between the strata of dimensions $0 \leqslant k < d + 1$ associated with corresponding faces of τ and τ'. These implies that these local isomorphism paste and provide an isomorphism $I : Z' \longrightarrow \mathcal{Z}_\Sigma$ which preserves the toroidal embedding structure. Since $p \circ I = \pi$ it follows that the isomorphism above is in fact an isomorphism of the pairs (Z', S') and $(\mathcal{Z}_\Sigma, S_\Sigma)$. □

5.5. An example.

We build an example for the quasi-ordinary surface germ S defined by $f = 0$ where $f = (Y^2 - X_1^3)^2 - X_1^4 X_2 Y^2$. The polynomial $f \in \mathbb{C}\{X_1, X_2\}[Y]$ is quasi-ordinary and irreducible. The characteristic exponents and integers are $\lambda_1 = (\frac{3}{2}, 0)$, $\lambda_2 = (2, \frac{1}{2})$ and $n_1 = n_2 = 2$. The associated semigroup is $\Gamma = \mathbb{Z}^2_{\geqslant 0} + \gamma_1 \mathbb{Z}_{\geqslant 0} + \gamma_2 \mathbb{Z}_{\geqslant 0}$ where $\gamma_1 = (\frac{3}{2}, 0)$ and $\gamma_2 = (\frac{7}{2}, \frac{1}{2})$.

The embedding of S in $\mathbb{C}^4$ is defined by the vanishing of the polynomials

$$H_1 := U_1^2 - X_1^3 - U_2, \quad H_2 := U_2^2 - X_1^4 X_2 U_1^2,$$

where $U_1 = Y$ and $U_2 = Y^2 - X_1^3$. We denote the coordinates of a vector in Δ with respect to the canonical basis by (v_1, v_2, w_1, w_2); (the cone ρ_0^g corresponds to $w_1 = w_2 = 0$ and we have $u_1 = (0,0,1,0)$ and $u_2 = (0,0,0,1)$ with the notations of the previous section). We denote by ℓ_2 the linear subspace orthogonal to the compact edge of $\mathcal{N}(H_2)$, by δ_1 the cone $\Delta \cap \ell_2 \cap \{v_1 = v_2 = 0\}$ and by δ_2 the cone $\Delta \cap \ell_2 \cap \{w_1 = 0\}$.

A suitable subdivision Σ of Δ has 4-dimensional cones:

$$\begin{aligned}
&\rho_2^2 + \delta_1 + u_2 \mathbb{R}_{\geqslant 0},\\
&\rho_2^2 + \delta_2 + u_2 \mathbb{R}_{\geqslant 0},\\
&\rho_2^2 + \rho_1^2 + \delta_1 + u_1 \mathbb{R}_{\geqslant 0},\\
&\rho_2^2 + \rho_1^2 + \rho_0^2 + \delta_2,\\
&\rho_1^2 + \rho_0^2 + u_1 \mathbb{R}_{\geqslant 0}.
\end{aligned}$$

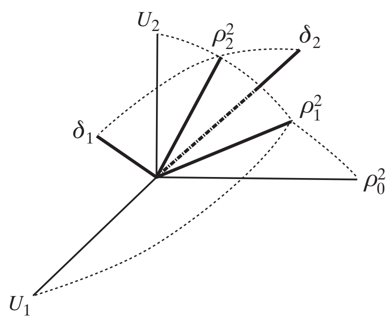

Figure 2. The diagram represents the suitable fan Σ.

We have (see formula (38)):

$$\begin{aligned}
\rho_1^2 &= \mathbb{R}_{\geqslant 0}(2,0,3,6) + \mathbb{R}_{\geqslant 0}(0,1,0,0)\\
\rho_2^2 &= \mathbb{R}_{\geqslant 0}(2,0,3,7) + \mathbb{R}_{\geqslant 0}(0,2,0,1)\\
\delta_1 &= \mathbb{R}_{\geqslant 0}(0,0,1,1)\\
\delta_2 &= \mathbb{R}_{\geqslant 0}(1,0,0,2) + \mathbb{R}_{\geqslant 0}(0,2,0,1).
\end{aligned}$$

The cone ρ_2^2 is regular, the normalization of the quasi-ordinary surface being smooth in this example. If Σ' is any resolution of the fan Σ it follows that the cone ρ_2^2 belongs to Σ and the strict transform of S by $\pi_{\Sigma'}$ only intersects the exceptional orbit corresponding to this cone.

We build a regular cone $\sigma \supset \rho_2^2$ of dimension four, which belongs to some resolution Σ', and we compute the strict transform of S by the toric morphism on the chart Z_σ. The strategy to build σ is to find a basis of the lattice $\ell_2 \cap \mathbb{Z}^4$ and then to use the equation of the hyperplane ℓ_2 to find a basis of $\mathbb{Z}^4$.

We find in this case

$$\sigma = \mathbb{R}_{\geqslant 0}(2,0,3,7) + \mathbb{R}_{\geqslant 0}(0,2,0,1) + \mathbb{R}_{\geqslant 0}(1,0,2,4) + \mathbb{R}_{\geqslant 0}(2,1,3,8),$$

the first three vectors defining a basis of $\ell_2 \cap \mathbb{Z}^4$. The toric morphism $Z_{\Sigma'} \to \mathbb{C}^4$ on the chart is given by (see (2)):

$$\begin{array}{rcl} X_1 & = & V_1^2 V_3^2 V_4 \\ X_2 & = & V_2^2 V_3 \\ U_1 & = & V_1^3 V_3^3 V_4^2 \\ U_2 & = & V_1^3 V_2^7 V_3^8 V_4^4. \end{array}$$

The total transform of S is defined by

$$V_1^6 V_3^6 V_4^3 \left(V_4 - 1 - V_1 V_2 V_3^2 V_4\right) = 0$$
$$V_1^{14} V_2^2 V_3^{15} V_4^8 \left(V_3 - 1\right) = 0.$$

The strict transform, defined by the vanishing of $V_4 - 1 - V_1 V_2 V_3^2 V_4$ and $V_3 - 1$, is clearly smooth and transversal to the exceptional divisor.

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