TAMPERE UNIVERSITY OF TECHNOLOGY

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LOGIC FILTERS

Master of Science Thesis

Examiner: professor Esko Turunen
Examiner and topic approved by the Computing and Electrical Engineering Faculty Council Meeting
November 5th, 2014

ABSTRACT

TAMPERE UNIVERSITY OF TECHNOLOGY
Master's Degree Programme in Information Technology
SHCHERBAN, ANASTASIA: Logic Filters
Master of Science Thesis, 51 pages
October 2014
Major: Mathematics
Examiner: professor Esko Turunen, Ph.D
Keywords: Lindenbaum algebra, BL-algebra, MV-algebra, lattice filters, deductive system, quotient algebra

Being a two valued logic, classical logic, i.e. Boolean logic associates with each proposition one of the two values: true or false. In contrast to the classical logics many-valued logics were introduced. In many valued logics any proposition could have various values for the truth, from total false to complete truth. Polish mathematicians Adolf Lindenbaum (1904-1941) and Alfred Tarski (1901-1983) investigated an approach for establishing correspondence between logics and an algebraic structures. First logic for which such a correspondence was established was a Boolean algebra that models a classical logic. Similarly, BL-algebras are algebraic structures associated with basic fuzzy logic. MV-algebras rise as Lindenbaum algebras from the Łukasiewicz logic, G-algebras rise from Gödel logic and eventually, product algebras rise from the product logic.

The notion of the construction of the Lindenbaum-Tarski algebra is related to the factoring the algebra with the congruence relation. Based on this idea we can derive a quotient algebra for the particular algebraic structure. In our thesis we study quotient algebras generated by the logic filters and deductive systems. Additionally, we study different types of filters, their properties and conditions under which they correspond to each other.

We begin by introducing algebraic structures and their properties. Then we describe filters and their properties. Additionally we talk about the deductive systems and their similarity with the lattice filters and filters (of BL-algebras). Moreover, we observe that deductive systems and filters coincide in the BL-algebras. Next by introducing the notion of the quotient algebra we observe the algebraic structures as a quotient algebras generated by the filters and deductive systems. We show that filter of the BL-algebra generates a quotient algebra, which is a BL-algebra [6], maximal deductive system of a BL-algebra generates a quotient algebra, which is an MV-algebra [6]. We finish with observing the implicative filters. We also show other results of the quotient algebras generated by filters and deductive systems from [6], [5] and [7].

PREFACE

This work has been conducted at the Department of Mathematics of Tampere University of Technology. I would like to express the deep gratitude to my supervisor Professor Esko Turunen for the guidance, for his comments and encouragement throughout the whole process of writing thesis. I would like also to thank my family and all of my friends for their constant moral support and encouragement that I am able to perform this work.

Anastasia Shcherban
Tampere, November 2014

CONTENTS

TERMS AND DEFINITIONS

xRy	binary relation R between x and y elements
$\leq$	partial order
$\sim$	equivalence relation
$\vert x \vert$	equivalence classes of x
$\vee$	join lattice opeartion
$\wedge$	meet lattice opeartion
$\odot$	product operation of the residuated lattice
$\rightarrow$	residum of the product operation of the residuated lattice
x^*	lattice complement of x
$\mathbf{1}$	lattice top element
$\mathbf{0}$	lattice bottom element
$\max(A)$	maximum of the set A
$\min(A)$	minimum of the set A
$\sup(A)$	the least element of the set A
$\cdot$	standard product of the real numbers
$ord(x)$	order of the element x in BL-algebra
$A \times A$	product set
$A \cup B$	the union of the set A and the set B
$A \cap B$	the intersection of set A and the set B
$\bigcap_{i \in I} A_i$	the intersection of the sets A_i (where $i = 1, 2, ...$)
A^c	complement of the set A
$A \subseteq B$	A is a subset of B
$\emptyset$	empty set
$\mathbb{N}$	the set of natural numbers (1, 2, ...)

1. INTRODUCTION

Logic as a language for reasoning has been observed over centuries. The history of its establishing dates back to the ancient times. Formal logic was developed in ancient times in China, India, and Greece. Greek logic, particularly Aristotelian logic, was established as a formal discipline by Aristotle, who gave it a fundamental place in philosophy. It was studied in several ancient civilizations, including India, China, Persia and Greece and was further developed by Islamic and Christian philosophers in the Middle Ages, reaching a high point in the mid-fourteenth century. The significant development of logic from the mid-thirteenth to the mid-fourteenth century was made in the Medieval Logic, which is a form of the Aristotelian logic. It was related to the development of the three areas, such as: the theory of supposition, the theory of syncategoremata and the theory of consequences. The period between the fourteenth century and the beginning of the nineteenth century is generally regarded as barren by historians of logic.

Logic was reviewed in the mid-nineteenth century (1850 - 1920) with the rise of the modern logic. The transition to the modern logics began with the discoveries of the Gottfried Wilhelm Leibniz (1646 - 1716), who was the first to formulate the notion of a broadly applicable system of mathematical logic. The study of the modern logic started at the end of the nineteenth century and can be characterized with the several relevant discoveries in algebraic, logistic and mathematical schools of logic. A big impact to the algebraic school was made with the works of George Boole (1815 - 1864). The major names of the logistic school, which aim was to incorporate the logic into a single unified system, were Bertrand Russell (1872 - 1970) and Alfred North Whitehead (1861 - 1947). These mathematicians made a relevant contribution to the mathematical logic. Additionally, a German mathematician and philosopher Gottlob Frege (1848 - 1925) laid the foundation in formal logic with his works. The third school, the mathematical school, can be characterized with the works of Giuseppe Peano (1858-1932), David Hilbert (1862 - 1943), Ernst Zermelo (1871 - 1953). Its goal was the axiomatization of certain branches of mathematics, including geometry, arithmetic, analysis, and set theory. A significant contribution to the mathematical logic was done in the twentieth century, particularly from the 1950s onwards, with the works devoted to the multi-valued logics of a polish mathematician Jan Łukasiewicz (1878-1956) and further with the works of Gödel and Tarski in the field of modal logic, temporal logic, deontic logic, and relevance logic.

Classic logic gave rise to the more sophisticated fuzzy logic framework. Many-valued logic as a separate part of logic was created by the works of a polish mathematician Jan Łukasiewicz (1878 - 1956) and a American mathematician Post (1897 - 1954) in the beginning of 1920. Fuzzy propositional calculus is an extension of the classical one in the way that each proposition can obtain infinitely many values, i.e. degrees of truth that vary from the total false to absolute truth. A common structure for expressing standard logical conjunction and implication in the many valued logics is a continuous triangular norm (t-norm, in short) and its residuum, which can be verified in different ways depending on the type of logic. T-norms are defined as a cartesian product in the real unit interval. Basic propositional fuzzy logic (BL in short), Łukasiewicz fuzzy logic, Gödel fuzzy logic, and product fuzzy logics are examples of fuzzy propositional logics.

Fuzzy propositional logics provides a sophisticated tool which can be applied in different areas, such as logics, hardware design, artificial intelligence, mathematics, etc. An example is an application of the many-valued logics to logic itself. This is already familiar example, namely previously described Lukasiewcz logic. Furthermore in the hardware design field the many-valued switching could be considered. More precisely, for this field the many-valued logics can be used for the implementation of the electrical circuits. Whereas the electrical circuits based on the classical propositional logics built up from switche with two stable states, the n-valued logics allow building circuits based on the switches with n stable states. However the most diverse and essential area of the application of the many-valued logics is the artificial intelligence. There exist several kinds of application in this particular area, but we will take a brief look on the automatization of data and knowledge mining, where the clustering methods are related to the utilization of the fuzzy sets notion. Additionally, in order to operate with the vague information in databases and knowledge-based systems or to model the vague notions of the expert systems fuzzy sets and fuzzy logic is utilized.

In logics two connectives are of the special interest: connective of implication and connective of equivalence. The first one is related to the notion of inference in logic. The second one says that two sentences are interderivable on the basis of a given logic. A logic and its system of axioms generates a certain algebraic structure, which can be seen as a model for a logic. Given a logical theory, the Lindenbaum algebra is a quotient algebra, that is is generated by equivalence classes of sentences of this theory. Thus Lindenbaum algebra establishes a correspondence between a logic and an algebraic structure. The Lindenbaum-Tarski algebra named for logicians Adolf Lindenbaum (1904-1941) and Alfred Tarski (1901-1983), was introduced by Tarski in 1935. This approach, based on Lindenbaum-Tarski algebras, is essential since it bridges the gap between logic and algebra.

The general algebraic structure used for modeling the truth values of logics is lattice. Being a partially ordered set, on which reflexive, transitive and antisymmetric binary relation is defined, lattice has top and bottom elements. For every two elements in lattice there defined a join and meet operations, which are idempotent, commutative, associative, consistent and isotone. Additionally it has the absorption property. Furthermore in lattice there exist a least upper bound and the greatest lower bound for all subsets of lattice. If the join and meet operations of lattice are distributive over each other, the lattice is called a distributive lattice. Furthermore a distributive lattice is a boolean algebra if each element is associated with the lattice complement. The top and bottom elements of the distributive lattice are defined via the l-complement elements.

The lattice structure could be seen as a good example for modeling the truth values of the logic. However, even if lattice structure well suited for the two-valued classical logic, is not enough to model the sophisticated fuzzy fenomena. Therefore a concept of the extended version of lattice, namely the residuated lattice has been introduced. A residuated lattice is a lattice on which the associative, commutative and isotone binary operation and its residuum is defined via the Galois correspondence. Residuated lattice forms a BL algebras under certain assumptions. The three examples of the algebraic structures of continuous t-norm are BL-lgebras: Łukasiewicz structure, Gödel structure and product structure. Furthermore, a BL-algebra is an MV-algebra if the law of double negation holds. Additionally, locally finite BL-algebras are MV-algebras, which is proved in [6].

Furthermore the other algebraic structures corresponding to a certain lattice structure, are defined which include lattice filters, filters of BL-algebra (filters in short) and deductive systems. Filters defined on the lattice structure in general are nonempty subsets of lattice, that can be of such types as proper, prime and maximal. It can be seen that the lattice is itself a filter. Moreover, the set containing the top element of lattice, i.e. $\{\mathbf{1}\}$, is again a lattice filter. In the similar manner as the lattice filters, filters of BL-algebras are formed on the corresponding BL-algerba. Again, from the definition of filters it could be seen that BL-algebra and the set containing its top element $\{\mathbf{1}\}$ are itself filters. Together with the lattice filters and filters of BL-algebras, a notion of deductive system (ds, for short) of the BL-algebra has been introduced. It was shown in [6] that there is a correspondence between a ds of BL-algebra and a lattice filters in such a way that each ds of a BL-algebra is a lattice filter of BL-algebra. Moreover, it was proved in [7] that any subset of a BL algebra is a ds if and only if this subset forms a filter of BL-algebra. Therefore, is could be seen that deductive systems and filters coincide in BL-algebra. Any BL-algebra contains prime deductive systems and any proper ds can be extended to a maximal one, which is prime [8].

Construction of the Lindenbaum algebra enables finding an association between a logical theory and an algebraic structure. The first application of the Lindenbaum algebra was aimed on establishing correspondence between classical propositional calculus and Boolean algebra. However, the Lindenbaum-Tarski algebra is a significant construction, which carries over to many logics and many algebras other than Boolean algebras. For a the logical theory T, given the sentence symbols, a longer sentences can be built from them, using the common logical connectives. These connectives can be such as disjunction, conjunction, and negation. We say that two sentences are equivalent if the theory T proves that each implies the other. In order to find a correspondence for many valued logics, more complicated algebraic structures, other than Boolean algebra, for modeling the truth degrees should be considered.

A lot of papers with the significant results on the study of the logics and the construction of the Lindenbaim algebra has been published. It was observed that BL algebras rise as Lindenbaum algebras from the basic propositional fuzzy logic (BL in short). BL-algebras were invented by Hájek [3] in order to prove the completeness theorem of the many valued logics. In the similar manner, MV-algebras introduced by C.C. Chang [1] do form the logical axioms of the infinite-valued Lukasiewicz logic. In his paper C.C. Chang gives some applications of his results to the study of completeness of the many-valued logics. Whereas it is known that every Boolean algebra will be a an MV-algebra, the converse is not true [1]. Furthermore, locally finite BL-algebras are MV-algebras [6]. The class of Lindenbaum-Tarski algebras corresponding to intuitionistic logic coincides with the class of Heyting algebras. In turn, G-algebras are generated from the Gödel fuzzy logic and similarly, product algebras are generated from the product fuzzy logic. The above listed algebraic structures use Łukasiewicz t-norm, Gödel t-norm and a product t-norm, respectively, for expressing standard conjunction and implication.

Since Lindenbaum-Tarski algebra is the quotient algebra obtained by factoring the algebra of formulas by the congruence relation, we can fix the equivalence relation on the certain algebraic structure and derive the quotient sets for this particular algebra. Thus we can observe a quotient algebra, induced by operations of Bl-algebra and generated by a filter or deductive system. It is shown in [6] that there is a one-to-one correspondence between deductive system and congruence relations of BL-algebra. The induced quotient algebra is a BL-algebra and is linear if and only if the ds is prime. Given a BL-algebra and a filter, the quotient obtained by factoring the BL-algebra is a BL-algebra. Furthermore, if the filter is prime, the corresponding algebraic structure is a linear BL-algebra. And finally, if the filter is prime, the quotient algebra is an MV-algebra. If we consider the maximal deductive systems of the BL-algebra, the obtained quotient algebra is an MV-algebra. In addition to lattice filters, a notion of the implicative filters was introduced in [5]. It is shown that the quotient algebra generated by the implicative filter is a Godel algebra and, in addition, that a filter of a BL-algbra is a positive filter if and only if every filter of a quotient algebra is a positive implicative filter. Moreover, given a BL algebra and a positive implicative filter of it, the generated quotient algebra is a Boolean algebra [5].

2. PRELIMINARIES

Material for this chapter is taken from [6], [5] and [9].

2.1 Lattice

A *binary relation* R on a nonempty set A is a subset of the product set $A \times A$, i.e. a collection of ordered pairs (x, y), where $x, y \in A$. If $(x, y) \in R$, we write xRy. A binary relation R on A, for all $x, y, z \in A$, is

1. *Reflexive*, when xRx holds;
2. *Transitive*, when if xRy and yRz, then xRz holds;
3. *Anti - symmetric*, when if xRy and yRx, then $x = y$ holds;
4. *Symmetric*, when if xRy, then yRx holds.

A binary relation R satisfying (1) and (2) is called a *quasi - order* (denoted $\leq$), satisfying (1) - (3) is called a *partial order* and satisfying (1), (2) and (4) is called an *equivalence relation* (denoted by $\sim$). Thus a *partially ordered set* (or *poset*) A is set, in which the partial order relation is defined. If, for any x,$y \in A$, either $x \leq y$ or $y \leq x$ holds, then the relation $\leq$ is *total order* and A is called *linearly ordered* set (or a *chain*).

Let A be a poset and $a \in A$. If, for any other element $b \in A$, $a > b$ holds, then a is called *top element* of A. Conversely, if for any other element $b \in A$, $a \leq b$ holds, then a is called *bottom element* of A.

Definition 2.1. *Given an equivalence relation E on a set A, the set of* equivalence classes *of an element $x \in A$ is defined as $|x| = \{y \in A,\ yEx\}$. Consequently the set of all equivalence classes $|x|$ forms a quotinent set A/E.*

Lemma 2.1. *Given a quasi - order R on A, there is an equivalence relation E such that*

$$xEy \textit{ iff } xRy \text{ and } yRx, \tag{2.1}$$

for all $x, y \in A$.

Proof. Since R is reflexive, then from xRx and xRx we obtain xEx and hence E is reflexive. Let xEy and yEz. Then xRy, yRx and yRz, zRy hold. Therefore by transitivity of R from zRy, yRx and xRy, yRz we obtain zRx and xRz. Hence xEz and E is transitive. The symmetry of E follows from (2.1). $\square$

Given a subset X of a nonempty poset A, the *upper bound* of X is an element $a \in A$ such that $x \leq a$ for every $x \in X$. The *least upper bound* (l.u.b.) is an upper bound a of X such that $a \leq b$ for any other upper bound b of X, $b \in A$. The least upper bound of a set $X \subseteq A$, if it exists, is denoted by $\bigvee\{x \mid x \in X\}$. Similarly an element $c \in A$ is the *lower bound* of X if $c \leq x$, for every $x \in X$. The *greatest lower bound* (g.l.b.) is thus a lower bound c of X if $w \leq c$ for any other lower bound w of X, where $w \in A$. The greatest lower bound of a set $X \subseteq A$ is denoted by $\bigwedge\{x \mid x \in X\}$.

Definition 2.2. *A* lattice *$\langle L, \leq, \vee, \wedge \rangle$ is a partially ordered set, where for any $x, y \in L$ there exist binary operations $x \vee y$ and $x \wedge y$, called* join *and* meet *of x and y, respectively.*

For a lattice L and for any $x, y \in L$ the following equations hold

$$x \wedge x = x, \quad x \vee x = x \text{ (idempotency)} \tag{2.2}$$

$$x \wedge y = y \wedge x, \quad x \vee y = y \vee x \text{ (commutativity)} \tag{2.3}$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \text{ (associativity)} \tag{2.4}$$

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x \text{ (absorbtion)} \tag{2.5}$$

$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y \text{ (consistency)} \tag{2.6}$$

Lemma 2.2. *In a lattice L the binary operations $\vee$ and $\wedge$ are isotone, i.e.*

$$\textit{if } y \leq z, \quad \textit{then } x \wedge y \leq x \wedge z,\ x \vee y \leq x \vee z \tag{2.7}$$

Proof. Since $y \leq z$, then by (2.2), (2.6) and (2.3) we reason that

$$x \wedge y = (x \wedge x) \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z).$$

Therefore by (2.6) we conclude that $(x \wedge y) \leq (x \wedge z)$. Similarly we obtain

$$(x \vee z) = (x \vee x) \vee z = (x \vee x) \vee (y \vee z).$$

Therefore $(x \vee y) \leq (x \vee z)$. $\square$

2.2 Boolean algebra

Lemma 2.3. *In a lattice $\langle L, \leq, \vee, \wedge \rangle$ the following holds*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2.8}$$

iff

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{2.9}$$

Proof. Let (2.9) hold in a lattice L. Then, for all $x, y, z \in L$

$$\begin{aligned}
x \wedge (y \vee z) &= [x \wedge (x \vee y)] \wedge (y \vee z) && \text{by (2.5)}\\
&= x \wedge [(x \vee y) \wedge (y \vee z)] && \text{by (2.4)}\\
&= x \wedge [y \vee (x \wedge z)] && \text{by (2.9)}\\
&= [x \vee (x \wedge z)] \wedge [y \vee (x \wedge z)] && \text{by (2.5)}\\
&= (x \wedge z) \vee (y \wedge z) && \text{by (2.9)}
\end{aligned}$$

By the similar proof we obtain that if (2.8) holds in a lattice L, then (2.9) also holds. □

Definition 2.3. *A lattice $\langle L, \leq, \vee, \wedge \rangle$ is* distributive *if, for all $x, y \in L$, (2.8) and consequently (2.9) holds.*

Definition 2.4. *A* Boolean algebra *$\langle L, \leq, \vee, \wedge, ^*, \mathbf{1}, \mathbf{0} \rangle$ is a distributive lattice, in which with every element $x \in L$ a lattice complement (l-complement) $x^* \in L$ is associated such that, for each $x, y \in L$*

$$(x \wedge x^*) \vee y = y, \tag{2.10}$$

$$(x \vee x^*) \wedge y = y \tag{2.11}$$

hold.

Lemma 2.4. *In a Boolean algebra L $x \vee x^*$ and $x \wedge x^*$ are top and bottom elements of L, respectively, i.e. for every $x \in L$*

$$x \vee x^* = \mathbf{1}, \tag{2.12}$$

$$x \wedge x^* = \mathbf{0} \tag{2.13}$$

hold.

Proof. Since L is a Boolean algebra, then from $(x \wedge x^*) \vee y = y = (x \vee x^*) \wedge y$ by (2.7) we infer that $(x \wedge x^*) \leq y \leq (x \vee x^*)$, for every $y \in L$. Hence $x \vee x^*$ and $x \wedge x^*$ are bottom and top elements of L, respectively. □

Lemma 2.5. *In a Boolean algebra L the l-complement $x^* \in L$ of $x \in L$ is unique.*

Proof. Suppose there are another l-complements $y, z \in L$ of x. From (2.10) follows

$$\begin{aligned}
y = (x \wedge z) \vee y &= (x \vee y) \wedge (z \vee y) && \text{by (2.9)}\\
&= \mathbf{1} \wedge (z \vee y) && \text{by (2.12)}\\
&= (z \vee y) && \text{by (2.7)}
\end{aligned}$$

Similarly from (2.11) follows $y = (z \wedge y)$. From the isotonicity of meet and join (2.7) we obtain $z \leq y \leq z$. Therefore $y = z = x^*$. □

2.3 Residuated lattice

Definition 2.5. *A* residuated lattice $\langle L, \leq, \vee, \wedge, \odot, \rightarrow, \mathbf{1}, \mathbf{0}\rangle$ *is a lattice, in which there exists an associative, commutative and isotone binary operation* $\odot$ *and its residuum* $\rightarrow$*, together called an* adjoint couple $\langle \odot, \rightarrow\rangle$ *such that, for all* $x, y, z \in L$*, the following holds*

$$(x \odot z) \leq y \text{ iff } z \leq x \rightarrow y \text{ (Galois correspondence)}. \tag{2.14}$$

Operation $\rightarrow$ *defined via*

$$x \rightarrow y = \bigvee\{z \mid z \odot x \leq y\} \tag{2.15}$$

is unique.

In a residuated lattice L, for all $x \in L$ and a natural number n, define $x^* = x \rightarrow \mathbf{0}$, $(x^*)^* = x^{**}$, $x^0 = \mathbf{1}$ and, for $n \geq 1$, $x^n = x \odot ... \odot x$. Additionally there holds

$$(x \odot \mathbf{1}) = x \text{ and } (x \odot \mathbf{0}) = \mathbf{0}. \tag{2.16}$$

Proposition 2.1. *Let* L *be a residuated lattice. Then for every* $x, y, z \in L$ *the following equations hold in* L

$$\mathbf{1} \rightarrow x = x, \tag{2.17}$$
$$x \rightarrow x = \mathbf{1}, \tag{2.18}$$
$$(x \odot y) \leq x, y, \tag{2.19}$$
$$(x \odot y) \leq x \wedge y, \tag{2.20}$$
$$x \leq y \text{ iff } x \rightarrow y = \mathbf{1}, \tag{2.21}$$
$$x \rightarrow \mathbf{1} = \mathbf{1}, \tag{2.22}$$
$$\mathbf{0} \rightarrow x = \mathbf{1}, \tag{2.23}$$
$$y \leq (y \rightarrow x) \rightarrow x, \tag{2.24}$$
$$x \odot (x \rightarrow y) \leq y, \tag{2.25}$$
$$x \leq y \rightarrow (x \odot y), \tag{2.26}$$
$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z), \tag{2.27}$$
$$\text{If } x \leq y, \text{ then } (x \odot z) \leq (y \odot z), \tag{2.28}$$
$$x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \tag{2.29}$$

$$x \to y \leq (y \to z) \to (x \to z), \tag{2.30}$$

$$(x \to y) \odot (y \to z) \leq (x \to z), \tag{2.31}$$

$$\textit{If } x \leq y, \textit{ then } (z \to x) \leq (z \to y), \tag{2.32}$$

$$\textit{If } x \leq y, \textit{ then } (y \to z) \leq (x \to z), \tag{2.33}$$

$$x \to (y \to z) = (x \odot y) \to z, \tag{2.34}$$

$$x \to (y \to z) = y \to (x \to z), \tag{2.35}$$

$$x \to (y \to x) = \mathbf{1}, \tag{2.36}$$

$$x_1 \to y_1 \leq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]. \tag{2.37}$$

Proof. (2.17): Since $\mathbf{1} \to x = \bigvee\{z \mid z \odot 1 \leq x\} = \bigvee\{z \mid z \leq x\}$.
(2.18): Since $\mathbf{1} \odot x \leq x$, then $\mathbf{1} \leq x \to x \leq \mathbf{1}$. By anti-symmetric property of $\leq$, obtain $x \to x = \mathbf{1}$.
(2.19): By (2.18) $x \leq \mathbf{1} = y \to x$ and $y \leq \mathbf{1} = x \to x$, which implies by the Galois correspondence that $(x \odot y) \leq x$ and consequently, $(x \odot y) \leq y$. Hence, $x \odot y \leq x,\ y$.
(2.20): From $x \wedge y \leq x$,y and (2.19) obtain $x \odot y \leq x \wedge y$.
(2.21): Since $x \leq y$, then $x = x \odot \mathbf{1} \leq y \odot \mathbf{1} = y$ iff $\mathbf{1} \leq x \to y \leq \mathbf{1}$.
(2.22): Since $x \leq \mathbf{1}$, then by (2.21) $x \to \mathbf{1} = \mathbf{1}$. Similarly for (2.23) we reason that $\mathbf{0} \leq x$ and by applying (2.21), obtain $\mathbf{0} \to x = \mathbf{1}$.
(2.24): From $(y \to x) \leq (y \to x)$ by (2.14) and commutativity of $\odot$ we reason that $y \odot (y \to x) \leq x$ and $y \leq (y \to x) \to x$.
(2.25) and (2.26): Follow imidiately from $(x \to y) \leq (x \to y)$ and $(x \odot y) \leq (x \odot y)$, respectively, by applying (2.14).
(2.27): By (2.25) and (2.26) we have $x \odot (x \to y) \leq y$ and $y \leq z \to (y \odot z)$. Therefore $(x \to y) \odot x \leq z \to (y \odot z)$ iff $(x \to y) \odot (x \odot z) \leq (y \odot z)$ iff

$$x \to y \leq (x \odot z) \to (y \odot z).$$

(2.28): If $x \leq y$, then from (2.27) by (2.21) we conclude $\mathbf{1} \odot (x \odot z) \leq (y \odot z)$.
(2.29): By (2.14) we deduce that $x \to y \leq (z \to x) \to (z \to y)$ iff

$$(x \to y) \odot (z \to x) \leq (z \to y) \text{ iff } [(z \to x) \odot z] \odot (x \to y) \leq y,$$

which holds by applying two times (2.25): $[z \odot (z \to x)] \odot (x \to y) \leq x \odot (x \to y) \leq y$.
(2.30): By (2.14) we deduce that $x \to y \leq (y \to z) \to (x \to z)$ iff $(x \to y) \odot (y \to z) \leq (x \to z)$ iff $(x \to y) \odot (y \to z) \odot x \leq z$, which holds in the similar manner as (2.29): $[x \odot (x \to y)] \odot (y \to z) \leq y \odot (y \to z) \leq z$. Applying (2.14), obtain(2.31).
(2.32): If $x \leq y$ then from (2.29) by (2.21) we obtain $\mathbf{1} \odot (z \to x) \leq (z \to y)$. In the similar manner from (2.30) we infer (2.33).
(2.34): On one hand by (2.14) and (2.25) $[x \to (y \to z)] \odot (x \odot y) \leq (y \to z) \odot y \leq z$

and therefore $x \to (y \to z) \le (x \odot y) \to z$. On the other hand $(x \odot y) \to z \le (x \odot y) \to z$ iff $[(x \odot y) \to z] \odot (x \odot y) \le z$ iff $[(x \odot y) \to z] \odot x \le y \to z$ and consequently $(x \odot y) \to z \le x \to (y \to z)$. (2.35): From (2.34) we deduce

$$\begin{aligned} x \to (y \to z) &= (x \odot y) \to z \\ &= (y \odot x) \to z \\ &= y \to (x \to z). \end{aligned}$$

(2.36): From (2.35) deduce

$$\begin{aligned} x \to (y \to x) &= y \to (x \to x) \\ &= y \to \mathbf{1} \\ &= \mathbf{1}. \end{aligned}$$

(2.37): From $x_1 \to y_1 \le (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$ follows

$$(x_1 \to y_1) \odot (y_2 \to x_2) \odot (y_1 \to y_2) \odot x_1 \le x_2,$$

which holds since, by applying several times (2.25), we obtain

$$\begin{aligned} (x_1 \to y_1) \odot (y_2 \to x_2) \odot (y_1 \to y_2) \odot x_1 &\le (y_2 \to x_2) \odot (y_1 \to y_2) \odot y_1 \\ &\le (y_2 \to x_2) \odot y_2 \\ &\le x_2. \end{aligned}$$

□

In addition to the above equations, in a residuated lattice the following equations hold

$$x \le x^{**}, \tag{2.38}$$

$$x^* = x^{***}, \tag{2.39}$$

$$x^* \odot x = \mathbf{0}, \tag{2.40}$$

$$(x \vee y)^* = x^* \wedge y^*, \tag{2.41}$$

$$\text{If } x \vee x^* = \mathbf{1}, \text{ then } x \wedge x^* = \mathbf{0}, \tag{2.42}$$

$$x \to y \le y^* \to x^*. \tag{2.43}$$

Definition 2.6. *A residuated lattice $\langle L, \le, \vee, \wedge, \odot, \to, \mathbf{1}, \mathbf{0}\rangle$ is* linearly ordered *if A is a linearly ordered set.*

2.4 BL-algebra

Definition 2.7. *A residuated lattice L forms a* BL algebra $\langle L, \leq, \vee, \wedge, \odot, \rightarrow, \mathbf{1}, \mathbf{0}\rangle$ *if, for all x,y,z $\in L$, it satisfies*

$$x \wedge y = x \odot (x \rightarrow y), \tag{2.44}$$

$$(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}. \tag{2.45}$$

In BL-algebra the equations (2.17)-(2.37) hold.

Proposition 2.2. *Let A be a BL-algebra. Then the following equations hold in A for every $x, y, z \in A$*

$$(x \vee y) \odot z = (x \odot z) \vee (y \odot z), \tag{2.46}$$

$$(y \vee z) \rightarrow x = (y \rightarrow x) \wedge (z \rightarrow x). \tag{2.47}$$

Proof. (2.46): Since x, $y \leq x \vee y$, then by (2.28) we have that

$$x \odot z \leq (x \vee y) \odot z \text{ and } y \odot z \leq (x \vee y) \odot z.$$

Therefore $(x \odot z) \vee (y \odot z) \leq (x \vee y) \odot z$. Similarly from $(x \odot z)$,$(y \odot z) \leq (x \odot z) \vee (y \odot z)$ we infer

$$x \leq z \rightarrow [(x \odot z) \vee (y \odot z)] \text{ and } y \leq z \rightarrow [(x \odot z) \vee (y \odot z)],$$

respectively. Hence $x \vee y \leq z \rightarrow [(x \odot z) \vee (y \odot z)]$. Eventually

$$(x \vee y) \odot z \leq [(x \odot z) \vee (y \odot z)].$$

(2.47): Since y, $z \leq y \vee z$, then by (2.33)

$$(y \vee z) \rightarrow x \leq y \rightarrow x \text{ and } (y \vee z) \rightarrow x \leq z \rightarrow x.$$

Hence $(y \vee z) \rightarrow x \leq (y \rightarrow x) \wedge (z \rightarrow x)$ (1). On the other hand, from $(y \rightarrow x) \wedge (z \rightarrow x) \leq (y \rightarrow x)$,$(z \rightarrow x)$ we infer

$$y \odot [(y \rightarrow x) \wedge (z \rightarrow x)] \leq x \text{ and } z \odot [(y \rightarrow x) \wedge (z \rightarrow x)] \leq x,$$

whence

$$y \leq [(y \rightarrow x) \wedge (z \rightarrow x)] \rightarrow x \text{ and } z \leq [(y \rightarrow x) \wedge (z \rightarrow x)] \rightarrow x.$$

Thus $y \vee z \leq [(y \rightarrow x) \wedge (z \rightarrow x)] \rightarrow x$, and then $(y \vee z) \odot [(y \rightarrow x) \wedge (z \rightarrow x)] \leq x$.

Eventually $(y \to x) \wedge (z \to x) \leq (y \vee z) \to x$ (2). With (1) and (2) we obtain that (2.47) holds. □

Lemma 2.6. *In each BL-algebra A the following holds for all x,y,$z \in A$:*

$$x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x], \tag{2.48}$$

$$x \to (y \wedge z) = (x \to y) \wedge (x \to z). \tag{2.49}$$

Proof. By applying (2.45), (2.46) and (2.25) we obtain

$$\begin{aligned}
[(x \to y) \to y] \wedge [(y \to x) \to x] &= [...] \odot [(x \to y) \vee (y \to x)] \\
&= ([...] \odot (x \to y)) \vee ([...] \odot (y \to x)) \\
\leq [((x \to y) \to y) \odot (x \to y)] &\vee [((y \to x) \to x) \odot (y \to x)] \\
&\leq y \vee x.
\end{aligned}$$

On the other hand, applying (2.46), (2.25) and (2.19) we obtain

$$(x \to y) \odot (x \vee y) = (x \odot (x \to y)) \vee (y \odot (x \to y)) \leq y \vee y = y.$$

From the latter by (2.14) we obtain $(x \vee y) \leq [(x \to y) \to y]$. Similarly from $(y \to x) \odot (x \vee y) \leq x$ we infer $(x \vee y) \leq [(y \to x) \to x]$. Therefore we conclude that $(x \vee y) \leq [(x \to y) \to y] \wedge [(y \to x) \to x]$.

For proving (2.49), from $(x \to y) \wedge (x \to z) \leq (x \to y), (x \to z)$ by Galois correspondence deduce $x \odot [(x \to y) \wedge (x \to z)] \leq y$, $x \odot [(x \to y) \wedge (x \to z)] \leq z$ and consequently that $x \odot [(x \to y) \wedge (x \to z)] \leq y \wedge z$. Therefore, by Galois correspondence we obtain $[(x \to y) \wedge (x \to z)] \leq x \to (y \wedge z)$ (1). On the other hand from $y \wedge z \leq y, z$ by (2.32) deduce $x \to (y \wedge z) \leq x \to y$, $x \to (y \wedge z) \leq x \to z$. Consequently, $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$ (2). With (1) and (2) we obtain $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ □

The binary operation $\odot$ on a BL-algebra defined on the real unit interval [0,1] is called *t-norm*, which is continious mapping $\odot$: $[0,1]\times[0,1]\to[0,1]$. The binary operation $\to$ is called the *residuum* of a t-norm $\odot$.

The following algebraic structures of continious t-norm are BL-algebras.

$$\text{Łukasiewizc structure:} \begin{cases} x \odot y = \max(0, x+y-1) \\ x \to y = (1 - x + y) \end{cases} \tag{2.50}$$

$$\text{Gödel structure:} \begin{cases} x \odot y = \min(x, y) \\ x \to y = y \end{cases} \tag{2.51}$$

$$\text{Product structure:} \begin{cases} x \odot y = x \cdot y \\ x \to y = y/x \end{cases} \tag{2.52}$$

Here the residuum of all three t-norms is defined for $x > y$, since for $x \le y$, $x \to y = \mathbf{1}$.

Proposition 2.3. *A linearly ordered residuated lattice L is a BL-algebra iff it satisfies the condition (2.44).*

Proof. Let L be a linear ordered residuated lattice and x,$y \in L$. By the definition of a BL-algebra, a residuated lattice should satisfy (2.45) to be a BL-algebra. Since in L holds $x \le y$ or $y \le x$, which imply $x \to y = \mathbf{1}$ or $y \to x = \mathbf{1}$ by (2.21), then

$$(x \to y) \vee (y \to x) = \mathbf{1},$$

i.e. (2.45) holds. □

Theorem 2.1. *A BL-algebra is a distributive lattice.*

Proof. Let A be a BL-algebra and x,y,$z \in A$. In order A to be a distributive lattice, it must satisfy (2.8) or (2.9), which hold due to the following inference. On one hand we have

$$\begin{aligned} x \wedge (y \vee z) &= (y \vee z) \odot [(y \vee z) \to x] && \text{by (2.44)} \\ &= (y \vee z) \odot [(y \to x) \wedge (z \to x)] && \text{by (2.48)} \\ &= (y \odot [(y \to x) \wedge (z \to x)]) \vee (z \odot [(y \to x) \wedge (z \to x)]) && \text{by (2.46)} \\ &\le [y \odot (y \to x)] \vee [z \odot (z \to x)] \\ &= (x \wedge y) \vee (x \wedge z) && \text{by (2.44)} \end{aligned}$$

On the other hand, since $(y \wedge x) \le y$,x and $(z \wedge x) \le z$,x, then $(y \wedge x)$, $(z \wedge x) \le x \wedge (y \vee z)$ by (2.44). Therefore $(y \wedge x) \vee (z \wedge x) \le x \wedge (y \vee z)$. □

2.5 MV-algebra

Definition 2.8. *In each BL-algebra A the* order $ord(x)$ *of an element $x \in A$ is the least integer n in $x^n = x \odot ... \odot x$ such that $x^n = \mathbf{0}$, if such an integer n exists. Otherwise, $ord(x) = \infty$.*

Consequently we have

Definition 2.9. *A BL-algebra is locally finite if all $x \in A$ such that $x < \mathbf{1}$ are of finite order.*

Proposition 2.4. *Locally finite BL-algebras are linear.*

Proof. Let A be a locally finite BL-algebras. Then by a definition, there is an integer n such that $x^n = \mathbf{0}$ for any element $x \in A$. Let $x \vee y = \mathbf{1}$ and $x \neq \mathbf{1}$, where x, $y \in A$. Then by (2.48) $\mathbf{1} = [(x \to y) \to y] \wedge [(y \to x) \to x] \leq [(x \to y) \to y]$, $[(y \to x) \to x]$. Obtain $\mathbf{1} \leq (x \to y) \to y$. Therefore by Galois correspondence, $(x \to y) \leq y$. Next we deduce from $x \odot y \leq y$ that $y \leq x \to y$ and thus $(x \to y) = y$. Now by (2.34) we have

$$\begin{aligned} y = (x \to y) &= x \to (x \to y) = x \odot x \to y \\ &= x^2 \to y = ... = x^n \to y = \mathbf{0} \to y = \mathbf{1}. \end{aligned}$$

Consequently we conclude $x \vee y = \mathbf{1}$ iff $x = \mathbf{1}$ or $y = \mathbf{1}$. Since for all elements x, $y \in A$, the condition (2.45) holds, we have $(x \to y) \vee (y \to x) = \mathbf{1}$ iff $x \to y = \mathbf{1}$ or $y \to x = \mathbf{1}$. Hence $x \leq y$ or $y \leq x$.

□

Proposition 2.5. *In a locally finite BL-algebra A, for all $x \in A$ the following holds*

$$\mathbf{0} < x < \mathbf{1} \textit{ iff } \mathbf{0} < x^* < \mathbf{1}, \tag{2.53}$$

$$x^* = \mathbf{0} \textit{ iff } x = \mathbf{1}, \tag{2.54}$$

$$x^* = \mathbf{1} \textit{ iff } x = \mathbf{0}. \tag{2.55}$$

Proof. Ommited. □

Proposition 2.6. *In a BL-algebra A, for all x,y,$z \in A$, the following equation holds*

$$\textit{If } z \to x = z \to y \textit{ and } x,\ y \leq z, \textit{ then } x = y \tag{2.56}$$

Proof. Since x,$y \leq z$ then, by assumption

$$x = x \wedge z = z \odot (z \to x) = z \odot (z \to y) = y \wedge z = y.$$

□

Proposition 2.7. *In any linear BL-algebra A, for all elememts x,y,$z \in A$, the following condition holds*

$$\textit{If } z \to x = z \to y \neq \mathbf{1}, \textit{ then } x = y \tag{2.57}$$

Proof. If $z \to x = z \to y \neq \mathbf{1}$, then $z \nleq x$ and $z \nleq y$. Since A is linear, we have x,$y \leq z$. Therefore by Proposition (2.6), $x = y$. □

Definition 2.10. *A BL-algebra A is an* MV-algebra *iff for all $x \in A$,*

$$x = x^{**} \tag{2.58}$$

Theorem 2.2. *Locally finite BL-algebras are MV-algebras.*

Proof. Let A be a locally finite BL-algebra. We show that the condition (2.58) is satisfied for any $\mathbf{0} < x < \mathbf{1}$, where $x \in A$. By (2.53) we have that $\mathbf{0} < x^* < \mathbf{1}$. Again by (2.53), $\mathbf{0} < x^{**} < \mathbf{1}$. Applying (2.34) and letting $z = \mathbf{0}$ we obtain $x \to (y \to z) = x \to (y \to \mathbf{0}) = x \to y^*$ and $x \to (y \to \mathbf{0}) = (x \odot y) \to \mathbf{0} = (x \odot y)^*$, i.e. $x \to y^* = (x \odot y)^*$ (1). From (2.38), $x \leq x^{**}$ and (2.44) we obtain that $x = x^{**} \wedge x = x^{**} \odot (x^{**} \to x)$ and consequently that $x^* = [x^{**} \odot (x^{**} \to x)]^*$. Applying the above obtained result (1), we deduce $[x^{**} \odot (x^{**} \to x)]^* = x^{**} \to (x^{**} \to x)^*$. On the other hand by Propositon 2.4, A is linear and by (2.39), $x^* = x^{***} = x^{**} \to \mathbf{0}$ holds in A. Thus $x^{**} \to \mathbf{0} = x^{**} \to (x^{**} \to x)^* \neq \mathbf{1}$, which by Proposition (2.7) implies that $(x^{**} \to x)^* = \mathbf{0}$. Therefore by (2.54) we obtain that $x^{**} \to x = \mathbf{1}$, which by (2.21) implies $x^{**} \leq x$. Moreover $x \leq x^{**}$, by (2.38). Hence $x = x^{**}$. $\square$

Lemma 2.7. *In [5] it is shown that in a MV-algebra the following are equivalent*

(i) $(x \to y) \to x = x$,

(ii) $x \wedge x^* = \mathbf{0}$, *where* $x^* = x \to \mathbf{0}$,

(iii) $x \vee x^* = \mathbf{1}$,

(iv) $x^* \to x = x$,

(v) $x \to x^* = x^*$,

(vi) $(y^* \to x) \to x = y \to x$,

(vii) $(y \to x) \to x = y^* \to x$,

(viii) $z^* \to x \leq (y \to x) \to [(z \to y) \to x]$.

Proof. Let (i) hold. First we show that (ii) holds. By (2.44) and (2.40) we deduce $(x \wedge x^*) = x^* \odot (x^* \to x) = x^* \odot [(x \to \mathbf{0}) \to x] = x^* \odot x = \mathbf{0}$.

Now we show that from (ii) we can infer (iii). By (2.58) and (2.18) follows $x^* \vee x = (x^* \vee x)^{**} = ((x^* \vee x)^*)^* = (x^{**} \wedge x^*)^* = (x^* \wedge x)^* = (x^* \wedge x) \to \mathbf{0} = (x^* \wedge x) \to (x^* \wedge x) = \mathbf{1}$.

Next we show that (v) holds given (iv). By applying (iv), (2.34) and (2.16) we get

$$
\begin{aligned}
x \to x^* &= (x^* \to x) \to x^* = [x^* \to (x \to x^*)] \to x^* \\
&= [(x^* \odot x) \to x^*] \to x^* = [(x^* \odot x) \to (x \to \mathbf{0})] \to x^* \\
&= [((x^* \odot x) \odot x) \to \mathbf{0})] \to x^* = [(\mathbf{0} \odot x) \to \mathbf{0})] \to x^* \\
&= (\mathbf{0} \to \mathbf{0}) \to x^* = \mathbf{1} \to x^* = x^*
\end{aligned}
$$

In order to prove that (vii) holds given (vi) we infer that $(y \to x) \to x = ((y^*)^* \to x) \to x = y^* \to x$.

Next assume (vii) holds. Then from $y \le (y \to x) \to x$ by (2.32) we have $(z \to y) \to y \le (z \to y) \to [(y \to x) \to x]$ and consequently $z^* \to y \le (y \to x) \to [(z \to y) \to x]$. Therefore (viii) holds.

We omit the proof of (vi) given (v), (i) given (viii) and (iv) from (iii). $\square$

Additionally, we have proved that the following are equivalent.

Lemma 2.8. *In a MV-algebra the following are equivalent:*

(i) $(x \to y) \to x = x$,

(ii) $x \vee x^* = \mathbf{1}$,

(iii) $(x^* \to x) = x$.

Proof. First we assume that (i) holds. Then by (2.41),(2.44), (2.34) and (2.18) we deduce that

$$\begin{aligned} x \vee x^* &= (x \vee x^*)^{**} = ((x \vee x^*)^*)^* = (x^* \wedge x^{**})^* \\ &= (x^* \wedge x)^* = (x^* \odot (x^* \to x))^* \\ &= [x^* \odot ((x \to \mathbf{0}) \to x)]^* \\ &= (x^* \odot x)^* = (x^* \odot x) \to \mathbf{0} \\ &= (x^* \to (x \to \mathbf{0})) = x^* \to x^* = \mathbf{1}. \end{aligned}$$

For proving (iii) given (ii) we show that $x^* \to x \le x$ and $x \le x^* \to x$, which implies $x^* \to x = x$. From $x \odot x^* \le x, x^*$ by Galois correspondence $x \le x^* \to x$. Therefore, the first part holds. Further by (2.26), (2.44) and (2.42) we have

$$x^* \to x \le x^* \to (x^* \odot (x^* \to x)) = x^* \to (x^* \wedge x) = x^* \to \mathbf{0} = (x^*)^* = x^{**} = x.$$

Now we assume that *(iii)* holds. We prove that (i) holds by showing that $x \le (x \to y) \to x$ (1) and $(x \to y) \to x \le x$ (2).

By (2.19) we reason that $x \odot (x \to y) \le x, (x \to y)$ by Galois correspondence we deduce that $x \le (x \to y) \to x$. Therefore (1) holds.

In order to prove (2) from $\mathbf{0} \le y$ by (2.32) we deduce that $(x \to \mathbf{0}) \le (x \to y)$, $x^* \le (x \to y)$ and further by (2.33) that $(x \to y) \to x \le x^* \to x = x$. Therefore, (2) and consequently (i) holds.

$\square$

3. DEDUCTIVE SYSTEMS

Material for this chapter is taken from [6], [9] and [4].

3.1 Lattice filters

Definition 3.1. *Let L be a lattice $\langle L, \leq, \vee, \wedge \rangle$ and $F \subseteq L$ be a nonempty subset of L. Then F is a* lattice filter *if*

$$x \wedge y \in F \text{ iff } x,\, y \in F. \tag{3.1}$$

This definition can be replaced with the following one

Remark 3.1. *Let L be a lattice $\langle L, \leq, \vee, \wedge \rangle$ and $F \subseteq L$ be a nonempty subset of L. Then F is a lattice filter if*

$$\text{if } x,y \in F, \text{ then } x \wedge y \in F, \tag{3.2}$$

$$\text{if } x \in F \text{ and } x \leq y, \text{ then } y \in F. \tag{3.3}$$

hold.

Proof. Assume first that (3.1) holds. Then $x,y \in F$ implies $x \wedge y \in F$ and (3.2) holds. Moreover if $x \in F$ and $x \leq y$, then $x \wedge y = x \in F$, which in turn implies that $y \in F$, by (3.1). Hence (3.3) holds. Conversely, assume (3.2) and (3.3) hold. Then $x \wedge y \in F$. From $x \wedge y \leq x,y$ by (3.3) we deduce $x,y \in F$, which means that (3.1) holds. □

Remark 3.2. *The condition (3.3) can be replaced by the following condition*

$$\text{if } x \in F \text{ and } y \in L, \text{ then } x \vee y \in F. \tag{3.4}$$

Proof. Suppose (3.3) holds. Then from $x \in F$ and $x \leq x \vee y$ by (3.3) we infer $x \vee y \in F$. Hence (3.4) holds. Conversely, suppose (3.4) holds. Then $x \vee y \in F$. Let $x \leq y$. Then $x \vee y = y \in F$. Therefore $y \in F$ and (3.3) holds. □

It is easy to see that L itself is a lattice filter. Moreover, given an element $x \in L$, the set $F = \{y \in L \mid x \leq y\}$ is again a lattice filter. Additionally if L contains $\mathbf{1}$ as an element, then $\{\mathbf{1}\}$ is also a lattice filter of L. If L contains $\mathbf{1}$, then by (3.3), $\mathbf{1}$ is included in every filter of L.

A partially ordered set of filters of a lattice L is any non-empty family of filters of L. In this case a partial ordering is defined by a set theoretical inclusion $\subseteq$. A non-empty family of lattice filters of L forms a *chain of filters* if, for any filters F,$G \subseteq L$ in this family, either $G \subseteq F$ or $F \subseteq G$.

A filter F of L is called a *proper filter* if $F \neq L$. If a lattice L contains the least element $\mathbf{0}$, then F is proper iff $\mathbf{0} \notin F$. A proper filter F is called *maximal* if, for any other filter G of L, where $F \subseteq G$, either $F = G$ or $G = L$ holds. Thus a maximal filter is a maximal element of a poset of all proper filters of L. Additionally a proper filter F is called *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for any x,$y \in L$.

Proposition 3.1. *Let L be a lattice. Then the union of every chain of proper lattice filters of L, containing zero element, is a proper lattice filter of L.*

Proof. Let L be a lattice such that $\mathbf{0} \in L$, F_1,...,F_n be a chain of proper lattice filters of L and assume

$$\mathcal{F} = \cup\{F_i \mid F_1 \subseteq F_2.... \subseteq F_n... \subseteq L\}$$

be the union of this chain. Then x,$y \in \mathcal{F}$ iff x,$y \in F_i$ for some i iff $x \wedge y \in F_i \subseteq \mathcal{F}$. Hence $x \wedge y \in \mathcal{F}$ and $\mathcal{F}$ is a lattice filter. Since F_i for some i is proper iff $\mathbf{0} \notin L$, then F_i and consequently $\mathcal{F}$ do not contain $\mathbf{0}$ as an element. Hence $\mathcal{F}$ is a proper lattice filter of L. □

Zorn's lemma. If there is an upper bound in every chain of elements of a poset A, then A contains a maximal element, i.e. for every $y \in A$ there exists a maximal element $x \in A$ such that $y \leq x$ holds.

Since the proof of zorn's lemma requires additional topics, far from those of the current work, we will not cover it.

Proposition 3.2. *Let L be a lattice, containing a zero element. Then every proper lattice filter F of L is contained in a maximal lattice filter.*

Proof. Let $\mathcal{F}$ be a poset of all proper filters of L. By Proposition 3.1 each chain of elements has an upper bound in $\mathcal{F}$. By Zorn's lemma this implies that there is a maximal filter G of L such that, for any other filter F of L, $F \subseteq G$. □

Proposition 3.3. *If a lattice L contains a zero element, then for every $x \neq \mathbf{0}$ of L there is a maximal filter G on L such that $x \in G$.*

Proof. Since $F = \{y \in L \mid x \leq y\}$ is a proper filter of L, then, by Proposition 3.2, F is contained in a maximal filter. □

Proposition 3.4. *Let L be a non-degenerate lattice, i.e. the lattice with at least two different elements, provided that $\mathbf{0} \in L$. Then by taking $x \neq \mathbf{0}$ of L, L has a maximal filter.*

Proof. Follows from the Theorem 3.3. □

Proposition 3.5. *Each maximal lattice filter F of a distributive lattice L is prime.*

Proof. We show that F is a prime lattice filter of L by assuming the reverse and coming to a contradicton.

Let F be a maximal lattice filter such that it is not prime and L be a distributive lattice. Then there exist a,$b \in L$ such that $a \vee b \in F$, but $a \notin F$ and $b \notin F$. Define a set

$$G = \{x \in L \mid a \wedge c \leq x, \text{ for } c \in F\}.$$

First we show that G is a lattice filter of L. Let $x, y \in G$. Then for some c, $d \in F$ there exist $a \wedge c \leq x$ and $a \wedge d \leq y$. Therefore $a \wedge (c \wedge d) \leq x$,$y$, whence $x \wedge y \leq x$,y. Thus $a \wedge (c \wedge d) \leq x \wedge y$. By (3.2), from $c \in F$ and $d \in F$ we deduce that $c \wedge d \in F$. Consequently $x \wedge y \in G$. Conversely the assumption $x \wedge y \in G$ leads to $a \wedge c \leq x \wedge y \leq x$,$y$ for some $c \in F$ and therefore to x,$y \in G$. By (3.1) this implies that G is a lattice filter of L.

Next we show that G is a proper lattice filter. Letting $b \in G$, we obtain that for some $c \in F$, $a \wedge c \leq b$. Since $b \leq a \vee b$, then $a \vee b \in F$. Moreover $c \vee b \in F$. Since L is a distributive lattice, then $b = (a \wedge c) \vee b = (a \vee b) \wedge (c \vee b) \in F$, which contradicts the initial assumption. Hence $b \notin G$ and we conclude that G is a proper filter. By definition of maximal filter, for some other filter G, either $F = G$ or $G = L$ holds. Since $a \wedge \mathbf{1} \leq a$ and $\mathbf{1} \in F$, then $a \in G$, provided that $a \notin F$ by the initial assumption. Hence neither $F = G$ nor $G = L$ holds. This imlies that F is not a maximal filter, which is a contradiction. Therefore we conclude that F must be prime. □

If we talk about a BL-algebra, then the other definition of a filter appears.

Definition 3.2. *Let A be a BL-algebra $\langle L, \leq, \vee, \wedge \rangle$ and $F \subseteq A$ be a nonempty subset of A. Then, F is a* filter *of BL-algebra if for all x,$y \in A$*

$$\text{if } x,y \in F, \text{ then } x \odot y \in F, \tag{3.5}$$

$$\text{if } x \in F \text{ and } x \leq y, \text{then } y \in F. \tag{3.6}$$

hold.

It is also easy to see that, like in lattice filters, A itself is a filter. Moreover, given an element $x \in A$, the set $F = \{y \in A \mid x \leq y\}$ is again a filter. Since A contains $\mathbf{1}$ as an element, $\{\mathbf{1}\}$ is also a filter of A. Additionally by (3.6), $\mathbf{1}$ is included in every filter of A.

The definitions of proper, prime and maximal lattice filters are also valid for filters of BL-algebra A.

We can easily see that the Proposition 3.1 is also valid for filters of BL-algebras A. Letting F_1,...,F_n be a chain of proper filters of A and

$$\mathcal{F} = \cup\{F_i \mid F_1 \subseteq F_2 .... \subseteq F_n ... \subseteq A\}$$

be the union of this chain, x,$y \in \mathcal{F}$ iff x,$y \in F_i$ for some i iff $x \odot y \in F_i \subseteq \mathcal{F}$. Hence $x \odot y \in \mathcal{F}$ and $\mathcal{F}$ is a filter. By the same argument as in Proposition 3.1, $\mathcal{F}$ is a proper filter of A. Since $x \odot y \in \mathcal{F}$ and $x \odot y \leq x \wedge y$ by (2.20), then $x \wedge y \in \mathcal{F}$, which validates the propositions 3.1 - 3.4 for filters also.

3.2 Proper deductive system

Definition 3.3. *A subset $D \subseteq A$ is a deductive system (ds) of A if for all x,$y \in A$ the following holds:*

$$\mathbf{1} \in D, \tag{3.7}$$

$$\textit{if } x \in D \textit{ and } x \to y \in D, \textit{ then } y \in D. \tag{3.8}$$

Obviously A as well as $\{\mathbf{1}\}$ are deductive systems of A. Given that $D \subseteq A$ is a ds of A and $x \in D$ such that $x \leq y$, we immediately obtain $x \to y = \mathbf{1} \in D$. Hence $y \in D$.

Now we realize that there is similarity between deductive systems and lattice filters as well as filters of BL-algebra. The following two theorems observe this similarity.

Proposition 3.6. *A deductive system D of a BL-algebra A is a lattice filter of A.*

Proof. Let D be a deductive system of A and x,$y \in D$. By the definition of a ds, $\mathbf{1} \in D$. By (2.26) we have $x \leq y \to (x \odot y)$ from which by (2.21) we obtain $x \to [y \to (x \odot y)] = \mathbf{1} \in D$. Thus we deduce $y \to (x \odot y) \in D$ and consequently $(x \odot y) \in D$. Hence by (2.20) we conclude that $(x \odot y) \to (x \wedge y) = \mathbf{1} \in D$ and therefore $x \wedge y \in D$, which shows that (3.2) holds.

Next we prove (3.4). By (2.24) we have $x \leq (x \to y) \to y$ and $y \leq (y \to x) \to x$. Consequently, $x \odot y \leq x \wedge y \leq [(x \to y) \to y] \wedge [(y \to x) \to x] = (x \vee y)$ by (2.20) and (2.48). Therefore $(x \odot y) \to (x \vee y) = \mathbf{1} \in D$ and further $(x \vee y) \in D$ which verifies (3.4).

□

Proposition 3.7. *A subset D of a BL-algebra A is a deductive system of A iff D is a filter of A.*

Proof. Let D be a deductive system of A and x,$y \in D$. By the definition of a ds, $\mathbf{1} \in D$. By (2.26) we have $x \leq y \to (x \odot y)$ from which by (2.21) we obtain

$x \to [y \to (x \odot y)] = \mathbf{1} \in D$. Thus we deduce $y \to (x \odot y) \in D$ and consequently $(x \odot y) \in D$ which implies that (3.5) holds. To verify (3.6), assume $x \leq y$, given $x \in D$. Now $x \to y = \mathbf{1} \in D$ Therefore $y \in D$ and D is a filter.

Conversely, let D be a filter of A and $x \in D$. Since $x \leq \mathbf{1}$, then $\mathbf{1} \in D$, which proves (3.7). Next to prove (3.8), assume $x \in D$ and $x \to y \in D$. Since D is a filter, then by (3.6), $x \odot (x \to y) \in D$, which by (2.25) is $x \odot (x \to y) \leq y$ and hence $y \in D$. Therefore D is a ds of A. □

A deductive sustem D of BL-algebra A is *proper* if $D \neq A$ or if there is no element $x \in A$ such that x,$x^* \in D$. Since A contains the zero element, a ds D is proper iff $\mathbf{0} \notin D$.

Proposition 3.8. *If X,Y are two deductive systems of BL-algebra such that $D_{X \cup Y} = \{x \in A \mid y_1 \odot ... \odot y_n \leq x$ for some $y_1, y_2,..., y_n \in X \cup Y\}$, then $D_{X \cup Y}$ is a ds of A such that $X \cup Y \subseteq D_{X \cup Y}$.*

Proof. To prove that $D_{X \cup Y}$ is a ds, it should satisfy (3.7) and (3.8). First we show that $\mathbf{1} \in D_{X \cup Y}$. Since $\mathbf{1}$ is in every ds of A, then also $\mathbf{1} \in X \cup Y$ such that $y_1 \odot ... \odot y_n \leq \mathbf{1}$ for some $y_1, y_2,..., y_n \in X \cup Y$, then $\mathbf{1} \in D_{X \cup Y}$.

Next assume z, $z \to w \in D_{X \cup Y}$. Then there exist some $y_1, y_2,..., y_n \in X \cup Y$ and $q_1, q_2,...,q_n \in X \cup Y$ such that $q_1 \odot ... \odot q_n \leq z \to w$ and $y_1 \odot ... \odot y_n \leq z$. Since $q_1 \odot ... \odot q_n \odot z \leq w$, we conclude that $(q_1 \odot ... \odot q_n) \odot (y_1 \odot ... \odot y_n) \leq w$ and consequently that $w \in D_{X \cup Y}$. Thus (3.8) is valid. Additionally since $y_i \in X \cup Y$ for any i in $\overline{1,n}$ and $y_i \leq y_i$, then $X \cup Y \subseteq D_{X \cup Y}$. □

3.2.1 Quotient algebra

As can be seen from the previous subsection, deductive systems and filters coincide in BL-algebra. In this section a quotient algebra, induced by operations of BL-algebra and generated by a filter, is observed. The following theorems, valid for filters of BL-algebra, are also valid for deductive systems of BL-algebra.

Recalling $x^n = x \odot ... \odot x$ (n times), we have

Lemma 3.1. *Let F be a filter of a BL-algebra A. Then for any $x \in A$, $x \in F$ iff $x^n \in F$.*

Proof. Assume $x \in F$. By Proposition 3.7, F is a ds of A. Hence $\mathbf{1} \in F$. By (2.28) from (2.20), $x \leq x \to (x \odot x)$, we have

$$x \to (x \to (x \odot x)) = \mathbf{1} \in F.$$

Thus we deduce $x \to (x \odot x) \in F$ and consequently $(x \odot x) \in F$. Therefore by (3.5) we obtain that $x \odot (x \odot x) \in F$. We continue in the similar manner and get $x^n \in F$. Conversely, assume $x^n \in D$. Since by (2.19) $x^n \leq x$, then $x \in F$. □

Proposition 3.9. *In a BL-algebra A, if $\sim$ is congruence relation on A then the set $D = \{x \in A \mid x \sim \mathbf{1}\}$ is a deductive system.*

Proof. We show that D is a filter, which by Proposition 3.7 is also a ds of A. Assume x,$y \in D$. Thus by the assumption that $\sim$ is a congruence on A from $x \sim \mathbf{1}$, $y \sim \mathbf{1}$ we obtain $(x \odot y) \sim (\mathbf{1} \odot \mathbf{1}) = \mathbf{1}$. Consequently $x \odot y \in D$. Thus (3.5) holds. Now let $x \in D$, $x \leq y$ for some $y \in A$. Therefore $x \sim \mathbf{1}$. Since $y \sim y$ and $\sim$ is a congruence on A, then $x \to y \sim \mathbf{1} \to y = y$. Since $\mathbf{1} \sim \mathbf{1}$, then $\mathbf{1} \in D$ and consequntly $x \to y = \mathbf{1} \in D$. Hence $y \sim \mathbf{1}$ and $y \in D$. Therefore (3.6) holds and D is a filter and a ds of A. □

Theorem 3.1. *Let A be a BL-algebra and F be a filter of A. Then by defining $x \sim y$ iff $(x \to y) \odot (y \to x) \in F$ we obtain an equivalence relation on A.*

Proof. First of all let us show that $\sim$ is an equivalence relation. In order $\sim$ to be an equivalence relation, it should reflexive, transitive and symmetric.

Let $x \sim x$. Then $(x \to x) \odot (x \to x) = \mathbf{1} \in F$ and $\sim$ is reflexive. In order to prove the symmetricity of $\sim$ we assume that $x \sim y$, which by definition of $\sim$ and commutativity of $\odot$ operation implies $(x \to y) \odot (y \to x) = (y \to x) \odot (x \to y) \in F$. Therefore, $y \sim x$ and $\sim$ is symmetric. Next we show that it is transitive. Assume x,y,$z \in A$ and $x \sim y$, $y \sim z$. Then

$$(x \to y) \odot (y \to x) \in F \text{ and } (y \to z) \odot (z \to y) \in F.$$

Applying (2.19) we obtain $(x \to y) \odot (y \to x) \leq (x \to y)$,$(y \to x)$ and similarly, $(y \to z) \odot (z \to y) \leq (y \to z)$,$(z \to y)$. Since F is a filter, then by (3.6) we deduce that $(x \to y) \in F$, $(y \to x) \in F$ and $(y \to z) \in F$, $(z \to y) \in F$, whence $(x \to y) \odot (y \to z) \in F$ and $(z \to y) \odot (y \to x) \in F$. By (2.31) we have $(x \to y) \odot (y \to z) \leq x \to z$ and $(z \to y) \odot (y \to x) \leq z \to x$. Therefore $(x \to z) \in F$ and $(z \to x) \in F$. By (3.5) we obtain $(x \to z) \odot (z \to x) \in F$, which proves that $x \sim z$. □

Theorem 3.2. *In a BL-algebra A, congruence relation on A and a filter are in one-to-one correspondence.*

Proof. Ommited. □

Let A/F be a quotient algebra, i.e. an induced set of equivalent classes

$$\{|x| \; |x \in A\},$$

where A is a BL-algebra and F is a filter of A. Define $|x| = |\mathbf{1}|$ iff $x \in F$. Thus for some element $x \in A$, $x/F = 1/F$ iff $x \in F$.

Theorem 3.3. *Let A be a BL-algebra and F be a filter of A. Then, an equivalence relation $\sim$ is a congruence on A.*

Proof. Let us show that $\sim$ is a congruence relation with respect to operations. First of all consider $\rightarrow$ operation. Let $x \sim y$ and $z \sim w$. Then by (2.19) and (3.6) we deduce

$$(x \rightarrow y) \odot (y \rightarrow x) \leq (x \rightarrow y),\ (y \rightarrow x)$$

and

$$(w \rightarrow z) \odot (z \rightarrow w) \leq (w \rightarrow z),\ (z \rightarrow w)$$

and consequently that $(x \rightarrow y) \in F$ and $(w \rightarrow z) \in F$. By (2.37) we have

$$x \rightarrow y \leq (w \rightarrow z) \rightarrow [(y \rightarrow w) \rightarrow (x \rightarrow z)],$$

from which we deduce

$$(x \rightarrow y) \odot (w \rightarrow z) \leq (y \rightarrow w) \rightarrow (x \rightarrow z),$$

where $(x \rightarrow y) \odot (w \rightarrow z) \in F$. Hence $(y \rightarrow w) \rightarrow (x \rightarrow z) \in F$. Similarly we obatin $(x \rightarrow z) \rightarrow (y \rightarrow w) \in F$. Therefore

$$[(y \rightarrow w) \rightarrow (x \rightarrow z)] \odot [(x \rightarrow z) \rightarrow (y \rightarrow w)] \in F.$$

Hence $(x \rightarrow z) \sim (y \rightarrow w)$.

Next consider $\odot$ operation. Assume $x \sim y, z$. By the similar reasoning as in the first part of our proof from the definition of the equivalence relation we deduce that $(x \rightarrow y), (y \rightarrow x) \in F$, $(x \rightarrow z), (z \rightarrow x) \in F$. By (2.27) we have that $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and similarly $y \rightarrow x \leq (y \odot z) \rightarrow (x \odot z)$, which by (3.6) implies that $(x \odot z) \rightarrow (y \odot z) \in F$ and $(y \odot z) \rightarrow (x \odot z) \in F$. Hence

$$[(x \odot z) \rightarrow (y \odot z)] \odot [(y \odot z) \rightarrow (x \odot z)] \in F$$

and therefore

$$(x \odot z) \sim (y \odot z).$$

Next assume $x \sim y$ and $z \sim w$. Since $(y \odot z) \sim (z \odot y)$, then we have

$$(x \odot z) \sim (y \odot z) \sim (z \odot y) \sim (w \odot y).$$

Now consider $\vee$ and $\wedge$ operations. Assume $x \sim y, z$. By the similar reasoning as in the first part of our proof from the definition of the equivalence relation we deduce that $(x \rightarrow y), (y \rightarrow x) \in F$, $(x \rightarrow z), (z \rightarrow x) \in F$. Further, from x,$z \leq x \vee z$

applying (2.33) and (3.6) we obtain $x \to z \leq x \to (x \vee z) \in F$ and $y \to x \leq y \to (x \vee z) \in F$. Since F is a filter, then by (3.5) we have that $[x \to (x \vee z)] \odot [y \to (x \vee z)] \in F$, which in turn by (2.19), (2.47) and (3.6) is

$$[x \to (x \vee z)] \odot [y \to (x \vee z)] \leq [x \to (x \vee z)] \wedge [y \to (x \vee z)] = [(x \vee y) \to (x \vee z)] \in F$$

Further from $x,y \leq x \vee y$ we deduce that $x \to y \leq x \to (x \vee y) \in F$ and $z \to x \leq z \to (x \vee y) \in F$, which implies in the similar manner $[(x \vee z) \to (x \vee y)] \in F$. Combining the two parts with the property (3.5) of the filter we obtain that

$$[(x \vee y) \to (x \vee z)] \odot [(x \vee z) \to (x \vee y)] \in F.$$

Therefore, $\sim$ is congruent with respect to the $\vee$ operation.

Further, from $x \wedge y \leq x,y$ applying (2.34) and (3.6) we obtain $x \to z \leq (x \wedge y) \to z \in F$ and $y \to x \leq (x \wedge y) \to x \in F$. Since F is a filter, then by (3.5) we have that $[(x \wedge y) \to z] \odot [(x \wedge y) \to x] \in F$, which in turn by (2.19), (2.49) and (3.6) is

$$[(x \wedge y) \to x] \odot [(x \wedge y) \to z] \leq [(x \wedge y) \to x] \wedge [(x \wedge y) \to z] = [(x \wedge y) \to (x \wedge z)] \in F.$$

By a similar reasoning we deduce that $[(x \wedge z) \to (x \wedge y)] \in F$. Combining the two parts with the property (3.5) of the filter we obtain that

$$[(x \wedge y) \to (x \wedge z)] \odot [(x \wedge z) \to (x \wedge y)] \in F$$

and consequently $\sim$ is congruent with respect to the $\wedge$ operation. $\square$

Now let us show that the *quotient algebra* A/F, where F is a filter of A, is a BL-algebra. First consider the theorem, defining the order relation on A/F.

Theorem 3.4. *Let F be a filter of a BL-algebra A. For all $x,y \in A$ on a quotient set A/F define $|x| \leq |y|$ iff $x \to y \in F$. Then $\leq$ is the order relation on A/F and $|\mathbf{1}|$, $|\mathbf{0}|$ are top and bottom elements, respectively.*

Proof. Since $x \to x = \mathbf{1} \in F$, then $|x| \leq |x|$ for all $|x| \in A/F$. Now let us show that $\leq$ is transitive. Let $|x| \leq |y|$ and $|y| \leq |z|$. Then $x \to y \in F$ and $y \to z \in F$. Thus by (3.5) we have $(x \to y) \odot (y \to z) \in F$, which by (2.32) and (3.6) implies $(x \to y) \odot (y \to z) \leq (x \to z) \in F$. Coonsequently $|x| \leq |z|$.

Next we show that $\leq$ is anti-symmetric. Assume $|x| \leq |y|$ and $|y| \leq |x|$. Thus $x \to y \in F$ and $y \to x \in F$ and therefore $(x \to y) \odot (y \to x) \in F$. This leads to $x \sim_F y$ and consequently to $|x| = |y|$. Thus we conclude that $\leq$ is an order relation on a quotient set A/F. By (2.23) and (2.22), $\mathbf{0} \to x = \mathbf{1} \in F$ and $x \to \mathbf{1} = \mathbf{1} \in F$. Hence $|\mathbf{0}| \leq |x| \leq |\mathbf{1}|$. $\square$

Now we are ready to show that a quotient algebra A/F is a BL-algebra.

Theorem 3.5. *If F is a filter of a BL-algebra A, then the obtained algebra*

$$\langle L/F, \leq, \vee, \wedge, \odot, \rightarrow, |\mathbf{1}|, |\mathbf{0}|\rangle$$

is a BL-algebra, in which the operations $\vee$ and $\wedge$ are defined via

$$|x \wedge y| = |x| \wedge |y|,$$
$$|x \vee y| = |x| \vee |y|,$$
$$|x \odot y| = |x| \odot |y|,$$
$$|x \rightarrow y| = |x| \rightarrow |y|.$$

Proof. First of all we show that $\langle L/F, \leq, \vee, \wedge\rangle$ is a lattice. Let $x,y \in A$. Since $\mathbf{1} \in F$ and by (2.21) from $x \wedge y \leq x,y$ we obtain that $(x \wedge y) \rightarrow x = \mathbf{1} \in F$ and $(x \wedge y) \rightarrow y = \mathbf{1} \in F$. Therefore $|x \wedge y| \leq |x|, |y|$.

Now we show that $|x \wedge y|$ is a greatest lower bound of $|x|$ and $|y|$. Suppose $|z| \leq |x|,|y|$. Then $z \rightarrow x \in F$ and $z \rightarrow x \in F$. Since F is a filter, then we have that $(z \rightarrow x) \odot (z \rightarrow y) \in F$. Consequently by (2.20)

$$(z \rightarrow x) \odot (z \rightarrow y) \leq (z \rightarrow x) \wedge (z \rightarrow y).$$

Then $(z \rightarrow x) \wedge (z \rightarrow y) \in F$. By (2.49) we obtain that

$$(z \rightarrow x) \wedge (z \rightarrow y) = z \rightarrow (x \wedge y),$$

which in turn implies that $|z| \leq |x \wedge y|$. This proves that $|x \wedge y|$ is a g.l.b. of $|x|$ and $|y|$.

Similarly we show that $|x \vee y|$ is a least upper bound of $|x|$ and $|y|$. Since $x,y \leq x \vee y$, then $x \rightarrow x \vee y = \mathbf{1} \in F$ and $y \rightarrow x \vee y = \mathbf{1} \in F$. Letting $|x|,|y| \leq |z|$ we have $(x \rightarrow z),(y \rightarrow z) \in F$ and consequently $(x \rightarrow z) \odot (y \rightarrow z) \in F$ by (3.5). Further reasoning that $(x \rightarrow z) \odot (y \rightarrow z) \leq (x \rightarrow z) \wedge (y \rightarrow z) = (x \vee y) \rightarrow z \in F$ we conclude that $|x \vee y| \leq |z|$ and therefore that $|x \vee y|$ is a l.u.b. of $|x|$ and $|y|$.

Next we show that L/F is a residuated lattice. First we examine the $\odot$ operation. Assume $|x| \leq |y|$ holds, which implies that $x \rightarrow y \in F$. Suppose $w \in A$. Then by (2.26), $y \leq w \rightarrow (y \odot w)$, which by (2.32) implies $x \rightarrow y \leq x \rightarrow [w \rightarrow (y \odot w)] = (x \odot w) \rightarrow (y \odot w)$. Since $x \rightarrow y \in F$, by (3.6) condition of a filter, we have $(x \odot w) \rightarrow (y \odot w) \in F$. Thus $|x| \odot |z| \leq |y| \odot |z|$. Moreover $\odot$ on L/F is commutative, associative operation such that $|x| \odot |\mathbf{1}| = |x \odot \mathbf{1}| = x$. Hence we realize that $\odot$ operation is isotone and it coincides with those in residuated lattice. Now we examine $\rightarrow$ operation. Since $|x| \odot |y| \leq |z|$ iff $(x \odot y) \rightarrow z$ iff $x \rightarrow (y \rightarrow z)$ iff

$|x| \leq |y| \to |z|$. Finally we conclude that L/D is a residuated lattice. The properties of L/D to be a BL-algebra follows from (2.44) and (2.45) of A. □

3.3 Prime deductive systems

A proper ds D is called *prime* if $x \vee y \in D$ implies $x \in D$ or $y \in D$ for any x,$y \in A$, where A is a BL-algebra.

Remark 3.3. *A deductive system D of a BL-algebra A is prime iff $(x \to y) \in D$ or $(y \to x) \in D$ for all x, $y \in A$.*

Proof. Let D be a prime ds of A. Assume $x \vee y \in D$. Since D is prime this implies that x or $y \in D$. By (2.37) we deduce that $x \to (y \to x) = \mathbf{1} \in D$ or $y \to (x \to y) = \mathbf{1} \in D$. Therefore $y \to x \in D$ or $x \to y \in D$. Conversely, let $(x \to y) \in D$ or $(y \to x) \in D$, where D is a ds of A (and in addition a filter of A). Let $x \vee y \in D$, $(x \to y) \in D$. Since

$$x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x] \leq (x \to y) \to y \in D,$$

then $y \in D$. Therefore, D is prime. □

Theorem 3.6. *Let A be a non-degenerate BL-algebra algebra, i.e. if $\mathbf{0} \neq \mathbf{1}$. Then it contains a prime deductive system.*

Proof. Since A is a non-degenerate algebra, then by Proposition 3.4, it has a maximal filter P, which by Proposition 3.5 is prime. Define such a set

$$\hat{P} = \bigcap_{y \in P^c} \{x \mid x \to y \in P^c\},$$

where $P^c = A \backslash P$ is a complement of P in A. We show that $\hat{P}$ is a prime ds.

First of all we deduce that $\hat{P}$ is a deductive system, i.e. (3.7) and (3.8) hold. To begin with we realize that $P^c \neq \emptyset$. Next for some $y \in P^c$, $y = \mathbf{1} \to y \in P^c$. Thus we deduce that $\mathbf{1} \in \hat{P}$ and therefore (3.7) holds. Now let $x \in \hat{P}$ and hence $x \to y \in P^c$ for some $y \in P^c$. Letting $x \to z \in \hat{P}$ we obtain $(x \to z) \to (x \to y) \in P^c$ for some $y \in P^c$. By (2.29) we have $z \to x \leq (x \to z) \to (x \to y)$. If we assume that $z \to x \in P$, then $(x \to z) \to (x \to y) \in P$, which contradicts the previously obtained result. Therefore $z \to x \in P^c$ for some $y \in P^c$ and consequently $z \in \hat{P}$, which validates the condition (3.8). Hence this proves that $\hat{P}$ is a ds.

Next we realize that $\hat{P}$ is proper, as for some $y \in P^c$, $y \to y = \mathbf{1} \in P$ and thus $y \notin \hat{P}$. This also proves that $\hat{P} \subseteq P$.

Finally we show that $\hat{P}$ is prime. Let x, $y \in A$, $x \vee y \in \hat{P}$, but $x \notin \hat{P}$ and $y \notin \hat{P}$. Suppose there are some z, $w \in P^c$ such that $x \to z$,$y \to w \in P$. Therefore

$z \vee w \in P^c$ and consequently $(x \vee y) \to (z \vee w) \in P^c$. Now from z, $w \leq (z \vee w)$ we deduce that $x \to z \leq x \to (z \vee w) \in P$ and $y \to z \leq y \to (z \vee w) \in P$, by (2.32). Therefore by (2.47) we obtain $[x \to (z \vee w)] \wedge [y \to (z \vee w)] \in P$, which contradicts the previously made assumption. Hence $x \in \hat{P}$ and $x \in \hat{P}$ and we conclude that proper ds $\hat{P}$ is prime. □

Theorem 3.7. *Let A be a BL-algebra and P be a prime ds of A. If there is a proper ds D such that $P \subseteq D \subseteq A$, then D is also a prime ds.*

Proof. Since P is a prime ds, then there are some x, $y \in A$ such that $x \to y \in P$ or $y \to x \in P$. Suppose $x \to y \in P$ and $x \vee y \in D$. Therefore $x \to y \in D$. Since $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x] \leq [(x \to y) \to y]$, $[(y \to x) \to x]$ then $x \vee y \leq (x \to y) \to y$, which implies that $y \in D$. Similarly the assumption that $y \to x \in P$ implies $x \in D$. This proves that P is prime. □

Theorem 3.8. *Any proper deductive system of a non-degenerate BL-algebra can be extended to a prime deductive system.*

Proof. Let A be a non-degenerate BL-algebra algebra and F be a proper ds. We know that F is also a lattice filter on A, by Proposition 3.6, and it is contained in a maximal filter P, by Proposition 3.3, which by Theorem 3.6 defines a prime ds $\hat{P}$, where $\hat{P} \subseteq P$. Recall that $P^c = A \backslash P$ is a complement of P in A and that $\hat{P}$ is defined such that $\hat{P} = \bigcap_{y \in P^c} \{x \mid x \to y \in P^c\}$. Now we have two proper deductive systems $\hat{P}$ and F, which leads by Theorem 3.8 to the conclusion, that the set $D_{F \cup \hat{P}}$ is also a ds of A and $F \cup \hat{P} \subseteq D_{F \cup \hat{P}}$.

We show that $D_{F \cup \hat{P}}$ is a proper ds. Let $x \in D_{F \cup \hat{P}}$ (1). Then by Proposition 3.8 for some y_1, y_2,..., $y_n \in \hat{P}$ and z_1, z_2,..., $z_n \in F$ we have

$$(y_1 \odot ... \odot y_n) \odot (z_1 \odot ... \odot z_n) \leq x,$$

where

$$z_1 \to (... \to (z_n \to (y_1 \to (... \to (y_n \to x)...)))...) = \mathbf{1} \in F.$$

Hence we conclude that $(y_1 \to (... \to (y_n \to x)...) \in F \subseteq P$.

Next assume $x \in P^c$. Then we obtain the following result. Since $\hat{P}$ is defined by Theorem 3.6, then $y_n \to x \in P^c$ and consequently $(y_1 \to (... \to (y_n \to x)...) \in P^c$, which is a contradiction. Hence $x \in P$ (2). By (1) and (2) we conclude that $D_{F \cup \hat{P}} \subseteq P$ is a proper ds. Finally since $\hat{P} \subseteq D_{F \cup \hat{P}}$ and $\hat{P}$ is prime, then by Proposition 3.7 $\hat{P}$ is prime. □

Proposition 3.10. *A BL-algebra A is linear iff any proper ds of A is prime.*

Proof. Assume first that A is linear BL-algebra. Then for all x,$y \in A$ either $x \vee y = x$ or $x \vee y = y$ holds, i.e. $x \vee y \in D$ iff $x \in D$ or $y \in D$, where D be a proper ds of A.

Hence D is prime. Conversely, assume that any proper ds of A is prime. Then $\{\mathbf{1}\}$ is a prime ds. Let x, $y \in A$. By (2.45), $(x \to y) \vee (y \to x) = \mathbf{1} \in \{\mathbf{1}\}$, which implies $(x \to y) \in \{\mathbf{1}\}$ or $y \to x \in \{\mathbf{1}\}$. Finally we conclude that $x \leq y$ or $y \leq x$ and A is linear. □

Theorem 3.9. *The quotient algebra A/F is linear iff F is prime.*

Proof. Let F be a prime filter. Then by Remark 3.3 we obtain $|x| \leq |y|$ or $|y| \leq |x|$ for all x,$y \in A$. Thus A/F is linear. Conversely, assume A/F is linear. Hence $|x| \leq |y|$ or $|y| \leq |x|$ implies either $x \to y \in F$ or $y \to x \in F$, for all x,$y \in A$. Let $x \vee y \in F$. Thus

$$x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x] \leq [(x \to y) \to y], [(y \to x) \to x].$$

Suppose $|x| \leq |y|$ holds. Thus $x \to y \in F$. Therefore $(x \to y) \to y \in F$ and $y \in F$. By assuming $y \to x \in F$ we obtain $(y \to x) \to x \in F$ and consequently $x \in F$. This proves that F is prime. □

3.3.1 Maximal deductive systems

A proper ds D is called *maximal* if for any other ds G of A, where $D \subseteq G$, either $D = G$ or $G = A$ holds.

Lemma 3.2. *If we define a partially ordered set of proper deductive systems of a BL-algebra A, containing a given a prime ds P, it is a totally ordered set with respect to the set theoretical inclusion $\subseteq$, i.e. given two proper deductive systems G and D either $G \subseteq D$ or $D \subseteq G$ holds.*

Proof. Given a prime ds P and two proper deductive systems G and D such that $P \subseteq G$ and $P \subseteq D$, assume $D \nsubseteq G$ and $G \nsubseteq D$. Therefore there exist x, $y \in A$ such that $x \in G$, $x \notin D$ and conversely, $y \in D$, $y \notin G$. Since P is prime, then by remark 3.3 either $x \to y \in P$ or $y \to x \in P$. Assume first $x \to y \in P \subseteq G$. Then $y \in G$. Conversely the assumption $y \to x \in P \subseteq D$ implies $x \in D$. Thus these both assumptions lead to a contradiction. Hence either $D \subseteq G$ or $G \subseteq D$. □

Theorem 3.10. *Let A be a non-degenerate BL-algebra. Then any proper ds of A can be extended to a maximal, prime ds.*

Proof. Let $D \subseteq A$ be a proper ds on A, which can be extended to a prime ds, by Theorem 3.8. Define $\mathcal{F} = \{G \mid D \subseteq G\}$ and $M = \bigcup\{G \mid G \in \mathcal{F}\}$, where G is a proper ds on A. By Lemma 3.2, $\mathcal{F}$ is a totally ordered set and by Theorem 3.7 any proper ds $G \in \mathcal{F}$ is a prime ds.

Now we show that M is a maximal ds. First we realize that $\mathbf{1} \in M$. Thus (3.7) holds. Next assume x, $x \to y \in M$. According to the construction of M, we conclude that x, $x \to y \in G$ for some $G \in \mathcal{F}$ and consequently that $y \in G \subseteq M$. Thus (3.8) holds and M is a ds.

Next we show that M is prime. Since G is a proper ds, then $\mathbf{0} \notin G$ for all $G \in \mathcal{F}$. Moreover $\mathbf{0} \notin M$. Therefore M is proper and, by Theorem 3.8, prime. It is also clear that M is maximal, which follows from the construction of it. □

Theorem 3.11. *If M is a maximal ds of a BL-algebra A, then for any $x \notin M$ there is some natural number $n \in \mathbb{N}$ such that $(x)^n \in M$.*

Proof. Let M be a maximal ds of a BL-algebra A and $D \subseteq A$ be a subset of A such that $D = \{z \in A \mid y \odot x^n \leq z\}$ for some $y \in M$, $n \in \mathbb{N}$ and $x \notin M$.

First we show that D is a ds. Since for any $y \in M$, $n \in \mathbb{N}$, $y \odot x^n \leq \mathbf{1}$, then $\mathbf{1} \in D$ and (3.7) holds. Next assume z,$z \to w \in D$. Then by construction of D there exists some y_1,$y_2 \in M$ and n,$m \in \mathbb{N}$ such that $y_1 \odot x^n \leq z$ and $y_2 \odot x^m \leq z \to w$. Hence by (2.29)

$$(y_1 \odot x^n) \odot (y_2 \odot x^m) = (y_1 \odot y2) \odot (x^{n+m}) \leq z \odot (z \to w) \leq z.$$

Since the condition y_1,$y_2 \in M$ by (3.8) implies that $y_1 \odot y_2 \in M$, then from the construction of D we conclude that $(y_1 \odot y2) \odot (x^{n+m}) \in D$. Consequently from the above result we conclude $z \in D$. Therefeore D is a deductive system. Since for any $y \in M$, $y \odot x \leq y$, then $y \in D$ and therefore $M \subseteq D$. On the other hand, since $\mathbf{1} \in M$ and $\mathbf{1} \odot x \leq x$, then $x \in D$. Along with the assumption $x \notin M$ and since M is a maximal ds we conclude that $D = A$, which follows from the definition of maximal ds. Since $\mathbf{0} \in A$, then $\mathbf{0} \in D$, which implies that $y \odot x^n \leq \mathbf{0}$ for some $y \in M$, $n \in \mathbb{N}$. This is immediately followed by the fact that $y \leq x^n \to \mathbf{0}$ and consequently that $y \leq (x^n)^*$. Therefore $(x^n)^* \in M$. □

Theorem 3.12. *Let A be a BL-algebra and M a maximal ds of A. Then the generated quotient algebra A/M is an MV-algebra.*

Proof. Let $|x|_M \in A/M$ such that $|x|_M \neq |\mathbf{1}|_M$. Recalling that $|x|_M = |\mathbf{1}_M|$ iff $x \in M$, then from $|x|_M \neq |\mathbf{1}|_M$ we deduce that $x \notin M$. This by Theorem 3.11 implies that there is some integer $n \in \mathbb{N}$ such that $(x^n)^* \in M$. Thus $|x^n|_M^* = |(x^n)^*|_M = |\mathbf{1}|_M$. Moreover since $|x^n|_M \leq |(x^n)^{**}| = |\mathbf{1}^*|_M = |\mathbf{0}|_M$, then $|x^n|_M = |\mathbf{0}|_M$ and L/M is a locally finite algebra. Since the generated quotient algebra L/M satisfies $|x|^{**} = |x|$ for $|x| \in A/M$, then by Theorem 2.2 A/M is an MV-algebra. □

4. BOOLEAN DEDUCTIVE SYSTEMS

Material for this chapter is taken from [7].

4.1 Implicative deductive systems

Definition 4.1. *Let A be a BL-algebra. Then any deductive system D is called* weakly implicative *if for all x,y,$z \in A$ the following holds*

$$\text{If } x \to (z^* \to y) \in D \text{ and } y \to z \in D, \text{ then } x \to (z^* \to z) \in D. \tag{4.1}$$

Proposition 4.1. *In a BL-algebra A any deductive system D is weakly implicative.*

Proof. Suppose $x \to (z^* \to y) \in D$ and $y \to z \in D$. Since D is a filter, by (2.34) and (2.31) we have

$$\begin{aligned}
[x \to (z^* \to y)] \odot (y \to z) &= ((x \odot z^*) \to y) \odot (y \to z) \in D \\
&\leq x \odot z^* \to z \\
&= x \to (z^* \to z) \in D.
\end{aligned}$$

□

Definition 4.2. *Let A be a BL-algebra. Then a deductive system is called a* Boolean ds *if $x \vee x^* \in D$ for all $x \in A$.*

Definition 4.3. *Letting A be a BL-algebra, a deductive system D is* implicative *if for all x,y,$z \in A$ the following holds*

$$\text{If } x \to (z^* \to y) \in D \text{ and } y \to z \in D, \text{ then } x \to z \in D. \tag{4.2}$$

Lemma 4.1. *Let D be an implicative ds and $x \to z^{**} \in D$. Then $x \to z \in D$ holds.*

Proof. Let A be a BL-algebra. From $x \to z^{**} \in D$ we deduce $x \to (z^* \to \mathbf{0}) \in D$. Since D is an implicative system and $\mathbf{0} \to z = \mathbf{1} \in D$, for any $z \in A$, then $x \to z \in D$. □

Theorem 4.1. *Let A be a BL-algebra. Then the following equations are equivalent*

(i) D is implicative ds,

(ii) D is Boolean ds,

(iii) A/D is a Boolean algebra.

Proof. First we show that an implicative ds is a Boolean ds. Let D be an implicative ds of BL-algebra A. Since $\mathbf{1} \in D$, then for some $x \in A$

$$(x^* \to x) \to [x^* \to (x^* \to x) \odot x^*] = [(x^* \to x) \odot x^*] \to [(x^* \to x) \odot x^*] = \mathbf{1} \in D.$$

Additionally by (2.42) and (2.23)

$$[(x^* \to x) \odot x^*] \to x = (x^* \wedge x) \to x = \mathbf{0} \to x = \mathbf{1} \in D.$$

Hence we realize that (4.2) holds and thus $(x^* \to x) \to x \in D$. Moreover $(x^{**} \to x^*) \to x^* \in D$. From $(x^{**} \to x^*) \to x^* \leq (x^{**} \to x^*) \to x^*$ by Galois correspondence we deduce that

$$(x^{**} \to x^*) \odot [(x^{**} \to x^*) \to x^*] \leq x^*,$$

which further by (2.28) implies

$$x^{**} \odot \{(x^{**} \to x^*) \odot [(x^{**} \to x^*) \to x^*]\} \leq x^{**} \odot x^* = \mathbf{0}.$$

Consequently we have

$$\begin{aligned}
(x^{**} \to x^*) \to x^* &\leq [x^{**} \odot (x^{**} \to x^*)] \to \mathbf{0} \\
&= [x^{**} \odot (x^{**} \to x^*)]^* \\
&= (x^{**} \wedge x^*)^* \\
&= (x^* \vee x)^{**}.
\end{aligned}$$

Since $(x^{**} \to x^*) \to x^* \in D$ and D is a ds, then the latter result implies

$$[(x^{**} \to x^*) \to x^*] \to (x^* \vee x)^{**} = \mathbf{1} \in D,$$

which in turn implies $[(x^{**} \to x^*) \to x^*] \to (x^* \vee x) \in D$, by Lemma 4.1. Therefore $(x^* \vee x) \in D$, which shows that the condition for a ds to be a Boolean ds holds.

Conversely, assume D is a Boolean ds. We show that (4.2) holds. Since D is a ds, then $x \to (z^* \to y) \in D$ and $y \to z \in D$ imply that

$$\begin{aligned}
[x \to (z^* \to y)] \odot (y \to z) &= [x \odot z^* \to y] \odot (y \to z) && \text{by (2.34)} \\
&\leq z^* \odot x \to z \in D && \text{by (2.31)} \\
&= z^* \to (x \to z) \in D && \text{by (2.34)}
\end{aligned}$$

At this point we realize that $z \to (x \to y) = \mathbf{1} \in D$, by (2.36). Therefore by (2.47) we conclude that

$$[z^* \to (x \to y)] \wedge [z \to (x \to y)] = (z^* \vee z) \to (x \to z) \in D.$$

Since D is a Boolean ds, then $(z^* \vee z) \in D$ and the latter equation implies that $(x \to z) \in D$. This proves that D is an implicative ds.

Next we show that if D is a Boolean deductive system, then the corresponding quotient algebra L/D is a Boolean algebra. By Theorem 3.5 A/D is a BL-algebra, which by Theorem 2.1 is a distributive lattice. Recall that for some element $x \in A$, $x/D = \mathbf{1}/D$ iff $x \in D$. Then $\mathbf{1}/D$ and $\mathbf{0}/D$ are the largest and least elements, respectively. By (2.42) if $\mathbf{1}/D = [x/D] \vee [x/D]^* = [x \vee x^*]/D$, then

$$\mathbf{0}/D = [x/D] \wedge [x/D]^* = [x \wedge x^*]/D.$$

Therefore A/D is a Boolean algebra. □

4.2 Bipartite BL-algebra

Theorem 4.2. *Let A be a BL-algebra. Then the following equations are equivalent*

(i) D is maximal and Boolean ds,

(ii) D is prime and Boolean ds,

(iii) D is proper ds and $x \in D$ or $x^* \in D$, for all $x \in A$.

Proof. First we show that (i) and (ii) are equivalent. Assume D is maximal and Boolean ds. By Theorem 3.5 any maximal ds is prime. Hence D is prime and Boolean ds and (ii) holds.

Next we show that (ii) and (iii) are equivalent. (ii) implies that $x \vee x^* \in D$, for all $x \in A$. Since D is prime, then either $x \in D$ or $x^* \in D$. Thus (iii) holds.

Finally from (iii) follows (i). Since $x \in D$ or $x^* \in D$, then from x,$x^* \leq x \vee x^*$ for all $x \in A$ we deduce $x \vee x^* \in D$. Therefore D is a Boolean deductive system. Assume from (iii) that $x \notin D$, while $x^* \in D$. Now we realize that D is also maximal. This follows from the fact that by Theorem 3.11, if $x \notin D$, then there exist an integer $n \in \mathbb{N}$, namely $n = \mathbf{1}$ such that $(x^n)^* \in D$ or consequently that $x^* \in D$. Therefore D is maximal ds. □

Proposition 4.2. *Let D be a proper deductive system of a BL-algebra A. Then $D \cup D^*$ is a subalgebra of A, where*

$$D^* = \{x \in A \mid x \leq y^* \text{ for some } y \in D\}.$$

Proof. By defining D and D^* in such a way we obtain that $\{\mathbf{0},\mathbf{1}\} \subseteq D \cup D^*$ and $D \cap D^* = \emptyset$. Otherwise the condition $x \in D \cap D^*$ imply that $x \in D$, $x \leq y^*$ for some $y \in D$, which in turn means $x \odot y \in D$ since D is a filter. Hence $\mathbf{0} \in D$, which is a contradiction since D is a proper ds and $\mathbf{0} \notin D$. Thus $D \cap D^* = \emptyset$. In order to show that the set $D \cup D^*$ is closed with respect to the BL-algebra operations $\odot$,$\rightarrow$,*,$\wedge$,$\vee$ we consider the 'Boolean style truth table'.

x	y	x^*	$x \odot y$	$x \wedge y$	$x \vee y$	$x \rightarrow y$
D	D	D^*	D	D	D	$^{(3)}D$
D	D^*	D^*	D^*	D^*	D	$^{(4)}D^*$
D^*	D	$^{(1)}D$	D^*	D^*	D	$^{(3),(5)}D$
D^*	D^*	D	D^*	D^*	$^{(2)}D^*$	$^{(5)}D$

We show that (1) holds. Let $x \in D^*$. Then there exists some $y \in D$ such that $x \leq y^*$. Therefore $\mathbf{1} = x \rightarrow y^* \leq y^{**} \rightarrow x^* \leq \mathbf{1}$, by (2.43). Thus $y^{**} \rightarrow x^* = \mathbf{1}$, which implies that $y^{**} \leq x^*$. Additionally by (2.38) we conclude $y \leq y^{**} \leq x^*$. Since $y \in D$, then the latter implies that $x^* \in D$.

To justify the column corresponding to $x \odot y$ we reason in the following way. Since $x \odot y \leq x$,y, then the conditions $x \in D$, $y \in D^*$ imply that there is some $a \in D^*$ such that $y \leq a^*$ and consequently that $x \odot y \leq y \leq a^*$. Thus $x \odot y \in D^*$. Similarly the conditions $x \in D^*$, $y \in D$ imply $x \odot y \in D^*$. Finally letting $x \in D$ and $y \in D$ we obtain $x \odot y \in D$, since D is a filter.

To justify the column, corresponding to $x \wedge y$ we reason reason in the following way. Assume $x \in D$ and $y \in D^*$. Then there is an element $a \in D$ such that $y \leq a^*$. Then we reason that $x \wedge y = x \odot (x \rightarrow y) \leq y \leq a^*$ and therefore $x \wedge y \in D^*$. Now we verify the last row of this column. Assume $x \in D^*$ and $y \in D^*$. Then there exist such a, $b \in D$ such that $x \leq a^*$ and $y \leq b^*$. Since D is a lattice filter by (3.6), then by 3.4 we know that $a \vee b \in D$. Thus we deduce $x \wedge y \leq a^* \wedge b^* = (a \vee b)^*$ and consequently $x \wedge y \in D^*$. Finally letting $x \in D$ and $y \in D$ we obtain $x \wedge y \in D$, since D is a lattice filter.

Next we show how the results are obtained in the column, corresponding to $x \vee y$. Assume first that $x \in D$ and $y \in D^*$. Then there is an element $a \in D$ such that $y \leq a^*$. Since $x \leq (x \vee y)$ we have that $x \rightarrow (x \vee y) = \mathbf{1} \in D$ and consequently that $x \vee y \in D$. Now let $x \in D$ and $y \in D$. Then we obtain $x \wedge y \in D$, since D is a lattice filter and consequently that $x \wedge y \leq x$, $y \leq x \vee y \in D$ by the same reasoning. Next we show that (2) is valid. Let x,$y \in D^*$. Then there exist some a,$b \in D$ such that $x \leq a^*$ and $y \leq b^*$, where $x \vee y \leq a^* \vee b^*$ and $a \wedge b \in D$ by the truth table. From $a \leq a^{**}$, $b \leq b^{**}$ we deduce

$$a \wedge b \leq a^{**} \wedge b^{**} = (a^* \vee b^*)^* = (a^* \vee b^*) \rightarrow \mathbf{0}$$

iff

$$(a \wedge b) \odot (a^* \vee b^*) \leq \mathbf{0}$$

iff

$$(a^* \vee b^*) \leq (a \wedge b) \to \mathbf{0} = (a \wedge b)^*$$

Therefore $x \vee y \leq a^* \vee b^* \leq (a \wedge b)^*$. Hence $x \vee y \in D^*$. The remaining proofs for the operation $\vee$ are related to the fact that x,$y \leq x \vee y$, which implies that $x \vee y \in D$.

Finally we show that the operations in the last column corresponding to $x \to y$ are valid. (3) holds since from $x \odot y \leq y$ iff $y \leq x \to y$, where $y \in D$ follows that $x \to y \in D$. Next we derive (4). We have $x \in D$, $y \in D^*$. Then $y \leq a^*$ for some $a \in D$, which implies $x \odot a \in D$. Moreover by (2.32) we have

$$x \to y \leq x \to a^* = x \to (a \to \mathbf{0}) = (x \odot a) \to \mathbf{0} = (x \odot a)^*.$$

Therefore $x \to y \in D^*$. To prove (5) we have $x \in D^*$, which implies that there exists $a \in D$ such that $x \leq a^*$. Thus $\mathbf{1} = x \to a^* \leq a^{**} \to x^* \leq \mathbf{1}$ and consequently $a^{**} \to x^* = \mathbf{1}$, which implies that $a^{**} \leq x^*$. Hence $a \leq a^{**} \leq x^* = x \to \mathbf{0} \leq x \to y$. Consequently $x \to y \in D$. Therefore we conclude that $D \cup D^*$ is a subalgebra of BL-algebra. □

Now we realize that D is a maximal ds of the BL-algebra $D \cup D^*$. Moreover for all $x \in D \cup D^*$, $x \vee x^* \in D$, which proves that D is a Boolean ds of $D \cup D^*$. This leads to the following definitions

Definition 4.4. *A BL-algebra A is* bipartite *if $A = M \cup M^*$, where M is some maximal ds of A.*

Definition 4.5. *A BL-algebra A is* strongly bipartite *if $A = M \cup M^*$, where M is any maximal ds of A.*

From the definitions we can easily see that a strongly bipartite BL-algebra is bipartite.

Theorem 4.3. *A BL-algebra A has a proper Boolean ds iff A is bipartite.*

Proof. First let A be bipartite. Then $A = M \cup M^*$ for some maximal and, trivially, proper ds of A. Therefore for some $x \in A$ if $x \notin M$, $x \in M^*$, whence $x^* \in M$. Thus by x,$x^* \leq x \vee x^*$ we deduce $x \vee x^* \in M$ and M is a Boolean ds. Conversely, let M be a proper Boolean ds of A. Since by Theorem 3.10 any proper ds can be extended to a maximal ds, M is a maximal Boolean ds. Let $x \in A$. The fact that M is a maximal Boolean ds by Theorem 4.2 implies that M is a proper ds and if $x \notin M$ then $x^* \in M$. Now we realize that $x \leq x^{**} = (x^*)^*$ for some $x^* \in M$. Hence $x \in M^*$, which proves that $A = M \cup M^*$.

□

Proposition 4.3. *Let A be a BL-algebra. Then by defining*

$$B(A) = \bigcap\{D \mid D \text{ is a Boolean ds of } A\}$$

and

$$\langle supA\rangle = \{x \in A \mid y_1 \odot ... \odot y_n \leq x \text{ for some } y_1, ..., y_n \in sup\ A\},$$

where

$$supA = \{z \vee z^* \mid z \in A\}$$

obtain that $B(A) = \langle supA\rangle$.

Proof. First we show that $B(A) \subseteq \langle \text{sup}A\rangle$. From the definition of $B(A)$ we can easily see that $B(A) \neq \emptyset$. Moreover it is the smallest Boolean ds. From the definition of sup$A$ we realize that if it is a ds, then it is Boolean. Therefore $B(A) \subseteq \text{sup}A$. Additionally by Theorem 3.8, we have that $\langle \text{sup}A\rangle$ is also a ds of A. It is clear that $\langle \text{sup}A\rangle$ is Boolean. Hence $B(A) \subseteq \langle \text{sup}A\rangle$.

Now we show that $\langle \text{sup}A\rangle \subseteq B(A)$. Assume $x \in \langle \text{sup}A\rangle$. Then

$$(z_1 \vee z_1^*) \odot ... \odot (z_n \vee z_n^*) \leq x$$

for some $z_1, ..., z_n \in A$. Since $B(A)$ is a Boolean ds, then

$$(z_1 \vee z_1^*) \odot ... \odot (z_n \vee z_n^*) \in B(A),$$

which implies that $x \in B(A)$. Therefore $\langle \text{sup}A\rangle \subseteq B(A)$. Finally we conclude $B(A) = \langle \text{sup}A\rangle$. □

Theorem 4.4. *In any BL-algebra A the following are equivalent*

(i) A is strongly bipartite,

(ii) any maximal ds is boolean

(iii) $B(A) \subseteq M(A)$, where $M(A) = \bigcap\{M \mid M \text{ is a maximal ds of } A\}$.

Proof. Assume (i) holds. Then $A = M \cup M^*$ for any maximal ds of A. Let M be a maximal ds of A and $x \in A$. Then the condition $x \notin M$ implies that $x \in M^*$ and therefore $x^* \in M$. Hence from x,$x^* \leq x \vee x^*$ we conclude $x \vee x^* \in M$. Consequently M is a Boolean ds and (ii) holds. Conversely, let (ii) hold. By Theorem 3.10 any maximal ds is prime. Therefore as M is a Boolean ds and for all $x \in A$ the condition $x \vee x^* \in M$ implies either $x \in M$ or $x^* \in M$. Assume $x \notin M$. Then $x^* \in M$. Recalling that $M^* = \{x \in A \mid x \leq y^* \text{ for some } y \in M\}$, the condition $x \leq (x^*)^* = x^{**}$ implies $x \in M^*$. Thus for any maximal ds M we conclude that $L = M \cup M^*$, i.e. (i) holds.

Now assume (ii) holds. Let $x \in \langle \sup A \rangle = B(A)$. Then there are some $z_1, ..., z_n \in A$ such that $(z_1 \vee z_1^*) \odot ... \odot (z_n \vee z_n^*) \leq x$. Since by (ii) any maximal ds is Boolean, then $(z_1 \vee z_1^*) \odot ... \odot (z_n \vee z_n^*) \in M$ implying $x \in M$, for all maximal ds M. Thus $x \in M(A)$ and consequently $B(A) \subseteq M(A)$, i.e. (iii) holds. Conversely, let (iii) hold. Therefore $B(A) \subseteq M(A) \subseteq M$, for any maximal ds M. Since $B(A)$ is a Boolean ds, M is also Boolean. $\square$

5. IMPLICATIVE FILTERS

Material for this chapter are taken from [5].

Definition 5.1. *A non-empty subset F of a BL-algebra A is called an* implicative filter *of A if it satisfies the following conditions:*

$$\mathbf{1} \in F, \tag{5.1}$$

$$\textit{if } x \to (y \to z) \in F \textit{ and } x \to y \in F, \textit{ then } x \to z \in F, forallx, y, z \in F \tag{5.2}$$

Theorem 5.1. *Any implicative filter of a BL-algebra A is a filter, but the converse is not true.*

Proof. Let x,$x \to y \in F$ and F be an implicative filter. We prove that F is a deductive system and thus is a filter. At first by the definition of the implicative filter $\mathbf{1} \in F$. Then $\mathbf{1} \to (x \to y) \in F$ and $\mathbf{1} \to x \in F$ and consequently $\mathbf{1} \to y \in F$. Moreover as $\mathbf{1} \to y = y$, then $y \in F$. Hence (3.5) and (3.6) hold, i.e. F is a filter.

Next we show that the converse generaly is not true. In order to do this consider the following example.

Example 1. Let $B = \{\mathbf{0}, a, b, \mathbf{1}\}$. The following binary operations $\odot$ and $\to$ are

$\odot$	$\mathbf{0}$	a	b	$\mathbf{1}$	$\to$	$\mathbf{0}$	a	b	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a	$\mathbf{0}$	$\mathbf{0}$	a	a	a	a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	$\mathbf{0}$	a	b	b	b	$\mathbf{0}$	a	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	a	b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	a	b	$\mathbf{1}$

called a local peculiar BL-algebra. We can easily see that $F = \{b, \mathbf{1}\}$ is a filter, whilst it is not an implicative filter, as $a \to (a \to \mathbf{0}) = \mathbf{1} \in F$, $a \to a = \mathbf{1} \in F$, but $a \to \mathbf{0} = a \notin F$.

□

We can easilily notice that A itself is an implicative filter. Then for some element $x \in A$ the set $F = \{y \in A \mid x \leq y\}$ is an implicative filter, i.e. the interval $[x, \mathbf{1}]$ is an implicative filter. This can be additionally proved by the fact that F is a filter of A.

Theorem 5.2. *Let F be a non-void subset of a BL-algebra A. Then the following are equivalent*

(i) F is an implicative filter,

(ii) F is a filter $y \to (y \to x) \in F$ implies $y \to x \in F$, for all x,$y \in F$,

(iii) F is a filter, if $z \to (y \to x) \in F$, then $(z \to y) \to (z \to x) \in F$, for all x,y,$z \in F$,

(iv) $\mathbf{1} \in F$, if $z \to (y \to (y \to x)) \in F$ and $z \in F$, then $y \to x \in F$, for all x,y,$z \in A$.

Proof. Assume (i) holds. At first we realize that by Theorem 5.1, F is a filter. Next since F is an implicative filter letting $y \to (y \to x) \in F$ and realizing that $y \to y = \mathbf{1} \in F$, we deduce that $y \to x \in F$. Hence (ii) holds.

Next assume (ii) holds. Let $z \to (y \to x) \in F$. Applying (2.29) and (2.30), obtain

$$y \to x \leq (z \to y) \to (z \to x)$$

and

$$z \to (y \to x) \leq z \to [(z \to y) \to (z \to x)] = z \to [z \to ((z \to y) \to x)],$$

respectively. Since F is a filter, then $z \to (y \to x) \in F$ implies

$$z \to [z \to ((z \to y) \to x)] \in F.$$

By (ii) we deduce $z \to ((z \to y) \to x) = (z \to y) \to (z \to x) \in F$. Thus (iii) holds.

Now let (iii) hold. First of all we realize that $\mathbf{1} \in F$, as F is a filter. Let $z \to [y \to (y \to x)] \in F$ and $z \in F$, which implies that $y \to (y \to x) \in F$. Hence by (iii) we deduce that $(y \to y) \to (x \to y) \in F$. Since

$$y \to x = \mathbf{1} \to (y \to x) = (y \to y) \to (y \to x) \in F,$$

we conclude $y \to x \in F$.

Finally assume (iv). Let $z \to (y \to x) \in F$ and $z \to y \in F$. Since F is a filter, then $z \to (y \to x) \in F$ implies

$$z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to [z \to (z \to x)] \in F.$$

Since $z \to y \in F$, then by (iv) we deduce $z \to x \in F$, i.e. F is an implicative filter. □

Proposition 5.1. *Let A be a BL-algebra and F be an implicative filter of A. If there is a filter G such that $F \subseteq G \subseteq A$, then G is also an implicative filter.*

Proof. Let $w = z \to (y \to x) \in G$. Then applying two times (2.35), obtain

$$\begin{aligned} \mathbf{1} &= w \to [z \to (y \to x)] \\ &= z \to [w \to (y \to x)] \\ &= z \to [y \to (w \to x)]. \end{aligned}$$

Since F is a filter and $\mathbf{1} \in F$, then $z \to [y \to (w \to x)] \in F$, which by Theorem 5.2 (iii) implies $(z \to y) \to [z \to (w \to x)] \in F \subseteq G$. Moreover

$$\begin{aligned} (z \to y) \to [z \to (w \to x)] &= (z \to y) \to [w \to (z \to x)] \\ &= w \to [(z \to y) \to (z \to x)]. \end{aligned}$$

Therefore $w \to [(z \to y) \to (z \to x)] \in F \subseteq G$. Since G is also a ds, then $w \in G$ implies $(z \to y) \to (z \to x) \in G$. By Theorem 5.2, G is an implicative filter. □

Proposition 5.2. *In a BL-algebra A the following are equivalent*

(i) A is a Gödel algebra,

(ii) Any filter of A is an implicative filter,

(iii) $\{\mathbf{1}\}$ is an implicative filter of A.

Proof. First of all we recall that a Gödel algebra is a BL-algebra such that $x \odot x = x$ holds. We show that from (i) follows (ii). Let F be an arbitrary filter of A, where A is a Gödel algebra. Assume $y \to (y \to x) \in F$. Then

$$y \to (y \to x) = y^2 \to x = y \to x \in F,$$

which by Theorem 5.2 implies that F is an implicative filter.

Next we realise that by (ii) any filter of A is an implicative filter. Hence $\{\mathbf{1}\}$ is an implicative filter. Thus from (ii) we deduce (iii).

Now from (iii) we infer (i). Since $\{\mathbf{1}\}$ is an implicative filter and for all $x \in A$ we have

$$\mathbf{1} = (x \odot x) \to (x \odot x) = x \to (x \to x \odot x) = x \to (x \to x^2) \in F$$

as well as $x \to x = \mathbf{1} \in F$, we conclude that $x \to x^2 = \mathbf{1} \in F$. Hence $x \leq x^2$. In addition we know that $x^2 = x \odot x \leq x$. Therefore we obtain $x = x^2$, which proves that A is a Gödel algebra. □

The Theorem 5.1 helps us to realize that since an implicative filter is a filter of BL-algerba, then the corresponding quotient algebra generated by an implicative filter is a BL-algebra, by Theorem 3.5. Now we can observe a BL-algebra generated by an implicative filter. Thus we have

Theorem 5.3. *A filter F of a BL-algebra A is implicative iff A/F is a Gödel algebra.*

Proof. Let F be an implicative filter if a BL-algebra A and let $y \to (y \to x) \in F$. Then by Theorem 5.2 we have that $y \to x \in F$. Recalling that $x/F = \mathbf{1}/F$ iff $x \in F$ for all $x \in A$, we obtain

$$y/F \to (y/F \to x/F) = (y \to (y \to x))/F = \mathbf{1}/F$$

and

$$y/F \to x/F = (y \to x)/F = \mathbf{1}/F$$

for all x,$y \in A$. Since $\mathbf{1} \in F$, then $\mathbf{1} \in \{\mathbf{1}/F\}$ and $\{\mathbf{1}/F\}$ is an implicative filter of A/F, which by Proposition 5.2 implies that A/F is a Gödel algebra. Conversely, assume A/F is a Gödel algebra. Then by Theorem 5.2 any filter of A/F is implicative and $\{\mathbf{1}/F\}$ is an implicative filter of A/F. Let $(y \to (y \to x)) \in F$. Therefore

$$y/F \to (y/F \to x/F) = (y \to (y \to x))/F = \mathbf{1}/F \in \{\mathbf{1}/F\}.$$

Since $\{\mathbf{1}/F\}$ is an implicative filter, then by Theorem 5.2 the latter condition implies $y/F \to x/F \in \{\mathbf{1}/F\}$ and $y/F \to x/F = \mathbf{1}/F$. Hence $y \to x \in F$ and F is an implicative filter. □

5.1 Positive implicative filters

Definition 5.2. *A non-void subset F of a BL-algebra A is a* positive implicative filter *if for all x,y,$z \in A$ it satisfies the following conditions*

$$\mathbf{1} \in F, \tag{5.3}$$

$$\text{if } x \to ((y \to z) \to y) \in F \text{ and } x \in F, \text{ then } y \in F. \tag{5.4}$$

Like for implicative filters we can find out the connection between positive implicative filters, implicative filters and and filters. The following theorems observe this connection.

Theorem 5.4. *Any positive implicative filter of a BL-algebra A is a filter.*

Proof. Let F be a filter of a BL-algebra A. Since filters and ds coincide in BL-algebra, it is enough to show that F is a ds of A. Fisrt of all we realize that $\mathbf{1} \in F$, which validates the condition (3.7). Now let x,$x \to y \in F$. We show that this implies $y \in F$. Since

$$x \to y = x \to (\mathbf{1} \to y) = x \to [(y \to \mathbf{1}) \to y] \in F$$

and as F is a positive implicative filter, then $y \in F$. This proves that F is a ds (filter) of BL-algebra. □

In order to be a positive implicative filter, a filter should satisfy the additional condition. Thus we have

Theorem 5.5. *A filter F of a BL-algebra A is a positive implicative filter iff $(y \to z) \to y \in F$ implies $y \in F$.*

Proof. Let F be a positive implicative filter of BL-algebra A and $(y \to z) \to y \in F$. Since

$$(y \to z) \to y = \mathbf{1} \to [(y \to z) \to y] \in F$$

and as F is a positive implicative filter, then $y \in F$.

Conversely, assume F is a filter of A and $(y \to z) \to y \in F$ implies $y \in F$. We show that F is a positive implicative filter. Additionally let $x \in F$ and also that $x \to ((y \to z) \to y) \in F$. Then $(y \to z) \to y \in F$, as F is a filter. This by our assumption implies that $y \in F$. Therefore F is a positive implicative filter. □

Theorem 5.6. *Any positive implicative filter of a BL-algebra A is an implicative filter.*

Proof. Let F be a positive implicative filter, $x \to (y \to z) \in F$ and $x \to y \in F$. In order F to be an implicative filter, the condition $x \to z \in F$ sholud hold. We deduce that

$$y \to z \leq (x \to y) \to (x \to z),$$

$$x \to (y \to z) \leq x \to [(x \to y) \to (x \to z)] = (x \to y) \to [x \to (x \to z)].$$

Since $x \to (y \to z) \in F$ and F is a filter by Theorem 5.4, then

$$(x \to y) \to [x \to (x \to z)] \in F.$$

Additionally as $x \to y \in F$, then $x \to (x \to z) \in F$. By (2.24) and (2.29) we infer

$$x \to z \leq [(x \to z) \to z] \to z,$$

$$x \to (x \to z) \leq x \to \{[(x \to z) \to z] \to z\}.$$

From the previous result $x \to (x \to z) \in F$ we deduce that

$$x \to \{[(x \to z) \to z] \to z\} \in F.$$

Moreover we realize that

$$\mathbf{1} \to \{[(x \to z) \to z] \to (x \to z)\} = [(x \to z) \to z] \to (x \to z) \in F.$$

Since $\mathbf{1} \in F$ and F is a positive implicative filter this implies that $x \to z \in F$ and F is an implicative filter.

Next we show that the converse is not true. In order to do this consider the following example.

Example 2. Let A be a Gödel algebra. The binary operations $\odot$ and $\to$ are defined as in (2.51), i.e.

$$\text{Gödel structure: } \begin{cases} x \odot y = \min(x, y) \\ \qquad x \to y = y \end{cases}$$

Here the residuum of a t-norm is defined for $x > y$ as for $x \leq y$, $x \to y = \mathbf{1}$. Then the interval $F = [\frac{2}{3}, \mathbf{1}]$ is an implicative filter. Now we can easily see that

$$\begin{aligned} \frac{3}{4} \to [(\frac{1}{4} \to \frac{1}{12}) \to \frac{1}{4}] &= \frac{3}{4} \to (\frac{1}{12} \to \frac{1}{4}) \\ &= \frac{3}{4} \to \mathbf{1} = \mathbf{1} \in F \end{aligned}$$

and $\frac{3}{4} \in F$, while $\frac{1}{4} \notin F$ and F is not a positive implicative filter.

□

However an implicative filter can be a positive implicative filter if it meet the certain conditions. In other words we have

Theorem 5.7. *An implicative filter F of BL-algebra A is a positive implicative filter iff for all x,y $\in F$, $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$.*

Proof. Let F be an implicative filter and z,$z \to ((x \to y) \to x) \in F$. Additionally we let $(x \to y) \to y \in F$ imply $(y \to x) \to x \in F$. In order F to be a positive implicative filter, the condition $x \in F$ should hold. By (2.24) and (2.32) we have

$$x \leq (x \to y) \to y,$$

$$(x \to y) \to x \leq (x \to y) \to [(x \to y) \to y].$$

Since by Theorem 5.4 F is a filter and $(x \to y) \to y \in F$, then this implies that $(x \to y) \to [(x \to y) \to y] \in F$. Moreover as F is an implicative filter, the latter result along with $(x \to y) \to (x \to y) = \mathbf{1} \in F$ implies that $(x \to y) \to y \in F$. This by our assumption means that $(y \to x) \to x \in F$ (1). Next by (2.19), (2.14) and (2.33) we reason in the following way

$$y \odot x \leq y, \; y \leq x \to y \text{ and } (x \to y) \to x \leq y \to x.$$

Again applying (2.19) and (2.14) we have

$$z \odot (y \to x) \leq (y \to x) \text{ and } (y \to x) \leq z \to (y \to x).$$

Therefore we obtain

$$(x \to y) \to x \leq y \to x \leq z \to (y \to x). \;\; (2)$$

Since by Theorem 5.4 F is a filter, the assumpltion z,$z \to ((x \to y) \to x) \in F$ stated at the begining of the proof implies $(x \to y) \to x \in F$. Thus from (2) we infer $z \to (y \to x) \in F$. Moreover as $z \in F$, then $(y \to x) \in F$, which by (1) implies that $x \in F$.

Conversely, let F be a positive implicative filter and $(x \to y) \to y \in F$. We can easily see that

$$\begin{aligned} y \to x &\leq \mathbf{1}, \\ y \to x &\leq x \to x, \\ x \odot (y \to x) &\leq x, \\ x &\leq (y \to x) \to x. \end{aligned}$$

Therefore by (2.33) we obtain

$$[(y \to x) \to x] \to y \leq x \to y. \;\; (1)$$

Additionally by (2.24), (2.32), (1) and (2.33) from

$$y \leq (y \to x) \to x$$

we obtain

$$\begin{aligned} (x \to y) \to y &\leq (x \to y) \to [(y \to x) \to x] \\ &\leq ([(y \to x) \to x] \to y) \to [(y \to x) \to x]. \end{aligned}$$

Since by Theorem 5.4 F is a filter, the latter result and the assumption $(x \to y) \to$

$y \in F$ implies that $([(y \to x) \to x] \to y) \to [(y \to x) \to x] \in F$. Moreover

$$\mathbf{1} \to \left\{\left[\Big((y \to x) \to x\Big) \to y\right] \to \Big((y \to x) \to x\Big)\right\} =$$
$$= \left[\Big((y \to x) \to x\Big) \to y\right] \to \Big((y \to x) \to x\Big) \in F$$

and $\mathbf{1} \in F$ along with the fact that F is a positive imlicative filter implies that $(y \to x) \to x \in F$. □

Proposition 5.3. *Let A be a BL-algebra and F be a positive implicative filter of A. If there is a filter G such that $F \subseteq G \subseteq A$, then G is also a positive implicative filter.*

Proof. Let F be a positive implicative filter of a BL-algebra A and G be a filter such that $F \subseteq G \subseteq A$. By Theorem 5.6 we know that F is an implicative filter and by Theorem 5.1 we have that G is an implicative filter. By Theorem 5.7 in order G to be a positive implicative filter the condition $(y \to x) \to x \in G$ must imply $(x \to y) \to y \in G$. So, let $(y \to x) \to x \in G$, where $w = (y \to x) \to x$. Since F is an implicative filter, as we realised above, and therefore a filter, then by Theorem 5.2 (iii) the condition $w \to [(y \to x) \to x] = \mathbf{1} \in F$ implies

$$[w \to (y \to x)] \to (w \to x) = [y \to (w \to x)] \to (w \to x) \in F.$$

Since F is an implicative filter, which is also a positive one, then the condition in Theorem 5.7 holds. Hence

$$[y \to (w \to x)] \to (w \to x) \in F$$

implies

$$[(w \to x) \to y] \to y \in F.$$

Since $F \subseteq G$, then $[(w \to x) \to y] \to y \in G$ (1). Now by (2.24) and two times (2.30) we reason in the following way

$$\begin{aligned}
(y \to x) \to x &\leq \Big\{\big[(y \to x) \to x\big] \to x\Big\} \to x \\
&= (w \to x) \to x \\
&\leq (x \to y) \to [(w \to x) \to y] \\
&\leq \Big\{\big[(w \to x) \to y\big] \to y\Big\} \to \big[(x \to y) \to y\big].
\end{aligned}$$

Therefore as G is a filter, then $\Big\{\big[(w \to x) \to y\big] \to y\Big\} \to \big[(x \to y) \to y\big] \in G$.

Moreover by (1) we conclude that $(x \to y) \to y \in G$ and G is a positive implicative filter. $\square$

Theorem 5.8. *In any BL-algebra A the following are equivalent*

(i) $\{\mathbf{1}\}$ *is a positive implicative filter,*

(ii) Every filter of A is a positive implicative filter,

(iii) For some $w \in A$ the set $F(w) = \{x \in A \mid w \leq x\}$ is a positive implicative filter,

(iv) $(x \to y) \to x = x$ *for all* $x, y \in A$,

(v) A is a Boolean algebra.

Proof. First of all we show that (i) implies (ii). Let A be a BL-algebra and $\{\mathbf{1}\}$ be a positive implicative filter and F be any filter of A. Since $\{\mathbf{1}\}$ is a positive implicative filter and $\{\mathbf{1}\} \subseteq F$, then by Theorem 5.3 we have that F is a positive implicative filter and (ii) is valid.

Now assume (ii) holds. We realize that $\{\mathbf{1}\}$ is a positive implicative filter, which is an implicative filter by Theorem 5.6. Now for some $w \in A$ define a set

$$F(w) = \{x \in A \mid w \leq x\}.$$

We show that the set $F(w)$ is a positive implicative filter. Since $y \leq \mathbf{1}$ for every $y \in A$, then $\mathbf{1} \in F$. Assume x, $x \to y \in F$. By the definition of F we obtain that $w \leq x$ and $w \leq x \to y$, which implies that

$$w \to x = \mathbf{1} \text{ and } w \to (x \to y) = \mathbf{1}.$$

Since $\{\mathbf{1}\}$ is an implicative filter, the latter result implies $w \to y = 1$.Therefore $w \leq y$ and consequently $y \in F$. This proves that F is a filter. By (ii) every filter is a positive implicative filter. Hence we conclude that F is a positive implicative filter and (iii) holds.

Now let (iii) be valid. Assume $w = (x \to y) \to x$ and $x \in F(w)$. Then

$$(x \to y) \to x \leq x. \tag{1}$$

Additionally we have that

$$\begin{aligned} (x \to y) &\leq \mathbf{1}, \\ (x \to y) &\leq x \to x, \\ (x \to y) \odot x &\leq x, \\ x &\leq (x \to y) \to x. \end{aligned} \tag{2}$$

Therefore by (1) and (2) we obtain that $x = (x \to y) \to x$ and (iv) holds.

Next we show that from (iv) we can deduce (v). Let (iv) be valid. Then we substitute $x = (x \to y) \to x$ in

$$(y \to x) \to x = (y \to x) \to [(x \to y) \to x]. \quad (1)$$

Next we reason in the similar manner as in the previous proof and deduce that $x \leq (x \to y) \to x$. By (2.30) we obtain

$$(x \to y) \to y \leq (y \to x) \to [(x \to y) \to x]. \quad (2)$$

Therefore by (1) and (2) we obtain $(x \to y) \to y \leq (y \to x) \to x$. Moreover in the similar manner by $y = (y \to x) \to y$ and (2.30) we deduce

$$(x \to y) \to y = (x \to y) \to [(y \to x) \to y]$$

and

$$(y \to x) \to x \leq (x \to y) \to [(y \to x) \to y].$$

Therefore we obtain $(y \to x) \to x \leq (x \to y) \to y$. Consequently we conclude that $(y \to x) \to x = (x \to y) \to y$. By Lemma 2.7 (i)-(iii) this means that

$$\begin{aligned} (x \wedge x^*) \vee y = \mathbf{0} \vee y = y, \\ (x \vee x^*) \wedge y = \mathbf{1} \wedge y = y, \end{aligned}$$

which in turn implies that A is a Boolean algebra, by the definition of a Boolean algebra. Thus (v) is valid.

Now we show that (v) validates (iv). Since A is a Boolean algebra, then there are top and bootom elements $x \vee x^* = \mathbf{1}$ and $x \wedge x^* = \mathbf{0}$, respectively. By Lemma 2.7 this implies that $(x \to y) \to x = x$. Thus (iv) is valid.

Finally we show that (iv) implies (i). Let $\{\mathbf{1}\}$ be a filter of A and let also $(x \to y) \to x \in \{\mathbf{1}\}$. Since we know that $(x \to y) \to x = x$, then $x \in \{\mathbf{1}\}$. By Theorem 5.5 this means that $\{\mathbf{1}\}$ is a positive implicative filter. Therefore (i) holds. □

Theorem 5.9. *A filter F of a BL-algebra A is a positive implicative filter iff every filter of a quotient algebra A/F is a positive implicative filter.*

Proof. Let F be a positive implicative filter and let $(x/F \to y/F) \to x/F = \mathbf{1}/F$ for all x,$y \in A$. We show that $\{\mathbf{1}/F\}$ is a positive implicative filter by showing that the latter assumption implies $x/F = \mathbf{1}/F$.

Recalling that $a/F = \mathbf{1}/F$ iff $a \in F$ for all $a \in A$, from

$$(x/F \to y/F) \to x/F = [(x \to y) \to x]/F = \mathbf{1}/F,$$

we deduce that $(x \to y) \to x \in F$, which by Theorem 5.5 implies $x \in F$, as F is a positive implicative filter. Therefore $x/F = \mathbf{1}/F$ and $\{\mathbf{1}/F\}$ is a positive implicative filter. Thus by Theorem 5.8 (i),(ii) every filter of A/F is a positive implicative filter.

Conversely, assume that every filter of Bl-algebra A/F is a positive implicative filter and for all x,$y \in A$, $(x \to y) \to x \in F$, where F is a filter of BL-algebra A. We show that F is a positive implicative filter. In order F to be so, the latter assumption must imply $x \in F$ by Theorem 5.5.

From our assumption $(x \to y) \to x \in F$ we deduce

$$[(x \to y) \to x]/F = (x/F \to y/F) \to x/F = \mathbf{1}/F.$$

Since $\{\mathbf{1}/F\}$ is a positive implicative filter, by Theorem 5.8 (iv) we have that

$$(x/F \to y/F) \to x/F = x/F = \mathbf{1}/F,$$

Therefore we obtain that $x \in F$ and F is a positive implicative filter. □

Theorem 5.10. *Let A be a BL-algebra and F a a positive implicative filter of A. Then the generated quotient algebra A/F is an Boolean algebra.*

Proof. Omitted. □

6. CONCLUSIONS

The approach based on the construction Lindenbaum-Tarski algebras is essential since it bridges the gap between logic and algebra. The idea that we can link a certain logic with the certain algebraic structure allows application of the logic in different fields. Thus by finding the corresponding algebraic structure to the classical logic, which is a Boolean algebra, we can further apply further the system of axioms of logic in such fields as, computer programming, and mathematical logic, and in other areas of mathematics such as set theory and statistics. Similarly, by associating many-valued logics with the certain, more complicated algebraic structures, these logics can be further applied in the areas where the degree of truth of a propostion can vary over even infinite values. Moreover, the idea of the construction of Lindenbaum-Tarski algebra could be applied on the algebraic structures itselves.

In the first chapter we have observed particular algebraic structures, their properties, the binary operations these structures utilizes, the ways they differ from each other and the way they correspond to a certain logic. We have observed such algebraic structures as lattice, distributive lattice, boolean algebra. Additionally, since modeling of the the many-valued logics assumes that the algebraic structure should have more complicated structure for modeling its truth values, we have described a residuated lattice and its operations. Moreover we have observed the conditions by which a residuated lattice forms a BL-algebra, and further under which circumstances BL-algebra forms and MV-algebra.

In the next chapter we observe lattice filters, filters and deductive systems and their similarity. Additionally here we introduce the notion of the quotient algebra and observe the algebraic structures resulting from factoring the algebra of formulas by the congruence relation. The obtained quotient algebra induced by operations of an algebra and generated by a filter or deductive system.

Next we examine the quotient algebras generated by the prime filters and systems. Further, we observe boolean deductive systems are observed and the quotient algebra that they generate in chapter 4. In chapter 5 implicative filters are introduced.

In this work we observe several significant results obtained in the study of the deductive systems and filters and the quotient algebra generated by them. The first important result shows that a BL-algebra is linear if and only if any proper deductive system of it is prime. The other result is related to the study of the Boolean deductive systems. It says that a BL-algebra has a proper Boolean deductive system if and only if BL-algebra is bipartite. Additionally we observe the relevant results obtained in the study of implicative filters.

In conclusion I would like to say that approach proposed by Lindenbaum and Tarski is a powerful approach that can be applied not only for establishing the correspondence between a logic and an algebraic structure, but that could also be extended for the algebraic structures. This gives rise to the diverse field of research and application. Nowadays it is highly important to investigate and study new structures that could allow more sophisticated application in different fields. And since many-valued logics and classical logics as well are widely used in various fields, it is essential to study and analyze the corresponding algebraic structures.

REFERENCES

[1] Chang, C. (1958). Algebraic analysis of many valued logics. Transactions of the American Mathematical society, pp.467–490.

[2] Czelakowski, J. (2003). Logics and operators. Logic and Logical Philosophy, 3, pp.87–100.

[3] Hájek, P. (1998). Basic fuzzy logic and BL-algebras. Soft computing, 2(3), pp.124–128.

[4] Hájek, P. (1998). Metamathematics of fuzzy logic. Dordrecht: Kluwer.

[5] Haveshki, M., Saeid, A. and Eslami, E. (2006). Some types of filters in BL algebras. Soft computing, 10(8), pp.657–664.

[6] Turunen, E. (1999). Mathematics behind fuzzy logic. Heidelberg: Physica-Verlag.

[7] Turunen, E. (2001). Boolean deductive systems of BL-algebras. Archive for Mathematical Logic, 40(6), pp.467–473.

[8] Turunen, E., Sessa, S. and a di Architettura, F. (2001). Local BL-algebras1.

[9] Turunen, E. (2008). BL-algebras of basic fuzzy logic. Mathware and soft computing, 6(1), pp.49–61.