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Scattering invariants in Euler's two-center problem

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Abstract. The problem of two fixed centers was introduced by Euler as early as in 1760. It plays an important role both in celestial mechanics and in the microscopic world. In the present paper we study the spatial problem in the case of arbitrary (both positive and negative) strengths of the centers. Combining techniques from scattering theory and Liouville integrability, we show that this spatial problem has topologically non-trivial scattering dynamics, which we identify as scattering monodromy. The approach that we introduce in this paper applies more generally to scattering systems that are integrable in the Liouville sense.

Keywords: Action-angle coordinates; Hamiltonian system; Liouville integrability; Scattering map; Scattering monodromy.

MSC: 37J35, 34L25, 57R22, 70F99, 70H05

1. Introduction

The problem of two fixed centers, also known as the Euler 3-body problem, is one of the most fundamental integrable problems of classical mechanics. It describes the motion of a point particle in Euclidean space under the influence of the Newtonian force field

$$F = -\nabla V, \quad V = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}. \quad (1)$$

Here r_i are the distances of the particle to the two fixed centers and μ_i are the strengths (the masses or the charges) of these centers. We note that the Kepler problem corresponds to the special cases when the centers coincide or when one of the strengths is zero.

The (gravitational) Euler problem was first studied by L. Euler in a series of works in the 1760s [21–23]. He discovered that this problem is integrable by putting the equations of motion in a separated form. Elliptic coordinates, which separate the problem and which are now commonly used, appeared in his later paper [23] and, at about the same

time, in the work of Lagrange [41]. The systematic use of elliptic coordinates in classical mechanics was initiated by Jacobi, who used a more general form of these coordinates to integrate, among other systems, the geodesic flow on a triaxial ellipsoid; see [32] for more details.

Since the early works of Euler and Lagrange the Euler problem and its generalizations have been studied by many authors. First classically and then, since the works of Pauli [52] and Niessen [50] in the early 1920s, also in the setting of quantum mechanics. We indicatively mention the works [5, 12, 16, 20, 55, 59, 60, 62]. For a historical overview we refer to [28, 51].

In the present work we will be interested in the spatial Euler problem. For us, it will be important that this problem is a Hamiltonian system with two additional structures: it is a *scattering system* and it is also *integrable in the Liouville sense*. The structure of a scattering system comes from the fact that the potential

$$V(q) \rightarrow 0, \quad \|q\| \rightarrow \infty,$$

decays at infinity sufficiently fast (is of *long range*). It allows one to compare a given set of initial conditions at $t = -\infty$ with the outcomes at $t = +\infty$. An introduction to the general theory of scattering systems can be found in [13, 37]. Liouville integrability comes from the fact that the system is *separable*; the three commuting integrals of motion are:

- the energy function H — the Hamiltonian,
- the separation constant G ; see Subsection 2.1,
- the angular momentum L_z about the axis connecting the two centers.

An introduction to the general theory of Liouville integrable systems can be found in [3, 9, 37].

Separately these two different structures of the Euler problem were discussed in the literature. Scattering was studied, for instance, in [35, 55]. The corresponding Liouville fibration was studied in [60] — from the perspective of Fomenko theory [3, 27], action coordinates and *Hamiltonian monodromy* [14]. We will consider both of the structures together and show that the Euler problem has non-trivial scattering invariants, which we will call *pure scattering* and *mixed scattering monodromy*, cf. the works [2, 15, 18, 36, 45]. The standard case of Hamiltonian monodromy [14] and its difference to the scattering case will also be discussed. We note that even though the main goal of this paper is to study the Euler problem, the approach to scattering monodromy that we develop here applies more generally to any Hamiltonian system that is both scattering and integrable; this will become clear from the discussion in Section 5.

In the following we explain our approach without going into technical details and summarize the main results on the Euler problem. As we mentioned earlier, the Euler problem is an integrable system. Let

$$F = (H, L_z, G): M \rightarrow \mathbb{R}^3$$

denote the corresponding *integral map*, which consists of the three commuting integrals of motion. Here M is the regularized phase space; see Subsection 2.2. For sufficiently large positive energies h , the corresponding *bifurcation diagram*, that is, the set of critical values of F , is shown in Fig. 1.

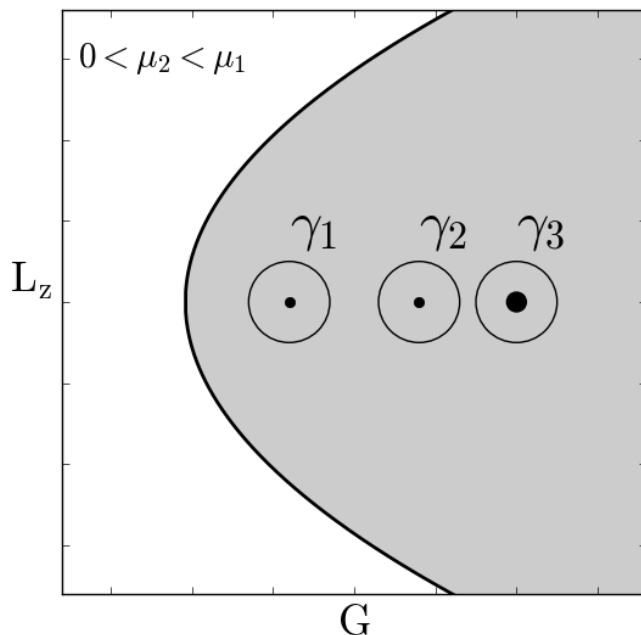


Figure 1: Energy slice of the image the integral map F (shaded region) for a sufficiently large energy $h > 0$; the case of attractive centers. The energy slice of the bifurcation diagram consists of the boundary and three isolated points.

From the bifurcation diagram we see that the set of the regular values $R \subset \text{image}(F)$ of F is not simply connected. In fact, the fundamental group $\pi_1(R)$ is a free group on three generators $\gamma_1, \gamma_2, \gamma_3$; see Fig. 1. Following Duistermaat [14], one can compute Hamiltonian monodromy around γ_i , that is, determine the topology of the bundles $F: F^{-1}(\gamma_i) \rightarrow \gamma_i$. We will do this in Subsection 6.3. For now, we note that the fibers $F^{-1}(h, g, l)$ of these bundles are 3-dimensional cylinders $T^2 \times \mathbb{R}$, where \mathbb{R} corresponds to the flow of the Hamiltonian H . It follows that each bundle $F: F^{-1}(\gamma_i) \rightarrow \gamma_i$ is a direct product of \mathbb{R} with some compact manifold. Let us denote this by

$$F^{-1}(\gamma_i) = B_i \times \mathbb{R}.$$

We see that the topology of $F^{-1}(\gamma_i)$ does not manifest any non-triviality in scattering, as the non-compact \mathbb{R} direction always splits. In order to take scattering into account, we need another ingredient, namely, a *reference system* H_r . We note that since the Euler problem is integrable, the choice of H_r is not arbitrary; we assume that H_r preserves F at infinity; see Definition 5.1.

To define *scattering* monodromy, we will thus use a pair of systems given by the Hamiltonians H and H_r ; this is a typical setting in classical scattering theory [13, 36, 37]. More specifically, we will use the so-called scattering map ([36], Section 4)

$$S_i: B_i \rightarrow B_i,$$

which is defined by following the dynamics of H and H_r as the time t goes to $\pm\infty$. The scattering map can be used to identify the asymptotic states of the Hamiltonian $H|_{F^{-1}(\gamma_i)}$ at $t = -\infty$ with the corresponding asymptotic states of $H|_{F^{-1}(\gamma_i)}$ at $t = +\infty$ (note that \mathbb{R} is naturally parametrized by the time t). This identification results in a new total space $F_c^{-1}(\gamma_i)$ and a new fibration

$$F_c: F_c^{-1}(\gamma_i) \rightarrow \mathbb{R}^n,$$

which does not have to split as a direct product. This new fibration is a 3-torus bundle. Hamiltonian monodromy of this torus bundle will be called *scattering monodromy* of F with respect to H_r . We note that the fibration F_c splits as a direct product if one takes $H_r = H$ as the reference system; in this case, the Hamiltonian monodromy of the original fibration F is recovered.

Later we shall discuss the construction of F_c and its relation to other scattering invariants, such as Knauf's degree [36], in more detail and generality. We now conclude the introduction with stating the obtained results on the Euler problem.

In order to define scattering monodromy for the Euler problem, we need to choose a reference system. Typically, one takes a Kepler problem with the strength $\mu_1 + \mu_2$, where μ_1 and μ_2 as in Eq. 1. However, such a Kepler problem is not a reference system in our sense (it does not satisfy Definition 5.1). We have the following result.

Theorem 1.1. *Among all Kepler Hamiltonians only*

$$H_{r_1} = \frac{1}{2}p^2 - \frac{\mu_1 - \mu_2}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2}p^2 - \frac{\mu_2 - \mu_1}{r_2}$$

are reference Hamiltonians of $F = (H, L_z, G)$. In particular, the free Hamiltonian is a reference Hamiltonian of F only in the case $\mu_1 = \mu_2$.

In the case when $0 < \mu_1 < \mu_2$ the scattering monodromy of F with respect to H_{r_1} is given in the following theorem. The general case can be found in Subsection 6.4.

Theorem 1.2. *The scattering monodromy matrices M_i along γ_i with respect to H_{r_1} are conjugate in $SL(3, \mathbb{Z})$ to*

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that in the case of the curve γ_3 , Hamiltonian monodromy is trivial, while scattering monodromy (for both of the reference Hamiltonians H_{r_1} and H_{r_2}) is not; we

refer to this case as *pure scattering monodromy*. In the case of the curve γ_2 , Hamiltonian monodromy and scattering monodromy (with respect to H_{r_1} only) are both non-trivial. We call this type of monodromy *mixed scattering*. In the case of the curve γ_1 , we have non-trivial scattering monodromy with respect to H_{r_2} ; this case is also mixed scattering. The proofs of these results are given in Section 6 and Appendix C.

The paper is organized as follows. The Euler problem is introduced in Section 2. Bifurcation diagrams are given in Section 3. In Section 4 we discuss classical potential scattering theory. In Section 5 we adapt the discussion of Section 4 to the context of scattering systems that are integrable in the Liouville sense. In particular, we give a definition of a reference system for integrable systems. For the Euler problem, scattering monodromy is discussed in detail in Section 6. Hamiltonian monodromy is addressed in Subsection 6.3. The main part of the paper is concluded with a discussion in Section 7. Additional details are presented in the Appendix.

2. Preliminaries

We start with the 3-dimensional Euclidean space \mathbb{R}^3 and two distinct points in this space, denoted by o_1 and o_2 . Let $q = (x, y, z)$ be Cartesian coordinates in \mathbb{R}^3 and let $p = (p_x, p_y, p_z)$ be the conjugate momenta in $T_q^*\mathbb{R}^3$. The *Euler two-center problem* can be defined as a Hamiltonian system on $T^*(\mathbb{R}^3 \setminus \{o_1, o_2\})$ with a Hamiltonian function H given by

$$H = \frac{\|p\|^2}{2} + V(q), \quad V(q) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}, \quad (2)$$

where $r_i: \mathbb{R}^3 \rightarrow \mathbb{R}$ is the distance to the center o_i . The strengths of the centers μ_i can be both positive and negative; without loss of generality we assume that the center o_1 is stronger, that is, $|\mu_2| \leq |\mu_1|$.

Remark 2.1. When $\mu_i > 0$ (resp., $\mu_i < 0$) the center o_i is attractive (resp., repulsive). The cases $\mu_1 \neq \mu_2 = 0$ and $\mu_2 \neq \mu_1 = 0$ correspond to a Kepler problem. In the case $\mu_1 = \mu_2 = 0$ the dynamics is trivial and we have the free motion $(t, q_0, p_0) \mapsto (q_0 + tp_0, p_0)$.

2.1. Separation and integrability

Without loss of generality we assume $o_i = (0, 0, (-1)^i a)$ for some $a > 0$, so that, in particular, the fixed centers o_1 and o_2 are located on the z -axis in the configuration space. Rotations around the z -axis leave the potential function V invariant. It follows that (the z -component of) the angular momentum

$$L_z = xp_y - yp_x \quad (3)$$

commutes with H , that is, L_z is a first integral. It is known [20, 62] that there exists another first integral given by

$$G = H + \frac{1}{2}(L^2 - a^2(p_x^2 + p_y^2)) + a(z + a)\frac{\mu_1}{r_1} - a(z - a)\frac{\mu_2}{r_2}, \quad (4)$$

where $L^2 = L_x^2 + L_y^2 + L_z^2$ is the squared angular momentum. The expression for the integral G can be obtained using separation in elliptic coordinates, as described below. It will follow from the separation procedure that the function G commutes both with H and with L_z , which means that the problem of two fixed centers is *Liouville integrable*.

Consider prolate ellipsoidal coordinates (ξ, η, φ) :

$$\xi = \frac{1}{2a}(r_1 + r_2), \quad \eta = \frac{1}{2a}(r_1 - r_2), \quad \varphi = \text{Arg}(x + iy). \quad (5)$$

Here $\xi \in [1, \infty)$, $\eta \in [-1, 1]$, and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. Let $p_\xi, p_\eta, p_\varphi = L_z$ be the conjugate momenta and l be the value of L_z . In the new coordinates the Hamiltonian H has the form

$$H = \frac{H_\xi + H_\eta}{\xi^2 - \eta^2}, \quad (6)$$

where

$$H_\xi = \frac{1}{2a^2}(\xi^2 - 1)p_\xi^2 + \frac{1}{2a^2} \frac{l^2}{\xi^2 - 1} - \frac{\mu_1 + \mu_2}{a} \xi$$

and

$$H_\eta = \frac{1}{2a^2}(1 - \eta^2)p_\eta^2 + \frac{1}{2a^2} \frac{l^2}{1 - \eta^2} + \frac{\mu_1 - \mu_2}{a} \eta.$$

Multiplying Eq. 6 by $\xi^2 - \eta^2$ and separating we get the first integral

$$G = \xi^2 H - H_\xi = \eta^2 H + H_\eta.$$

In original coordinates G has the form given in Eq. 4. Since $L_z = p_\varphi$, the function G commutes both with H and with L_z .

2.2. Regularization

We note that when one of the strengths μ_i is attractive, collision orbits are present and, consequently, the flow of H on $T^*(\mathbb{R}^3 \setminus \{o_1, o_2\})$ is not complete. This complication is, however, not essential for our analysis since the collision orbits can be regularized in an essentially unique way [35] (though the form of the regularization may be different †). More specifically, there is the following result.

Theorem 2.2. ([35]) *Let $H: T^*(\mathbb{R}^3 \setminus \{o_1, o_2\}) \rightarrow \mathbb{R}$ be the Hamiltonian of the Euler two-center problem. Then there exist a 6-dimensional symplectic manifold (P, ω) and a smooth Hamiltonian function \tilde{H} on P such that*

- (i) $(T^*(\mathbb{R}^3 \setminus \{o_1, o_2\}), dq \wedge dp)$ is symplectically embedded in (P, ω) ,
- (ii) $H = \tilde{H}|_{T^*(\mathbb{R}^3 \setminus \{o_1, o_2\})}$,
- (iii) The flow of \tilde{H} on P is complete.

The integrals L_z and G can also be extended to P .

† For the Kepler problem, the well-known regularizations schemes are the Kustaanheimo-Stiefel, Moser, Souriau, and Ligon-Schaaf regularizations; see [7, 29, 57] and references therein. For the Euler problem, we refer to the work [35]; see also [61].

Proof. This result was proven in [35, Proposition 2.3] for the gravitational planar problem. The spatial case follows from the planar case since collisions occur only when the momentum $L_z = 0$. The case of arbitrary strengths is similar (note that collisions with a repulsive center are not possible). \square

One important property of this regularization is that the extensions of the integrals to P , which will be also denoted by H , L_z and G , form a *completely integrable system*. In particular, the Arnol'd-Liouville theorem [1] applies. In what follows we shall work on the regularized space P .

3. Bifurcation diagrams

Before we move further and discuss scattering in the Euler problem, we shall compute the *bifurcation diagrams* of the integral map $F = (H, L_z, G)$, that is, the set of the critical values of this map. We distinguish two cases, depending on whether L_z is zero or different from zero. The bifurcation diagrams are obtained by superimposing the critical values found in these two cases. By a choice of units we assume that $a = 1$.

3.1. The case $L_z = 0$

Since $L_z = 0$, the motion is planar. We assume that it takes place in the xz -plane. Consider the elliptic coordinates $(\lambda, \nu) \in \mathbb{R} \times S^1[-\pi, \pi]$ defined by

$$x = \sinh \lambda \cos \nu, \quad z = \cosh \lambda \sin \nu.$$

The level set of constant $H = h, L_z = l = 0$ and $G = g$ in these coordinates is given by the equations

$$\begin{aligned} p_\lambda^2 &= 2h \cosh^2 \lambda + 2(\mu_1 + \mu_2) \cosh \lambda - 2g, \\ p_\nu^2 &= -2h \sin^2 \nu - 2(\mu_1 - \mu_2) \sin \nu + 2g, \end{aligned}$$

where p_λ and p_ν are the momenta conjugate to λ and ν . The value $(h, 0, g)$ is critical when the Jacobian matrix corresponding to these equations does not have a full rank. Computation yields lines

$$\begin{aligned} \ell_1 &= \{g = h + \mu_2 - \mu_1, l = 0\}, \quad \ell_2 = \{g = h + \mu_1 - \mu_2, l = 0\} \quad \text{and} \\ \ell_3 &= \{g = h + \mu, l = 0\}, \quad \mu = \mu_1 + \mu_2, \end{aligned}$$

and two curves

$$\begin{aligned} \{g = \mu \cosh \lambda/2, h = -\mu/2 \cosh \lambda, l = 0\}, \\ \{g = (\mu_1 - \mu_2) \sin \nu/2, h = (\mu_2 - \mu_1)/2 \sin \nu, l = 0\}. \end{aligned}$$

Points that do not correspond to any physical motion must be removed from the obtained set (allowed motion corresponds to the regions where the squared momenta are positive).

Remark 3.1. The corresponding bifurcation diagrams in the planar problem are given in Appendix B; see Fig. B1 and B2. We note that they were already computed in [12], see also [54, 60]. We observe that in the planar case the set of the regular values of the map F consists of contractible components and hence the topology of the regular part of the Liouville fibration is trivial. Interestingly, this is not the case if the dimension of the configuration space is $n = 3$.

We note that the singular Liouville foliation has non-trivial topology already in the planar case. The corresponding bifurcations, in the sense of Fomenko theory [3, 4, 25–27], have been studied in [34, 60].

3.2. The case $L_z \neq 0$

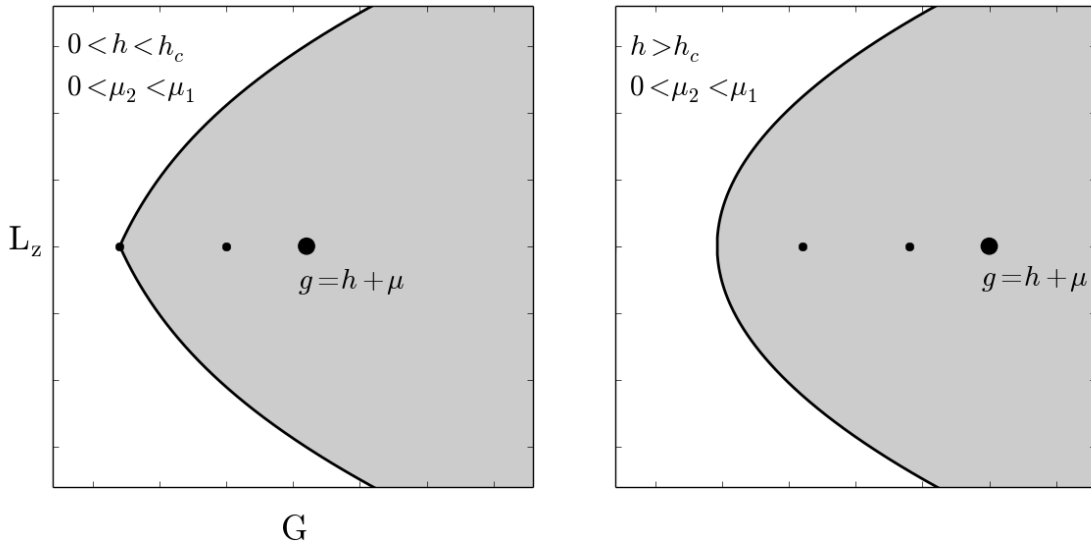


Figure 2: Positive energy slices of the bifurcation diagram for the spatial Euler problem, attractive case. The black points correspond to the critical lines ℓ_i .

In order to compute the critical values in this case it is convenient to use the ellipsoidal coordinates (ξ, η) . (We note that for $L_z \neq 0$ the z -axis is inaccessible, so (ξ, η) are non-singular.) The level set of constant $H = h, L_z = l$ and $G = g$ in these coordinates is given by the equations

$$p_\xi^2 = \frac{(\xi^2 - 1)(2h\xi^2 + 2(\mu_1 + \mu_2)\xi - 2g) - l^2}{(\xi^2 - 1)^2},$$

$$p_\eta^2 = \frac{(1 - \eta^2)(-2h\eta^2 - 2(\mu_1 - \mu_2)\eta + 2g) - l^2}{(1 - \eta^2)^2}.$$

The value (h, l, g) with $l \neq 0$ is critical when the corresponding Jacobian matrix does not have a full rank. Computation yields the following sets of critical values:

$$\left\{ g = h(2\xi^2 - 1) + \frac{(\mu_1 + \mu_2)(3\xi^2 - 1)}{2\xi}, l^2 = -\frac{(\mu_1 + \mu_2 + 2h\xi)(-1 + \xi^2)^2}{\xi} \right\},$$

$$\left\{ g = h(2\eta^2 - 1) + \frac{(\mu_1 - \mu_2)(3\eta^2 - 1)}{2\eta}, l^2 = -\frac{(\mu_1 - \mu_2 + 2h\eta)(-1 + \eta^2)^2}{\eta} \right\},$$

where $\xi > 1$ and $-1 < \eta < 1$. As above, points that do not correspond to any physical motion must be removed.

Representative positive energy slices in the gravitational case $0 < \mu_2 < \mu_1$ are given in Fig. 2. The case of arbitrary strengths μ_i is similar. The structure of the corresponding diagrams can partially be deduced from the diagrams obtained in the planar case; see Appendix Appendix B.

4. Classical scattering theory

In this section we discuss certain qualitative aspects of scattering theory following [36, 37]. In Section 5 we explain how the theory can be adapted to the context of scattering systems that are integrable in the Liouville sense, with the Euler problem as the leading example.

4.1. Preliminary remarks

Classical scattering theory goes back to the works of Cook [6], Hunziker [31] and Simon [56]. Since then it has received considerable interest and has been actively developed in several directions; see [2, 13, 15, 30, 36].

In the framework of classical scattering one considers two Hamiltonian functions H and H_r such that their flows become similar ‘at infinity’. This allows one can compare a given distribution of particles, that is, initial conditions, at $t = -\infty$ with their final distribution at $t = +\infty$. To be more specific, consider a pair of Hamiltonians on $T^*\mathbb{R}^n$ given by

$$H = \frac{1}{2}\|p\|^2 + V(q) \quad \text{and} \quad H_r = \frac{1}{2}\|p\|^2 + V_r(q),$$

where the (singular) potentials V and V_r are assumed to satisfy a certain decay assumption; see Subsection 4.2. For scattering Hamiltonians the comparison will be achieved in two steps. First we shall parametrize the possible initial and final distributions using the flow of the ‘free’ Hamiltonian $H_0 = \frac{1}{2}\|p\|^2$. Then, for a given invariant manifold, we shall construct the *scattering map*, where only H and H_r are compared.

Remark 4.1. One reason for such a procedure is the following. As we shall see later in Section 5 and Appendix Appendix C, the ‘free’ Hamiltonian is not a natural reference Hamiltonian for the Euler problem, unless the strengths $\mu_1 = \mu_2$. However, the ‘free’ Hamiltonian will be convenient for the definition of the asymptotic states.

Remark 4.2. In what follows we sometimes refer to H, H_r as *scattering Hamiltonians* and to H_r is a *reference Hamiltonian* for H . We note that the ‘reference’ dynamics of H_r is usually chosen to be simpler than the ‘original’ dynamics of H .

4.2. Decay assumptions

In classical potential scattering theory, the potential function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ of a scattering Hamiltonian $H = \frac{1}{2}\|p\|^2 + V(q)$ (the case of V_r is similar) is assumed to decay according to one of the following estimates:

1. Finite-range: $\text{supp}(V) \subset \mathbb{R}^n$ is compact;
2. Short-range case: $|\partial_k V(q)| < c(\|q\| + 1)^{-|k|-1-\varepsilon}$;
3. Long-range case: $|\partial_k V(q)| < c(\|q\| + 1)^{-|k|-\varepsilon}$.

Here c and ε are positive constants, $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ is a multi-index, $|k| = k_1 + \dots + k_n$ is a norm of k and $\|q\|$ denotes the Euclidean norm of q . For instance, any Kepler potential is of long range and the same is true of the potential found in the Euler problem.

In what follows we shall assume that the original potential V and the reference potential V_r satisfy the following assumptions.

Assumption 4.3. We assume there exist functions \tilde{V} and \tilde{V}_r such that

- (i) \tilde{V} and \tilde{V}_r are rotationally symmetric and long-range;
- (ii) $V - \tilde{V}$ and $V_r - \tilde{V}_r$ are short-range.

In other words, we assume that V and V_r are short-range with respect to some long-range rotationally symmetric potentials \tilde{V} and \tilde{V}_r , respectively.

Remark 4.4. The auxiliary potential function \tilde{V} and \tilde{V}_r are needed to guarantee that the asymptotic directions and the impact parameters, which are defined below, exist and parametrize the scattering trajectories in a continuous way. This is known to be the case for short-range potentials; see [37]. The case of the potentials V and V_r reduces to the case of long-range symmetric potentials since the differences $V - \tilde{V}$ and $V_r - \tilde{V}_r$ are short-range; the existence of the parametrization in the symmetric case follows from the conservation of the angular momentum.

4.3. Asymptotic states

The Hamiltonian flow $g_H^t: P \rightarrow P$ of H partitions the (regularized) phase space P into the following invariant subsets:

$$b^\pm = \{x \in P \mid \sup_{t \in \mathbb{R}^\pm} \|g_H^t(x)\| < \infty\} \quad \text{and} \quad s^\pm = \{x \in P \mid H(x) > 0\} \setminus b^\pm.$$

The invariant subsets

$$b = b^+ \cap b^-, \quad s = s^+ \cap s^- \quad \text{and} \quad \text{trp} = (b^+ \setminus b^-) \cup (b^- \setminus b^+)$$

are the sets of the *bound*, the *scattering* and the *trapped* states, respectively. We note that s^-, s^+ and hence $s = s^- \cap s^+$ are open subsets of P .

If the potential V satisfies the decay assumptions (see Assumptions 4.3), then the following limits

$$\hat{p}^\pm(x) = \lim_{t \rightarrow \pm\infty} p(t, x) \quad \text{and} \quad q_\perp^\pm(x) = \lim_{t \rightarrow \pm\infty} \left(q(t, x) - \langle q(t, x), \hat{p}^\pm(x) \rangle \frac{\hat{p}^\pm(x)}{2h} \right),$$

where $h = H(x) > 0$ is the energy of $g_H^t(x)$, are defined for any $x \in s^\pm$ and depend continuously on x . These limits are called the *asymptotic direction* and the *impact parameter* of the trajectory $g_H^t(x)$, respectively. We note that an asymptotic direction is always orthogonal to the corresponding impact parameter. Due to the g_H^t -invariance of \hat{p}^\pm and q_\perp^\pm , we have the maps

$$A^\pm = (\hat{p}^\pm, q_\perp^\pm): s/g_H^t \rightarrow AS$$

from s/g_H^t to the asymptotic states $AS \subset \mathbb{R}^n \times \mathbb{R}^n$. Here s/g_H^t is the space of trajectories of s , that is, it is a quotient space of s by the equivalence relation where two points are considered equivalent if and only if they belong to a single trajectory $g_H^t(x)$. Similarly, one can construct the maps

$$A_r^\pm = (\hat{p}^\pm, q_\perp^\pm): s_r/g_{H_r}^t \rightarrow AS$$

for the ‘reference’ Hamiltonian $H_r = \frac{1}{2}p^2 + V_r(q)$.

4.4. Scattering map

Using the maps A^\pm and A_r^\pm constructed in Subsection 4.3, we can now define the notion of a *scattering map* for a given invariant submanifold R of s .

Definition 4.5. Let R be a g_H^t -invariant submanifold of s and let $B = R/g_H^t$. Assume that the composition map

$$S = (A^-)^{-1} \circ A_r^- \circ (A_r^+)^{-1} \circ A^+$$

is well defined and maps B to itself. The map S is called the *scattering map* (w.r.t. H, H_r and B).

Remark 4.6. Due to the decay assumptions the maps

$$A^\pm: s/g_H^t \rightarrow AS \quad \text{and} \quad A_r^\pm: s_r/g_{H_r}^t \rightarrow AS$$

are homeomorphisms onto their images in AS . It follows that the scattering map $S: B \rightarrow B$ is a homeomorphism as well. Here the sets s/g_H^t , $s_r/g_{H_r}^t$ and B are endowed with the quotient topology.

4.5. Knauf's topological degree

To get qualitative information about the scattering it is useful to look at topological invariants of the scattering map. An important example in the context of general scattering theory is *Knauf's topological degree*, the notion of which was introduced in [36] and later extended in [38, 39]. We shall now recall its definition.

Consider the case when the potential V is short-range relative to $V_r = 0$. Let $h > 0$ be a *non-trapping energy*, that is, a positive energy such that the energy level $H^{-1}(h)$ contains no trapping states, and let $R = H^{-1}(h) \cap s$ be the intersection of the level $H^{-1}(h)$ with the set s of the scattering states. There is the following result.

Theorem 4.7. ([13, 36]) *The scattering manifold $B = R/g_H^t$ is the cotangent bundle T^*S^{n-1} , where S^{n-1} is the sphere of asymptotic directions. The corresponding scattering map*

$$S_h: B \rightarrow B$$

*is a symplectic transformation of T^*S^{n-1} .*

Knauf's topological degree is defined as a topological invariant of S_h . Specifically, let $\text{Pr}: T^*S^{n-1} \rightarrow S^{n-1}$ be the canonical projection and

$$S_p^{n-1} = T_p^*S^{n-1} \cup \{\infty\}$$

be the one-point compactification of the cotangent space $T_p^*S^{n-1}$.

Definition 4.8. (Knauf, [36]) *The degree $\deg(h)$ of the energy h scattering map S_h is defined as the topological degree of the map*

$$\text{Pr} \circ S_h: S_p^{n-1} \rightarrow S^{n-1}.$$

Remark 4.9. We note that by continuity $\deg(h)$ is independent of the choice of the initial direction $p \in S^{n-1}$; see [36].

The following theorem shows that for regular (that is, everywhere smooth) potentials $\deg(h)$ is either 0 or 1, depending on the value of the energy h ; see Fig. 3. We note that for singular potentials, such as the Kepler potential, values different from 0 and 1 may appear.

Theorem 4.10. (Knauf-Krapf, [38]) *Let V be a regular short-range potential and $h > 0$ be a non-trapping energy. Then*

$$\deg(h) = \begin{cases} 0, & \text{for } h \in (\sup V, \infty), \\ 1, & \text{for } h \in (0, \sup V). \end{cases}$$

Remark 4.11. For the Euler problem with $\mu_1\mu_2 \neq 0$, Knauf's degree is not defined (every positive energy h is trapping). Moreover, the free flow is not a proper reference unless $\mu_1 = \mu_2$; see Section 5. Nonetheless, as we shall show in Sections 5 and 6, for a proper choice of a reference Hamiltonian and a scattering manifold, an analogue of Knauf's degree can be defined.

5. Scattering in integrable systems

The goal of the present section is to recast the above theory of scattering in the context of Liouville integrability. The approach developed in the present section will be applied to the Euler problem in Section 6.

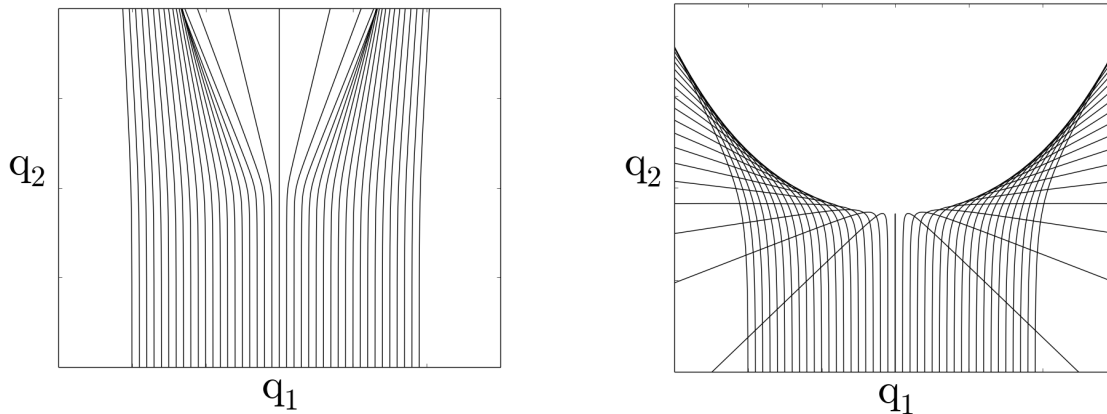


Figure 3: Scattering at different energies. At high energies $\deg(h) = 0$ (left), at low energies $\deg(h) = 1$ (right).

5.1. Reference systems

As we have seen in Section 4, reference systems can be used to define a scattering map, which is a map between the asymptotic states at $t = -\infty$ and $t = +\infty$ of a given invariant manifold. For integrable systems, natural invariant manifolds are the fibers of the corresponding integral map F and various unions of these fibers. It is thus natural to require that the flow of a reference Hamiltonian maps the set of asymptotic states of a given fiber of F to the set of asymptotic states of the same fiber. This leads to the following definition.

Definition 5.1. Consider a scattering Hamiltonian H which gives rise to an integrable system $F: P \rightarrow \mathbb{R}^n$. A scattering Hamiltonian H_r will be called a *reference* Hamiltonian for this system if

$$F \left(\lim_{t \rightarrow +\infty} g_{H_r}^t(x) \right) = F \left(\lim_{t \rightarrow -\infty} g_{H_r}^t(x) \right) \quad (7)$$

for every scattering trajectory $t \mapsto g_{H_r}^t(x)$.

Remark 5.2. Definition 5.1 can be generalized to the setting of scattering and integrable systems defined on abstract symplectic manifolds. For the purposes of the present work, it is sufficient to assume that H and H_r are as in Section 4. We note that Eq. 7 (for the fixed reference Hamiltonian $H_r = \frac{1}{2}\|p\|^2$) appeared in a related context in [33].

Remark 5.3. In scattering theory it is usually assumed that a reference Hamiltonian preserves asymptotic states of energy levels, which is a less restrictive assumption. Our point of view is that in the case of scattering integrable systems, conserved quantities, such as the angular momentum, should also be taken into account.

A series of examples of scattering integrable systems with a reference Hamiltonian in the above sense can be obtained by considering rotationally symmetric potentials V

and V_r . This follows from the conservation of the angular momentum. Another example is the Euler problem. Recall that the Hamiltonian of this problem is given by

$$H = \frac{\|p\|^2}{2} - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}.$$

Let F be the corresponding integral map; see Section 2. We have the following result.

Theorem 5.4. *Among all Kepler Hamiltonians only*

$$H_{r_1} = \frac{1}{2}p^2 - \frac{\mu_1 - \mu_2}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2}p^2 - \frac{\mu_2 - \mu_1}{r_2}$$

are reference Hamiltonians of the Euler problem $F = (H, L_z, G)$. In particular, the free Hamiltonian is a reference Hamiltonian of the Euler problem only in the case $\mu_1 = \mu_2$.

Proof. See Appendix Appendix C. □

Remark 5.5. It follows from Theorem 5.4 that a Kepler Hamiltonian with the strength $\mu_1 + \mu_2$ is not a reference of $F = (H, L_z, G)$, no matter where the center of attraction, resp., repulsion, is located. For the strength $\mu_1 + \mu_2$ and only for this strength, the difference between the potentials is short-range. This implies that the *Møller transformations* (or the *wave transformations*) [13, 37] are not defined with respect to the reference Hamiltonians H_{r_i} , unless the reference flow is appropriately modified. We note that the existence of Møller transformations is important for the study of quantum scattering in this problem.

5.2. Scattering invariants

Consider the Liouville fibration $F: s \rightarrow \mathbb{R}^n$. Let H_r be a reference Hamiltonian for F such that $A^\pm(s) \subset A^\pm(s_r)$ holds. Setting $R = s$, we get the scattering map

$$S: B \rightarrow B, \quad B = R/g_H^t.$$

The scattering map S allows to identify the asymptotic states of s at $t = +\infty$ with the asymptotic states at $t = -\infty$. This results in a new total space s_c . We observe that under this identification the asymptotic states of a given fiber of $F: s \rightarrow \mathbb{R}^n$ are mapped to the asymptotic states of the same fiber. This implies that s_c is naturally fibered by F . The resulting fibration will be denoted by

$$F_c: s_c \rightarrow \mathbb{R}^n.$$

Invariants of this new fibration F_c contain essential information about the scattering dynamics. In what follows, we shall discuss one concrete example of a such an invariant, namely, *scattering monodromy*.

Definition 5.6. Assume that

$$F_c: s_c \rightarrow \mathbb{R}^n$$

is a torus bundle. The (usual) monodromy of this torus bundle will be called *scattering monodromy* of the fibration F .

For the first time, the notion of scattering monodromy was introduced in [2] for a two degree of freedom hyperbolic oscillator and, at about the same time, in the work [15] for planar scattering systems with a repulsive rotationally symmetric potential. In [18], a more general notion of *non-compact monodromy*, which is defined for not necessarily scattering systems, was proposed. A related billiard-type approach, which is based on separation, was discussed in [11, 48, 53]. What has been missing until now for scattering monodromy, is a definition which makes an explicit connection to scattering theory and which is applicable to general scattering and integrable systems; in particular, to systems that have many degrees of freedom and which are not necessarily (rotationally) symmetric. Definition 5.6 meets these properties. Moreover, it can even be generalized to scattering systems that are not integrable.

Below in this section we connect Definition 5.6 with the original definition in terms of the deflection angle [2, 15]. We will discuss the Euler problem in Section 6.

5.3. Planar potential scattering

Here we shall discuss the case $n = 2$ of planar scattering systems. The goal is to relate our notion of scattering monodromy to the existing definition in terms of the deflection angle [2, 15] and to make an explicit connection to the scattering map.

Assume that V and V_r are rotationally symmetric, that is,

$$V(q) = W(\|q\|) \quad \text{and} \quad V_r(q) = W_r(\|q\|) \quad \text{for some } W, W_r: \mathbb{R}_+ \rightarrow \mathbb{R}.$$

Then the angular momentum $L_z = xp_y - yp_x$ is conserved. Let $F = (H, L_z)$ be the integral map of the original system and N be an arbitrary submanifold of the non-trapping set

$$NT = \{(h, l) \in \text{image}(F) \mid F^{-1}(h, l) \subset s\}. \quad (8)$$

The manifold $F^{-1}(N)$ is an invariant submanifold of the phase space P , which contains no trapping states (it consists of scattering states only).

Consider the case when $N = \gamma$ is a regular simple closed curve in NT . Let $R = F^{-1}(\gamma)$ and $S: B \rightarrow B$, $B = F^{-1}(\gamma)/g_H^t$, denote the corresponding scattering map. Then we have the following result.

Theorem 5.7. *The following statements are equivalent.*

- (1) *The scattering monodromy along γ is a Dehn twist of index m ;*
- (2) *The variation of the deflection angle along γ equals $2\pi m$;*
- (3) *The scattering map S is a Dehn twist of index m .*

Remark 5.8. By a Dehn twist of index m we mean a homeomorphism of a 2-torus such that its push-forward map is given by (the conjugacy class of) the matrix

$$M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

We note that the scattering manifold B is a 2-torus in this case.

Remark 5.9. The *total deflection angle* of a trajectory $g_H^t(x) = (q(t), p(t))$ is defined by

$$\Phi = \int_{-\infty}^{+\infty} \frac{d\varphi(q(t))}{dt} dt,$$

where φ is the polar angle in the configuration xy -plane. The *deflection angle* is defined as the difference of the total deflection angles for the original and the reference trajectories. We note that (2) is essentially the definition of scattering monodromy due to [2, 15].

Proof. (1) \Leftrightarrow (2). Let (a, b) be homology cycles on the fiber $F_c^{-1}(\gamma(t_0))$ such that b corresponds to the circle action given by L_z . Transporting the cycles along γ we get $b \mapsto b$ and $a \mapsto a + mb$ for some integer m . But the difference

$$\Phi - \Phi_r = \int_{-\infty}^{+\infty} \frac{d\varphi(q(t))}{dt} dt - \int_{-\infty}^{+\infty} \frac{d\varphi(q_r(t))}{dt} dt,$$

where $g_{H_r}^t = (q_r(t), p_r(t))$ is a reference trajectory with the same energy and angular momentum, can be seen as the rotation number on the fibers of F_c . It follows that the variation of $\Phi - \Phi_r$ along γ equals $2\pi m$.

(2) \Leftrightarrow (3). The scattering map S allows one to consider the compactified torus bundle

$$\text{Pr}: F^{-1}(\gamma)^c \rightarrow S^1 = \mathbb{R} \cup \{\infty\},$$

where \mathbb{R} corresponds to the time. The torus bundle considered in (1) has the same total space, but is fibered over γ . Suppose that the monodromy of this bundle is given by the matrix

$$M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

Then the monodromy of $\text{Pr}: F^{-1}(\gamma)^c \rightarrow S^1$ is the same, for otherwise the total spaces would be different. The result follows. \square

Remark 5.10. We note that in the original definition of [15] the potential V is assumed to be repulsive and $V_r = 0$. In this case, (1) \Leftrightarrow (2) follows from the results of [18].

The three equivalent statements in Theorem 5.7 can be viewed as three alternative definitions of monodromy in the case of scattering integrable systems in the plane. We observe that for the original definition in terms of the deflection angle (statement (2) in Theorem 5.7), it is important that the scattering takes place in the plane. On the other hand, from Section 4 and the present section it follows that statements (1) and (3) in Theorem 5.7 are suitable for scattering integrable systems with many degrees of freedom, such as the Euler problem.

We note that Definition (3), similarly to Knauf's degree, can be naturally applied to scattering systems even without integrability. For a discussion of the relation between scattering monodromy and Knauf's degree, see [43, 45].

6. Scattering in the Euler problem

In this section we study scattering in the Euler problem using the reference Kepler Hamiltonians identified in the previous section. We will show that the Euler problem has non-trivial scattering monodromy of two different kinds: purely scattering monodromy and another kind, where both scattering and Hamiltonian monodromy are non-trivial. The latter kind can be observed only if the number of degrees of freedom $n \geq 3$. Purely Hamiltonian monodromy is also present in the problem; it survives the limiting cases of vanishing μ_i , including the free flow. Scattering monodromy (of both kinds) is trivial for the free flow. However, scattering monodromy of the second kind is still present in the Kepler problem.

6.1. Scattering map

Let $F = (H, L_z, G)$ denote the integral map of the Euler problem. Let N be a submanifold of

$$NT = \{(h, l, g) \in \text{image}(F) \mid F^{-1}(h, l, g) \subset s\}. \quad (9)$$

The manifold $F^{-1}(N)$ is an invariant submanifold of the phase space P , which contains scattering states only. Following the construction in Sections 4 and 5, we can define the scattering maps $S: B \rightarrow B$ with respect to H , the reference Kepler Hamiltonian $H_r = H_{r_1}$ or $H_r = H_{r_2}$, where

$$H_{r_1} = \frac{1}{2}p^2 - \frac{\mu_1 - \mu_2}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2}p^2 - \frac{\mu_2 - \mu_1}{r_2},$$

and $B = F^{-1}(N)/g_H^t$ as in Subsection 4.4.

Remark 6.1. We recall that the scattering map S is defined by

$$S = (A^-)^{-1} \circ A_r^- \circ (A_r^+)^{-1} \circ A^+,$$

where

$$A^\pm = (\hat{p}^\pm, q_\perp^\pm): s^\pm/g_H^t \rightarrow AS \quad \text{and} \quad A_r^\pm = (\hat{p}^\pm, q_\perp^\pm): s_r^\pm/g_H^t \rightarrow AS$$

map $s^\pm \subset P$ and s_r^\pm to the asymptotic states AS . Here the index r refers to a reference system (H_{r_1} or H_{r_2} in our case).

Remark 6.2. We note that the potential

$$V = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}$$

of the Euler problem is short-range relative to $\tilde{V}(q) = -(\mu_1 + \mu_2)/\|q\|$, which is a Kepler potential. The reference potentials are Kepler potentials and are therefore rotationally symmetric. It follows that Assumptions 4.3 are met.

6.2. Scattering monodromy

First we consider the case of a gravitational problem ($0 < \mu_2 < \mu_1$) with $H_r = H_{r_2}$ as the reference Kepler Hamiltonian. The other cases can be treated similarly; see Subsection 6.4.

For sufficiently large h_0 the $h = h_0$ slice of the bifurcation diagram has the form shown in Fig. 4. Let $\gamma_i, i = 1, 2, 3$, be a simple closed curve in

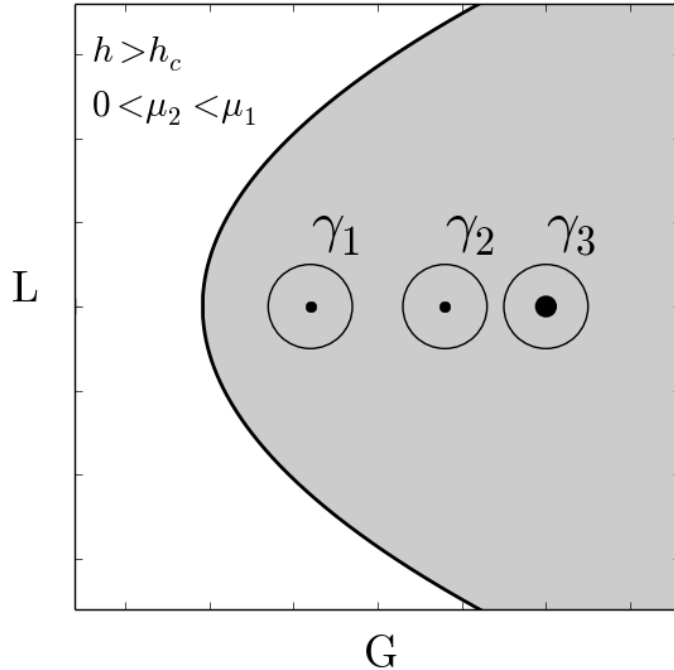


Figure 4: Energy slice of the bifurcation diagram for the spatial Euler problem; $h > h_c$, attractive case.

$$NT_{h_0} = \{(h, g, l) \in NT \mid h = h_0\}$$

that encircles the critical line ℓ_i , where

$$\begin{aligned} \ell_1 &= \{g = h + (\mu_2 - \mu_1), l = 0\}, \ell_2 = \{g = h + (\mu_1 - \mu_2), l = 0\} \text{ and} \\ \ell_3 &= \{g = h + (\mu_1 + \mu_2), l = 0\}. \end{aligned}$$

For each γ_i , consider the torus bundle $F_i: E_i \rightarrow \gamma_i$, where the total space E_i is obtained by gluing the ends of the fibers of F over γ_i via the scattering map S . We recall that *scattering monodromy* along γ_i with respect to H_r is defined as the usual monodromy of the torus bundle $F_i: E_i \rightarrow \gamma_i$; see Definition 5.6 and Appendix Appendix A.

Remark 6.3. Alternatively, one can define $F_i: E_i \rightarrow \gamma_i$ by gluing the fibers of the original and the reference integral maps at infinity. Both definitions are equivalent in the sense that the monodromy of the resulting torus bundles are the same.

Consider a starting point $\gamma_i(t_0) \in \gamma_i$ in the region where $l > 0$. We choose a basis $(c_\xi, c_\eta, c_\varphi)$ of the first homology group $H_1(F_i^{-1}(\gamma_i(t_0))) \simeq \mathbb{Z}^3$ as follows. The cycle $c_\xi = c_\xi^o \cup c_\xi^r$ is obtained by gluing the non-compact ξ -coordinate lines c_ξ^o for the original and c_ξ^r for the reference systems at infinity. In other words, for we glue the lines

$$p_\xi^2 = \frac{(\xi^2 - 1)(2h\xi^2 + 2(\mu_1 + \mu_2)\xi - 2g) - l^2}{(\xi^2 - 1)^2}$$

on $F^{-1}(\gamma_i(t_0))$, $\gamma_i(t_0) = (h, g, l)$, and

$$p_\xi^2 = \frac{(\xi^2 - 1)(2h\xi^2 + 2(\mu_2 - \mu_1)\xi - 2g) - l^2}{(\xi^2 - 1)^2}$$

on the reference fiber $F_r^{-1}(\gamma_i(t_0))$ at the limit points $\xi = \infty$, $p_\xi = \pm\sqrt{2h}$. The cycles c_η and c_φ are such that their projections onto the configuration space coincide with coordinate lines of η and φ , respectively. In other words, the cycle c_η on $F^{-1}(\gamma_i(t_0))$ is given by

$$p_\eta^2 = \frac{(1 - \eta^2)(-2h\eta^2 - 2(\mu_1 - \mu_2)\eta - 2g) - l^2}{(1 - \eta^2)^2}$$

and c_φ is an orbit of the circle action given by the Hamiltonian flow of the momentum L_z . We have the following result.

Theorem 6.4. *The monodromy matrices M_i of $E_i \rightarrow \gamma_i$ with respect to the natural basis $(c_\xi, c_\eta, c_\varphi)$ have the form*

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. **Case 1**, loop γ_1 . First we note that the cycle c_φ is preserved under the parallel transport along γ_1 . This follows from the fact that L_z generates a free fiber-preserving circle action on E_i . The cycles c_ξ and c_η can be naturally transported only in the regions where $l \neq 0$. We thus need to understand what happens at the critical plane $l = 0$.

Let $R > 1$ be a sufficiently large number. Then

$$E_{1,R} = \{x \in E_1 \mid \xi(x) > R\}$$

has exactly two connected components, which we denote by $E_{1,R}^+$ and $E_{1,R}^-$. We define a 1-form α on (a part of) E_i by the formula

$$\alpha = pdq - \chi(\xi)p_\xi(h, g, l, \xi)d\xi,$$

where $\chi(\xi)$ is a bump function such that

- (i) $\chi(\xi) = 0$ when $\xi < R$;
- (ii) $\chi(\xi) = 1$ when $\xi > 1 + R$.

The square root function $p_\xi(h, g, l, \xi)$ is assumed to be positive on $E_{1,R}^+$ and negative on

$E_{1,R}^-$. By construction, the 1-form α is well-defined and smooth on E_i outside collision points. Since

$$d\alpha = dp \wedge dq = -\omega \quad \text{on} \quad F^{-1}(\gamma_i) \cup F_r^{-1}(\gamma_i) \subset E_i,$$

we have that $d\alpha = 0$ on each fiber of F_i .

Consider the modified actions with respect to the form α :

$$I_\varphi = \frac{1}{2\pi} \int_{c_\varphi} \alpha, \quad I_\eta = \frac{1}{2\pi} \int_{c_\eta} \alpha \quad \text{and} \quad I_\xi^{mod} = \frac{1}{2\pi} \int_{c_\xi} \alpha.$$

The modified actions are well defined and, in view of $d\alpha = 0$, depend only on the homology classes of c_ξ, c_η and c_φ . It follows that I_φ and I_η coincide with the ‘natural’ actions (defined as the integrals over the usual 1-form pdq). We note that the ‘natural’ ξ -action

$$I_\xi = \frac{1}{2\pi} \int_{c_\xi} pdq$$

diverges, cf. [15]. From the continuity of the modified actions at $l = 0$ it follows that the corresponding scattering monodromy matrix has the form

$$M_1 = \begin{pmatrix} 1 & 0 & m_1 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the modified actions do not have to be smooth at $l = 0$, the integers m_1 and m_2 are not necessarily zero. In order to compute these integers we need to compare the derivatives $\partial_l I_\eta$ and $\partial_l I_\xi$ at $l \rightarrow \pm 0$. A computation of the corresponding residues gives

$$\lim_{l \rightarrow \pm 0} \partial_l I_\eta = \lim_{l \rightarrow \pm 0} \frac{1}{2\pi} \partial_l \int_{c_\eta} pdq = \begin{cases} 0, & \text{for } g < h + \mu_2 - \mu_1, \\ \mp 1/2, & \text{for } \mu_2 - \mu_1 < g - h < \mu_1 - \mu_2, \end{cases}$$

and

$$\begin{aligned} \lim_{l \rightarrow \pm 0} \partial_l I_\xi^{mod} &= \lim_{l \rightarrow \pm 0} \left(\frac{1}{2\pi} \partial_l \int_{c_\xi^0} pdq - \frac{1}{2\pi} \partial_l \int_{c_\xi^r} pdq \right) - \\ &\quad \lim_{l \rightarrow \pm 0} \frac{1}{2\pi} \int_{c_\xi} \chi(\xi) p_\xi(h, g, l, \xi) d\xi = 0 \end{aligned}$$

(for the two ranges of g). It follows that $m_1 = 0$ and $m_2 = 1$.

Case 2, loop γ_2 . This case is similar to **Case 1**. The corresponding limits are given by

$$\lim_{l \rightarrow \pm 0} (\partial_l I_\eta, \partial_l I_\xi^{mod}) = \begin{cases} (\mp 1/2, 0), & \text{for } \mu_2 - \mu_1 < g - h < \mu_1 - \mu_2, \\ (\mp 1, \pm 1/2), & \text{for } \mu_1 - \mu_2 < g - h < \mu_1 + \mu_1. \end{cases}$$

Case 3, loop γ_3 . The computation in this case is also similar to **Case 1**. The corresponding limits are given by

$$\lim_{l \rightarrow \pm 0} (\partial_l I_\eta, \partial_l I_\xi^{mod}) = \begin{cases} (\mp 1, \pm 1/2), & \text{for } \mu_1 - \mu_2 < g - h < \mu_1 + \mu_2, \\ (\mp, 0), & \text{for } h + \mu_1 + \mu_2 < g. \end{cases}$$

□

Remark 6.5. One difference between **Case 3** and the other cases is the topology of the critical fiber, around which scattering monodromy is defined. In **Case 3** the critical fiber is the product of a pinched cylinder and a circle, whereas in the other cases it is the product of a pinched torus and a real line. This implies, in fact, that **Case 3** is purely scattering, whereas in the other cases Hamiltonian monodromy is present; see Subsection 6.3 for details.

Remark 6.6. Theorem 6.4 admits another, geometric, proof in the pure scattering case.

Proof for Case 3 of Theorem 6.4. The action

$$I'_\eta = \begin{cases} I_\eta, & \text{for } l \geq 0 \\ I_\eta - 2l, & \text{for } l < 0 \end{cases}$$

is smooth and globally defined (over γ_3). Moreover, the corresponding circle action extends to a free action in $F_3^{-1}(D_3)$, where $D_3 \subset NT_{h_0}$ is a 2-disk such that $\partial D_3 = \gamma_3$. Since there is also a circle action given by I_φ , the result can be also deduced from the general theory developed in [19, 44]. □

From the last proof it follows that the specific choice of a reference Hamiltonian does not affect the result in the case of pure scattering monodromy. This agrees with the point of view presented in [18]. For the curves γ_1 and γ_2 , when monodromy is mixed scattering, the two reference Kepler Hamiltonians give different results; see Table 1.

As a corollary, we get the following result for the scattering map in the purely scattering case of the curve γ_3 .

Theorem 6.7. *The scattering map $S: B_3 \rightarrow B_3$, where $B_3 = F^{-1}(\gamma_3)/g_H^t$, is a Dehn twist. The push-forward map is conjugate in $SL(3, \mathbb{Z})$ to*

$$S_\star = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. The proof is similar to the proof of the equivalence (2) \Leftrightarrow (3) given in Theorem 5.7. The scattering map S allows one to consider the compactified torus bundle

$$\text{Pr}: F^{-1}(\gamma_3)^c \rightarrow S^1 = \mathbb{R} \cup \{\infty\},$$

where \mathbb{R} corresponds to the time. The torus bundle $F_3: E_3 \rightarrow \gamma_3$ has the same total space, but is fibered over γ_3 . By Theorem 6.4, the monodromy of the bundle $F_3: E_3 \rightarrow \gamma_3$ is given by the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the monodromy of the first bundle $\text{Pr}: F^{-1}(\gamma_3)^c \rightarrow S^1$ is the same, for otherwise the total spaces would be different. The result follows. \square

Remark 6.8. It follows from the proof and Subsection 6.4 that Theorem 6.7 holds for any $\mu_i \neq 0$ and for any regular closed curve $\gamma \subset NT$ such that

1. The energy value h is positive on γ ;
2. γ encircles the critical line $\{g = h + \mu_1 + \mu_2, l = 0\}$ exactly once and does not encircle any other line of critical values;
3. γ does not cross critical values of F .

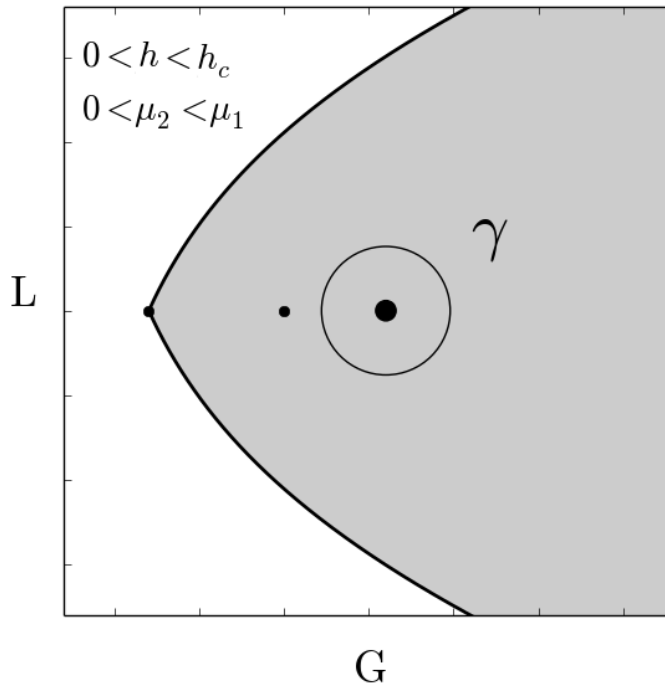


Figure 5: Energy slice of the bifurcation diagram for the spatial attractive Euler problem; $0 < h < h_c$. The curve γ encircles the ‘scattering’ line $\ell_3 = \{g = h + \mu_1 + \mu_2, l = 0\}$.

It can be shown that such a curve γ always exists; an example is given in Fig. 5. We note that the third condition can be weakened in the case $-\mu_1 < \mu_2 < 0$. In this case the attraction of μ_1 dominates the repulsion of μ_2 and, as a result, bound motion

coexists with unbound motion for a range of positive energies. Instead of $F^{-1}(\gamma)$ one may consider its unbounded component.

6.3. Topology

As we have noted before, alongside scattering monodromy, the Euler problem admits also another type of invariant — Hamiltonian monodromy. Here we consider the generic case of $|\mu_1| \neq |\mu_2| \neq 0$ in the case of positive energies. The case of negative energies is similar — it has been discussed in detail in [60]. The critical cases can be easily computed from the generic case by considering curves that encircle more than one of the singular lines

$$\begin{aligned} \ell_1 &= \{g = h + (\mu_2 - \mu_1), l = 0\}, \ell_2 = \{g = h + (\mu_1 - \mu_2), l = 0\} \text{ and} \\ \ell_3 &= \{g = h + (\mu_1 + \mu_2), l = 0\}. \end{aligned}$$

Let γ_i be a closed curve that encircles only the critical line ℓ_i ; see Fig. 5. The fibration

$$F: F^{-1}(\gamma_i) \rightarrow \gamma_i$$

is a $T^2 \times \mathbb{R}$ -bundle. As we show in Theorems 6.9 and 6.10, the Hamiltonian monodromy of this bundle (see Appendix Appendix A) is non-trivial for $i = 1, 2$ and is trivial in the other case $i = 3$.

Theorem 6.9. *The Hamiltonian monodromy of $F: F^{-1}(\gamma_i) \rightarrow \gamma_i$, $i = 1, 2$, is conjugate in $SL(2, \mathbb{Z}) \subset SL(3, \mathbb{Z})$ to*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the right-bottom 2×2 block acts on T^2 and the left-top 1×1 block acts on \mathbb{R} .

Proof. The result follows from the proof of Theorem 6.4. For completeness, we give an independent proof below.

After the reduction of the surface $H^{-1}(h)$ with respect to the flow g_H^t we get a singular T^2 torus fibration over a disk D_i , $\partial D_i = \gamma_i$, with exactly one focus-focus point. The result then follows from [42, 47, 64]. This argument applies to both of the lines ℓ_1 and ℓ_2 . Since the flow of L_z gives a global circle action, the monodromy matrix M is the same in both cases; see [10]. \square

Theorem 6.10. *The Hamiltonian monodromy of $F: F^{-1}(\gamma_3) \rightarrow \gamma_3$ is trivial.*

Proof. Observe that the Hamiltonian flows of I_φ ,

$$I'_\eta = \begin{cases} I_\eta, & \text{for } l \geq 0 \\ I_\eta - 2l, & \text{for } l < 0 \end{cases}$$

and H generate a global $\mathbb{T}^2 \times \mathbb{R}$ action on $F^{-1}(\gamma_3)$. It follows that the bundle $F: F^{-1}(\gamma_3) \rightarrow \gamma_3$ is principal. Since γ_3 is a circle, it is also trivial. \square

We note that Hamiltonian monodromy is an intrinsic invariant of the Euler problem, related to the non-trivial topology of the integral map F . Interestingly, it is also present in the critical cases:

- (1) $\mu_1 = \mu_2$ (symmetric Euler problem) [60],
- (2) μ_1 or $\mu_2 = 0$ (Kepler problem) [17] and
- (3) $\mu_1 = \mu_2 = 0$ (the free flow).

In the case of bound motion (1) and (2) are due to [60] and [17], respectively. From the scattering perspective Hamiltonian monodromy is recovered if one considers the original Hamiltonian H also as a reference.

6.4. General case

Here we consider the case of arbitrary strengths μ_i . We observe that the scattering monodromy matrices with respect to the reference Kepler Hamiltonians H_{r_1} and H_{r_2} are necessarily of the form

$$\begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

for some integers m and n . These integers (for different choices of the strengths μ_i and the critical lines ℓ_i) are given in Table 1.

Remark 6.11. We note that one can compute the monodromy matrices in the critical cases from the matrices found in the generic cases. Specifically, it is sufficient to consider the curves that encircle more than one critical line ℓ_i and multiply the monodromy matrices found around each of these lines. For instance, the monodromy matrix around the curve $g = h$ in the free flow equals the product of the three monodromy matrices found in (any) generic Euler problem.

7. Discussion

In the present paper we have shown that the spatial Euler problem, alongside non-trivial Hamiltonian monodromy [60], has non-trivial scattering monodromy of two different types: pure and mixed scattering monodromy. The first type reflects the presence of a special periodic orbit — a collision orbit that bounces between the two centers — and the associated trapping trajectories. In the spatial case one can go around these trajectories and compare the flow at infinity to an appropriately chosen Kepler problem. Scattering monodromy of the second type is related to the difference in dynamics of the original and the reference systems; here in addition to scattering monodromy also Hamiltonian monodromy is present. Interestingly, scattering monodromy of the second type survives vanishing of one of the centers: it can be also observed in the limiting case of attractive and repulsive Kepler problems

$$H_{r_1} = \frac{1}{2}p^2 - \frac{\mu}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2}p^2 + \frac{\mu}{r_2}.$$

	γ_1	γ_2	γ_3
	Scattering monodromy w.r.t. H_{r_1}		
Generic			
$ \mu_1 \neq \mu_2 \neq 0$	$m = -1, n = 1$	$m = 0, n = 1$	$m = 1, n = 0$
Critical			
$-\mu_1 = \mu_2 < 0$	$m = -1, n = 1$	$m = 0, n = 1$	$n = 1, n = 0$
$0 < \mu_1 = \mu_2$	$m = -1, n = 2$		$m = 1, n = 0$
$\mu_1 = \mu_2 < 0$	$m = -1, n = 2$		$m = 1, n = 0$
$\mu_1 = \mu_2 = 0$	$m = 0, n = 2$		
$0 = \mu_2 < \mu_1$	$n = 1$	$m = 0, n = 1$	$m = 0,$
$\mu_1 < \mu_2 = 0$	$m = -1, n = 1$	$m = 1, n = 1$	
	Scattering monodromy w.r.t. H_{r_2}		
Generic			
$ \mu_1 \neq \mu_2 \neq 0$	$m = 0, n = 1$	$m = -1, n = 1$	$m = 1, n = 0$
Critical			
$-\mu_1 = \mu_2 < 0$	$m = 0, n = 1$	$m = -1, n = 1$	$m = 1, n = 0$
$0 = \mu_2 < \mu_1$	$n = 1$	$m = -1, n = 1$	$m = 1,$
$\mu_1 < \mu_2 = 0$	$m = 0, n = 1$	$m = 0, n = 1$	

Table 1: Scattering monodromy, general case.

Hamiltonian monodromy is present not only in the Kepler problem [17], but also in the free flow. The purely scattering monodromy is special to the genuine Euler problem; we conjecture that this invariant is also present in the restricted three-body problem.

8. Acknowledgements

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Appendix A. Hamiltonian monodromy

Consider an integrable Hamiltonian system

$$F = (F_1 = H, F_2, \dots, F_n)$$

on a $2n$ -dimensional symplectic manifold (M, ω) . If the fibers of the integral map F are *compact* and *connected*, then according to the classical Arnol'd-Liouville theorem [1] a tubular neighborhood of each regular fiber is a trivial torus bundle $D^n \times T^n$ admitting action-angle coordinates. Hence

$$F: F^{-1}(R) \rightarrow R,$$

where $R \subset \text{image}(F)$ is the set of regular values of F , is a locally trivial torus bundle. This bundle is, however, not necessary globally trivial even from the topological viewpoint. One geometric invariant that measures this non-triviality was introduced by Duistermaat in [14] and is called *Hamiltonian monodromy*. Specifically, Hamiltonian monodromy is defined as a representation

$$\pi_1(R, \xi_0) \rightarrow \text{Aut } H_1(F^{-1}(\xi_0)) \simeq \text{GL}(n, \mathbb{Z})$$

of the fundamental group $\pi_1(R, \xi_0)$ in the group of automorphisms of the integer homology group $H_1(F^{-1}(\xi_0)) \simeq \mathbb{Z}^n$. Each element $\gamma \in \pi_1(R, \xi_0)$ acts via parallel transport of integer homology cycles [14].

Since the pioneering work of Duistermaat, Hamiltonian monodromy and its quantum counterpart [8, 58] have been observed in many integrable systems of physics and mechanics. General results are known that allow to compute this invariant in specific examples. It has been shown in [42, 47, 64] that in the typical case of $n = 2$ degrees of freedom non-trivial Hamiltonian monodromy is manifested by the presence of the so-called *focus-focus* points of the map F . In the case of a global circle action Hamiltonian monodromy (and, more generally, *fractional monodromy* [49]) can be computed in terms of the singularities of the circle action [19, 44].

Remark Appendix A.1. A notion of monodromy can be defined for torus bundles that do not necessarily come from an integrable system and also in the case of bundles with non-compact fibers (for instance, in the case of cylinder bundles). Specifically, consider a bundle $F: F^{-1}(\gamma) \rightarrow \gamma$, $\gamma = S^1$. It can be obtained from a direct product $[0, 1] \times F^{-1}(\gamma(t_0))$ by gluing the boundaries via a non-trivial homeomorphism f , called the *monodromy* of the bundle. We call this monodromy *Hamiltonian* if F comes from a completely integrable system. In this case the push-forward map f_* coincides with the automorphism given by the parallel transport.

We note that non-compact fibrations appear in the Euler problem in the case of positive energies and in various other integrable systems. We mention the works [24, 40, 46] and [2, 15, 18, 63]. For systems that are both scattering and integrable scattering monodromy and Hamiltonian monodromy coincide if the reference is given by the original Hamiltonian H .

Appendix B. Bifurcation diagrams for the planar problem

In this section we give *bifurcation diagrams* of the planar Euler problem in the case of arbitrary strengths μ_i . The computation has been performed in Section 3; more details can be found in [12, 54, 60].

The computation of Section 3 yields the following critical lines

$$\ell_1 = \{g = h + \mu_2 - \mu_1\}, \ell_2 = \{g = h + \mu_1 - \mu_2\} \quad \text{and} \quad (\text{B.1})$$

$$\ell_3 = \{g = h + \mu\}, \mu = \mu_1 + \mu_2, \quad (\text{B.2})$$

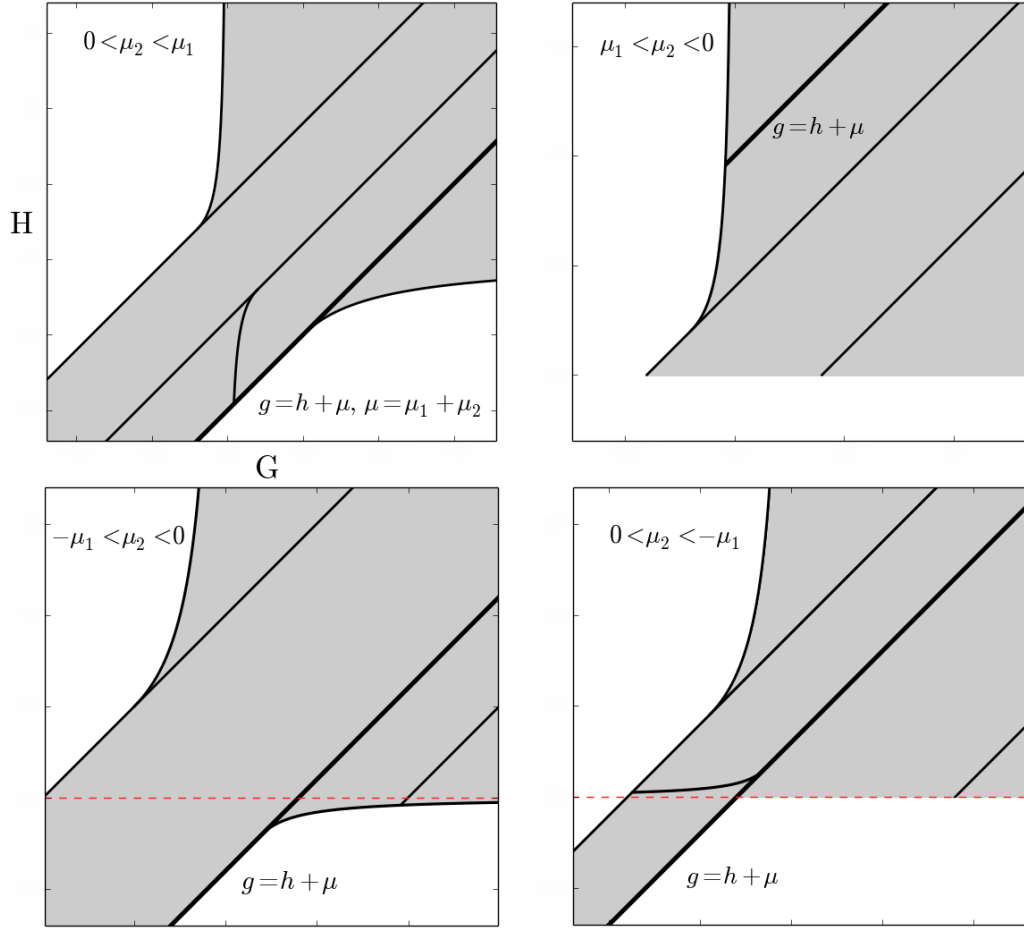


Figure B1: Bifurcation diagrams for the planar problem, generic cases $|\mu_1| \neq |\mu_2| \neq 0$. Top: attractive (left), repulsive (right). Bottom: mixed.

and the critical curves

$$\begin{aligned} \{g = \mu \cosh \lambda/2, h = -\mu/2 \cosh \lambda\}, \\ \{g = (\mu_1 - \mu_2) \sin \nu/2, h = (\mu_2 - \mu_1)/2 \sin \nu\}. \end{aligned}$$

Points that do not correspond to any physical motion must be removed from the obtained set. The resulting diagrams are given in Figs. B1 and B2. Here we distinguish two cases: generic case when the strengths $|\mu_1| \neq |\mu_2| \neq 0$ and the remaining critical cases.

We note that the critical cases occur when $|\mu_1| = |\mu_2|$ or when $\mu_1 \mu_2 = 0$. In the case $\mu_1 = -\mu_2 \neq 0$ the attraction of one of the centers equalizes the repulsion of the other center, making the bifurcation diagram qualitatively different from the cases when $-\mu_1 < \mu_2 < 0$ or $0 < \mu_2 < -\mu_1$. However, we still have the three different critical lines ℓ_1, ℓ_2 and ℓ_3 . In the other critical cases collisions of the critical lines ℓ_i occur. For instance, $\mu_1 = 0$ implies that $\ell_1 = \ell_3$ and so on. The same situation takes place in the spatial problem.

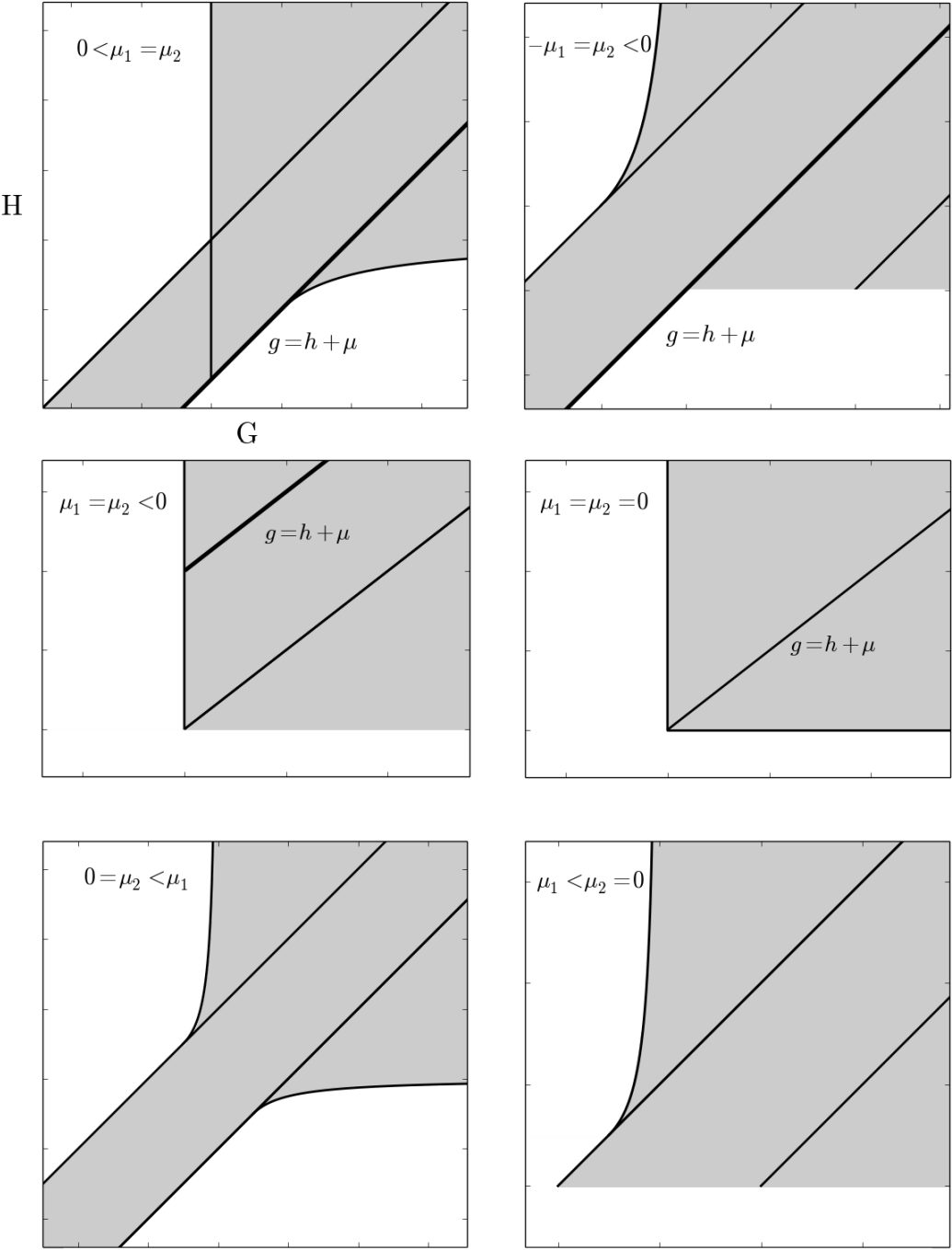


Figure B2: Bifurcation diagrams for the planar problem, non-generic cases $|\mu_1| = |\mu_2|$ or $\mu_1\mu_2 = 0$. From left to right, from top to bottom: symmetric attractive, anti-symmetric, symmetric repulsive, free flow, attractive Kepler problem, repulsive Kepler problem.

Appendix C. Proof of Theorem 5.4

We shall show that the Euler problem has two natural reference Hamiltonians when $\mu_1 \neq \mu_2$ and one otherwise.

Theorem Appendix C.1. *Among all Kepler Hamiltonians only*

$$H_{r_1} = \frac{1}{2}p^2 - \frac{\mu_1 - \mu_2}{r_1} \quad \text{and} \quad H_{r_2} = \frac{1}{2}p^2 - \frac{\mu_2 - \mu_1}{r_2}$$

are reference Hamiltonians of $F = (H, L_z, G)$. In particular, the free Hamiltonian is a reference Hamiltonian of F only in the case $\mu_1 = \mu_2$.

Proof. Sufficiency. Consider the Hamiltonian H_{r_1} . Let

$$G_{r_1} = H_{r_1} + \frac{1}{2}(L^2 - a^2(p_x^2 + p_y^2)) + a(z + a)\frac{\mu_1 - \mu_2}{r_1}.$$

From Section 2.1 (see also Eq. 4) it follows that the functions H_{r_1}, L_z and G_{r_1} Poisson commute. This implies that any trajectory $g_{H_{r_1}}^t(x)$ belongs to the common level set of $F_{r_1} = (H_{r_1}, L_z, G_{r_1})$. For a scattering trajectory we thus get

$$F_{r_1} \left(\lim_{t \rightarrow +\infty} g_{H_{r_1}}^t(x) \right) = F_{r_1} \left(\lim_{t \rightarrow -\infty} g_{H_{r_1}}^t(x) \right).$$

A straightforward computation of the limit shows that also

$$F \left(\lim_{t \rightarrow +\infty} g_{H_{r_1}}^t(x) \right) = F \left(\lim_{t \rightarrow -\infty} g_{H_{r_1}}^t(x) \right).$$

The case of H_{r_2} is completely analogous.

Necessity. Without loss of generality $\mu_2 \leq \mu_1$. Let

$$H_r = \frac{1}{2}p^2 - \frac{\mu}{r},$$

where $r: \mathbb{R}^3 \setminus \{o\} \rightarrow \mathbb{R}$ is the distance to some point $o \in \mathbb{R}^3$, be a reference Hamiltonian of F . We have to show that

1. $\mu > 0$ implies $o = o_1$ and $\mu = \mu_1 - \mu_2$;
2. $\mu < 0$ implies $o = o_2$ and $\mu = \mu_2 - \mu_1$;
3. $\mu = 0$ implies $\mu_1 = \mu_2$.

Case 1. First we show that o belongs to the z axis. If this is not the case, then, due to rotational symmetry, we have a reference Hamiltonian H_r with $o = (-b_0, 0, z_0)$ for some $b_0, z_0 \in \mathbb{R}$, $b_0 \neq 0$. This reference Hamiltonian H_r has a trajectory $t \mapsto g_{H_r}^t(x)$ that (in the configuration space) has the form shown in Figure C1. But for such a trajectory

$$L_z \left(\lim_{t \rightarrow +\infty} g_{H_{r_1}}^t(x) \right) = 0 \neq \sqrt{2h} \cdot b_0 = L_z \left(\lim_{t \rightarrow -\infty} g_{H_{r_1}}^t(x) \right),$$

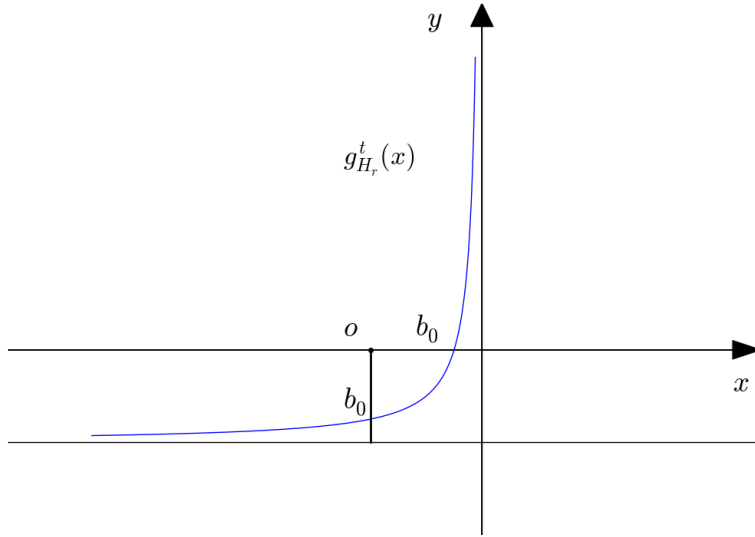


Figure C1: Kepler trajectory $g_{H_r}^t(x)$ in the $z = z_0$ plane.

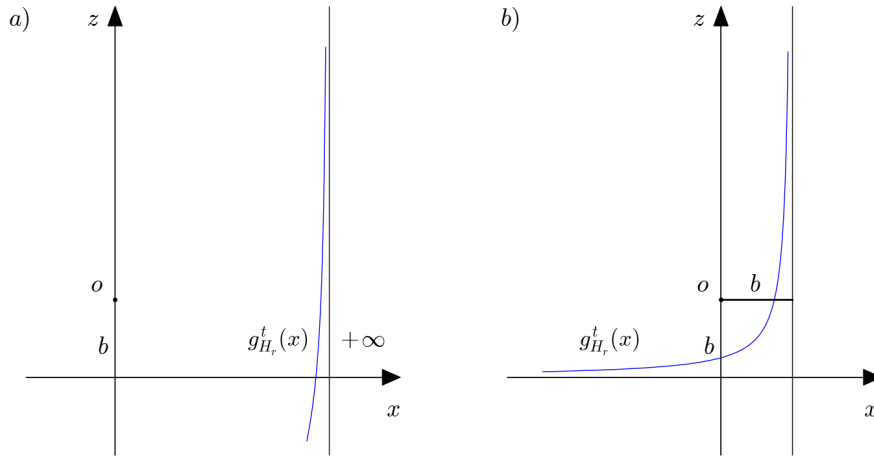


Figure C2: Kepler trajectories in the $y = 0$ plane.

where $h = H_r(x) > 0$ is the energy of $g_{H_r}^t(x)$. We conclude that $o = (0, 0, b)$ for some $b \in \mathbb{R}$.

Next we show that $b\mu = a(\mu_1 - \mu_2)$. Consider a trajectory $g_{H_r}^t(x)$ of H_r that has the form shown in Figure C2a. It follows from Eq. 4 that the function

$$G_r = H_r + \frac{1}{2}(L^2 - b^2(p_x^2 + p_y^2)) + b(z + b)\frac{\mu}{r}$$

is constant along this trajectory. Thus, for H_r to be a reference Hamiltonian we must have

$$(G - G_r) \left(\lim_{t \rightarrow +\infty} g_{H_r}^t(x) \right) = (G - G_r) \left(\lim_{t \rightarrow -\infty} g_{H_r}^t(x) \right). \quad (\text{C.1})$$

In the configuration space, $g_{H_r}^t(x)$ is asymptotic to the ray $x = c$, $y = 0$, $z \geq 0$ at $t = +\infty$. The other asymptote at $t = -\infty$ gets arbitrarily close to the ray $x = c$, $y = 0$, $z \leq 0$ when $c \rightarrow +\infty$. It follows that Eq. C.1 is equivalent to

$$a(\mu_1 - \mu_2) - b\mu = b\mu - a(\mu_1 - \mu_2) + \varepsilon,$$

where $\varepsilon \rightarrow 0$ when $c \rightarrow +\infty$.

The remaining equality $b = a$ can be proven using a trajectory $g_{H_r}^t(x)$ that has the form shown in Figure C2b.

Case 2. In this case trajectories $g_{H_r}^t(x)$ of the repulsive Kepler Hamiltonian H_r do not project to the curves shown in Figs. C1, C2a and C2b. However, each of these curves is a branch of a hyperbola. The ‘complementary’ branches are (projections of) trajectories of H_r ; see Fig. C3. If the latter branches are used, the proof becomes similar to **Case 1**.

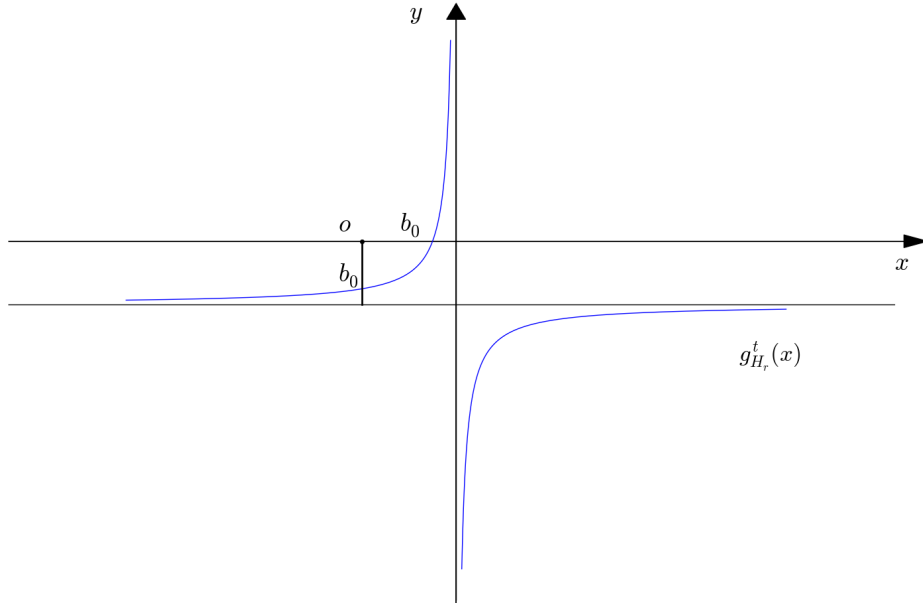


Figure C3: The two branches ($z = z_0$ plane). In the repulsive case $\mu > 0$ a Kepler trajectory is represented by the convex branch.

Case 3. In this case H_r generates the free motion. Let

$$g_{H_r}^t(x) = (q(t), p(t)), \quad q(t) = (c, 0, t), \quad p(t) = (0, 0, 1).$$

Since L^2 and (p_x, p_y, p_z) are conserved,

$$G\left(\lim_{t \rightarrow +\infty} g_{H_{r_1}}^t(x)\right) = G\left(\lim_{t \rightarrow -\infty} g_{H_{r_1}}^t(x)\right)$$

implies $a(\mu_1 - \mu_2) = a(\mu_2 - \mu_1)$ and hence $\mu_1 = \mu_2$. \square

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