Published for SISSA by Springer

Received: *April 4, 2012*
Revised: *June 12, 2012*
Accepted: *June 13, 2012*
Published: *July 6, 2012*

On the cubic interactions of massive and partially-massless higher spins in (A)dS

Euihun Joung, Luca Lopez and Massimo Taronna

Scuola Normale Superiore and INFN,
Piazza dei Cavalieri 7, 56126 Pisa, Italy

E-mail: euihun.joung@sns.it, luca.lopez@sns.it, massimo.taronna@sns.it

Abstract: Cubic interactions of massive and partially-massless totally-symmetric higher-spin fields in any constant-curvature background of dimension greater than three are investigated. Making use of the ambient-space formalism, the consistency condition for the traceless and transverse parts of the parity-invariant interactions is recast into a system of partial differential equations. The latter can be explicitly solved for given $s_1-s_2-s_3$ couplings and the $2-2-2$ and $3-3-2$ examples are provided in detail for general choices of the masses. On the other hand, the general solutions for the interactions involving massive and massless fields are expressed in a compact form as generating functions of all the consistent couplings. The Stückelberg formulation of the cubic interactions as well as their massless limits are also analyzed.

Keywords: Gauge Symmetry, Bosonic Strings, Field Theories in Higher Dimensions, Space-Time Symmetries

ArXiv ePrint: 1203.6578

doi:10.1007/JHEP07(2012)041

JHEP07(2012)041

Contents

1 Introduction

One of the key lessons of intense studies in higher-spin (HS) field theories is the need to abandon many of the beliefs inherited by years of extraordinary results devoted to understanding their lower-spin counterparts. For instance, the higher-derivative nature of the couplings as well as the need of introducing infinitely many HS fields are clear signals that the standard frameworks are not sufficient. These features naturally surface in String Theory (ST), where the presence of an infinite tower of massive higher-spin excitations brings about most of its remarkable properties. Besides being responsible for planar duality, open-closed duality and modular invariance, the plethora of massive HS particles is what makes the high-energy behavior of string amplitudes softer than in any local quantum field theory. To wit, although its lower-spin truncations are in general non-renormalizable, ST is believed to be finite. The field-theoretical reason for this difference is the contribution of an infinite number of massive HS fields to quantum corrections. Moreover, since the massive HS spectrum becomes massless in the tensionless limit, it has long been conjectured that ST may describe a broken phase of an underlying HS gauge theory. Therefore, in order to better understand the quantum properties of ST as well as other of its remarkable features, it would be important to investigate the dynamics of HS gauge fields and their links to massive counterparts on more general field-theoretical grounds.

Over the years, finding consistent interactions of HS gauge fields has proven to be a very challenging task.[1] A long-recognized difficulty concerns the inconsistency of the gravitational minimal couplings in flat space-time [11]. As shown in [12, 13], this problem can be solved in (anti) de Sitter space-time ((A)dS). There HS gauge invariance, which is broken when one replaces ordinary partial derivatives by the gravitational covariant ones, is restored by adding a chain of higher-derivative interactions sized by negative powers of the cosmological constant. Interestingly, this way of solving the minimal interaction problem is similar to the one used for massive HS fields in the Stückelberg formulation. More precisely, one can restore the Stückelberg gauge invariance of the HS fields by adding higher-derivative interactions sized by inverse powers of the mass.[2] This analogy between the roles of cosmological constant and masses suggests that a systematic study of massive HS theories in (A)dS can provide new insights on both Vasiliev's HS gauge theory[3] in (A)dS and ST, and eventually shed some light on their relations. However, although both of them have been known for many years, extracting their interaction vertices remains a very difficult program that only recently have been pushed forward by some new but yet not conclusive steps. The present work aims to constructing all consistent cubic interactions of totally symmetric HS relying on the Noether procedure. The cubic results are expected to be further constrained by the higher-order consistency leading eventually to ST and Vasiliev's system and possibly to other consistent theories. We hope our work to be a first step towards HS systematics. Let us mention as well that the construction of consistent interacting massive HS theories is also relevent from a phenomenological prespective. Indeed, they provide an effective description for hadronic resonances in certain regimes.

Free massive HS particles can be described by the Fierz system [18] consisting of dynamical field equations together with the traceless and transverse (TT) constraints. The latter constraints guarantee the propagation of the correct number of physical degrees of freedom (DoF). The Lagrangian reproducing the Fierz system was first obtained in [19, 20] in flat space, and further studied in [21–40] in flat or (A)dS background. In dS, the mass spectrum in the unitary region presents a discrete series of mass values, called partially-massless points [21, 41–56], for which the fields acquire gauge symmetries and the corresponding representations become shorter. It is worth noticing that the interac-

[1] For recent reviews on the subject, see e.g. the proceeding [1] (which includes [2–6]) and [7–10].

[2] See e.g. [14] for the EM interaction of spin 2 and [15] for the gravitational interaction of spin 3.

[3] Vasiliev's equation provides at present the only known fully non-linear consistent description of an infinite number of HS gauge fields of all spins [16, 17].

JHEP07(2012)041

tions of these partially-massless fields might play some important role in the inflationary cosmology.

As noticed some time ago, the introduction of interactions for massive HS fields might either spoil the TT constraints, thus leading to the appearance of unphysical DoF [57], or violate causality [58–60]. See e.g. [14, 15, 61–74] for some recent works on the consistency of the electromagnetic (EM) and gravitational couplings to massive HS fields.[4] It is worth noticing that, as shown for spin 2 in [76, 77] and for arbitrary spins in [78], ST provides a solution for the case of constant EM background. See also [79, 80] for an analysis of HS interactions in the open bosonic string and [81–85] for studies on scattering amplitudes of HS states in superstring and heterotic string theories. Other works on cubic interactions of massive HS fields in (A)dS can be found in [86, 87].

Traceless and transverse part of the interactions. The aforementioned difficulties in finding consistent interactions manifest themselves only at the full *off-shell* level,[5] while they can be circumvented restricting the attention to the physical DoF. Indeed, relying on the light-cone formalism, Metsaev constructed all consistent cubic interactions involving massive and massless HS fields in flat space-time [88, 89]. In this approach, what is left is to find the *complete* expressions associated with those vertices. Starting from the TT parts of the interactions, that can be viewed as the covariant versions of Metsaev's lightcone vertices, the corresponding complete forms within the Fronsdal setting were obtained recently in [80, 90]. Moreover, the computation of (tree-level) correlation functions does not require the full vertices but only their TT parts.[6] Therefore, although they ought to be completed, the TT parts of the vertices are also interesting in their own right. Motivated by this observation, recently the TT parts of the cubic interactions of massless HS fields in (A)dS were identified in [92].[7] In the present paper, we extend this approach to the cases of massive and partially-massless fields in (A)dS.

Radial reduction with delta function. A way of obtaining massive theories is via dimensional reduction of a $(d+1)$-dimensional massless theory [53, 94–97].[8] However, when applied to cubic interactions, the conventional Kaluza-Klein (KK) reduction method imposes some restrictions. In the case of flat-space interactions, these rule out the possibility of reproducing most of the known examples of massive HS interactions, notably those appearing in ST [79, 80]. Notice that, after all, the consistency of the KK reduction does not hold if one considers only a part of the KK spectrum.[9] In this paper we avoid this

[4] See also [75] for the study of EM interactions of partially-massless spin 2 fields.

[5] By *off-shell* we mean the entire Lagrangian including traces and divergences of fields, as opposed to its TT part.

[6] See e.g. [80, 91] for the analysis of higher-order interactions of massless particles in flat space.

[7] See [93] for the frame-like approach to the same problem.

[8] The Singh-Hagen massive HS Lagrangian [19, 20] can be obtained through dimensional reduction of Fronsdal's massless one [98, 99] after gauge fixing. However, the gauge fixing procedure is non-trivial if one starts with the Fronsdal action and a more convenient one can be found in [100]. Let us also mention that the analysis in [53] is carried out within the unconstrained setting of [101, 102] bypassing all the problems related to the constrained Fronsdal formulation. Furthermore, let us mention that similar results can be also recovered starting from the tractor approach [38, 54, 103].

[9] The only consistent truncation is the massless one which is not the main goal of the present paper.

restriction working within the ambient-space formulation of (A)dS fields [22, 24, 104–106] with an insertion of a delta function of the radial coordinate into the $(d+1)$-dimensional action.[10] This means that we are actually dealing with a d-dimensional action but in a $(d+1)$-dimensional representation. On the other hand, the gauge consistency requires particular attention in treating the total-derivative terms that, because of the insertion of the delta function, do not vanish any longer.

Taking into account the aforementioned subtleties, we translate the consistency condition for the vertices into a set of differential equations. The latter can be explicitly solved for given $s_1-s_2-s_3$ couplings and the $2-2-2$ and $3-3-2$ examples are provided in detail for all different combinations of the masses. In the following we summarize our results for arbitrary spins. Let us stress that our analysis is independent of space-time dimensionality, however subtleties arise in three and four dimensions due to the appearance of some identities. More precisely, in three dimensions our analysis is not complete while in four dimensions some parts of the vertices can vanish identically.[11]

Massive and massless interactions. Cubic interactions involving massive and massless fields can be expressed in a compact form via generating functions of all consistent couplings. Depending on the number of massless fields entering the latter, the corresponding vertices are given by functions $\mathcal{K}$ of subsets of the following building blocks:

$$\begin{aligned}
\tilde{Y}_i &= \partial_{U_i}\cdot\partial_{X_{i+1}} + \alpha_i\,\partial_{U_i}\cdot\partial_X\,,\\
Z_i &= \partial_{U_{i+1}}\cdot\partial_{U_{i-1}}\,,\\
\tilde{G} &= (\partial_{U_1}\cdot\partial_{X_2} + \beta_1\,\partial_{U_1}\cdot\partial_X)\,\partial_{U_2}\cdot\partial_{U_3} + (\partial_{U_2}\cdot\partial_{X_3} + \beta_2\,\partial_{U_2}\cdot\partial_X)\,\partial_{U_3}\cdot\partial_{U_1}\\
&\quad + (\partial_{U_3}\cdot\partial_{X_1} + \beta_3\,\partial_{U_3}\cdot\partial_X)\,\partial_{U_1}\cdot\partial_{U_2}\,,\\
\tilde{H}_i &= \partial_{X_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{U_{i-1}}\cdot\partial_{U_{i+1}} - \partial_{X_{i+1}}\cdot\partial_{U_{i-1}}\,\partial_{X_{i-1}}\cdot\partial_{U_{i+1}}\,,
\end{aligned} \tag{1.1}$$

that are differential operators acting on ambient-space HS fields

$$\Phi(X_i,U_i) = \sum_{s=0}^{\infty}\frac{1}{s!}\,\Phi^{(s)}_{M_1\ldots M_s}(X_i)\,U_i^{M_1}\cdots U_i^{M_s}\,. \tag{1.2}$$

Here $\partial_{X^M} = \partial_{X_1^M} + \partial_{X_2^M} + \partial_{X_3^M}$ denotes total derivatives, while the α_i's and the β_i's are parameterized as

$$\begin{aligned}
\alpha_1 &= \alpha\,, & \alpha_2 &= -\frac{1}{\alpha+1}\,, & \alpha_3 &= -\frac{\alpha+1}{\alpha}\,,\\
\beta_1 &= \beta\,, & \beta_2 &= -\frac{\beta+1}{\alpha+1}\,, & \beta_3 &= -\frac{\alpha-\beta}{\alpha}\,.
\end{aligned} \tag{1.3}$$

Finally, the TT parts of the cubic interactions for massive and massless HS fields in (A)dS can be expressed as

$$\int d^{d+1}X\,\delta\big(\sqrt{\epsilon\,X^2}-L\big)\;\mathcal{K}\;\Phi(X_1,U_1)\,\Phi(X_2,U_2)\,\Phi(X_3,U_3)\,\Big|_{\substack{X_i=X\\U_i=0}}\,, \tag{1.4}$$

[10] A similar delta-function calculus has been used in the framework of 2T-physics (see [107] and references therein).

[11] For instance, in four dimensions the Gauss-Bonnet identity allows to rewrite the coupling of three partially-massless spin 2 fields with at most four derivatives as a coupling with at most two derivatives.

where ϵ is a sign, positive for dS and negative for AdS. The flat-space interactions can be smoothly recovered as limits of the AdS ones.

Partially-massless interactions. Although at present we are not able to derive a generating function encompassing all possible interactions of partially-massless fields (which are unitary only in dS), this can be done for a class of highest-derivative couplings. As a result, whenever the i-th field is at one of its partially-massless points $\mu_i \in \{0\,, \ldots\,, s_i{-}1\}$, the corresponding vertices are consistent provided the condition

$$\mu_i - |\mu_{i+1} - \mu_{i-1}| \in 2\,\mathbb{N}_{\geq 0}\,, \quad [i \simeq i+3]\,, \tag{1.5}$$

holds. Here, the μ_i's are numbers parameterizing the mass-squared values

$$M_i^2 = -\frac{1}{L^2}\left[(\mu_i - s_i + 2)(\mu_i - s_i - d + 3) - s_i\right], \tag{1.6}$$

of the spin s_i fields. It is conceivable that the aforementioned pattern does not change in the general case, giving rise to an enhancement of the number of consistent couplings whenever (1.5) is satisfied. This is indeed the case for all the examples that we have analyzed explicitly, although arriving at a definite conclusion on this issue would require more efforts so that we leave this problem for future work.

Stückelberg-field formulation. For the purpose of getting the full vertices, one would need to implement gauge symmetry also for massive fields. Then, as in the massless case, the remaining parts of the interactions could be recursively determined relying on the gauge invariance of the vertices. Massive HS fields acquire gauge symmetries in the Stückelberg formulation, wherein one introduces new fields and gauge symmetries into the massive theory in such a way not to alter it. The advantage of such a formulation is that it allows to properly analyze the massless limit of a massive theory that, in general, turns out to be very delicate. A renowned example is the vDVZ discontinuity [108, 109], related to the fact that the massless limit of a massive spin 2 is not simply a massless spin 2 but involves a massless vector and a massless scalar too.[12] The analysis preserving the number of DoF in the massless limit can be carried out within the Stückelberg formulation. Let us mention here a key difference between the massless limit in flat and in AdS space. While in flat space a massive spin s splits into a collection of massless fields of spin from s down to 0, in AdS it gives rise to a massless spin s and a massive spin $s-1$ field [21, 46, 48, 50, 51, 97]. With the aim of extending the analysis of the massless limit to the cubic level, we also provide the Stückelberg formulation of the cubic interactions. The latter can be obtained making use of the Stückelberg shift encoded in the following generating functions:

$$\mathcal{K}(\boldsymbol{Y},\boldsymbol{Z})\,P(w_1\,X_1\cdot\partial_{U_1})\,P(w_2\,X_2\cdot\partial_{U_2})\,P(w_3\,X_3\cdot\partial_{U_3})\Big|_{w_i=0}\,, \tag{1.7}$$

where the $\boldsymbol{Y}_i$'s and the $\boldsymbol{Z}_i$'s are given by

$$\begin{aligned}\boldsymbol{Y}_i &= Y_i + \partial_{X_i}\cdot\partial_{X_{i+1}}\,\partial_{w_i}\,,\\ \boldsymbol{Z}_i &= Z_i + \partial_{U_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{w_{i-1}} + \partial_{U_{i-1}}\cdot\partial_{X_{i+1}}\,\partial_{w_{i+1}} + \partial_{X_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{w_{i+1}}\,\partial_{w_{i-1}}\,,\end{aligned} \tag{1.8}$$

[12] Let us mention that the vDVZ discontinuity is absent in (A)dS [49, 53, 110–112].

and $P(z) = {}_0F_1(-\mu\,;\,-L\,z)$ is a hypergeometric function. Under the assumption that all mass parameters of the theory scale uniformly in the massless limit, we find that in AdS the leading terms of the interactions are massive couplings involving all the massive spin $s-1$ components of the original spin s fields. On the other hand, when some of the leading parts are absent, then the new dominant ones start to involve the massless spin s components. Finally, performing the massless limit in flat space one recovers consistent massless vertices in agreement with the aforementioned pattern.

Organization of the paper. Section 2 is devoted to the formulation of the free theories of massive and (partially-) massless fields in the ambient-space formalism. In section 3 we provide the solutions to the Noether procedure for the corresponding cubic interactions. We then extend the previous results to the Stückelberg formulation and study the massless limit of the massive couplings in section 4. Our results as well as some outlook are summarized and discussed in section 5. Appendix A contains some identities and mathematical tools used in our construction. In appendix B we provide the detailed examples of $2-2-2$ and $3-3-2$ interactions, while in appendix C we discuss a class of interactions containing the highest number of derivatives. Finally, appendices D and E include further details on the massless limit in flat space and on the ST interactions, respectively.

2 Free HS fields in (A)dS

In this section we present the free theories of massive and (partially-)massless totally-symmetric HS fields in (A)dS.[13] After providing an intrinsic formulation, we introduce the ambient-space formalism in which the construction of the cubic vertices becomes considerably simpler.

A massive spin-s boson in (A)dS can be described in terms of a totally-symmetric rank-s tensor field $\varphi^{(s)}_{\mu_1\ldots\mu_s}$. In the following, we use the generating functions of such fields:

$$\varphi^{A}(x,u) := \sum_{s=0}^{\infty}\,\frac{1}{s!}\;\varphi^{A\,(s)}_{\mu_1\ldots\mu_s}(x)\;u\cdot e^{\mu_1}(x)\,\cdots\,u\cdot e^{\mu_s}(x)\,, \tag{2.1}$$

where the contraction with the flat auxiliary variables u^α is via the inverse (A)dS vielbein $e_\alpha{}^\mu(x)$: $u\cdot e^\mu(x) = u^\alpha\, e_\alpha{}^\mu(x)$, and A is a *color* index associated with the Chan-Paton factors. The massive representations of the (A)dS isometry group correspond to HS fields satisfying the Fierz system:

$$(D^2 - M^2)\,\varphi^A = 0\,, \qquad \partial_u\cdot e^\mu\, D_\mu\,\varphi^A = 0\,, \qquad \partial_u^2\,\varphi^A = 0\,, \tag{2.2}$$

where M is the mass operator defined by $M^2\,\varphi^{(s)} := m_s^2\,\varphi^{(s)}$, and D_μ is the covariant derivative:

$$D_\mu := \nabla_\mu + \frac{1}{2}\,\omega_\mu^{\alpha\beta}(x)\,u_{[\alpha}\partial_{u^{\beta]}}\,. \tag{2.3}$$

Here ∇_μ is the usual (A)dS covariant derivative and $\omega_\mu^{\alpha\beta}$ is the (A)dS spin connection, so that the (A)dS Laplacian operator is given simply by D^2.

[13]Throughout this paper, by (A)dS we refer to any constant-curvature background including flat space.

The quadratic action for HS fields reproducing the Fierz system (2.2) can be written as

$$S^{(2)} = \frac{1}{2}\int d^dx\sqrt{-g}\left[\delta_{A_1A_2}\, e^{\partial_{u_1}\cdot\,\partial_{u_2}}\, \varphi^{A_1}(x_1,u_1)\left(D_2^2 - M_2^2\right)\varphi^{A_2}(x_2,u_2) + \,\ldots\right]_{\substack{x_i=x\\ u_i=0}}, \quad (2.4)$$

where the ellipsis denote, henceforth, terms proportional to divergences and traces of the fields as well as possible auxiliary fields. Since we focus on the TT parts of the cubic interactions, such terms are not relevant for our discussion although they must be taken into account in order to construct the full theory.[14] The Lagrangian equations are

$$(D^2 - M^2)\,\varphi^A + \,\ldots\, \approx 0\,, \quad (2.5)$$

together with possible equations for the auxiliary fields.

A massless spin-s boson in (A)dS corresponds to the mass-squared value:

$$m_s^2 = \frac{(-\epsilon)}{L^2}\left[(s-2)(s+d-3) - s\right], \quad (2.6)$$

where L is the (A)dS radius and ϵ is a sign, negative for AdS and positive for dS. For this value of the mass, the action (2.4) admits the gauge symmetries:

$$\delta^{(0)}\,\varphi(x,u) = u\cdot e^{\mu}\, D_\mu\, \varepsilon(x,u)\,, \quad (2.7)$$

with the gauge parameter ε traceless in the Fronsdal's formulation [104] and traceful in the unconstrained ones [102, 113]. For simplicity, in this paper we disregard the issue of trace constraints keeping the unconstrained formulation in mind. However, since we focus on the TT parts of the Lagrangian such a distinction is irrelevant.

2.1 Ambient-space formalism

It is well known that the d-dimensional Euclidean AdS or Lorentzian dS space can be embedded in the $(d+1)$-dimensional flat space with metric:

$$ds^2_{\text{Amb}} = \eta_{MN}\, dX^M\, dX^N\,, \qquad \eta = (-,+,\ldots,+)\,. \quad (2.8)$$

The (A)dS space is then defined as the hyper-surface $X^2 = \epsilon\, L^2$, where, as before, ϵ is a sign, negative for AdS and positive for dS. We concentrate on the region of the ambient space with $\epsilon\, X^2 > 0$, and consider the generating function of totally-symmetric tensor fields $\Phi_{M_1\ldots M_s}$ given by

$$\Phi(X,U) = \sum_{s=0}^{\infty}\frac{1}{s!}\,\Phi^{(s)}_{M_1\ldots M_s}(X)\, U^{M_1}\cdots U^{M_s}\,. \quad (2.9)$$

These fields are equivalent to totally-symmetric tensor fields in (A)dS if they are homogeneous in X^M and tangent to constant X^2 surfaces. At the level of the generating function, the latter conditions translate into

$$\text{Homogeneity}: \qquad (X\cdot\partial_X - U\cdot\partial_U + 2 + \mu)\,\Phi(X,U) = 0\,, \quad (2.10)$$

$$\text{Tangentiality}: \qquad X\cdot\partial_U\,\Phi(X,U) = 0\,, \quad (2.11)$$

[14] See [19, 20, 24–27, 29, 31–34, 37, 38] for the precise forms of the free action.

where μ is a parameter related to the (A)dS mass. In order to identify the ambient-space fields with the (A)dS ones, we parameterize the $\epsilon\, X^2 > 0$ region with the radial coordinates (R, x) given by

$$X^M = R\,\hat{X}^M(x)\,, \qquad \hat{X}^2(x) = \epsilon\,, \tag{2.12}$$

and rotate the auxiliary U^M-variables as

$$U^M = \hat{X}^M(x)\,v + L\,\frac{\partial \hat{X}^M}{\partial x^\mu}(x)\,e_\alpha{}^\mu(x)\,u^\alpha\,. \tag{2.13}$$

With this change of variables from (X, U) to $(R, x; v, u)$, the homogeneity and tangentiality conditions (2.10, 2.11) are solved by the (A)dS intrinsic generating functions as

$$\Phi(R, x; v, u) = \left(\frac{R}{L}\right)^{u\cdot\partial_u - 2 - \mu} \varphi(x, u)\,, \tag{2.14}$$

and the action (2.4) can be written as

$$S^{(2)} = \frac{1}{2}\int d^{d+1}X\ \delta\Big(\sqrt{\epsilon\,X^2} - L\Big)\Big[\,\delta_{A_1 A_2}\,e^{\partial_{U_1}\cdot\,\partial_{U_2}}\,\Phi^{A_1}(X_1, U_1)\,\partial^2_{X_2}\,\Phi^{A_2}(X_2, U_2) + \ldots\Big]_{\substack{X_i = X\\ U_i = 0}}\,. \tag{2.15}$$

In the ambient space, the Lagrangian equation (2.5) reads

$$\partial_X^2\,\Phi + \ldots \approx 0\,, \tag{2.16}$$

where the ambient-space d'Alembertian is related to the (A)dS one as

$$\partial_X^2\,\Phi = \left(\frac{R}{L}\right)^{u\cdot\partial_u - 4 - \mu}(D^2 - M^2)\,\varphi\,. \tag{2.17}$$

Here, the mass-squared is given in terms of μ by

$$M^2 = \frac{(-\epsilon)}{L^2}\,\big[(\mu - u\cdot\partial_u + 2)(\mu - u\cdot\partial_u - d + 3) - u\cdot\partial_u\big]\,. \tag{2.18}$$

Notice that for dS, where $\epsilon = 1$, the parameter μ is in general a complex number, hence, in order for the fields to be real one has to add the complex conjugate in eq. (2.14). Making a comparison with (2.6), one can also see that $\mu = 0$ corresponds to the massless case.

Flat-space limit. The flat-space limit $L \to \infty$ can be considered keeping the ambient-space point of view. In order to do that, we first need to place the origin of the ambient space in a point on the hyper-surface $X^2 = \epsilon\,L^2$ by translating the coordinate system as

$$X^M \quad\to\quad X^M + L\,\hat{N}^M\,. \tag{2.19}$$

Here, $\hat{N}$ is a constant vector in the ambient space satisfying $\hat{N}^2 = \epsilon$. After this shift, taking the $L \to \infty$ limit one gets

$$\delta\Big(\sqrt{\epsilon\,X^2} - L\Big) \quad \underset{L\to\infty}{\longrightarrow} \quad \epsilon\,\delta(\hat{N}\cdot X)\,, \tag{2.20}$$

so that the hyper-surface $X^2 = \epsilon L^2$ becomes the hyperplane $\hat{N} \cdot X = 0$, defining the d-dimensional flat space embedded in the ambient space. Moreover, the homogeneity and tangentiality conditions (2.10, 2.11) admit a well-defined limit:

$$\left(\hat{N} \cdot \partial_X - \sqrt{-\epsilon}\, M\right) \Phi(X,U) = 0\,, \qquad \hat{N} \cdot \partial_U\, \Phi(X,U) = 0\,, \tag{2.21}$$

provided one first divides them by L and redefines μ in terms of M according to (2.18). The latter equations are solved by

$$\Phi(X,U) = e^{-\sqrt{-\epsilon}\, M\, \rho}\, \varphi(x,u)\,, \tag{2.22}$$

where $\rho := \hat{N} \cdot X$ and (x,u) are coordinates on the hyper-surface of constant $\hat{N} \cdot X$ and $\hat{N} \cdot U$. Since the flat limit from dS presents some issues related to the partially-massless points, in the following we only consider the limit starting from AdS ($\epsilon = -1$).

Let us conclude this section with a few remarks about the role of the delta function. Notice that without the latter the ambient-space action (2.15) would contain a diverging factor coming from the radial integral. The insertion of the delta function precisely cures this divergence. On the other hand, one may wonder whether we could have avoided such insertion by taking the extra dimension to be compact. For instance, in flat space one can consider a compact coordinate $\rho \sim \rho + L$ together with a harmonic ρ-dependence of the fields: $\Phi = e^{i\,\frac{2\pi}{L}\, m\, \rho}\, \varphi$. Then, the ρ-integral gives an orthogonality condition:

$$\int_0^L d\rho\; e^{i\,\frac{2\pi}{L}\, m_1\, \rho}\; e^{-i\,\frac{2\pi}{L}\, m_2\, \rho} = L\, \delta_{m_1\,,\, m_2}\,. \tag{2.23}$$

Although this KK reduction works well at the free level, it turns out to be problematic or at least too restrictive at the cubic level since one gets in this case an undesired mass equality:

$$\int_0^L d\rho\; e^{i\,\frac{2\pi}{L}\, m_1\, \rho}\; e^{i\,\frac{2\pi}{L}\, m_2\, \rho}\; e^{-i\,\frac{2\pi}{L}\, m_3\, \rho} = L\, \delta_{m_1 + m_2\,,\, m_3}\,. \tag{2.24}$$

The latter forbids many interactions, notably those arising in ST, and can be avoided via the insertion of a delta function $\delta(\hat{N} \cdot X)$.

2.2 Gauge symmetries in the ambient-space formalism

As we have seen, in the intrinsic formulation, HS fields whose mass-squared is given by (2.6), i.e. $\mu = 0$, admit the gauge symmetries (2.7). This gauge invariance of the massless theory can be seen also at the ambient space level. We first consider the linearized gauge symmetries:

$$\delta^{(0)}\, \Phi(X,U) = U \cdot \partial_X\, E(X,U)\,, \tag{2.25}$$

where E is the generating function of the gauge parameters. Since the action (2.15) does not contain any explicit mass term, the gauge invariance seems to be unrelated to the value of μ. This cannot be the case since it would imply the presence of gauge symmetries for massive theories in the absence of the Stückelberg fields. Indeed, as we show in the following, the homogeneity and tangentiality conditions (2.10, 2.11) are compatible with the gauge symmetry (2.25) only for particular values of μ.

2.2.1 Massless fields

Starting from eqs. (2.10) and (2.25), one can first derive the homogeneity degree of the gauge parameters:

$$(X \cdot \partial_X - U \cdot \partial_U + \mu)\, E(X, U) = 0\,. \tag{2.26}$$

Then, one has to impose the compatibility of the tangentiality condition (2.11) with the gauge transformations (2.25):

$$X \cdot \partial_U \;\delta^{(0)}\, \Phi(X, U) = (U \cdot \partial_X\, X \cdot \partial_U - \mu)\, E(X, U) = 0\,, \tag{2.27}$$

where we used eq. (2.26). When $\mu = 0$, any gauge parameter satisfying the tangentiality condition:

$$X \cdot \partial_U\, E(X, U) = 0\,, \tag{2.28}$$

is a solution of (2.27). Therefore, the ambient-space gauge parameter E is related to the intrinsic (A)dS one ε as

$$E(R, x; v, u) = \left(\frac{R}{L}\right)^{u\cdot\partial_u} \varepsilon(x, u)\,, \tag{2.29}$$

and the ambient-space gauge transformations (2.25) reduce to the (A)dS ones (2.7).

Massive fields. In the $\mu \neq 0$ case, eq. (2.27) implies

$$E(X, U) = \frac{1}{\mu}\, U \cdot \partial_X\, X \cdot \partial_U\, E(X, U)\,, \tag{2.30}$$

that in turn is compatible with the tangent condition provided

$$\left[\, (U \cdot \partial_X)^2\, (X \cdot \partial_U)^2 - 2\,\mu\,(\mu - 1)\,\right] E(X, U) = 0\,. \tag{2.31}$$

If $\mu \neq 1$, the latter gives

$$E(X, U) = \frac{1}{2\,\mu\,(\mu - 1)}\, (U \cdot \partial_X)^2\, (X \cdot \partial_U)^2\, E(X, U)\,. \tag{2.32}$$

Hence, when $[\mu]_r := \mu\,(\mu - 1) \cdots (\mu - r + 1) \neq 0$, one can iterate r times this procedure ending up with

$$\left[\, (U \cdot \partial_X)^r\, (X \cdot \partial_U)^r - r!\,[\mu]_r\,\right] E(X, U) = 0\,. \tag{2.33}$$

Since $(X \cdot \partial_U)^s\, E^{(s-1)} = 0$, whenever $[\mu]_s \neq 0$, the spin $s-1$ component of this equation implies that eqs. (2.10, 2.11) are compatible with the gauge symmetry only for vanishing $E^{(s-1)}$. In AdS all unitary representations have non-positive values of μ [114], therefore the gauge symmetry is allowed only in the massless case.

2.2.2 Partially-massless fields

In dS, the unitary representations [45, 51, 115, 116] include all positive integer values $\mu = r \in \mathbb{N}_{\geq 0}$. In those cases the iteration procedure stops whenever $r < s$. Therefore, non-vanishing solutions exist for the gauge parameters satisfying[15]

$$(X \cdot \partial_U)^{r+1}\, E(X, U) = 0\,. \tag{2.34}$$

[15] Similar constraints have been also exploited in [55, 56] keeping the necessary auxiliary fields in order to achieve an off-shell description.

Inverting (2.33), the initial gauge parameter E can be solved in terms of a new gauge parameter Ω as

$$E(X,U) = (U\cdot\partial_X)^r\,\Omega(X,U)\,, \tag{2.35}$$

where Ω satisfies the homogeneity and tangentiality conditions:

$$(X\cdot\partial_X - U\cdot\partial_U - r)\,\Omega(X,U) = 0\,, \qquad X\cdot\partial_U\,\Omega(X,U) = 0\,. \tag{2.36}$$

Thus, Ω can be reduced to the intrinsic dS gauge parameter ω as

$$\Omega(R,x;v,u) = \left(\frac{R}{L}\right)^{u\cdot\partial_u + r}\,\omega(x,u)\,. \tag{2.37}$$

Finally, the gauge transformations[16]

$$\delta^{(0)}\,\Phi = (U\cdot\partial_X)^{r+1}\,\Omega(X,U)\,, \tag{2.38}$$

become the dS intrinsic ones:

$$\delta^{(0)}\,\varphi(x,u) = \left[(u\cdot D)^{r+1} + \ldots\right]\omega(x,u)\,, \tag{2.39}$$

whose form has been obtained recursively in [21, 53].

3 Cubic interactions of HS fields in (A)dS

In this section we construct the consistent parity-invariant cubic interactions of massive and partially-massless HS fields in (A)dS. More precisely, we focus on those pieces which do not contain divergences and traces of the fields (TT parts). We begin with the most general expression for the cubic vertices:[17]

$$\begin{aligned} S^{(3)} \;=&\; \frac{1}{3!}\int d^{d+1}X\;\delta\Big(\sqrt{\epsilon\,X^2}-L\Big)\;C_{A_1A_2A_3}(L^{-1}\,;\,\partial_{X_1},\partial_{X_2},\partial_{X_3}\,;\,\partial_{U_1},\partial_{U_2},\partial_{U_3})\,\times \\ &\times\,\Phi^{A_1}(X_1,U_1)\;\Phi^{A_2}(X_2,U_2)\;\Phi^{A_3}(X_3,U_3)\,\Big|_{\substack{X_i=X\\ U_i=0}} + \;\ldots\,. \end{aligned} \tag{3.1}$$

Here $C_{A_1A_2A_3}$ denotes the TT part of the vertices. The cubic interactions in (A)dS are in general inhomogeneous in the number of derivatives, the lower-derivative parts being dressed by negative powers of L compared to the highest-derivative one. Hence, the TT parts of the vertices can be expanded as

$$C_{A_1A_2A_3}(L^{-1}\,;\,\partial_X\,,\,\partial_U) = \sum_{n=0}^{\infty} L^{-n}\,C^{[n]}_{A_1A_2A_3}(Y,Z)\,, \tag{3.2}$$

where we have introduced the parity-preserving Lorentz invariants:

$$Y_i = \partial_{U_i}\cdot\partial_{X_{i+1}}\,,\qquad Z_i = \partial_{U_{i+1}}\cdot\partial_{U_{i-1}}\,,\qquad [i\simeq i+3]\,. \tag{3.3}$$

[16] An anolagous form of the gauge transformations has been obtained in the tractor approach [54].
[17] The dependence on the X^M in the ansatz can be neglected (see [92]).

Notice that we have dropped divergences, $\partial_{U_i}\cdot\partial_{X_i}$, traces, $\partial_{U_i}^2$ as well as terms proportional to $\partial_{X_i}\cdot\partial_{X_j}$'s. Indeed, being proportional to the field equations (2.16) up to total derivatives, the latter can be removed by proper field redefinitions. Moreover, since we have chosen a particular set of Y_i's, any ambiguity related to the total derivatives has been fixed.

In order to simplify the analysis, it is convenient to recast the expansion (3.2) in a slightly different, though equivalent form. First, let us notice that negative powers of L can be absorbed into derivatives of the delta function:

$$\delta^{(n)}(R-L)\; R^{\lambda} = \frac{(-2)^n\,[\lambda/2]_n}{L^n}\,\delta(R-L)\,R^{\lambda}\,. \tag{3.4}$$

where $\delta^{(n)}(R-L) = \left(\frac{L}{R}\,\frac{d}{dR}\right)^n\,\delta(R-L)$. Then, introducing $\hat{\delta}$ with the following prescription:

$$\delta^{(n)}(R-L) \equiv \delta(R-L)\,(\epsilon\,\hat{\delta})^n\,, \tag{3.5}$$

each coefficient of (3.2) can be redefined as

$$L^{-n}\,C^{[n]}_{A_1A_2A_3}(Y,Z) = \hat{\delta}^n\,C^{(n)}_{A_1A_2A_3}(Y,Z)\,. \tag{3.6}$$

Notice that $C^{[n]}_{A_1A_2A_3}$ and $C^{(n)}_{A_1A_2A_3}$ are different functions for $n\geq 1$. The entire couplings can be finally resummed as

$$C_{A_1A_2A_3}(\hat{\delta};Y,Z) = \sum_{n=0}^{\infty}\hat{\delta}^n\,C^{(n)}_{A_1A_2A_3}(Y,Z)\,, \tag{3.7}$$

where we have used the same notation for both $C_{A_1A_2A_3}(L^{-1};Y,Z)$ and $C_{A_1A_2A_3}(\hat{\delta};Y,Z)$ although they are different functions.

In order to make contact with the standard tensor notation, let us provide an explicit example. A vertex of the form

$$C(\hat{\delta};Y,Z) = (Y_1^2\,Y_2\,Y_3\,Z_1 + \text{cycl.}) - \frac{\hat{\delta}}{L}\,(Y_1\,Y_2\,Z_1\,Z_2 + \text{cycl.}) + \frac{3}{4}\left(\frac{\hat{\delta}}{L}\right)^2 Z_1\,Z_2\,Z_3\,, \tag{3.8}$$

which will turn out to be a consistent coupling involving three partially-massless spin 2 fields (see appendix B), gives

$$S^{(3)} = \frac{1}{2}\int d^{d+1}X\;\delta\Big(\sqrt{X^2}-L\Big)\Big[\partial_P\,\Phi^{MN}\,\partial_M\,\partial_N\,\Phi_{LQ}\,\partial^L\,\Phi^{PQ} \tag{3.9}$$

$$+\frac{d-5}{L^2}\,\Phi^M{}_N\,\partial_M\,\Phi_{LP}\,\partial^L\,\Phi^{NP} + \frac{(d-3)(d-5)}{4\,L^4}\,\Phi^M{}_N\,\Phi^N{}_P\,\Phi^P{}_M\Big]\,. \tag{3.10}$$

3.1 Consistent cubic interactions of massive and massless HS fields

So far, we have not specified whether the fields Φ^A are massive or massless. In the following we use $A=\alpha$ for massive fields and $A=a$ for massless ones. One can consider different cases depending on the number of massless and massive fields involved in the cubic interactions. The presence of massive fields does not impose any constraints on the vertices, while, whenever a massless field takes part in the interactions, the corresponding vertices must be compatible with the gauge symmetries of that field.

Gauge consistency can be studied order by order (in the number of fields), and at the cubic level gives

$$\delta_i^{(1)} S^{(2)} + \delta_i^{(0)} S^{(3)} = 0 \quad \Rightarrow \quad \delta_i^{(0)} S^{(3)} \approx 0\,, \tag{3.11}$$

where $\approx$ means equivalence modulo the free field equations (2.16) and $\delta_i^{(0)}$ is the linearized gauge transformation (2.25) associated with the massless field Φ^{a_i}. The key point of our approach is that the TT parts of the vertices can be determined from the Noether procedure (3.11) independently from the ellipses in (3.1). This amounts to quotient the Noether equation (3.11) by the Fierz systems of the fields Φ^{A_i} and of the gauge parameters E^{a_i}. In our notation, this is equivalent to impose, for $i = 1$,

$$\Big[C_{a_1 A_2 A_3}(\hat{\delta}; Y, Z)\,,\, U_1 \cdot \partial_{X_1} \Big] \Big|_{U_1=0} \approx 0\,, \tag{3.12}$$

modulo all the $\partial_{X_i}^2$'s, $\partial_{U_i}\!\cdot\partial_{X_i}$'s and $\partial_{U_i}^2$'s. Due to the presence of the delta function, the total derivative terms generated by the gauge variation do not simply vanish, but contribute as

$$\delta\Big(\sqrt{\epsilon\, X^2} - L\Big)\, \partial_{X^M} = -\,\delta\Big(\sqrt{\epsilon\, X^2} - L\Big)\, \hat{\delta}\, \frac{X_M}{L}\,. \tag{3.13}$$

Using the commutation relations (A.1) together with the identity (A.2), eq. (3.12) is equivalent to the following differential equation:

$$\left[Y_2 \partial_{Z_3} - Y_3 \partial_{Z_2} + \frac{\hat{\delta}}{L} \left(Y_2 \partial_{Y_2} - Y_3 \partial_{Y_3} - \frac{\mu_2 - \mu_3}{2} \right) \partial_{Y_1} \right] C_{a_1 A_2 A_3}(\hat{\delta}; Y, Z) = 0\,. \tag{3.14}$$

The consistent parity-invariant cubic interactions involving massive and massless HS fields in (A)dS can be obtained as solutions of the above equations. Since $C_{a_1 A_2 A_3}$ is a polynomial in $\hat{\delta}$, one can solve (3.14) iteratively starting from the lowest order in $\hat{\delta}$. To begin with, the zero-th order term $C^{(0)}_{a_1 A_2 A_3}$ in (3.7) is given by

$$C^{(0)}_{a_1 A_2 A_3} = C^{(0)}_{a_1 A_2 A_3}(Y_1, Y_2, Y_3, Z_1, G)\,, \tag{3.15}$$

where

$$G := Y_1\, Z_1 \, + Y_2\, Z_2 + Y_3\, Z_3\,. \tag{3.16}$$

On the other hand, when more than two massless fields are present, it becomes

$$C^{(0)}_{a_1 a_2 A_3} = C^{(0)}_{a_1 a_2 A_3}(Y_1, Y_2, Y_3, G)\,. \tag{3.17}$$

Notice that, while (3.15) is an arbitrary function of five arguments, the zero-th order solution (3.17) depends on four arguments. This is a consequence of the different number of differential equations imposed on the vertices. On the other hand, in the case of three massless fields, the third differential equation is redundant so that the number of arguments do not decrease further. Having obtained the zero-th order parts of the solution in eqs. (3.15, 3.17), what is left is to determine their higher order completions. Eq. (3.14) gives an inhomogeneous differential equation for $C^{(n\geq 1)}_{a_1 A_2 A_3}$, whose solutions are fixed up to a solution of the corresponding homogeneous equation. However, ambiguities of the interactions related to these solutions are nothing but redundancies as discussed in [92].

Before considering eq. (3.14), we first solve its flat limit $L \to \infty$. Once again, this is achieved via (2.19) after the replacement:

$$\lim_{L\to\infty} \frac{1}{L}\,\mu = -M\,. \tag{3.18}$$

The end result takes the following form:

$$\left[Y_2\partial_{Z_3} - Y_3\partial_{Z_2} + \frac{\hat{\delta}}{2}\,(M_2 - M_3)\,\partial_{Y_1}\right] C_{a_1 A_2 A_3}(\hat{\delta}; Y, Z) = 0\,, \tag{3.19}$$

so that the zero-th order parts of the solutions coincide with the (A)dS ones. Moreover, in flat space, the operator $\hat{\delta}$ appearing in (3.5) is simply given by

$$\hat{\delta} = \hat{N} \cdot \partial_X\,. \tag{3.20}$$

Notice also that in this case, for a given $C^{(0)}_{A_1 A_2 A_3}$, the lower-derivative parts of the vertices $C^{(n\geq 1)}_{A_1 A_2 A_3}$ can be recast into total-derivative terms, making them homogeneous in the number of derivatives. This observation makes it possible to write generic consistent vertices as arbitrary functions of some fixed building blocks.

Our strategy is as follows. We first solve the flat-space equation (3.19) and express the general solution in terms of homogeneous objects in the number of derivatives. In this way, we identify the building blocks of the flat-space cubic interactions. Then, we take as ansatz for the (A)dS building blocks the deformation of the flat-space ones with the addition of further total derivatives. Finally, we fix such ansatz requiring the latter to solve (3.12). In the following, we divide the analysis into four different cases: 3 massive, 1 massless and 2 massive, 2 massless and 1 massive and finally 3 massless fields. For each of them, we provide the most general solution as arbitrary functions of the corresponding building blocks.

A 3 massive. This case is rather trivial since no condition on $C_{\alpha_1\alpha_2\alpha_3}$ is imposed. Thus, the cubic interactions of three massive fields are given by

$$C_{\alpha_1\alpha_2\alpha_3} = \mathcal{K}_{\alpha_1\alpha_2\alpha_3}(Y_1, Y_2, Y_3, Z_1, Z_2, Z_3)\,. \tag{3.21}$$

This reflects the fact that we focused only on the TT parts of the vertices. Finding the remaining parts is in principle non-trivial but we expect that, working within the gauge invariant formulation à la Stückelberg (see section 4), those parts can be recursively determined from (3.21).

B 1 massless and 2 massive. When one massless ($A_1 = a_1$) and two massive HS fields are involved in the interactions, one needs to analyze separately the cases wherein the two fields have equal or different masses.

Equal mass. When $M_2 = M_3 = m \neq 0$, the M-dependent term in (3.19) vanishes and therefore the solution in flat space is given by its zero-th order part:

$$C_{a_1\alpha_2\alpha_3} = \mathcal{K}_{a_1\alpha_2\alpha_3}(Y_1, Y_2, Y_3, Z_1, G)\,. \tag{3.22}$$

Regarding the vertices in (A)dS, we make an ansatz by deforming the latter with total-derivative terms as

$$C_{a_1\alpha_2\alpha_3} = \mathcal{K}_{a_1\alpha_2\alpha_3}(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, Z_1, \tilde{G})\,, \tag{3.23}$$

where $\tilde{Y}_i$'s and $\tilde{G}$ are given by

$$\begin{aligned}\tilde{Y}_i &= Y_i + \alpha_i\,\partial_{U_i}\cdot\partial_X\,,\\ \tilde{G} &= (Y_1 + \beta_1\,\partial_{U_1}\cdot\partial_X)Z_1 + (Y_2 + \beta_2\,\partial_{U_2}\cdot\partial_X)Z_2 + (Y_3 + \beta_3\,\partial_{U_3}\cdot\partial_X)Z_3\,.\end{aligned} \tag{3.24}$$

Requiring the gauge invariance, one ends up with

$$\begin{aligned}(\alpha_1+1)\alpha_2 + 1 &= 0\,,\\ (\alpha_1+1)(\beta_2+1) + \alpha_1\,\beta_3 &= 0\,,\\ (\beta_1+1)(\beta_2+1) + \beta_3\,(\beta_1+\beta_2+1) &= 0\,,\end{aligned} \tag{3.25}$$

whose general solutions (see [92] for the details) are[18]

$$\begin{aligned}\alpha_1 &= \alpha\,, \qquad \alpha_2 = -\frac{1}{\alpha+1}\,, \qquad \alpha_3 = -\frac{\alpha+1}{\alpha}\,,\\ \beta_1 &= \beta\,, \qquad \beta_2 = -\frac{\beta+1}{\alpha+1}\,, \qquad \beta_3 = -\frac{\alpha-\beta}{\alpha}\,.\end{aligned} \tag{3.26}$$

As we have anticipated, the different values of the α_i's and the β_i's are related to the redundancies of the solutions.

Different masses. When $M_2 \neq M_3$, the zero-th order part of the solution $C^{(0)}_{a_1\alpha_2\alpha_3}$ (3.15) is an arbitrary function of the Y_i's, Z_1 and G. However, not all of these arguments admit a solution for $C^{(1)}_{a_1\alpha_2\alpha_3}$. In particular, $C^{(0)}_{a_1\alpha_2\alpha_3} = Y_2, Y_3$ and Z_1 are already consistent and do not need to be completed with $C^{(n\geq 1)}_{a_1\alpha_2\alpha_3}$, while

$$C^{(0)}_{a_1\alpha_2\alpha_3} = \quad Y_3\,Y_1\,, \quad Y_1\,Y_2\,, \tag{3.27}$$

involve next order contributions given by

$$C^{(1)}_{a_1\alpha_2\alpha_3} = \quad \frac{1}{2}\,(M_2 - M_3)\,Z_2\,, \quad \frac{1}{2}\,(M_3 - M_2)\,Z_3\,, \tag{3.28}$$

respectively, and $C^{(n\geq 2)}_{a_1\alpha_2\alpha_3} = 0$. Therefore, the flat-space solution can be written as

$$C_{a_1\alpha_2\alpha_3} = \mathcal{K}_{a_1\alpha_2\alpha_3}(Y_2, Y_3, Z_1, H_2, H_3)\,, \tag{3.29}$$

where the H_i's are given by

$$H_i := Y_{i+1}\,Y_{i-1} + \frac{1}{2}\,\hat{N}\cdot\partial_X\,(M_i - M_{i+1} - M_{i-1})\,Z_i\,. \tag{3.30}$$

[18] Notice that in the present conventions, the definitions of the α_i's and the β_i's differ from the ones used in [92]. The latter are recovered through the replacements: $\alpha_i \to (\alpha_i - 1)/2$ and of the $\beta_i \to (\beta_i - 1)/2$.

JHEP07(2012)041

Notice that, using the properties of the delta function (3.20), they can be recast in the form

$$\begin{aligned} H_i &= Y_{i+1}\, Y_{i-1} + \frac{1}{2}\left[M_i^2 - (M_{i+1}+M_{i-1})^2\right] Z_i \\ &= Y_{i+1}\, Y_{i-1} - \frac{1}{2}\,\partial_X \cdot (\partial_{X_i} - \partial_{X_{i+1}} - \partial_{X_{i-1}})\, Z_i\,. \end{aligned} \tag{3.31}$$

The first expression in eq. (3.31) does not contain any total-derivative part, thus one can trivially reduce it to d dimensions. On the other hand, the second one does not contain any explicit mass dependence, and this makes the deformation of arbitrary functions of the latter to (A)dS easier. Indeed, by adding proper total-derivative terms to the $Y_{i\pm1}$'s, one gets the (A)dS counterpart of (3.31):

$$\tilde{H}_i := Y_{i+1}\,(Y_{i-1} - \partial_X \cdot \partial_{U_{i-1}}) - \frac{1}{2}\,\partial_X \cdot (\partial_{X_i} - \partial_{X_{i+1}} - \partial_{X_{i-1}})\, Z_i\,. \tag{3.32}$$

Up to field redefinitions, the latter can be recast in a form where the gauge invariance is more transparent:

$$\tilde{H}_i \;\approx\; \partial_{X_{i+1}}\!\cdot \partial_{X_{i-1}}\, \partial_{U_{i-1}}\!\cdot \partial_{U_{i+1}} - \partial_{X_{i+1}}\!\cdot \partial_{U_{i-1}}\, \partial_{X_{i-1}}\!\cdot \partial_{U_{i+1}}\,. \tag{3.33}$$

Finally, the vertices in (A)dS are given by

$$C_{a_1\alpha_2\alpha_3} = \mathcal{K}_{a_1\alpha_2\alpha_3}(Y_2, Y_3, Z_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{3.34}$$

C 2 massless and 1 massive. This case can be recovered from the previous one as the intersection between the solutions:

$$C_{a_1A_2A_3} = \mathcal{K}_{a_1A_2A_3}(Y_2, Y_3, Z_1, \tilde{H}_2, \tilde{H}_3)\,, \qquad C_{A_1a_2A_3} = \mathcal{K}_{A_1a_2A_3}(Y_1, Y_3, Z_2, \tilde{H}_1, \tilde{H}_3)\,, \tag{3.35}$$

that is

$$C_{a_1a_2\alpha_3} = \mathcal{K}_{a_1a_2\alpha_3}(Y_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{3.36}$$

D 3 massless. This case is a combination of three equal mass cases:

$$C_{a_1a_2a_3} = \mathcal{K}_{a_1a_2a_3}(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{G})\,. \tag{3.37}$$

Here, the $\tilde{Y}_i$'s and $\tilde{G}$ are given by (3.24) with the α_i's and the β_i's satisfying eq. (3.25) and cyclic permutations thereof. Interestingly, the solutions (3.26) of (3.25) fulfill automatically also its cyclic counterparts.

At this stage we have completed the systematic constructions of the TT parts of the cubic interactions involving massive and massless HS fields in (A)dS. Before considering the partially-massless cases, let us make a few remarks. Similarly to what happens in the (A)dS massless case [92], all higher-order parts of the solutions $C^{(n)}_{A_1A_2A_3}$ are encoded via functions of simple building blocks that, being linear in ∂_{U_i} for any $i = 1, 2, 3$, describe the consistent couplings among fields of spin 1 and 0 only. These results resonate with the idea that spin 1 couplings can be used as building blocks of HS interactions [80, 91].

3.2 Consistent cubic interactions of partially-massless HS fields

In this section we focus on the cubic interactions in a dS background where, besides massive and massless fields, partially-massless fields also appear. As we have seen in section 2.2.2, partially-massless fields with homogeneities $\mu = r \in \{0\,,1\,,\ldots,\,s-1\}$ admit the gauge symmetries (2.38). Then, according to eq. (3.11), the cubic interactions ought to be compatible with those gauge symmetries leading to the following condition:

$$\Big[\,C_{A_1A_2A_3}(\hat{\delta};Y,Z)\,,\,(U_1\cdot\partial_{X_1})^{r_1+1}\Big]\Big|_{U_1=0}\approx 0\,. \tag{3.38}$$

Once again, neglecting all the $\partial_{X_i}^2$'s, $\partial_{U_i}\cdot\partial_{X_i}$'s and $\partial_{U_i}^2$'s, one ends up with

$$\begin{aligned}\sum_{\ell_1+\ell_2+\ell_3=r_1+1}&\binom{r_1+1}{\ell_1\,\ell_2\,\ell_3}\Big[Y_3\,\partial_{Y_3}-Y_2\,\partial_{Y_2}-2\,Z_3\,\partial_{Z_3}+\frac{r_1+\mu_2-\mu_3}{2}\Big]_{\ell_1}\times\\ \times\left(\frac{\hat{\delta}}{L}\,\partial_{Y_1}\right)^{\ell_1}&(Y_3\,\partial_{Z_2})^{\ell_2}\left(-Y_2\,\partial_{Z_3}+\frac{2\,\hat{\delta}}{L}\,Z_3\,\partial_{Z_3}\,\partial_{Y_1}\right)^{\ell_3}C_{A_1A_2A_3}(\hat{\delta};Y,Z)=0\,,\end{aligned} \tag{3.39}$$

where $[a]_n$ is the descending Pochhammer symbol we have introduced previously. Since (3.39) is an higher-order partial differential equation, solving it is a non-trivial task. However, if we restrict the attention to the $s_1-s_2-s_3$ couplings with fixed s_i's, then the solution is of the form:

$$C_{A_1A_2A_3}(\hat{\delta};Y,Z)\;=\sum_{\sigma_i+\tau_{i+1}+\tau_{i-1}=s_i}c_{A_1A_2A_3}^{\tau_1\,\tau_2\,\tau_3}(\hat{\delta})\;\;Y_1^{\sigma_1}\,Y_2^{\sigma_2}\,Y_3^{\sigma_3}\,Z_1^{\tau_1}\,Z_2^{\tau_2}\,Z_3^{\tau_3}\,, \tag{3.40}$$

where the number of the undetermined coefficients $c_{A_1A_2A_3}^{\tau_1\,\tau_2\,\tau_3}$ is of the order $N\sim s_1\,s_2\,s_3$. Hence, the coupling can be viewed as a vector in a N-dimensional space, and eq. (3.39) reduces to a set of linear equations for that vector. Then, the consistent couplings correspond to the solution space of such linear system. This procedure can be conveniently implemented in Mathematica. For instance, in the case of $4-4-2$ couplings between two spin 4 fields at their first partially-massless points ($\mu=1$) and a massless spin 2, we find one ten-derivative, two eight-derivative, two six-derivative and one four-derivative couplings:

$$\begin{aligned}C_1\;=\;&Y_1^4\,Y_2^4\,Y_3^2-12\,\hat{\delta}^2\,Y_1^2\,Y_2^2\,(Y_1\,Z_1+Y_2\,Z_2)^2+48\,\hat{\delta}^3\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3)\\&-24\,\hat{\delta}^4\,Z_3^2\,\big[6\,Y_1^2\,Z_1^2+6\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+2\,Y_1\,Z_1\,(7\,Y_2\,Z_2+2\,Y_3\,Z_3)\big]+96\,\hat{\delta}^5\,Z_1\,Z_2\,Z_3^3\,,\\C_2\;=\;&Y_1^3\,Y_2^3\,Y_3^2\,Z_3-3\,\hat{\delta}\,Y_1^2\,Y_2^2\,(Y_1\,Z_1+Y_2\,Z_2)^2+12\,\hat{\delta}^2\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3)\\&-6\,\hat{\delta}^3\,Z_3^2\,\big[6\,Y_1^2\,Z_1^2+6\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+2\,Y_1\,Z_1\,(7\,Y_2\,Z_2+2\,Y_3\,Z_3)\big]+24\,\hat{\delta}^4\,Z_1\,Z_2\,Z_3^3\,,\\C_3\;=\;&Y_1^3\,Y_2^3\,Y_3\,(Y_1\,Z_1+Y_2\,Z_2)+\hat{\delta}\,Y_1^2Y_2^2\,\big(6\,Y_1^2\,Z_1^2+11\,Y_1\,Y_2\,Z_1\,Z_2+6\,Y_2^2\,Z_2^2\big)\\&-18\,\hat{\delta}^2\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3)\\&+6\,\hat{\delta}^3\,Z_3^2\,\big[6\,Y_1^2\,Z_1^2+2\,Y_2\,Z_2\,(3\,Y_2\,Z_2+Y_3\,Z_3)+Y_1\,Z_1\,(15\,Y_2\,Z_2+2\,Y_3\,Z_3)\big]-12\,\hat{\delta}^4\,Z_1\,Z_2\,Z_3^3\,,\end{aligned}$$

$$
\begin{aligned}
C_4 \;=\; & -Y_1^2\,Y_2^2\,\big(Y_1^2\,Z_1^2+2\,Y_1\,Y_2\,Z_1\,Z_2+Y_2^2\,Z_2^2-Y_3^2\,Z_3^2\big)\\
& +4\,\hat{\delta}\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3)\\
& -2\,\hat{\delta}^2\,Z_3^2\,\big[6\,Y_1^2\,Z_1^2+6\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+2\,Y_1\,Z_1\,(7\,Y_2\,Z_2+2\,Y_3\,Z_3)\big]+8\,\hat{\delta}^3\,Z_1\,Z_2\,Z_3^3\,,\\
C_5 \;=\; & Y_1^2\,Y_2^2\,(Y_1\,Z_1+Y_2\,Z_2)\,(Y_1\,Z_1+Y_2\,Z_2+Y_3\,Z_3)\\
& -\hat{\delta}\,Y_1\,Y_2\,Z_3\,\big[6\,Y_1^2\,Z_1^2+2\,Y_2\,Z_2\,(3\,Y_2\,Z_2+2\,Y_3\,Z_3)+Y_1\,Z_1\,(13\,Y_2\,Z_2+4\,Y_3\,Z_3)\big]\\
& +2\,\hat{\delta}^2\,Z_3^2\,\big[3\,Y_1^2\,Z_1^2+Y_2\,Z_2\,(3\,Y_2\,Z_2+Y_3\,Z_3)+Y_1\,Z_1\,(8\,Y_2\,Z_2+Y_3\,Z_3)\big]-2\,\hat{\delta}^3\,Z_1\,Z_2\,Z_3^3\,,\\
C_6 \;=\; & Y_1\,Y_2\,Z_3\,(Y_1\,Z_1+Y_2\,Z_2+Y_3\,Z_3)^2\\
& -\hat{\delta}\,Z_3^2\,\big[3\,Y_1^2\,Z_1^2+3\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+4\,Y_1\,Z_1\,(2\,Y_2\,Z_2+Y_3\,Z_3)\big]\\
& +4\,\hat{\delta}^2\,Z_1\,Z_2\,Z_3^3\,,
\end{aligned}
\tag{3.41}
$$

where for simplicity we set $L=1$ while the L dependence can be recovered replacing $\hat{\delta}$ by $\hat{\delta}/L$. In appendix B, we also provide the examples of $2-2-2$ and $3-3-2$ couplings for any combinations of the masses.

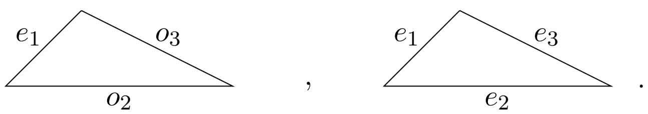
Remember that in the previous section the solutions were obtained in a compact form recasting the lower-derivative parts of the vertices into total derivatives. We expect this way of simplifying couplings to work in the partially-massless cases too. Indeed, the following class of highest-derivative couplings

$$
C_{A_1A_2A_3}=\mathcal{K}_{A_1A_2A_3}(\tilde{Y}_1,\tilde{Y}_2,\tilde{Y}_3)\,, \tag{3.42}
$$

is also compatible with the partially-massless gauge invariance provided the homogeneities of the fields satisfy

$$
r_i-|\mu_{i+1}-\mu_{i-1}|\in 2\,\mathbb{N}_{\geq 0}\,. \tag{3.43}
$$

Here the i-th field is at the r_i-th partially-massless point while the other two fields have generic homogeneities μ_{i+1} and μ_{i-1}. The proof of the conditions (3.43) can be found in appendix C. This implies that a partially-massless spin s field can interact with two scalars if and only if the masses of the latter satisfy (3.43). Moreover, when all the three fields are partially-massless, the r_i's satisfy a triangular inequality wherein one or three of them are even integers, $r_i=e_i$, while the others are odd, $r_i=o_i$:



Note that this triangular inequality is not imposed on the spins but on the homogeneities μ_i's which are related to the masses according to eq. (2.18). The conditions (3.43) reveal the systematics of the partially-massless vertices. Let us recall that whenever one massless field takes part to generic massive interactions, the vertices split into two categories according to whether the other two fields have equal or different masses. The condition (3.43) is a generalization of this pattern to the partially-massless cases. We expect that, as in the massless case (see the 1 massless and 2 massive case of section 3.1), whenever (3.43) holds

JHEP07(2012)041

we have $\tilde{G}$-like building blocks on top of the $\tilde{Y}_i$'s, otherwise, one is left with $\tilde{Y}_2$, $\tilde{Y}_3$, $\tilde{H}_2$-like and $\tilde{H}_3$-like building blocks.[19]

Despite at present we lack the building blocks for the interactions involving partially-massless fields, we can do a systematic analysis of the zero-th order parts of the solution $C^{(0)}_{A_1A_2A_3}$. In this case, eq. (3.39) reduces to

$$(Y_2\partial_{Z_3} - Y_3\partial_{Z_2})^{r_1+1}\, C^{(0)}_{A_1A_2A_3}(Y,Z) = 0\,, \tag{3.44}$$

whose corresponding solutions are given by

$$C^{(0)}_{A_1A_2A_3} = \sum_{m_2+m_3\leq r_1} Z_2^{m_2}\, Z_3^{m_3}\; \bar{C}^{(0)\,m_2m_3}_{A_1A_2A_3}(Y_1,Y_2,Y_3,Z_1,G)\,. \tag{3.45}$$

Notice that, compared to the massless case, some factors of Z_2 and Z_3 are also allowed increasing the number of possible ways of writing the couplings. However, when restricted to particular couplings with fixed spins, the number of solutions may be smaller than in the massless case.

4 Stückelberg formulation

In this section, we first consider the free theories of massive and massless HS fields in the Stückelberg formalism, and then extend the discussion to the cubic level. Once again, we focus on the TT parts of the vertices. It is worth stressing that, as in the massless case, working with a gauge invariant description for massive fields might give us a recipe in order to fix the remaining parts of the vertices. Moreover, as mentioned in the Introduction, Stückelberg formulation represents a convenient framework in order to study the massless limit of massive theories.

4.1 Free Stückelberg fields from dimensional reduction

The Stückelberg description of massive HS fields can be conveniently obtained through dimensional reduction of a $(d+1)$-dimensional massless theory. In the following we first provide the example of a spin 1 field and then generalize it to arbitrary-spin fields.

Spin 1. Let us consider the theory of a massive spin 1 field a_μ:

$$S = -\frac{1}{2}\int d^dx\sqrt{\epsilon\, g}\,\left(\frac{1}{2}\,f_{\mu\nu}\,f^{\mu\nu} + m^2\,a_\mu\,a^\mu\right)\,, \tag{4.1}$$

where $f_{\mu\nu} = \partial_\mu\,a_\nu - \partial_\nu\,a_\mu$. Because of the mass term this theory is not gauge invariant and describes the propagation of the DoF associated to a massive spin 1 particle. Notice that, performing the limit $m\to 0$ at this level, one ends up with a massless spin 1 field,

[19] In fact, it is even possible that the $\tilde{G}$ and the $\tilde{H}_i$'s defined for the massless case still work for the partially-massless cases. However, checking it requires non-trivial computations and we postpone this issue for future work.

loosing one DoF. On the other hand, before taking the massless limit, one can introduce a new scalar field α_1 via the Stückelberg shift:

$$a_\mu = \alpha_{0\,\mu} + \frac{1}{m}\,\partial_\mu\,\alpha_1\,, \tag{4.2}$$

in such a way that the resulting action acquires the gauge symmetries $\delta\alpha_{0\,\mu} = \partial_\mu\,\varepsilon_0$ and $\delta\alpha_1 = -m\,\varepsilon_0$. Then, the action becomes

$$S = -\frac{1}{2}\int d^dx\,\sqrt{\epsilon\,g}\,\left[\frac{1}{2}\,f_{0\,\mu\nu}\,f_0^{\mu\nu} + m^2\,\alpha_{0\,\mu}\,\alpha_0^\mu + \partial_\mu\alpha_1\,\partial^\mu\alpha_1 + 2\,m\,\alpha_0^\mu\,\partial_\mu\alpha_1\right], \tag{4.3}$$

which, in the massless limit, describes a massless spin 1 and spin 0 field, preserving the number of DoF.

The above discussion can be restated in the ambient space formalism. First of all, one can obtain the Stückelberg action through radial reduction of the massless ambient-space one:

$$S = -\frac{1}{4}\int d^{d+1}X\,\delta\big(\sqrt{\epsilon\,X^2} - L\big)\;F_{MN}\,F^{MN}\,, \tag{4.4}$$

where the spin-1 field is homogeneous and tangent:

$$(X\cdot\partial_X + \mu + 1)\,A_M = 0\,, \qquad X^M\,A_M = 0\,. \tag{4.5}$$

The tangentiality condition implies $A_d = 0$ and, after the identification $A_\mu = a_\mu$, one recovers the action (4.1). Remember that the gauge symmetry $\delta\boldsymbol{A}_M = \partial_M E$ of the action (4.4) is incompatible with the tangentiality condition when μ is different from zero. On the other hand, one can insist on a gauge invariant formulation also for $\mu \neq 0$ provided the tangentiality condition is relaxed. In this case one has to promote the tangent field A_M to a generic one $\boldsymbol{A}_M$ with non-vanishing radial part: $X^M\,\boldsymbol{A}_M \neq 0$. Then, after identifying

$$\boldsymbol{A}_\mu = \alpha_{0\,\mu}\,, \qquad \boldsymbol{A}_d = \alpha_1\,, \tag{4.6}$$

in (4.4), one recovers the Stückelberg action (4.3). Moreover, the usual Stückelberg symmetry is obtained by decomposing $\delta\boldsymbol{A}_M = \partial_M E$ into its tangent and radial parts. Such decomposition can be also carried out in terms of ambient-space fields as $\boldsymbol{A}_M = A_{0\,M} + A_1\,L\,X^M/X^2$, ending up with

$$\delta A_{0\,M} = \left(\delta_M^N - \frac{X_M\,X^N}{X^2}\right)\partial_N E\,, \qquad \delta A_1 = \frac{1}{L}\,X^M\,\partial_M E = -\frac{\mu}{L}\,E\,. \tag{4.7}$$

Finally, the Stückelberg shift (4.2) can be realized as well at the ambient space level as

$$A_M = \left(\delta_M^N + \frac{1}{\mu}\,\partial_{X^M}\,X^N\right)\boldsymbol{A}_N\,. \tag{4.8}$$

General spins. In the previous sections we have discussed how the quadratic action of massive HS fields (2.4) can be obtained through radial reduction of the ambient-space massless one (2.15). In the following, we introduce Stückelberg fields promoting the tangent ambient-space fields Φ to generic unconstrained ones $\boldsymbol{\Phi}$. In this case, after the radial reduction, one is led to

$$\boldsymbol{\Phi}(R,x;v,u)=\left(\frac{R}{L}\right)^{u\cdot\partial_u+v\partial_v-2-\mu}\boldsymbol{\varphi}(x;v,u)\,. \tag{4.9}$$

The $(d{+}1)$-dimensional tensor fields $\boldsymbol{\varphi}$ can be expanded into d-dimensional ones of different ranks as

$$\boldsymbol{\varphi}(x;v,u):=\sum_{r=0}^{\infty}\frac{v^r}{r!}\,\varphi_r(x,u)\,, \tag{4.10}$$

where the components φ_r with $r=1,2,\dots$ correspond to the Stückelberg fields. Although the action and the corresponding field equations for this system stay the same as in the unitary gauge ($\varphi_{r\geq 1}=0$), having relaxed the tangentiality condition, the theory acquires the gauge symmetries:

$$\delta^{(0)}\,\boldsymbol{\Phi}(X,U)=U\cdot\partial_X\,\boldsymbol{E}(X,U)\,, \tag{4.11}$$

with gauge parameters:

$$\boldsymbol{E}(R,x;v,u)=\left(\frac{R}{L}\right)^{u\cdot\partial_u+v\partial_v-\mu}\boldsymbol{\varepsilon}(x;v,u)\,. \tag{4.12}$$

The $(d+1)$-dimensional gauge parameters $\boldsymbol{\varepsilon}$ can be expanded into d-dimensional ones as

$$\boldsymbol{\varepsilon}(x;v,u):=\sum_{r=0}^{\infty}\frac{v^r}{r!}\,\varepsilon_r(x,u)\,. \tag{4.13}$$

Let us mention once again that, depending on the kind of formulation, the gauge fields as well as the gauge parameters can have trace constraints. However, since we focus on the TT part of the Lagrangian, they are not relevant for our discussion.

The radial reduction considered so far can be also restated in terms of ambient-space quantities as

$$\boldsymbol{\Phi}:=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{L\,X\cdot U}{X^2}\right)^r\Phi_r\,,\qquad \boldsymbol{E}:=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{L\,X\cdot U}{X^2}\right)^r E_r\,, \tag{4.14}$$

where

$$\Phi_r=\left(\frac{R}{L}\right)^{u\cdot\partial_u+2(r-1)-\mu}\varphi_r\,,\qquad E_r=\left(\frac{R}{L}\right)^{u\cdot\partial_u+2r-\mu}\varepsilon_r\,. \tag{4.15}$$

Decomposing the gauge transformation (4.11) into its tangent and normal parts, one gets

$$\delta^{(0)}\,\Phi_r=\left[U\cdot\partial_X+(\mu-2\,r)\,\frac{X\cdot U}{X^2}\right]E_r+\frac{L}{X^2}\left[U^2-\frac{(X\cdot U)^2}{X^2}\right]E_{r+1}-\frac{r\,(\mu-r+1)}{L}\,E_{r-1}\,. \tag{4.16}$$

From these gauge transformations, one can see that, when $\mu \neq 0$, all $\Phi_{r\geq 1}$'s can be gauge fixed to zero, going back to the unitary gauge. On the other hand, in the massless limit, one can gauge fix to zero only the $\Phi_{r\geq 2}$'s, ending up with a massless field Φ_0 together with a massive one Φ_1 (corresponding to $\mu = -2$). This differs from what happens in flat space where none of the Φ_r's can be gauged away. In other words, if we consider the massless limit of the flat-space Lagrangian of massive HS fields à la Stückelberg, it decomposes into the sum of massless ones: e.g. a massive spin s reduces to massless spin $s, s-1$, down to 0 fields. Therefore, the total number of physical DoF stays the same as in the massive case.

Similarly to the spin 1 case, it is possible to restate the Stückelberg shift in terms of ambient-space quantities as

$$\Phi(X,U) = \sum_{r=0}^{\infty} \frac{a_r}{r!}\,(U\cdot\partial_X)^r\, W^r\, \boldsymbol{\Phi}(X,U)\,, \qquad W := \frac{1}{L}\, X\cdot\partial_U\,. \tag{4.17}$$

Demanding either the compatibility with the unitary gauge, i.e. $\delta^{(0)}\,\Phi = 0$ under (4.11), or with the tangentiality condition (2.11), the coefficients a_r's are fixed as

$$a_r = \frac{L^r}{[\mu]_r}\,. \tag{4.18}$$

In the flat limit one gets

$$W = \hat{N}\cdot\partial_U\,, \qquad a_r = \frac{(-1)^r}{M^r}\,. \tag{4.19}$$

Notice that both the AdS and the flat-space results present a pole in the massless limit, while in dS further (partially-massless) poles appear at $\mu = 1, \ldots s-1$.

4.2 Cubic interactions of HS fields with Stückelberg symmetries

In this section we present the Stückelberg formulation of the consistent cubic interactions of massless and massive HS fields. Once again we restrict the attention to the TT parts of such vertices which are provided in terms of operators and fields in the ambient space formalism. The key point is that in this case the dependence on X^M cannot be neglected anymore and the possible $(d+1)$-dimensional cubic vertices are more general than the unitary gauge ones (3.1). In particular, as in section 3, we can simplify the ansatz for the cubic couplings making use of all scalar operators:

$$\begin{aligned} S^{(3)} = \frac{1}{3!}\int d^{d+1}X\ \delta\Big(\sqrt{\epsilon\, X^2} - L\Big)\ C_{A_1A_2A_3}(\hat{\delta}; Y, Z, W)\,\times \\ \times\ \boldsymbol{\Phi}^{A_1}(X_1,U_1)\ \boldsymbol{\Phi}^{A_2}(X_2,U_2)\ \boldsymbol{\Phi}^{A_3}(X_3,U_3)\ \Big|_{\substack{X_i=X\\ U_i=0}}\,, \end{aligned} \tag{4.20}$$

where, compared to the unitary gauge case, we have introduced the additional scalar quantities

$$W_i = \frac{1}{L}\, X_i\cdot\partial_{U_i}\,. \tag{4.21}$$

Gauge invariance under (4.11) imposes the following equation:

$$\big[\, C_{A_1A_2A_3}(\hat{\delta}; Y, Z, W)\,,\, U_i\cdot\partial_{X_i}\,\big] \approx 0\,, \tag{4.22}$$

which, once again can be solved modulo the Fierz system. However, in this case the non-commutativity between Y_i and W_{i+1} makes the analysis more involved. On the other hand, one can get the cubic vertices for the Stückerberg fields by exploiting the Stückelberg shift (4.17). Let us stress that we have explicitly checked the equivalence between the latter approach and resolution of eq. (4.22). The non-commutativity problem arises in this approach as well, and in order to deal with it we choose an ordering prescription where all the W_i's are placed after the Y_i's and the Z_i's. For this purpose, it is convenient to introduce a new variable w and write the Stückelberg shift (4.17) as

$$\Phi(X,U) = e^{U\cdot\partial_X\,\partial_w}\,P(w\,W)\,\boldsymbol{\Phi}(X,U)\Big|_{w=0}\,, \tag{4.23}$$

where

$$P(z) = \sum_{r=0}^{\infty}\frac{(L\,z)^r}{r!\,[\mu]_r} = {}_0F_1(-\mu\,;\,-L\,z)\,. \tag{4.24}$$

Then, the cubic vertices in the Stückelberg formulation can be obtained by shifting the unitary gauge ones as

$$C_{A_1A_2A_3} = \mathcal{K}_{A_1A_2A_3}(\boldsymbol{Y},\boldsymbol{Z})\,P(w_1\,W_1)\,P(w_2\,W_2)\,P(w_3\,W_3)\Big|_{w_i=0}\,, \tag{4.25}$$

where the $\boldsymbol{Y}_i$'s and the $\boldsymbol{Z}_i$'s are given by

$$\begin{aligned}
\boldsymbol{Y}_i &:= Y_i\;e^{U_i\cdot\partial_{X_i}\,\partial_{w_i}}\,\big|_{U_i=0} = Y_i + \partial_{X_i}\cdot\partial_{X_{i+1}}\,\partial_{w_i}\,,\\
\boldsymbol{Z}_i &:= Z_i\;e^{U_i\cdot\partial_{X_i}\,\partial_{w_i}}\,\big|_{U_i=0} = Z_i + \partial_{U_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{w_{i-1}} + \partial_{U_{i-1}}\cdot\partial_{X_{i+1}}\,\partial_{w_{i+1}}\\
&\qquad\qquad\qquad\qquad\quad + \partial_{X_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{w_{i+1}}\,\partial_{w_{i-1}}\,.
\end{aligned} \tag{4.26}$$

Depending on the number of massless fields involved in the interactions, one recovers a dependence of the vertices on the variables $\tilde{\boldsymbol{Y}}_i$, $\tilde{\boldsymbol{G}}$ and $\tilde{\boldsymbol{H}}_i$, which are defined as in eq. (4.26) starting from the $\tilde{Y}_i$'s, $\tilde{G}$ and the $\tilde{H}_i$'s, respectively.

4.3 Massless limit

As mentioned in the Introduction, the relation between massless and massive HS theories is of particular interest with regards to the possibility of having a better understanding of both ST and HS gauge theory in (A)dS. Although it is difficult to realize a mass generation mechanism for HS fields, one might get some hints for that by studying the massless limit of massive theories.

In the previous section we have shown that the cubic vertices in the Stückelberg formulation are given by arbitrary functions $\mathcal{K}_{A_1A_2A_3}$ of the $\tilde{\boldsymbol{Y}}_i$'s and of the $\tilde{\boldsymbol{Z}}_i$'s. Moreover, when some of the fields are massless, $\tilde{\boldsymbol{G}}$ and the $\tilde{\boldsymbol{H}}_i$'s also appear. In order to properly study the behavior of such vertices in the limit where some of the masses go to zero, one should know in principle how the coupling function $\mathcal{K}_{A_1A_2A_3}$ scales. However, as we will see in the following, interesting information can be also extracted considering generic behaviors in this limit. For simplicity, we consider the case where all the mass parameters of the theory scale uniformly with a mass scale μ:

$$\mu_i = \nu_i\,\mu\,. \tag{4.27}$$

Since the massless limit depends on the background, we analyze the AdS and the flat-space cases separately.

AdS case. As we have seen in section 4.1, in the massless limit $\mu \to 0$ one can gauge fix all the lower spin components up to spin $s-2$ ending up with:

$$\boldsymbol{\Phi} = \Phi_0 + \frac{L\, X\cdot U}{X^2}\, \Phi_1\,, \tag{4.28}$$

where Φ_0 and Φ_1 are a spin s massless field and a spin $s-1$ massive field, respectively. In this way, the Stückelberg shift (4.23) simplifies to

$$\Phi = \left(1 - \frac{1}{\mu}\, U\cdot\partial_X\, W\right)\, \boldsymbol{\Phi}\,. \tag{4.29}$$

Hence, the couplings (4.25) can be expanded as

$$\begin{aligned} C_{A_1A_2A_3} &= \mathcal{K}_{A_1A_2A_3}(Y,Z) + \frac{1}{\mu}\sum_{i=1}^{3} \mathcal{K}^{[i]}_{A_1A_2A_3}(\hat\delta;Y,Z)\, W_i \\ &\quad + \frac{1}{\mu^2}\sum_{i=1}^{3} \mathcal{K}^{[i+1,i-1]}_{A_1A_2A_3}(\hat\delta;Y,Z)\, W_{i+1}\, W_{i-1} + \frac{1}{\mu^3}\, \mathcal{K}^{[1,2,3]}_{A_1A_2A_3}(\hat\delta;Y,Z)\, W_1\, W_2\, W_3\,, \end{aligned} \tag{4.30}$$

where the $\mathcal{K}^{[\dots]}_{A_1A_2A_3}$'s are given by successive commutators of $\mathcal{K}_{A_1A_2A_3}$:

$$\mathcal{K}^{[\dots,i]}_{A_1A_2A_3} := \left[\mathcal{K}^{[\dots]}_{A_1A_2A_3}\,,\, -\frac{1}{\nu_i}\, U_i\cdot\partial_{X_i}\right]. \tag{4.31}$$

In the $\mu\to 0$ limit, the leading terms are massive couplings of the form $\mathcal{K}^{[1,2,3]}_{A_1A_2A_3}$ involving all the massive spin $s-1$ components $\Phi_1 = W\,\boldsymbol{\Phi}$. On the other hand, if some of leading parts of the couplings are absent, then the dominant ones contain less number of W_i's and consequently the interactions involve the corresponding massless fields.

Flat-space case. The situation in flat space is rather different from the one in AdS. First of all, in the massless limit one can not gauge fix the Stückelberg fields to zero so that the latter become all massless fields. Moreover, since the non-commutativity problem is absent, the Stückelberg vertices (4.25) can be simplified performing the w_i-contractions as

$$C_{A_1A_2A_3} = \mathcal{K}_{A_1A_2A_3}(\hat Y, \hat Z)\,, \tag{4.32}$$

where

$$\begin{aligned} \hat Y_i &= y_i - \frac{M_i^2 + M_{i+1}^2 - M_{i-1}^2}{2\,M_i}\,\partial_{v_i}\,, \\ \hat Z_i &= z_i + \frac{1}{M_{i-1}}\, y_{i+1}\,\partial_{v_{i-1}} - \frac{1}{M_{i+1}}\, y_{i-1}\,\partial_{v_{i+1}} + \frac{M_i^2 + M_{i-1}^2 - M_{i+1}^2}{2\,M_{i+1}\,M_{i-1}}\,\partial_{v_{i+1}}\,\partial_{v_{i-1}}\,. \end{aligned} \tag{4.33}$$

Here we have also performed the dimensional reduction providing the building blocks $\hat Y$ and $\hat Z$ in terms of the d-dimensional intrinsic ones:

$$y_i := \partial_{u_i}\cdot\partial_{x_{i+1}}\,, \qquad z_i := \partial_{u_{i+1}}\cdot\partial_{u_{i-1}}\,. \tag{4.34}$$

Then, under the assumption (4.27), one can observe the following behavior:

$$\hat{Y}_i = y_i + \mathcal{O}(\mu)\,, \qquad \mu\,\hat{Z}_i = \frac{1}{\nu_{i-1}}\,y_{i+1}\,\partial_{v_{i-1}} - \frac{1}{\nu_{i+1}}\,y_{i-1}\,\partial_{v_{i-1}} + \mathcal{O}(\mu)\,, \tag{4.35}$$

in the $\mu \to 0$ limit. Notice that the dominant terms contained in the $\hat{Z}_i$'s lead to consistent massless interactions and involve at least one Stückelberg field. The terms proportional to the z_i's, which can violate the gauge invariance, are contained in the subdominant $\mathcal{O}(\mu)$ part. Similarly, the variables $\hat{G}$ and $\hat{H}_i$'s behave as

$$\begin{aligned} \hat{G} &= g + \frac{\nu_2^2 + \nu_3^2 - \nu_1^2}{2\,\nu_2\,\nu_3}\,y_1\,\partial_{v_2}\,\partial_{v_3} + \text{cyclic}, \\ \hat{H}_i &= y_{i+1}\,y_{i-1} + \mathcal{O}(\mu)\,, \end{aligned} \tag{4.36}$$

where $g := y_1\,z_1 + y_2\,z_2 + y_3\,z_3$. Finally, the generic leading parts of the massive cubic vertices can be obtained by simply replacing all the variables by their leading terms (4.35, 4.36). The resulting vertices involve only the y_i's and g together with the ∂_{v_i}'s which encode the contribution of the Stückelberg fields. Hence, they are consistent with the gauge symmetries of the massless theory. For the sake of completeness, one should also analyze the cases where some of the leading parts cancel. This analysis can be found in appendix D.

5 Discussion

In this paper we have obtained the solutions to the cubic-interaction problem for massive and partially-massless HS fields in a constant-curvature background. This has been achieved through a dimensional reduction of a $(d+1)$-dimensional massless theory with a delta function insertion in the action.[20] For simplicity, the entire construction has been carried out focusing on the TT part of the Lagrangian. We expect that the completion of such vertices can be performed within the Stückelberg formulation, adding divergences and traces of the fields together with possible auxiliary fields.

Our studies are mainly motivated by ST whose very consistency rests on the presence of infinitely many HS fields. Conversely, string interactions may provide useful information on the systematics of the consistent HS couplings. In [79, 80], cubic vertices of totally-symmetric tensors belonging to the first Regge trajectory of the open bosonic string were investigated. Those vertices are encoded in the following generating function:

$$\begin{aligned} \frac{1}{\sqrt{G_N}}\,\mathcal{K}_{A_1A_2A_3} &= i\,\frac{g_o}{\alpha'}\,\mathrm{Tr}\left[T_{A_1}\,T_{A_2}\,T_{A_3}\right]\,\exp\left(i\sqrt{2\alpha'}\,(y_1+y_2+y_3)+z_1+z_2+z_3\right) \\ &\quad + i\,\frac{g_o}{\alpha'}\,\mathrm{Tr}\left[T_{A_2}\,T_{A_1}\,T_{A_3}\right]\,\exp\left(-i\sqrt{2\alpha'}\,(y_1+y_2+y_3)+z_1+z_2+z_3\right)\,, \end{aligned} \tag{5.1}$$

[20] Actually, any integrable function of the same argument is good. In particular one can consider the insertion of an Heavyside theta function, that is tantamount to introducing a cut-off for the diverging radial integral, or similarly, a boundary for the ambient space. Then, the total-derivative terms appearing in the interactions play the role of boundary actions which have to be taken into account whenever the base space-time has a non-empty boundary. See [117] for the recent construction of boundary actions for the free theory of massless HS fields in AdS.

where G_N denotes Newton's constant, g_o the open string coupling constant and α' the inverse string tension related to the masses of the string states as

$$M^2\,\varphi^{(s)}=\frac{s-1}{\alpha'}\,\varphi^{(s)}\,. \tag{5.2}$$

Remarkably, the Taylor coefficients of the exponential function and the spectrum (5.2) nicely combine to reproduce the right vertices belonging to the classification considered in section 3.1 (the details can be found in appendix E). In this respect, it would be interesting to understand how the exponential function (5.1) fits in with other ST properties and what its AdS counterpart may be. In particular, we believe that the choice of the exponential is crucial for the global symmetries as well as for the planar dualities of the theory. Let us mention however that in AdS an exponential couplings of the form:

$$e^{i\,\sqrt{2\alpha'}\,(\tilde{Y}_1+\tilde{Y}_2+\tilde{Y}_3)+Z_1+Z_2+Z_3}, \tag{5.3}$$

where the $\tilde{Y}_i$'s are any total-derivative deformations of the Y_i's, is incompatible with any spectrum containing a massless spin 1 field, reflecting the difficulties encountered in quantizing ST on (A)dS backgrounds [118–120]. From this perspective it is conceivable that a better understanding of the global symmetries of ST as well as of their implementation at the interacting level may shed some light on this issue. Moreover, coming back to flat space, Stückelberg fields can be also introduced into the vertices of the first Regge trajectory (5.1) using the $\hat{Y}_i$'s and the $\hat{Z}_i$'s in place of the y_i's and the z_i's. Clarifying their role is potentially interesting in view of a deeper comprehension of the states present in the lower Regge trajectories, to whom the Stückelberg fields may be related.

In the present paper we also studied the massless limit of the interactions focusing on the scaling of the masses leaving aside the behavior of the coupling functions. However, a complete analysis should take into account such behavior, which can depend in principle on more than one scale. For instance, conventional symmetry breaking scenarios, where masses are generated through interactions, need at least two mass scales: one related to the vev of the scalars (or more generally even-spin fields) and the other related to the coupling constants of the symmetric theory. Therefore, in order to properly address the mass generation issue in HS theories, it is necessary to have some control on the higher-order interactions and possibly on the full nonlinear theory. In this respect, if ST draws its origin from the spontaneous breaking of a HS gauge symmetry, one could expect that, besides the string tension, some new mass scales appear in the underlying fundamental description.

Finally, in order to complete the classification of the cubic interactions, it would be necessary to study the gauge deformations induced by the latter. Besides allowing us to address the issues related to the gravitational and the electromagnetic minimal couplings of HS fields, this would possibly shed some light on HS algebras and on their implications. Moreover, in order to get further insights into ST it would be interesting to extend the present analysis to fermionic and mixed-symmetry fields, and eventually to higher-order interactions along the lines of [80, 91, 121]. Last, let us stress that the ambient-space framework has proven particularly suitable in order to deal with interactions in curved backgrounds. For this reason, it is conceivable that this approach would give new insights into the AdS/CFT correspondence in relation to HS theories.

Acknowledgments

We are grateful to M. Bianchi, D. Francia and K. Mkrtchyan for helpful discussions, and especially to A. Sagnotti for reading the manuscript and for key suggestions. The present research was supported in part by Scuola Normale Superiore, by INFN and by the MIUR-PRIN contract 2009-KHZKRX.

A Useful identities

This appendix contains some identities and mathematical tools used in our construction of the cubic vertices. Basic commutation relations among the operators (3.3) are

$$\begin{aligned}
\big[\, Y_i \,,\, U_j \cdot \partial_{X_j} \,\big] &= \delta_{ij}\, \partial_{X_i} \cdot \partial_{X_{i+1}} \,, \\
\big[\, Z_i \,,\, U_{i+1} \cdot \partial_{X_{i+1}} \,\big] &= \partial_X \cdot \partial_{U_{i-1}} - Y_{i-1} \,, \\
\big[\, Z_i \,,\, U_{i-1} \cdot \partial_{X_{i-1}} \big] &= Y_{i+1} \,, \\
\big[\, X_i \cdot \partial_{U_i} \,,\, F(Y,Z) \,\big] &= -Z_{i+1}\, \partial_{Y_{i-1}}\, F(Y,Z) \,, \\
\big[\, X_i \cdot \partial_{X_i} \,,\, F(Y,Z) \,\big] &= -Y_{i-1}\, \partial_{Y_{i-1}}\, F(Y,Z) \,, \\
\big[\, F(Y,Z) \,,\, U_i \cdot \partial_{U_i} \,\big] &= \big(Y_i\, \partial_{Y_i} + Z_{i+1}\, \partial_{Z_{i+1}} + Z_{i-1}\, \partial_{Z_{i-1}}\big)\, F(Y,Z) \,.
\end{aligned} \tag{A.1}$$

Here i, j are defined modulo 3: $(i, j) \cong (i+3, j+3)$. Another identity used throughout all the paper concerns the commutator between an arbitrary function $f(A)$ of a linear operator A and an other linear operator B:

$$[\, f(A) \,,\, B \,] = \sum_{n=1}^{\infty} \frac{1}{n!}\, (\mathrm{ad}_A)^n B\, f^{(n)}(A) \,, \tag{A.2}$$

where $\mathrm{ad}_A\, B = [\, A \,,\, B \,]$ and $f^{(n)}(A)$ denotes the n-th derivative of f with respect to A. In order to prove the latter formula, we represent $f(A)$ as a Fourier integral so that the commutator appearing in (A.2) can be written as

$$[\, f(A) \,,\, B \,] = \int_{-\infty}^{\infty} dt\, [\, e^{itA} \,,\, B \,]\, f(t) \,. \tag{A.3}$$

Using the well-known identity

$$e^{itA}\, B\, e^{-itA} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\, (\mathrm{ad}_A)^n\, B \,, \tag{A.4}$$

eq. (A.3) becomes

$$[\, f(A) \,,\, B \,] = \sum_{n=1}^{\infty} \frac{1}{n!}\, (\mathrm{ad}_A)^n\, B \int_{-\infty}^{\infty} dt\, (it)^n\, e^{itA}\, f(t) = \sum_{n=1}^{\infty} \frac{1}{n!}\, (\mathrm{ad}_A)^n\, B\, f^{(n)}(A) \,. \tag{A.5}$$

Since our vertices are arbitrary functions of commuting operators, formula (A.2) applies independently to each of them.

B 2−2−2 and 3−3−2 partially-massless interactions

This appendix is devoted to the examples of 2−2−2 and 3−3−2 couplings involving at least one partially-massless field. The results are collected in the following tables in which we organized the solutions for given (μ_1, μ_2, μ_3) according to the maximal number of derivatives denoted by ∂. Arbitrary linear combinations of such solutions are consistent cubic couplings. Let us mention that in all cases we have checked, the number of solutions for the interactions involving massive fields is enhanced for those mass values satisfying eq. (3.43). For brevity, we consider such cases only in the 2−2−2 table (see e.g. $(\mu_1, \mu_2, \mu_3) = (1, 1, 2)\,,\ (1, \mu_3+1, \mu_3)\,,\ (1, \mu_3-1, \mu_3)\,)$. Moreover for simplicity we set $L = 1$ while the L dependence can be recovered replacing $\hat{\delta}$ by $\hat{\delta}/L$.

2-2-2 Couplings

$(\mu_1\,,\,\mu_2\,,\,\mu_3)$	∂	Couplings
$(1\,,\,1\,,\,1)$	6	$Y_1^2\,Y_2^2\,Y_3^2 - \frac{1}{4}\,\hat{\delta}^2\,(Y_1\,Y_2\,Z_1\,Z_2 + \text{cycl.}) + \frac{1}{4}\,\hat{\delta}^3\,Z_1\,Z_2\,Z_3$
	4	$(Y_1^2\,Y_2\,Y_3\,Z_1 + \text{cycl.}) - \hat{\delta}\,(Y_1\,Y_2\,Z_1\,Z_2 + \text{cycl.}) + \frac{3}{4}\,\hat{\delta}^2\,Z_1\,Z_2\,Z_3$
$(1\,,\,1\,,\,0)$	6	$Y_1^2\,Y_2^2\,Y_3^2$
	4	$Y_1\,Y_2\,Y_3^2\,Z_3 + \hat{\delta}\,(Y_1^2\,Z_1^2 + Y_2^2\,Z_2^2 + 2\,Y_1\,Y_2\,Z_1\,Z_2)$
	4	$Y_2^2\,Y_3\,Y_1\,Z_2 + Y_1^2\,Y_2\,Y_3\,Z_1 - \hat{\delta}\,Y_2\,Y_1\,Z_1\,Z_2$
	2	$Y_3^2\,Z_3^2 - Y_1^2\,Z_1^2 - Y_2^2\,Z_2^2 - 2\,Y_1\,Y_2\,Z_2\,Z_1$
	2	$Y_1^2\,Z_1^2 + Y_2^2\,Z_2^2 + 2\,Y_1\,Y_2\,Z_2\,Z_1 + Y_1\,Y_3\,Z_3\,Z_1 + Y_2\,Y_3\,Z_2\,Z_3$ $-\hat{\delta}\,Z_1\,Z_2\,Z_3$
$(1\,,\,1\,,\,\mu_3)$	6	$Y_1^2\,Y_2^2\,Y_3^2 + \frac{1}{4}\,\hat{\delta}^2\,\mu_3\,(\mu_3-2)\,(Y_1\,Y_3\,Z_1\,Z_3 + Y_2\,Y_3\,Z_2\,Z_3)$ $+\frac{1}{8}\,\hat{\delta}^3\,\mu_3\,(\mu_3-2)^2\,Z_1\,Z_2\,Z_3$
	4	$Y_1\,Y_2\,Y_3^2\,Z_3 + \frac{1}{2}\,\hat{\delta}\,(\mu_3-2)\,(Y_1\,Y_3\,Z_1\,Z_3 + Y_2\,Y_3\,Z_2\,Z_3)$ $+\frac{1}{4}\,\hat{\delta}^2\,(\mu_3-2)^2\,Z_1\,Z_2\,Z_3$
	4	$Y_1\,Y_2^2\,Y_3\,Z_2 - \frac{1}{2}\,\hat{\delta}\,\mu_3\,Y_2\,Y_3\,Z_2\,Z_3$
	4	$Y_1^2\,Y_2\,Y_3\,Z_1 - \frac{1}{2}\,\hat{\delta}\,\mu_3\,Y_1\,Y_3\,Z_1\,Z_3$
	2	$Y_3^2\,Z_3^2 + Y_1\,Y_3\,Z_1\,Z_3 + Y_2\,Y_3\,Z_2\,Z_3 + \frac{1}{2}\,\hat{\delta}\,(\mu_3-2)\,Z_1\,Z_2\,Z_3$
	2	$Y_1\,Y_2\,Z_1\,Z_2 - \frac{1}{2}\,\hat{\delta}\,\mu_3\,Z_1\,Z_2\,Z_3$
$(1\,,\,1\,,\,2)$	6	$Y_1^2\,Y_2^2\,Y_3^2$
	4	$Y_1\,Y_2\,Y_3^2 Z_3$
	4	$Y_1\,Y_2^2\,Y_3 Z_2 + \hat{\delta}\,Y_2^2 Z_2^2$
	4	$Y_1^2\,Y_2\,Y_3 Z_1 + \hat{\delta}\,Y_1^2 Z_1^2$
	2	$-Y_1^2 Z_1^2 - Y_2^2 Z_2^2 + Y_3^2 Z_3^2$
	2	$Y_2 Z_2\,(Y_2 Z_2 + Y_3 Z_3)$
	2	$Y_1 Z_1\,(Y_1 Z_1 + Y_3 Z_3)$
	2	$Y_1\,Y_2 Z_1 Z_2 - \hat{\delta} Z_1 Z_2 Z_3$
$(1\,,\,0\,,\,0)$	6	$Y_1^2\,Y_2^2\,Y_3^2 - \frac{1}{2}\,\hat{\delta}\,(3\,Y_1^2\,Y_2\,Y_3\,Z_1 + Y_1\,Y_2^2\,Y_3\,Z_2 + Y_1\,Y_2\,Y_3^2\,Z_3)$ $+\frac{1}{4}\,\hat{\delta}^2\,(3\,Y_1\,Y_2\,Z_1\,Z_2 + 3\,Y_1\,Y_3\,Z_1\,Z_3 + Y_2\,Y_3\,Z_2\,Z_3) - \frac{3}{8}\,\hat{\delta}^3\,Z_1\,Z_2\,Z_3$
$(1\,,\,0\,,\,\mu_3)$	6	$Y_1^2\,Y_2^2\,Y_3^2 + \hat{\delta}\,(\mu_3-1)\,Y_1^2\,Y_2\,Y_3\,Z_1$ $-\frac{1}{4}\,\hat{\delta}^2\,(\mu_3^2-1)\,(Y_3^2\,Z_3^2 + 2\,Y_1\,Y_3\,Z_1\,Z_3)$
	4	$Y_1^2\,Y_2\,Y_3\,Z_1 + Y_1\,Y_2\,Y_3^2\,Z_3 - \frac{1}{2}\,\hat{\delta}\,(\mu_3-1)\,Y_3^2\,Z_3^2$ $-\frac{1}{2}\,\hat{\delta}\,(\mu_3+1)\,Y_1\,Y_3\,Z_1\,Z_3$
	4	$Y_1\,Y_2^2\,Y_3\,Z_2 + \frac{1}{2}\,\hat{\delta}\,(\mu_3-3)\,Y_1\,Y_2\,Z_1\,Z_2 - \frac{1}{2}\,\hat{\delta}\,(\mu_3+1)\,Y_2\,Y_3\,Z_2\,Z_3$ $-\frac{1}{4}\,\hat{\delta}^2\,(\mu_3+1)\,(\mu_3-3)\,Z_1\,Z_2\,Z_3$
$(1\,,\,\mu_2\,,\,\mu_3)$	6	$Y_1^2\,Y_2^2\,Y_3^2 - \frac{1}{4}\,\hat{\delta}^2\,[(\mu_2-\mu_3)^2-1]\,Y_2^2\,Z_2^2$
	4	$Y_1\,Y_2\,Y_3^2\,Z_3 + \frac{1}{2}\,\hat{\delta}\,(\mu_2-\mu_3+1)\,Y_2^2\,Z_2^2$
	4	$Y_1\,Y_2^2\,Y_3\,Z_2 - \frac{1}{2}\,\hat{\delta}(\mu_2-\mu_3-1)\,Y_2^2\,Z_2^2$
	4	$Y_1^2\,Y_2\,Y_3\,Z_1 + \frac{1}{4}\,\hat{\delta}^2\,[(\mu_2-\mu_3)^2-1]\,Z_1\,Z_2\,Z_3$

JHEP07(2012)041

	2	$Y_3^2\, Z_3^2 - Y_2^2\, Z_2^2$
	2	$Y_2^2\, Z_2^2 + Y_2\, Y_3\, Z_2\, Z_3$
	2	$Y_1\, Y_3\, Z_1\, Z_3 - \frac{1}{2}\, \hat{\delta}\, (\mu_2 - \mu_3 + 1)\, Z_1\, Z_2\, Z_3$
	2	$Y_1\, Y_2\, Z_1\, Z_2 + \frac{1}{2}\, \hat{\delta}\, (\mu_2 - \mu_3 - 1)\, Z_1\, Z_2\, Z_3$
$(1\,,\, \mu_3 + 1\,,\, \mu_3)$	6	$Y_1^2\, Y_2^2\, Y_3^2$
	4	$Y_1\, Y_2\, Y_3^2 Z_3 + \hat{\delta}\, Y_2^2 Z_2^2$
	4	$Y_1\, Y_2^2\, Y_3 Z_2$
	4	$Y_1^2\, Y_2\, Y_3 Z_1$
	2	$-Y_2^2 Z_2^2 + Y_3^2 Z_3^2$
	2	$Y_2 Z_2\, (Y_2 Z_2 + Y_3 Z_3)$
	2	$Y_1\, Y_3 Z_1 Z_3 - \hat{\delta} Z_1 Z_2 Z_3,$
	2	$Y_1\, Y_2 Z_1 Z_2$
	2	$Y_1^2 Z_1^2$
$(1\,,\, \mu_3 - 1\,,\, \mu_3)$	6	$Y_1^2\, Y_2^2\, Y_3^2$
	4	$Y_1\, Y_2\, Y_3^2 Z_3$
	4	$Y_1\, Y_2^2\, Y_3 Z_2 + \hat{\delta}\, Y_2^2 Z_2^2$
	4	$Y_1^2\, Y_2\, Y_3 Z_1$
	2	$-Y_2^2 Z_2^2 + Y_3^2 Z_3^2$
	2	$Y_2 Z_2\, (Y_2 Z_2 + Y_3 Z_3)$
	2	$Y_1\, Y_3 Z_1 Z_3$
	2	$Y_1\, Y_2 Z_1 Z_2 - \hat{\delta} Z_1 Z_2 Z_3$
	2	$Y_1^2 Z_1^2$

3-3-2 Couplings

$(\mu_1\,,\, \mu_2\,,\, \mu_3)$	∂	Couplings
$(2\,,\, 2\,,\, 1)$	8	$Y_1^3\, Y_2^3\, Y_3^2 + \frac{1}{4}\hat{\delta}^2\, Y_1^3\, Y_2\, Z_1^2$ $-\frac{3}{8}\hat{\delta}^3\, (Y_1\, Z_1 + Y_2\, Z_2)\; Z_3\, (Y_1\, Z_1 + Y_3\, Z_3)$ $+\frac{3}{8}\hat{\delta}^4\, Z_1\, Z_2\, Z_3^2$
	6	$Y_1^2\, Y_2^2\, Y_3^2\, Z_3 - \frac{1}{4}\hat{\delta}^2\, Z_3\, (Y_2\, Y_3\, Z_2\, Z_3$ $+Y_1\, Z_1\, (Y_2\, Z_2 + Y_3\, Z_3)) + \frac{1}{4}\hat{\delta}^3\, Z_1\, Z_2\, Z_3^2$
	6	$Y_1^2\, Y_2^3\, Y_3\, Z_2,\, \frac{1}{2}\hat{\delta}\, Y_1^3\, Y_2\, Z_1^2$ $-\frac{3}{4}\hat{\delta}^2\, Z_3\, \left(Y_1^2\, Z_1^2 + Y_1\, Y_2\, Z_1\, Z_2 + Y_2\, Y_3\, Z_2\, Z_3\right),\, \frac{3}{8}\hat{\delta}^3\, Z_1\, Z_2\, Z_3^2$
	6	$Y_1^3\, Y_2^2\, Y_3\, Z_1 + \frac{1}{2}\hat{\delta}\, Y_1^3\, Y_2\, Z_1^2$ $-\frac{3}{4}\hat{\delta}^2\, Y_1\, Z_1\, Z_3\, (Y_1\, Z_1 + Y_2\, Z_2 + Y_3\, Z_3) + \frac{3}{8}\hat{\delta}^3\, Z_1\, Z_2\, Z_3^2$
	4	$Y_1\, Y_2\, Y_3^2\, Z_3^2 - \frac{1}{2}\hat{\delta}\, Y_3\, (Y_1\, Z_1 + Y_2\, Z_2)\; Z_3^2 + \frac{1}{4}\hat{\delta}^2\, Z_1\, Z_2\, Z_3^2$
	4	$Y_1\, Y_2^2\, Y_3\, Z_2\, Z_3 - \frac{1}{2}\hat{\delta}\, Y_2\, Z_2\, Z_3\, (Y_1\, Z_1 + Y_3\, Z_3) + \frac{1}{4}\hat{\delta}^2\, Z_1\, Z_2\, Z_3^2$
	4	$-Y_1^3\, Y_2\, Z_1^2 + Y_1\, Y_2^3\, Z_2^2 + \frac{3}{2}\hat{\delta}\, \left(Y_1^2\, Z_1^2 - Y_2^2\, Z_2^2\right)\; Z_3$
	4	$Y_1^2\, Y_2\, Y_3\, Z_1\, Z_3 - \frac{1}{2}\hat{\delta}\, Y_1\, Z_1\, Z_3\, (Y_2\, Z_2 + Y_3\, Z_3) + \frac{1}{4}\hat{\delta}^2\, Z_1\, Z_2\, Z_3^2$
	4	$Y_1^2\, Y_2\, Z_1\, (Y_1\, Z_1 + Y_2\, Z_2) - \frac{3}{2}\hat{\delta}\, Y_1\, Z_1\, (Y_1\, Z_1 + Y_2\, Z_2)\; Z_3$
$(2\,,\, 2\,,\, 0)$	8	$Y_1^3\, Y_2^3\, Y_3^2$
	6	$Y_1^2\, Y_2^2\, Y_3^2\, Z_3$
	6	$Y_1^2\, Y_2^2\, Y_3\, (Y_1\, Z_1 + Y_2\, Z_2) - \hat{\delta}\, Y_1^2\, Y_2^2\, Z_1\, Z_2$
	4	$Y_1\, Y_2\, Y_3^2\, Z_3^2 + \hat{\delta}\, (Y_1\, Z_1 + Y_2\, Z_2)^{\,2}\, Z_3$
	4	$Y_1\, Y_2\, Y_3\, (Y_1\, Z_1 + Y_2\, Z_2)\; Z_3 - \hat{\delta}\, Y_1\, Y_2\, Z_1\, Z_2\, Z_3$
	4	$Y_1\, Y_2\, (Y_1\, Z_1 + Y_2\, Z_2)^{\,2} - \hat{\delta}\, (Y_1\, Z_1 + Y_2\, Z_2)^{\,2}\, Z_3$
	2	$Z_3\, \left(-Y_1^2\, Z_1^2 - 2\, Y_1\, Y_2\, Z_1\, Z_2 - Y_2^2\, Z_2^2 + Y_3^2\, Z_3^2\right)$
	2	$(Y_1\, Z_1 + Y_2\, Z_2)\; Z_3\, (Y_1\, Z_1 + Y_2\, Z_2 + Y_3\, Z_3) - \hat{\delta}\, Z_1\, Z_2\, Z_3^2$
$(2\,,\, 1\,,\, 1)$	8	$Y_1^3\, Y_2^3\, Y_3^2$
	6	$Y_1^2\, Y_2^2\, Y_3^2\, Z_3$
	6	$Y_1^2\, Y_2^3\, Y_3\, Z_2 - \hat{\delta}\, Y_1^2\, Y_2^2\, Z_1\, Z_2 + 2\hat{\delta}^2\, Y_2\, Z_2\, (Y_1\, Z_1 + Y_2\, Z_2)\; Z_3$
	6	$Y_1^3\, Y_2^2\, Y_3\, Z_1 + 2\hat{\delta}\, Y_1^2\, Y_2\, Z_1\, (Y_1\, Z_1 + Y_2\, Z_2)$

JHEP07(2012)041

		$-2\hat{\delta}^2 Y_1 Z_1 Z_3 (2 Y_1 Z_1 + 2 Y_2 Z_2 + Y_3 Z_3)$
	4	$-Y_1 Y_2 \left(Y_1^2 Z_1^2 + Y_1 Y_2 Z_1 Z_2 - Y_3^2 Z_3^2\right) + 2\hat{\delta} Y_1 Z_1 (Y_1 Z_1 + Y_2 Z_2)\, Z_3$
	4	$Y_1 Y_2^2 Y_3 Z_2 Z_3 + \hat{\delta} Y_2^2 Z_2^2 Z_3$
	4	$Y_1 Y_2^2 Z_2 (Y_1 Z_1 + Y_2 Z_2) - 2\hat{\delta} Y_2 Z_2 (Y_1 Z_1 + Y_2 Z_2)\, Z_3$
	4	$Y_1^2 Y_2 Z_1 (Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3)$ $-\hat{\delta} Y_1 Z_1 Z_3 (2 Y_1 Z_1 + 2 Y_2 Z_2 + Y_3 Z_3)$
	2	$Z_3 \left(Y_1^2 Z_1^2 - Y_2^2 Z_2^2 + 2 Y_1 Y_3 Z_1 Z_3 + Y_3^2 Z_3^2\right)$
	2	$Y_2 Z_2 Z_3 (Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3) - \hat{\delta} Z_1 Z_2 Z_3^2$
(2 , 1 , 0)	8	$Y_1^3 Y_2^3 Y_3^2 - \hat{\delta} Y_1^3 Y_2^2 Y_3 Z_1$ $-\frac{1}{4}\hat{\delta}^2 Y_1 Y_2 \left(Y_1^2 Z_1^2 + 3 Y_1 Y_3 Z_1 Z_3 + 3 Y_3 Z_3 (Y_2 Z_2 + Y_3 Z_3)\right)$ $\frac{3}{8}\hat{\delta}^3 Z_3 \left(Y_1^2 Z_1^2 + Y_2 Y_3 Z_2 Z_3 + 3 Y_1 Z_1 (Y_2 Z_2 + Y_3 Z_3)\right) - \frac{9}{16}\hat{\delta}^4 Z_1 Z_2 Z_3^2$
	6	$Y_1^2 Y_2^2 Y_3^2 Z_3 - \frac{1}{2}\hat{\delta} Y_1 Y_2 Y_3 Z_3 (3 Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3)$ $\frac{1}{4}\hat{\delta}^2 Z_3 (Y_2 Y_3 Z_2 Z_3 + 3 Y_1 Z_1 (Y_2 Z_2 + Y_3 Z_3)) - \frac{3}{8}\hat{\delta}^3 Z_1 Z_2 Z_3^2$
	6	$Y_1^2 Y_2^2 Y_3 (Y_1 Z_1 + Y_2 Z_2)$ $-\frac{1}{2}\hat{\delta} Y_1 Y_2 \left(Y_1^2 Z_1^2 + 3 Y_2 Y_3 Z_2 Z_3 + 3 Y_1 Z_1 (Y_2 Z_2 + Y_3 Z_3)\right)$ $+\frac{3}{4}\hat{\delta}^2 Y_1 Z_1 (Y_1 Z_1 + 3 Y_2 Z_2)\, Z_3$
(2 , 0 , 1)	8	$Y_1^3 Y_2^3 Y_3^2 + \frac{3}{2}\hat{\delta} Y_1^3 Y_2^2 Y_3 Z_1$ $-\frac{3}{4}\hat{\delta}^2 Y_1 Y_2 \left(2 Y_1^2 Z_1^2 + 10 Y_1 Y_3 Z_1 Z_3 + 5 Y_3^2 Z_3^2\right)$ $+\frac{15}{8}\hat{\delta}^3 Y_1 Z_1 Z_3 (2 Y_1 Z_1 + 3 Y_3 Z_3)$
	6	$Y_1^2 Y_2^2 Y_3 (Y_1 Z_1 + Y_3 Z_3)$ $-\frac{1}{2}\hat{\delta} Y_1 Y_2 \left(Y_1^2 Z_1^2 + 8 Y_1 Y_3 Z_1 Z_3 + 3 Y_3^2 Z_3^2\right)$ $+\frac{1}{4}\hat{\delta}^2 Y_1 Z_1 Z_3 (5 Y_1 Z_1 + 9 Y_3 Z_3)$
	6	$Y_1^2 Y_2^3 Y_3 Z_2 - \frac{3}{2}\hat{\delta} Y_1 Y_2^2 Z_2 (Y_1 Z_1 + 2 Y_3 Z_3)$ $\frac{3}{4}\hat{\delta}^2 Y_2 Z_2 Z_3 (6 Y_1 Z_1 + Y_3 Z_3) - \frac{9}{8}\hat{\delta}^3 Z_1 Z_2 Z_3^2$
(2 , 0 , 0)	8	$Y_1^3 Y_2^3 Y_3^2 + 2\hat{\delta} Y_1^2 Y_2^3 Y_3 Z_2 + 2\hat{\delta}^2 Y_1 Y_2^3 Z_2^2$
	6	$Y_1^2 Y_2^2 Y_3 (Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3) + \hat{\delta} Y_1 Y_2 \left(Y_2^2 Z_2^2 - 2 Y_1 Y_3 Z_1 Z_3\right)$
	4	$Y_1 Y_2 (Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3)^2$ $-2\hat{\delta} Y_1 Z_1 Z_3 (Y_1 Z_1 + 2 Y_2 Z_2 + Y_3 Z_3)$
(1 , 1 , 1)	8	$Y_1^3 Y_2^3 Y_3^2 - \frac{1}{4}\hat{\delta}^2 Y_1 Y_2 (3 Y_3 Z_3 (2 Y_2 Z_2 + 5 Y_3 Z_3)$ $+Y_1 Z_1 (Y_2 Z_2 + 6 Y_3 Z_3)) + \frac{3}{4}\hat{\delta}^3 Z_3 (3 Y_2 Y_3 Z_2 Z_3$ $+Y_1 Z_1 (2 Y_2 Z_2 + 3 Y_3 Z_3)) - \frac{21}{16}\hat{\delta}^4 Z_1 Z_2 Z_3^2$
	6	$Y_1^2 Y_2^2 Y_3^2 Z_3 - \frac{1}{2}\hat{\delta} Y_1 Y_2 Y_3 Z_3 (Y_1 Z_1 + Y_2 Z_2 + 3 Y_3 Z_3)$ $\frac{1}{4}\hat{\delta}^2 Z_3 (3 Y_2 Y_3 Z_2 Z_3 + Y_1 Z_1 (Y_2 Z_2 + 3 Y_3 Z_3)) - \frac{3}{8}\hat{\delta}^3 Z_1 Z_2 Z_3^2$
	6	$Y_1^2 Y_2^3 Y_3 Z_2 - \frac{1}{2}\hat{\delta} Y_1 Y_2^2 Z_2 (Y_1 Z_1 + 6 Y_3 Z_3)$ $+\frac{3}{4}\hat{\delta}^2 Y_2 Z_2 Z_3 (2 Y_1 Z_1 + Y_3 Z_3) - \frac{3}{8}\hat{\delta}^3 Z_1 Z_2 Z_3^2$
	6	$Y_1^3 Y_2^2 Y_3 Z_1 - \frac{1}{2}\hat{\delta} Y_1^2 Y_2 Z_1 (Y_2 Z_2 + 6 Y_3 Z_3)$ $+\frac{3}{4}\hat{\delta}^2 Y_1 Z_1 Z_3 (2 Y_2 Z_2 + Y_3 Z_3) - \frac{3}{8}\hat{\delta}^3 Z_1 Z_2 Z_3^2$
(1 , 1 , 0)	8	$Y_1^3 Y_2^3 Y_3^2$
	6	$Y_1^2 Y_2^2 Y_3^2 Z_3$
	6	$Y_1^2 Y_2^2 Y_3 (Y_1 Z_1 + Y_2 Z_2)$ $+\hat{\delta} Y_1 Y_2 \left(2 Y_1^2 Z_1^2 + 3 Y_1 Y_2 Z_1 Z_2 + 2 Y_2^2 Z_2^2\right)$ $-2\hat{\delta}^2 (Y_1 Z_1 + Y_2 Z_2)\, Z_3 (2 Y_1 Z_1 + 2 Y_2 Z_2 + Y_3 Z_3) + 2\hat{\delta}^3 Z_1 Z_2 Z_3^2$
	4	$-Y_1 Y_2 \left(Y_1^2 Z_1^2 + 2 Y_1 Y_2 Z_1 Z_2 + Y_2^2 Z_2^2 - Y_3^2 Z_3^2\right)$ $+2\hat{\delta} (Y_1 Z_1 + Y_2 Z_2)^2 Z_3$
	4	$Y_1 Y_2 (Y_1 Z_1 + Y_2 Z_2)(Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3)$ $-\hat{\delta} Z_3 \left(2 Y_1^2 Z_1^2 + Y_2 Z_2 (2 Y_2 Z_2 + Y_3 Z_3) + Y_1 Z_1 (5 Y_2 Z_2 + Y_3 Z_3)\right) + \hat{\delta}^2 Z_1 Z_2 Z_3^2$
	2	$Z_3 (Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3)^2 - 2\hat{\delta} Z_1 Z_2 Z_3^2$
(1 , 0 , 1)	8	$Y_1^3 Y_2^3 Y_3^2 + 3\hat{\delta} Y_1^3 Y_2^2 Y_3 Z_1 + 6\hat{\delta}^2 \left(Y_1^3 Y_2 Z_1^2 - Y_1 Y_2^3 Z_2^2\right)$ $-6\hat{\delta}^3 Z_3 \left(3 Y_1^2 Z_1^2 - 3 Y_2^2 Z_2^2 + 3 Y_1 Y_3 Z_1 Z_3 + Y_3^2 Z_3^2\right)$
	6	$Y_1^2 Y_2^2 Y_3 (Y_1 Z_1 + Y_3 Z_3) + 2\hat{\delta} \left(Y_1^3 Y_2 Z_1^2 - Y_1 Y_2^3 Z_2^2\right)$ $-2\hat{\delta}^2 Z_3 \left(3 Y_1^2 Z_1^2 - 3 Y_2^2 Z_2^2 + 3 Y_1 Y_3 Z_1 Z_3 + Y_3^2 Z_3^2\right)$
	6	$Y_1^2 Y_2^3 Y_3 Z_2 + \hat{\delta} Y_1 Y_2^2 Z_2 (3 Y_1 Z_1 + 4 Y_2 Z_2)$

JHEP07(2012)041

		$-6\hat{\delta}^2 Y_2 Z_2 Z_3 (2Y_1Z_1+2Y_2Z_2+Y_3Z_3)+6\hat{\delta}^3 Z_1Z_2Z_3^2$
	4	$Y_1Y_2\left(Y_1^2Z_1^2-Y_2^2Z_2^2+2Y_1Y_3Z_1Z_3+Y_3^2Z_3^2\right)$ $-\hat{\delta} Z_3\left(3Y_1^2Z_1^2-3Y_2^2Z_2^2+4Y_1Y_3Z_1Z_3+Y_3^2Z_3^2\right)$
	4	$Y_1Y_2^2Z_2(Y_1Z_1+Y_2Z_2+Y_3Z_3)$ $-\hat{\delta} Y_2Z_2Z_3(4Y_1Z_1+3Y_2Z_2+2Y_3Z_3)+2\hat{\delta}^2 Z_1Z_2Z_3^2$
$(1, 0, 0)$	8	$Y_1^3Y_2^3Y_3^2-\frac{1}{2}\hat{\delta} Y_1^2Y_2^2Y_3(3Y_1Z_1+Y_2Z_2+6Y_3Z_3)$ $+\frac{3}{4}\hat{\delta}^2 Y_1Y_2(Y_3Z_3(2Y_2Z_2+Y_3Z_3)+Y_1Z_1(Y_2Z_2+6Y_3Z_3))$ $-\frac{3}{8}\hat{\delta}^3 Z_3(Y_2Y_3Z_2Z_3+3Y_1Z_1(2Y_2Z_2+Y_3Z_3))+\frac{9}{16}\hat{\delta}^4 Z_1Z_2Z_3^2$
$(0, 0, 1)$	8	$Y_1^3Y_2^3Y_3^2-\frac{1}{2}\hat{\delta} Y_1^2Y_2^2Y_3(Y_1Z_1+Y_2Z_2+10Y_3Z_3)$ $\frac{1}{4}\hat{\delta}^2 Y_1Y_2(5Y_3Z_3(2Y_2Z_2+3Y_3Z_3)+Y_1Z_1(Y_2Z_2+10Y_3Z_3))$ $-\frac{5}{8}\hat{\delta}^3 Z_3(3Y_2Y_3Z_2Z_3+Y_1Z_1(2Y_2Z_2+3Y_3Z_3))+\frac{15}{16}\hat{\delta}^4 Z_1Z_2Z_3^2$
$(2, 2, \mu_3)$	8	$Y_1^3Y_2^3Y_3^2+\frac{1}{8}\hat{\delta}^3 Y_3(Y_1Z_1+Y_2Z_2)\,Z_3^2\mu_3\left(-4+\mu_3^2\right)$ $+\frac{1}{16}\hat{\delta}^4 Z_1Z_2Z_3^2(-2+\mu_3)^2\mu_3(2+\mu_3)$
	6	$Y_1^2Y_2^2Y_3^2Z_3+\frac{1}{4}\hat{\delta}^2 Y_3(Y_1Z_1+Y_2Z_2)\,Z_3^2(-2+\mu_3)\mu_3$ $+\frac{1}{8}\hat{\delta}^3 Z_1Z_2Z_3^2(-2+\mu_3)^2\mu_3$
	6	$Y_1^2Y_2^3Y_3Z_2-\frac{1}{4}\hat{\delta}^2 Y_2Y_3Z_2Z_3^2\mu_3(2+\mu_3)$
	6	$Y_1^3Y_2^2Y_3Z_1-\frac{1}{4}\hat{\delta}^2 Y_1Y_3Z_1Z_3^2\mu_3(2+\mu_3)$
	4	$Y_1Y_2Y_3^2Z_3^2+\frac{1}{2}\hat{\delta} Y_3(Y_1Z_1+Y_2Z_2)\,Z_3^2(-2+\mu_3)$ $+\frac{1}{4}\hat{\delta}^2 Z_1Z_2Z_3^2(-2+\mu_3)^2$
	4	$Y_1Y_2^2Y_3Z_2Z_3-\frac{1}{2}\hat{\delta} Y_2Y_3Z_2Z_3^2\mu_3$
	4	$Y_1Y_2^3Z_2^2-\frac{1}{2}\hat{\delta} Y_2^2Z_2^2Z_3(2+\mu_3)$
	4	$Y_1^2Y_2Y_3Z_1Z_3-\frac{1}{2}\hat{\delta} Y_1Y_3Z_1Z_3^2\mu_3$
	4	$Y_1^2Y_2^2Z_1Z_2-\frac{1}{4}\hat{\delta}^2 Z_1Z_2Z_3^2\mu_3(2+\mu_3)$
	4	$Y_1^3Y_2Z_1^2-\frac{1}{2}\hat{\delta} Y_1^2Z_1^2Z_3(2+\mu_3)$
	2	$Y_3Z_3^2(Y_1Z_1+Y_2Z_2+Y_3Z_3)+\frac{1}{2}\hat{\delta} Z_1Z_2Z_3^2(-2+\mu_3)$
	2	$Y_1Y_2Z_1Z_2Z_3-\frac{1}{2}\hat{\delta} Z_1Z_2Z_3^2\mu_3$
$(2, 1, \mu_3)$	8	$Y_1^3Y_2^3Y_3^2-\frac{3}{4}\hat{\delta}^2 Y_1^3Y_2Z_1^2\left(-1+\mu_3^2\right)$ $-\frac{1}{8}\hat{\delta}^3 Z_3\left(-3Y_1^2Z_1^2+Y_3^2Z_3^2\right)(-1+\mu_3)(1+\mu_3)(3+\mu_3)$
	6	$Y_1^2Y_2^2Y_3^2Z_3-\hat{\delta} Y_1^3Y_2Z_1^2(-1+\mu_3)$ $+\frac{1}{4}\hat{\delta}^2 Z_3(-1+\mu_3)\left(-Y_3^2Z_3^2(1+\mu_3)+2Y_1^2Z_1^2(3+\mu_3)\right)$
	6	$Y_1^2Y_2^3Y_3Z_2, \frac{1}{4}\hat{\delta}^2 Y_2Z_2Z_3(1+\mu_3)(2Y_1Z_1(-3+\mu_3)-Y_3Z_3(3+\mu_3))$ $-\frac{1}{4}\hat{\delta}^3 Z_1Z_2Z_3^2(-3+\mu_3)(1+\mu_3)^2$
	6	$Y_1^3Y_2^2Y_3Z_1+\hat{\delta} Y_1^3Y_2Z_1^2(1+\mu_3)$ $-\frac{1}{4}\hat{\delta}^2 Y_1Z_1Z_3(2Y_1Z_1+Y_3Z_3)(1+\mu_3)(3+\mu_3)$
	4	$Y_1Y_2\left(-Y_1^2Z_1^2+Y_3^2Z_3^2\right)$ $+\frac{1}{2}\hat{\delta} Z_3\left(-Y_3^2Z_3^2(-1+\mu_3)+Y_1^2Z_1^2(3+\mu_3)\right)$
	4	$Y_1Y_2^2Y_3Z_2Z_3-\frac{1}{2}\hat{\delta} Y_2Z_2Z_3(-Y_1Z_1(-3+\mu_3)+Y_3Z_3(1+\mu_3))$ $-\frac{1}{4}\hat{\delta}^2 Z_1Z_2Z_3^2(-3+\mu_3)(1+\mu_3)$
	4	$Y_1^2Y_2Z_1(Y_1Z_1+Y_3Z_3)$ $-\frac{1}{2}\hat{\delta} Y_1Z_1Z_3(Y_3Z_3(1+\mu_3)+Y_1Z_1(3+\mu_3))$
	4	$Y_1^2Y_2^2Z_1Z_2-\hat{\delta} Y_1Y_2Z_1Z_2Z_3(1+\mu_3)$ $+\frac{1}{4}\hat{\delta}^2 Z_1Z_2Z_3^2\left(-1+\mu_3^2\right)$
$(2, 0, \mu_3)$	8	$Y_1^3Y_2^3Y_3^2+\frac{3}{2}\hat{\delta} Y_1^3Y_2^2Y_3Z_1\mu_3+\frac{3}{4}\hat{\delta}^2 Y_1^3Y_2Z_1^2\mu_3(2+\mu_3)$ $-\frac{1}{8}\hat{\delta}^3 Z_3\left(3Y_1^2Z_1^2+3Y_1Y_3Z_1Z_3+Y_3^2Z_3^2\right)\mu_3(2+\mu_3)(4+\mu_3)$
	6	$Y_1^2Y_2^2Y_3(Y_1Z_1+Y_3Z_3)+\hat{\delta} Y_1^2Y_2Z_1(Y_3Z_3(-2+\mu_3)+Y_1Z_1\mu_3)$ $-\frac{1}{4}\hat{\delta}^2 Z_3\mu_3\left(3Y_1Y_3Z_1Z_3(2+\mu_3)+Y_3^2Z_3^2(2+\mu_3)+2Y_1^2Z_1^2(4+\mu_3)\right)$
	6	$Y_1^2Y_2^3Y_3Z_2+\frac{1}{2}\hat{\delta} Y_1Y_2^2Z_2(Y_1Z_1(-4+\mu_3)-2Y_3Z_3(2+\mu_3))$ $\frac{1}{4}\hat{\delta}^2 Y_2Z_2Z_3(2+\mu_3)(-2Y_1Z_1(-4+\mu_3)+Y_3Z_3\mu_3)$ $+\frac{1}{8}\hat{\delta}^3 Z_1Z_2Z_3^2(-4+\mu_3)\mu_3(2+\mu_3)$
	4	$Y_1Y_2(Y_1Z_1+Y_3Z_3)^2$ $-\frac{1}{2}\hat{\delta} Z_3(Y_1Z_1+Y_3Z_3)(Y_3Z_3\mu_3+Y_1Z_1(4+\mu_3))$
$(1, 1, \mu_3)$	8	$Y_1^3Y_2^3Y_3^2+\frac{3}{4}\hat{\delta}^2 Y_1Y_2Y_3(Y_1Z_1+Y_2Z_2)\,Z_3\mu_3(2+\mu_3)$

JHEP07(2012)041

		$-\frac{1}{8}\hat{\delta}^3 Z_3\mu_3(2+\mu_3)(3Y_1Z_1(-Y_2Z_2(-2+\mu_3)+Y_3Z_3(2+\mu_3))$ $+Y_3Z_3(3Y_2Z_2(2+\mu_3)+Y_3Z_3(4+\mu_3)))$ $-\frac{3}{16}\hat{\delta}^4 Z_1Z_2Z_3^2(-2+\mu_3)\mu_3(2+\mu_3)^2$
	6	$Y_1^2Y_2^2Y_3^2Z_3+\hat{\delta}Y_1Y_2Y_3(Y_1Z_1+Y_2Z_2)Z_3\mu_3$ $-\frac{1}{4}\hat{\delta}^2Z_3\mu_3(Y_3Z_3(2Y_2Z_2+Y_3Z_3)(2+\mu_3)$ $+2Y_1Z_1(-Y_2Z_2(-2+\mu_3)+Y_3Z_3(2+\mu_3)))-\frac{1}{4}\hat{\delta}^3Z_1Z_2Z_3^2\mu_3\left(-4+\mu_3^2\right)$
	6	$Y_1^2Y_2^3Y_3Z_2-\hat{\delta}Y_1Y_2^2Y_3Z_2Z_3(2+\mu_3)+\frac{1}{4}\hat{\delta}^2Y_2Y_3Z_2Z_3^2\mu_3(2+\mu_3)$
	6	$Y_1^3Y_2^2Y_3Z_1-\hat{\delta}Y_1^2Y_2Y_3Z_1Z_3(2+\mu_3)+\frac{1}{4}\hat{\delta}^2Y_1Y_3Z_1Z_3^2\mu_3(2+\mu_3)$
	4	$Y_1Y_2Y_3Z_3(Y_1Z_1+Y_2Z_2+Y_3Z_3)$ $-\frac{1}{2}\hat{\delta}Z_3(Y_3Z_3(Y_3Z_3\mu_3+Y_2Z_2(2+\mu_3))$ $+Y_1Z_1(-Y_2Z_2(-2+\mu_3)+Y_3Z_3(2+\mu_3)))-\frac{1}{4}\hat{\delta}^2Z_1Z_2Z_3^2\left(-4+\mu_3^2\right)$
	4	$Y_1^2Y_2^2Z_1Z_2-\hat{\delta}Y_1Y_2Z_1Z_2Z_3(2+\mu_3)+\frac{1}{4}\hat{\delta}^2Z_1Z_2Z_3^2\mu_3(2+\mu_3)$
$(1,0,\mu_3)$	8	$Y_1^3Y_2^3Y_3^2+\frac{3}{2}\hat{\delta}Y_1^3Y_2^2Y_3Z_1(1+\mu_3)$ $-\frac{3}{4}\hat{\delta}^2Y_1Y_2Y_3Z_3(2Y_1Z_1+Y_3Z_3)(1+\mu_3)(3+\mu_3)$ $+\frac{1}{8}\hat{\delta}^3Y_3Z_3^2(1+\mu_3)(3+\mu_3)(2Y_3Z_3(-1+\mu_3)+3Y_1Z_1(1+\mu_3))$
	6	$Y_1^2Y_2^2Y_3(Y_1Z_1+Y_3Z_3)$ $-\hat{\delta}Y_1Y_2Y_3Z_3(Y_3Z_3(1+\mu_3)+Y_1Z_1(3+\mu_3))$ $+\frac{1}{4}\hat{\delta}^2Y_3Z_3^2(1+\mu_3)(Y_3Z_3(-1+\mu_3)+Y_1Z_1(3+\mu_3))$
	6	$Y_1^2Y_2^3Y_3Z_2+\frac{1}{2}\hat{\delta}Y_1Y_2^2Z_2(Y_1Z_1(-3+\mu_3)-2Y_3Z_3(3+\mu_3))$ $+\frac{1}{4}\hat{\delta}^2Y_2Z_2Z_3(3+\mu_3)(-2Y_1Z_1(-3+\mu_3)+Y_3Z_3(1+\mu_3))$ $+\frac{1}{8}\hat{\delta}^3Z_1Z_2Z_3^2(-3+\mu_3)(1+\mu_3)(3+\mu_3)$
$(2,\mu_2,1)$	8	$Y_1^3Y_2^2Y_3^3-\frac{1}{4}\hat{\delta}^2Y_1^3Y_3Z_1^2(-3+\mu_2)(-1+\mu_2)$ $-\frac{1}{8}\hat{\delta}^3Z_2\left(Y_1^2Z_1^2+Y_3(Y_1Z_1+Y_2Z_2)Z_3\right)(-5+\mu_2)(-3+\mu_2)(-1+\mu_2)$ $-\frac{1}{16}\hat{\delta}^4Z_1Z_2^2Z_3(-5+\mu_2)(-3+\mu_2)^2(-1+\mu_2)$
	6	$Y_1^2Y_2Y_3^3Z_3-\frac{1}{2}\hat{\delta}Y_1^3Y_3Z_1^2(-3+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2Z_2\left(Y_1^2Z_1^2+Y_3(Y_1Z_1+Y_2Z_2)Z_3\right)(-5+\mu_2)(-3+\mu_2)$ $-\frac{1}{8}\hat{\delta}^3Z_1Z_2^2Z_3(-5+\mu_2)(-3+\mu_2)^2$
	6	$Y_1^2Y_2^2Y_3^2Z_2$ $+\frac{1}{4}\hat{\delta}^2Y_3Z_2(Y_1Z_1+Y_2Z_2)Z_3(-3+\mu_2)(-1+\mu_2)$ $+\frac{1}{8}\hat{\delta}^3Z_1Z_2^2Z_3(-3+\mu_2)^2(-1+\mu_2)$
	6	$Y_1^3Y_2Y_3^2Z_1+\frac{1}{2}\hat{\delta}Y_1^3Y_3Z_1^2(-1+\mu_2)$ $+\frac{1}{4}\hat{\delta}^2Y_1Z_1Z_2(Y_1Z_1+Y_3Z_3)(-5+\mu_2)(-1+\mu_2)$
	4	$-Y_1^3Y_3Z_1^2+Y_1Y_3^3Z_3^2$ $-\frac{1}{2}\hat{\delta}Z_2\left(Y_1^2Z_1^2+Y_3(Y_1Z_1+Y_2Z_2)Z_3\right)(-5+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2Z_1Z_2^2Z_3(-5+\mu_2)(-3+\mu_2)$
	4	$Y_1Y_2Y_3^2Z_2Z_3+\frac{1}{2}\hat{\delta}Y_3Z_2(Y_1Z_1+Y_2Z_2)Z_3(-3+\mu_2)$ $+\frac{1}{4}\hat{\delta}^2Z_1Z_2^2Z_3(-3+\mu_2)^2$
	4	$Y_1Y_2^2Y_3Z_2^2-\frac{1}{2}\hat{\delta}Y_2Y_3Z_2^2Z_3(-1+\mu_2)$
	4	$Y_1^2Y_3Z_1(Y_1Z_1+Y_3Z_3)+\frac{1}{2}\hat{\delta}Y_1Z_1Z_2(Y_1Z_1+Y_3Z_3)(-5+\mu_2)$
	4	$Y_1^2Y_2Y_3Z_1Z_2-\frac{1}{2}\hat{\delta}Y_1Y_3Z_1Z_2Z_3(-1+\mu_2)$
	2	$Y_3Z_2Z_3(Y_1Z_1+Y_2Z_2+Y_3Z_3)+\frac{1}{2}\hat{\delta}Z_1Z_2^2Z_3(-3+\mu_2)$
	2	$Y_1Y_2Z_1Z_2^2-\frac{1}{2}\hat{\delta}Z_1Z_2^2Z_3(-1+\mu_2)$
$(1,\mu_2,1)$	8	$Y_1^3Y_2^2Y_3^3+\frac{1}{4}\hat{\delta}^2Y_1Y_3(-2+\mu_2)\left(-3Y_2^2Z_2^2(-4+\mu_2)+Y_1Y_3Z_1Z_3\mu_2\right)$ $+\frac{1}{4}\hat{\delta}^3Y_3Z_2Z_3(Y_1Z_1+2Y_2Z_2+Y_3Z_3)(-4+\mu_2)(-2+\mu_2)\mu_2$ $+\frac{1}{16}\hat{\delta}^4Z_1Z_2^2Z_3(-4+\mu_2)(-2+\mu_2)^2\mu_2$
	6	$Y_1^2Y_2Y_3^3Z_3+\frac{1}{2}\hat{\delta}Y_1Y_3\left(-2Y_2^2Z_2^2(-4+\mu_2)+Y_1Y_3Z_1Z_3(-2+\mu_2)\right)$ $+\frac{1}{4}\hat{\delta}^2Y_3Z_2Z_3(-4+\mu_2)(2Y_1Z_1(-2+\mu_2)$ $+2Y_3Z_3(-2+\mu_2)+Y_2Z_2(-2+3\mu_2))$ $+\frac{1}{8}\hat{\delta}^3Z_1Z_2^2Z_3(-4+\mu_2)(-2+\mu_2)^2$
	6	$Y_1^2Y_2^2Y_3^2Z_2+\hat{\delta}Y_1Y_2^2Y_3Z_2^2(-2+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2Y_3Z_2Z_3(2Y_2Z_2+Y_3Z_3)(-2+\mu_2)\mu_2$
	6	$Y_1^3Y_2Y_3^2Z_1-\frac{1}{2}\hat{\delta}Y_1^2Y_3^2Z_1Z_3\mu_2$

		$-\frac{1}{4}\hat{\delta}^2 Y_1 Y_2 Z_1 Z_2^2 (-6+\mu_2)(-4+\mu_2)+\frac{1}{8}\hat{\delta}^3 Z_1 Z_2^2 Z_3 (-6+\mu_2)(-4+\mu_2)\mu_2$
	4	$Y_1 Y_3 \left(-Y_2^2 Z_2^2+Y_3 Z_3 (Y_1 Z_1+Y_3 Z_3)\right)$ $+\hat{\delta} Y_3 Z_2 Z_3 (Y_1 Z_1 (-4+\mu_2)+Y_3 Z_3 (-4+\mu_2)+Y_2 Z_2 (-2+\mu_2))$ $+\frac{1}{4}\hat{\delta}^2 Z_1 Z_2^2 Z_3 (-4+\mu_2)(-2+\mu_2)$
	4	$Y_1 Y_2 Y_3 Z_2 (Y_2 Z_2+Y_3 Z_3)$ $-\frac{1}{2}\hat{\delta} Y_3 Z_2 Z_3 (Y_3 Z_3 (-2+\mu_2)+Y_2 Z_2 \mu_2)$
	4	$Y_1^2 Y_2 Y_3 Z_1 Z_2+\frac{1}{2}\hat{\delta} Y_1 Z_1 Z_2 (Y_2 Z_2 (-4+\mu_2)-Y_3 Z_3 \mu_2)$ $-\frac{1}{4}\hat{\delta}^2 Z_1 Z_2^2 Z_3 (-4+\mu_2)\mu_2$
$(0\,,\,\mu_2\,,\,1)$	8	$Y_1^3 Y_2^2 Y_3^3+\frac{3}{2}\hat{\delta} Y_1^2 Y_2^2 Y_3^2 Z_2 (-3+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2 Y_1 Y_3 (-1+\mu_2)\left(-2 Y_2^2 Z_2^2 (-5+\mu_2)\right.$ $\left.+2 Y_2 Y_3 Z_2 Z_3 (1+\mu_2)+Y_3^2 Z_3^2 (1+\mu_2)\right)$ $-\frac{1}{8}\hat{\delta}^3 Y_3 Z_2 Z_3 (Y_3 Z_3 (-7+\mu_2)+2 Y_2 Z_2 (-5+\mu_2))\left(-1+\mu_2^2\right)$
	6	$Y_1^2 Y_2 Y_3^2 (Y_2 Z_2+Y_3 Z_3)$ $+\frac{1}{2}\hat{\delta} Y_1 Y_3 \left(-8 Y_2 Y_3 Z_2 Z_3+Y_2^2 Z_2^2 (-5+\mu_2)-Y_3^2 Z_3^2 (-1+\mu_2)\right)$ $-\frac{1}{4}\hat{\delta}^2 Y_3 Z_2 Z_3 (Y_3 Z_3 (-7+\mu_2)(-1+\mu_2)+Y_2 Z_2 (-5+\mu_2)(1+\mu_2))$
	6	$Y_1^3 Y_2 Y_3^2 Z_1-\frac{1}{2}\hat{\delta} Y_1^2 Y_3 Z_1 (-2 Y_2 Z_2 (-5+\mu_2)+Y_3 Z_3 (1+\mu_2))$ $+\frac{1}{4}\hat{\delta}^2 Y_1 Z_1 Z_2 (-5+\mu_2)(Y_2 Z_2 (-3+\mu_2)-2 Y_3 Z_3 (1+\mu_2))$ $-\frac{1}{8}\hat{\delta}^3 Z_1 Z_2^2 Z_3 (-5+\mu_2)(-3+\mu_2)(1+\mu_2)$
$(2\,,\,\mu_2\,,\,0)$	8	$Y_1^3 Y_2^2 Y_3^3+\hat{\delta} Y_1^3 Y_2 Y_3^2 Z_1 (-2+\mu_2)+\frac{1}{4}\hat{\delta}^2 Y_1^3 Y_3 Z_1^2 (-2+\mu_2)\mu_2$ $+\frac{1}{8}\hat{\delta}^3 Z_2 (Y_1 Z_1+Y_3 Z_3)^2 (-4+\mu_2)(-2+\mu_2)\mu_2$
	6	$Y_1^2 Y_2 Y_3^2 (Y_1 Z_1+Y_3 Z_3)$ $+\frac{1}{2}\hat{\delta} Y_1^2 Y_3 Z_1 (Y_3 Z_3 (-4+\mu_2)+Y_1 Z_1 (-2+\mu_2))$ $+\frac{1}{4}\hat{\delta}^2 Z_2 (Y_1 Z_1+Y_3 Z_3)^2 (-4+\mu_2)(-2+\mu_2)$
	6	$Y_1^2 Y_2^2 Y_3^2 Z_2+\hat{\delta} Y_1^2 Y_2 Y_3 Z_1 Z_2 (-2+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2 Y_3 Z_2 Z_3 (2 Y_1 Z_1+Y_3 Z_3)(-2+\mu_2)\mu_2$
	4	$Y_1 Y_3 (Y_1 Z_1+Y_3 Z_3)^2+\frac{1}{2}\hat{\delta} Z_2 (Y_1 Z_1+Y_3 Z_3)^2 (-4+\mu_2)$
	4	$Y_1 Y_2 Y_3 Z_2 (Y_1 Z_1+Y_3 Z_3)-\frac{1}{2}\hat{\delta} Y_3 Z_2 Z_3 (Y_3 Z_3 (-2+\mu_2)+Y_1 Z_1 \mu_2)$
	4	$Y_1 Y_2^2 Y_3 Z_2^2+\frac{1}{2}\hat{\delta} Y_2 Z_2^2 (Y_1 Z_1 (-4+\mu_2)-Y_3 Z_3 \mu_2)-\frac{1}{4}\hat{\delta}^2 Z_1 Z_2^2 Z_3 (-4+\mu_2)\mu_2$
$(1\,,\,\mu_2\,,\,0)$	8	$Y_1^3 Y_2^2 Y_3^3+\hat{\delta} Y_1^3 Y_2 Y_3^2 Z_1 (-1+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2 Y_1 Y_3 (-1+\mu_2)\left(2 Y_2^2 Z_2^2 (-5+\mu_2)+Y_3 Z_3 (2 Y_1 Z_1+Y_3 Z_3)(1+\mu_2)\right)$ $-\frac{1}{4}\hat{\delta}^3 Y_2 Z_2^2 (-5+\mu_2)(-1+\mu_2)(Y_1 Z_1 (-3+\mu_2)-Y_3 Z_3 (1+\mu_2))$ $+\frac{1}{8}\hat{\delta}^4 Z_1 Z_2^2 Z_3 (-5+\mu_2)(-3+\mu_2)(-1+\mu_2)(1+\mu_2)$
	6	$Y_1^2 Y_2 Y_3^2 (Y_1 Z_1+Y_3 Z_3)$ $-\frac{1}{2}\hat{\delta} Y_1 Y_3 \left(Y_2^2 Z_2^2 (-5+\mu_2)+Y_3 Z_3 (Y_3 Z_3 (-1+\mu_2)+Y_1 Z_1 (1+\mu_2))\right)$ $-\frac{1}{4}\hat{\delta}^2 Y_2 Z_2^2 (-5+\mu_2)(Y_1 Z_1 (-3+\mu_2)-Y_3 Z_3 (1+\mu_2))$ $+\frac{1}{8}\hat{\delta}^3 Z_1 Z_2^2 Z_3 (-5+\mu_2)(-3+\mu_2)(1+\mu_2)$
	6	$Y_1^2 Y_2^2 Y_3^2 Z_2+\hat{\delta} Y_1 Y_2 Y_3 Z_2 (Y_1 Z_1+Y_2 Z_2)(-1+\mu_2)$ $-\frac{1}{4}\hat{\delta}^2 Z_2 (-1+\mu_2)(Y_3 Z_3 (2 Y_2 Z_2+Y_3 Z_3)(1+\mu_2)$ $+2 Y_1 Z_1 (-Y_2 Z_2 (-3+\mu_2)+Y_3 Z_3 (1+\mu_2)))$ $-\frac{1}{4}\hat{\delta}^3 Z_1 Z_2^2 Z_3 (-3+\mu_2)(-1+\mu_2)(1+\mu_2)$
	4	$Y_1 Y_2 Y_3 Z_2 (Y_1 Z_1+Y_2 Z_2+Y_3 Z_3)$ $-\frac{1}{2}\hat{\delta} Z_2 (Y_3 Z_3 (Y_3 Z_3 (-1+\mu_2)+Y_2 Z_2 (1+\mu_2))$ $+Y_1 Z_1 (-Y_2 Z_2 (-3+\mu_2)+Y_3 Z_3 (1+\mu_2)))$ $-\frac{1}{4}\hat{\delta}^2 Z_1 Z_2^2 Z_3 (-3+\mu_2)(1+\mu_2)$
$(2\,,\,\mu_2\,,\,\mu_3)$	8	$Y_1^3 Y_2^3 Y_3^2+\frac{1}{8}\hat{\delta}^3 Y_2^2 Z_2^2 Z_3 (-4+\mu_2-\mu_3)(-2+\mu_2-\mu_3)(\mu_2-\mu_3)$
	6	$Y_1^2 Y_2^2 Y_3^2 Z_3-\frac{1}{4}\hat{\delta}^2 Y_2^2 Z_2^2 Z_3 (-2+\mu_2-\mu_3)(\mu_2-\mu_3)$
	6	$Y_1^2 Y_2^3 Y_3 Z_2+\frac{1}{4}\hat{\delta}^2 Y_2^2 Z_2^2 Z_3 (-4+\mu_2-\mu_3)(-2+\mu_2-\mu_3)$
	6	$Y_1^3 Y_2^2 Y_3 Z_1-\frac{1}{8}\hat{\delta}^3 Z_1 Z_2 Z_3^2 (-4+\mu_2-\mu_3)(-2+\mu_2-\mu_3)(\mu_2-\mu_3)$
	4	$Y_1 Y_2 Y_3^2 Z_3^2+\frac{1}{2}\hat{\delta} Y_2^2 Z_2^2 Z_3 (\mu_2-\mu_3)$
	4	$Y_1 Y_2^2 Y_3 Z_2 Z_3+\frac{1}{2}\hat{\delta} Y_2^2 Z_2^2 Z_3 (2-\mu_2+\mu_3)$
	4	$Y_1 Y_2^3 Z_2^2+\frac{1}{2}\hat{\delta} Y_2^2 Z_2^2 Z_3 (-4+\mu_2-\mu_3)$
	4	$Y_1^2 Y_2 Y_3 Z_1 Z_3+\frac{1}{4}\hat{\delta}^2 Z_1 Z_2 Z_3^2 (-2+\mu_2-\mu_3)(\mu_2-\mu_3)$

	4	$Y_1^2\, Y_2^2\, Z_1\, Z_2 - \frac{1}{4}\hat\delta^2\, Z_1\, Z_2\, Z_3^2\,(-4+\mu_2-\mu_3)\,(-2+\mu_2-\mu_3)$
	4	$Y_1^3\, Y_2\, Z_1^2 + \frac{1}{2}\hat\delta\, Y_1^2\, Z_1^2\, Z_3\,(-4+\mu_2-\mu_3)$
	2	$-Y_2^2\, Z_2^2\, Z_3 + Y_3^2\, Z_3^3$
	2	$Y_2\, Z_2\, Z_3\,(Y_2\, Z_2 + Y_3\, Z_3)$
	2	$Y_1\, Y_3\, Z_1\, Z_3^2 + \frac{1}{2}\hat\delta\, Z_1\, Z_2\, Z_3^2\,(-\mu_2+\mu_3)$
	2	$Y_1\, Y_2\, Z_1\, Z_2\, Z_3 + \frac{1}{2}\hat\delta\, Z_1\, Z_2\, Z_3^2\,(-2+\mu_2-\mu_3)$
$(1\,,\,\mu_2\,,\,\mu_3)$	8	$Y_1^3\, Y_2^3\, Y_3^2 - \frac{3}{4}\hat\delta^2\, Y_1\, Y_2^3\, Z_2^2\,(-3+\mu_2-\mu_3)\,(-1+\mu_2-\mu_3)$ $-\frac{1}{4}\hat\delta^3\, Y_2\, Z_2\, Z_3\,(2\,Y_2\, Z_2 + Y_3\, Z_3)\,(-5+\mu_2-\mu_3)\,(-3+\mu_2-\mu_3)\,(-1+\mu_2-\mu_3)$
	6	$Y_1^2\, Y_2^2\, Y_3^2\, Z_3 + \hat\delta\, Y_1\, Y_2^3\, Z_2^2\,(-1+\mu_2-\mu_3)$ $+\frac{1}{4}\hat\delta^2\, Y_2\, Z_2\, Z_3\,(Y_2\, Z_2\,(-13+3\mu_2-3\mu_3) + 2\,Y_3\, Z_3\,(-3+\mu_2-\mu_3))\,(-1+\mu_2-\mu_3)$
	6	$Y_1^2\, Y_2^3\, Y_3\, Z_2 + \hat\delta\, Y_1\, Y_2^3\, Z_2^2\,(3-\mu_2+\mu_3)$ $-\frac{1}{4}\hat\delta^2\, Y_2\, Z_2\, Z_3\,(2\,Y_2\, Z_2 + Y_3\, Z_3)\,(-5+\mu_2-\mu_3)\,(-3+\mu_2-\mu_3)$
	6	$Y_1^3\, Y_2^2\, Y_3\, Z_1 + \frac{1}{4}\hat\delta^2\, Y_1\, Z_1\, Z_3\,(-3+\mu_2-\mu_3)\,(2\,Y_2\, Z_2\,(1+\mu_2-\mu_3)$ $+Y_3\, Z_3\,(5-\mu_2+\mu_3)) + \frac{1}{4}\hat\delta^3\, Z_1\, Z_2\, Z_3^2\,(1+\mu_2-\mu_3)\,(3-\mu_2+\mu_3)^{\,2}$
	4	$Y_1\, Y_2\,\left(-Y_2^2\, Z_2^2 + Y_3^2\, Z_3^2\right) + \hat\delta\, Y_2\, Z_2\, Z_3\,(Y_3\, Z_3\,(1-\mu_2+\mu_3) + Y_2\, Z_2\,(3-\mu_2+\mu_3))$
$(\mu_1\,,\,\mu_2\,,\,1)$	8	$Y_1^2\, Y_2^3\, Y_3^3 - \frac{1}{4}\hat\delta^2\, Y_2^3\, Y_3\, Z_2^2\,\left(-1+(\mu_1-\mu_2)^{\,2}\right)$
	6	$Y_1\, Y_2^2\, Y_3^3\, Z_3 + \frac{1}{2}\hat\delta\, Y_2^3\, Y_3\, Z_2^2\,(1+\mu_1-\mu_2)$
	6	$Y_1\, Y_2^3\, Y_3^2\, Z_2 + \frac{1}{2}\hat\delta\, Y_2^3\, Y_3\, Z_2^2\,(1-\mu_1+\mu_2)$
	6	$Y_1^2\, Y_2^2\, Y_3^2\, Z_1 - \frac{1}{4}\hat\delta^2\, Y_2^2\, Z_1\, Z_2^2\,\left(-1+(\mu_1-\mu_2)^{\,2}\right)$
	4	$-Y_2^3\, Y_3\, Z_2^2 + Y_2\, Y_3^3\, Z_3^2$
	4	$Y_2^2\, Y_3\, Z_2\,(Y_2\, Z_2 + Y_3\, Z_3)$
	4	$Y_1\, Y_2\, Y_3^2\, Z_1\, Z_3 + \frac{1}{2}\hat\delta\, Y_2^2\, Z_1\, Z_2^2\,(1+\mu_1-\mu_2)$
	4	$Y_1\, Y_2^2\, Y_3\, Z_1\, Z_2 + \frac{1}{2}\hat\delta\, Y_2^2\, Z_1\, Z_2^2\,(1-\mu_1+\mu_2)$
	4	$Y_1^2\, Y_2\, Y_3\, Z_1^2 + \frac{1}{4}\hat\delta^2\, Z_1^2\, Z_2\, Z_3\,\left(-1+(\mu_1-\mu_2)^{\,2}\right)$
	2	$Z_1\,\left(-Y_2^2\, Z_2^2 + Y_3^2\, Z_3^2\right)$
	2	$Y_2\, Z_1\, Z_2\,(Y_2\, Z_2 + Y_3\, Z_3)$
	2	$Y_1\, Y_3\, Z_1^2\, Z_3 - \frac{1}{2}\hat\delta\, Z_1^2\, Z_2\, Z_3\,(1+\mu_1-\mu_2)$
	2	$Y_1\, Y_2\, Z_1^2\, Z_2 + \frac{1}{2}\hat\delta\, Z_1^2\, Z_2\, Z_3\,(-1+\mu_1-\mu_2)$

C Highest-derivative partially-massless interactions

In this appendix we prove that the function (3.42) generates consistent highest-derivative interactions involving partially-massless fields provided the condition (3.43) holds. Our starting point is the Stückelberg version of (3.42) given in terms of the shifted variables $\tilde{\boldsymbol{Y}}_{\boldsymbol{i}}$ of eq. (4.26). Consistency of the partially-massless interactions is tantamount to the cancellation of the residues of the partially-massless poles. To this end, we need in principle to integrate by parts all the total-derivative terms contained in the shifted variables. However, one can simplify the computations considering the following ansatz:

$$\mathcal{K}(\tilde{\boldsymbol{Y}}_{\boldsymbol{1}}, \tilde{\boldsymbol{Y}}_{\boldsymbol{2}}, \tilde{\boldsymbol{Y}}_{\boldsymbol{3}}) = \mathcal{K}(\tilde{Y}_1 - \hat\delta\,\partial_{w_1}\partial_{\eta_1}\,,\,\tilde{Y}_2 - \hat\delta\,\partial_{w_2}\partial_{\eta_2}\,,\,\tilde{Y}_3 - \hat\delta\,\partial_{w_3}\partial_{\eta_3})\,f(\eta_1,\eta_2,\eta_3)\,, \qquad \text{(C.1)}$$

where the results of the integrations by parts are encoded in the function f. At this point, what is left is to impose the gauge invariance of the ansatz. Taking the α_i's in the $\tilde{Y}_i$'s to satisfy (3.25), one ends up with the following differential equation for the function f:

$$\begin{aligned}\Big\{\big[1-2(2\alpha_1+1)\eta_1+4\alpha_1(\alpha_1+1)\eta_1^2\big]\partial_{\eta_1}+\mu_1\big[2\alpha_1+1-4\alpha_1(\alpha_1+1)\eta_1\big]&\\ -2\big[\eta_2-2(\alpha_2+1)\eta_2^2\big]\partial_{\eta_2}+\mu_2\big[1-4(\alpha_2+1)\eta_2\big]&\\ +2\big[\eta_3-2\alpha_3\eta_3^2\big]\partial_{\eta_3}-\mu_3\big[1-4\alpha_3\eta_3\big]\Big\}f(\eta_1,\eta_2,\eta_3)&=0\,, \qquad \text{(C.2)}\end{aligned}$$

and cyclic permutations thereof. The solution of the latter differential equations is

$$\begin{aligned} f(\eta_1,\eta_2,\eta_3) \;=\; & \left[1-2\,(\alpha_1+1)\,\eta_1-2\,\alpha_2\,\eta_2\right]^{\frac{1}{2}(\mu_1+\mu_2-\mu_3)} \\ & \times\left[1-2\,(\alpha_2+1)\,\eta_2-2\,\alpha_3\,\eta_3\right]^{\frac{1}{2}(\mu_2+\mu_3-\mu_1)} \\ & \times\left[1-2\,(\alpha_3+1)\,\eta_3-2\,\alpha_1\,\eta_1\right]^{\frac{1}{2}(\mu_3+\mu_1-\mu_2)}\,, \end{aligned} \tag{C.3}$$

whose Taylor coefficients at the order $\eta_i^{\,r_i+1}$ correspond to the residues of the poles $\mu_i=r_i$ associated with $W_i^{\,r_i+1}$. Concentrating on the gauge consistency with respect to the i-th field, one can set $\eta_{i\pm1}=0$ so that the function f becomes the generating function of the Jacobi polynomials. One can then extract the residues as

$$\left(\partial_{\eta_i}^{r_i+1}f\right)(0,0,0)=(-2)^{r_i+1}\binom{\frac{1}{2}(r_i+\mu_{i+1}-\mu_{i-1})}{r_i+1}\,, \tag{C.4}$$

where the homogeneity of the i-th field is a positive integer r_i. The highest-derivative interactions (3.42) become consistent whenever (C.4) vanishes. The latter requirement is equivalent to (3.43).

D Massless limit in flat space

This appendix includes further details about the massless limit in flat space considered in section 4.3. It is important to notice that, in the generic analysis which led to eqs. (4.35, 4.36), g appears only through $\hat{G}$. However, in general, it can also appear whenever the leading terms cancel identically. More precisely, if the first n leading terms cancel then the $(n+1)$-th term becomes dominant and contains n-th powers of g. Therefore, for the sake of completeness, we consider all the cases in which the generic dominant terms cancel among each other. These situations can be systematically analyzed by focusing on the particular combinations of variables giving rise to the desired cancellations.

[i.] In the 2 massless and 1 massive case, the following combination

$$\mu^{-2}\left(\hat{H}_1\,\hat{H}_2-\hat{Y}_3^{\,2}\,\hat{H}_3\right)=-\frac{1}{2}\,\nu_3^2\,\left(g\,y_3+y_1\,y_2\,\partial_{v_3}^2\,\right)+\mathcal{O}(\mu)\,, \tag{D.1}$$

gives rise to the cancellation of the terms proportional to $\mu^{-2}\,y^4$.

[ii.] In the 1 massless and 2 equal massive case, there is no combination leading to the cancellation and g can show up only through $\hat{G}$ (4.36).

[iii.] In the 1 massless and 2 different massive case, the following combination

$$\begin{aligned} & \mu^{-2}\left[\hat{Y}_2\,\hat{Y}_3\,\hat{H}_2-\hat{Y}_3^{\,2}\,\hat{H}_3+\frac{1}{2}\,(M_2^2-M_3^2)\,\hat{Z}_1\,\hat{H}_2\right] \\ & \quad=\frac{1}{2}\,(\nu_2^2-\nu_3^2)\left[g\,y_3+\frac{\nu_2}{\nu_3}\,y_1\,y_3\,\partial_{v_3}\,\partial_{v_2}-\frac{1}{2}\,\frac{\nu_2^2-\nu_3^2}{\nu_3^2}\,y_1\,y_2\,\partial_{v_3}^2\right]+\mathcal{O}(\mu)\,, \end{aligned} \tag{D.2}$$

or, equivalently

$$\mu^{-2}\left[\hat{Y}_2\,\hat{Y}_3\,\hat{H}_3-\hat{Y}_2{}^2\,\hat{H}_2+\frac{1}{2}\left(M_3^2-M_2^2\right)\hat{Z}_1\,\hat{H}_3\right]$$
$$=\frac{1}{2}\left(\nu_3^2-\nu_2^2\right)\left[g\,y_2+\frac{\nu_3}{\nu_2}\,y_1\,y_2\,\partial_{v_3}\,\partial_{v_2}-\frac{1}{2}\,\frac{\nu_3^2-\nu_2^2}{\nu_2^2}\,y_1\,y_3\,\partial_{v_2}^2\right]+\mathcal{O}(\mu)\,, \qquad \text{(D.3)}$$

allow the cancellation of the dominant term proportional to $\mu^{-2}\,y^4$.

[iv.] In the 3 massive case, the following combination

$$\hat{Y}_1\,\hat{Z}_1+\hat{Y}_2\,\hat{Z}_2+\hat{Y}_3\,\hat{Z}_3$$
$$=g+\frac{\nu_2^2+\nu_3^2-\nu_1^2}{2\,\nu_2\,\nu_3}\,y_1\,\partial_{v_2}\,\partial_{v_3}+\frac{\nu_3^2+\nu_1^2-\nu_2^2}{2\,\nu_3\,\nu_1}\,y_2\,\partial_{v_3}\,\partial_{v_1}+\frac{\nu_1^2+\nu_2^2-\nu_3^2}{2\,\nu_1\,\nu_2}\,y_3\,\partial_{v_1}\,\partial_{v_2}+\mathcal{O}(\mu)\,, \qquad \text{(D.4)}$$

does not contain the dominant term proportional to $\mu^{-1}\,y^2\,\partial_v$.

Notice that all resulting massless vertices that involve g are decorated with the contributions of the Stückelberg fields.

E Cubic interactions of open strings in the first Regge trajectory

In this appendix we show that the string interactions encoded by (5.1) nicely fit in with the classification we have provided in section 3.1. For simplicity, let us drop Chan-Paton factors as well as the constant $i\sqrt{G_N}\,g_0/\alpha'$, and focus on the first term in (5.1). The latter can be expanded as

$$\mathcal{K}=\sum_{\sigma_i,\tau_i}\frac{1}{\sigma_1!\,\sigma_2!\,\sigma_3!\,\tau_1!\,\tau_2!\,\tau_3!}\,\left(-2\alpha'\right)^{\frac{\sigma_1+\sigma_2+\sigma_3}{2}}\,y_1^{\sigma_1}\,y_2^{\sigma_2}\,y_3^{\sigma_3}\,z_1^{\tau_1}\,z_2^{\tau_2}\,z_3^{\tau_3}\,, \qquad \text{(E.1)}$$

where the spins of the fields are

$$s_1=\sigma_1+\tau_2+\tau_3\,,\qquad s_2=\sigma_2+\tau_1+\tau_3\,,\qquad s_3=\sigma_3+\tau_1+\tau_2\,. \qquad \text{(E.2)}$$

Concentrating on particular choices of (s_1,s_2,s_3), we can extract consistent couplings for each of the five different categories. In particular, defining the d-dimensional counterpart h_i of H_i as

$$h_i=y_{i-1}\,y_{i+1}+\frac{1}{2}\,[M_i^2-(M_{i-1}+M_{i+1})^2]\,z_i\,, \qquad \text{(E.3)}$$

one ends up with the following five cases:

[i.] In the 3 massless case with $(s_1,s_2,s_3)=(1,1,1)$, one gets

$$\mathcal{K}=\left(-2\alpha'\right)^{\frac{3}{2}}\left(y_1\,y_2\,y_3-\frac{1}{2\alpha'}\,g\right). \qquad \text{(E.4)}$$

[ii.] In the 2 massless and 1 massive case with $(s_1, s_2, s_3) = (1, 1, s)$, one has

$$\mathcal{K} = -\frac{1}{(s-1)\, s!} \left(-2\alpha'\right)^{\frac{s+2}{2}} \; y_3^{s-2} \left(y_3^2\, h_3 - s\, h_1\, h_2\right) . \tag{E.5}$$

[iii.] In the 1 massless and 2 equal massive case with $(s_1, s_2, s_3) = (1, s, s)$, one finds

$$\mathcal{K} = \sum_{k=0}^{s} \frac{1}{k!\,[(s-k)!]^2} \left(-2\alpha'\right)^{\frac{2s+1}{2}-k} y_2^{s-k-1}\, y_3^{s-k-1}\, z_1^k \left(y_1\, y_2\, y_3 - \frac{(s-k)}{2\alpha'} \left(g - y_1\, z_1\right)\right) . \tag{E.6}$$

[iv.] In the 1 massless and 2 different massive case with $(s_1, s_2, s_3) = (1, s, s')$ with $s < s'$, one gets

$$\mathcal{K} = \sum_{k=0}^{s} \frac{1}{k!(s'-k)!(s-k)!} \left(-2\alpha'\right)^{\frac{s'+s+1}{2}-k} y_2^{s-k-1} y_3^{s'-k-1} z_1^k \left(\frac{s'-k}{s'-s} y_2 h_2 + \frac{s-k}{s-s'} y_3 h_3 \right) . \tag{E.7}$$

[v.] Finally, the 3 massive case trivially fits in with the classification.

References

[1] R. Argurio, G. Barnich, G. Bonelli and M. Grigoriev eds., *Higher-Spin Gauge Theories*, in proceedings of *First Solvay Workshop*, Brussels, 12–14 May 2004, International Solvay Institutes for Physics and Chemistry (2006) [http://www.ulb.ac.be/sciences/ptm/pmif/Solvay1proc.pdf].

[2] M. Bianchi and V. Didenko, *Massive higher spin multiplets and holography*, hep-th/0502220 [INSPIRE].

[3] D. Francia and C. Hull, *Higher-spin gauge fields and duality*, hep-th/0501236 [INSPIRE].

[4] N. Bouatta, G. Compere and A. Sagnotti, *An Introduction to free higher-spin fields*, hep-th/0409068 [INSPIRE].

[5] X. Bekaert, S. Cnockaert, C. Iazeolla and M. Vasiliev, *Nonlinear higher spin theories in various dimensions*, hep-th/0503128 [INSPIRE].

[6] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong Sp(2,R) condition*, hep-th/0501156 [INSPIRE].

[7] D. Sorokin, *Introduction to the classical theory of higher spins*, *AIP Conf. Proc.* **767** (2005) 172 [hep-th/0405069] [INSPIRE].

[8] D. Francia and A. Sagnotti, *Higher-spin geometry and string theory*, *J. Phys. Conf. Ser.* **33** (2006) 57 [hep-th/0601199] [INSPIRE].

[9] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, arXiv:1007.0435 [INSPIRE].

[10] A. Sagnotti, *Notes on Strings and Higher Spins*, arXiv:1112.4285 [INSPIRE].

[11] C. Aragone and S. Deser, *Consistency Problems of Hypergravity*, *Phys. Lett.* **B 86** (1979) 161 [INSPIRE].

[12] E. Fradkin and M.A. Vasiliev, *Cubic Interaction in Extended Theories of Massless Higher Spin Fields*, *Nucl. Phys.* **B 291** (1987) 141 [INSPIRE].

[13] E. Fradkin and M.A. Vasiliev, *On the Gravitational Interaction of Massless Higher Spin Fields*, *Phys. Lett.* **B 189** (1987) 89 [INSPIRE].

[14] Y. Zinoviev, *On massive spin 2 electromagnetic interactions*, *Nucl. Phys.* **B 821** (2009) 431 [arXiv:0901.3462] [INSPIRE].

[15] Y. Zinoviev, *On spin 3 interacting with gravity*, *Class. Quant. Grav.* **26** (2009) 035022 [arXiv:0805.2226] [INSPIRE].

[16] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Annals Phys.* **190** (1989) 59 [INSPIRE].

[17] M. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049] [INSPIRE].

[18] M. Fierz, *Force-free particles with any spin*, *Helv. Phys. Acta* **12** (1939) 3 [INSPIRE].

[19] L. Singh and C. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, *Phys. Rev.* **D 9** (1974) 898 [INSPIRE].

[20] L. Singh and C. Hagen, *Lagrangian formulation for arbitrary spin. 2. The fermion case*, *Phys. Rev.* **D 9** (1974) 910 [INSPIRE].

[21] Y. Zinoviev, *On massive high spin particles in AdS*, hep-th/0108192 [INSPIRE].

[22] T. Biswas and W. Siegel, *Radial dimensional reduction: Anti-de Sitter theories from flat*, *JHEP* **07** (2002) 005 [hep-th/0203115] [INSPIRE].

[23] R. Metsaev, *Massive totally symmetric fields in AdS(d)*, *Phys. Lett.* **B 590** (2004) 95 [hep-th/0312297] [INSPIRE].

[24] K. Hallowell and A. Waldron, *Constant curvature algebras and higher spin action generating functions*, *Nucl. Phys.* **B 724** (2005) 453 [hep-th/0505255] [INSPIRE].

[25] I. Buchbinder and V. Krykhtin, *Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimensions*, *Nucl. Phys.* **B 727** (2005) 537 [hep-th/0505092] [INSPIRE].

[26] A. Fotopoulos, K.L. Panigrahi and M. Tsulaia, *Lagrangian formulation of higher spin theories on AdS space*, *Phys. Rev.* **D 74** (2006) 085029 [hep-th/0607248] [INSPIRE].

[27] I. Buchbinder, V. Krykhtin, L. Ryskina and H. Takata, *Gauge invariant Lagrangian construction for massive higher spin fermionic fields*, *Phys. Lett.* **B 641** (2006) 386 [hep-th/0603212] [INSPIRE].

[28] I. Buchbinder, V. Krykhtin and P. Lavrov, *Gauge invariant Lagrangian formulation of higher spin massive bosonic field theory in AdS space*, *Nucl. Phys.* **B 762** (2007) 344 [hep-th/0608005] [INSPIRE].

[29] I. Buchbinder, V. Krykhtin and H. Takata, *Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields*, *Phys. Lett.* **B 656** (2007) 253 [arXiv:0707.2181] [INSPIRE].

[30] P.A. Horvathy, M.S. Plyushchay and M. Valenzuela, *Bosonized supersymmetry from the Majorana-Dirac-Staunton theory and massive higher-spin fields*, *Phys. Rev.* **D 77** (2008) 025017 [arXiv:0710.1394] [INSPIRE].

[31] D. Francia, *Geometric Lagrangians for massive higher-spin fields*, *Nucl. Phys.* **B 796** (2008) 77 [arXiv:0710.5378] [INSPIRE].

[32] D. Francia, *Geometric massive higher spins and current exchanges*, *Fortsch. Phys.* **56** (2008) 800 [arXiv:0804.2857] [INSPIRE].

[33] I. Buchbinder and A. Galajinsky, *Quartet unconstrained formulation for massive higher spin fields*, *JHEP* **11** (2008) 081 [arXiv:0810.2852] [INSPIRE].

[34] Y. Zinoviev, *Frame-like gauge invariant formulation for massive high spin particles*, *Nucl. Phys.* **B 808** (2009) 185 [arXiv:0808.1778] [INSPIRE].

[35] R. Metsaev, *CFT adapted gauge invariant formulation of massive arbitrary spin fields in AdS*, *Phys. Lett.* **B 682** (2010) 455 [arXiv:0907.2207] [INSPIRE].

[36] X. Bekaert, M. Rausch de Traubenberg and M. Valenzuela, *An infinite supermultiplet of massive higher-spin fields*, *JHEP* **05** (2009) 118 [arXiv:0904.2533] [INSPIRE].

[37] D. Ponomarev and M. Vasiliev, *Frame-Like Action and Unfolded Formulation for Massive Higher-Spin Fields*, *Nucl. Phys.* **B 839** (2010) 466 [arXiv:1001.0062] [INSPIRE].

[38] M. Grigoriev and A. Waldron, *Massive Higher Spins from BRST and Tractors*, *Nucl. Phys.* **B 853** (2011) 291 [arXiv:1104.4994] [INSPIRE].

[39] R. Metsaev, *Anomalous conformal currents, shadow fields and massive AdS fields*, arXiv:1110.3749 [INSPIRE].

[40] R. Metsaev, *Extended Hamiltonian Action for Arbitrary Spin Fields in Flat And AdS Spaces*, arXiv:1112.0976 [INSPIRE].

[41] S. Deser and R.I. Nepomechie, *Gauge invariance versus masslessness in de Sitter space*, *Annals Phys.* **154** (1984) 396 [INSPIRE].

[42] S. Deser and R.I. Nepomechie, *Anomalous propagation of gauge fields in conformally flat spaces*, *Phys. Lett.* **B 132** (1983) 321 [INSPIRE].

[43] A. Higuchi, *Forbidden mass range for spin-2 field theory in de Sitter space-time*, *Nucl. Phys.* **B 282** (1987) 397 [INSPIRE].

[44] A. Higuchi, *Symmetric tensor spherical harmonics on the N sphere and their application to the de Sitter group* $SO(N,1)$, *J. Math. Phys.* **28** (1987) 1553 [*Erratum ibid.* **43** (2002) 6385] [INSPIRE].

[45] A. Higuchi, *Massive symmetric tensor field in space-times with a positive cosmological constant*, *Nucl. Phys.* **B 325** (1989) 745 [INSPIRE].

[46] S. Deser and A. Waldron, *Gauge invariances and phases of massive higher spins in (A)dS*, *Phys. Rev. Lett.* **87** (2001) 031601 [hep-th/0102166] [INSPIRE].

[47] S. Deser and A. Waldron, *Stability of massive cosmological gravitons*, *Phys. Lett.* **B 508** (2001) 347 [hep-th/0103255] [INSPIRE].

[48] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, *Nucl. Phys.* **B 607** (2001) 577 [hep-th/0103198] [INSPIRE].

[49] S. Deser and A. Waldron, *(Dis)continuities of massless limits in spin 3/2 mediated interactions and cosmological supergravity*, *Phys. Lett.* **B 501** (2001) 134 [hep-th/0012014] [INSPIRE].

JHEP07(2012)041

[50] S. Deser and A. Waldron, *Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations*, Phys. Lett. **B 513** (2001) 137 [hep-th/0105181] [INSPIRE].

[51] S. Deser and A. Waldron, *Arbitrary spin representations in de Sitter from dS/CFT with applications to dS supergravity*, Nucl. Phys. **B 662** (2003) 379 [hep-th/0301068] [INSPIRE].

[52] E. Skvortsov and M. Vasiliev, *Geometric formulation for partially massless fields*, Nucl. Phys. **B 756** (2006) 117 [hep-th/0601095] [INSPIRE].

[53] D. Francia, J. Mourad and A. Sagnotti, *(A)dS exchanges and partially-massless higher spins*, Nucl. Phys. **B 804** (2008) 383 [arXiv:0803.3832] [INSPIRE].

[54] A. Gover, A. Shaukat and A. Waldron, *Weyl Invariance and the Origins of Mass*, Phys. Lett. **B 675** (2009) 93 [arXiv:0812.3364] [INSPIRE].

[55] K.B. Alkalaev and M. Grigoriev, *Unified BRST description of AdS gauge fields*, Nucl. Phys. **B 835** (2010) 197 [arXiv:0910.2690] [INSPIRE].

[56] K. Alkalaev and M. Grigoriev, *Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type*, Nucl. Phys. **B 853** (2011) 663 [arXiv:1105.6111] [INSPIRE].

[57] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, Proc. Roy. Soc. Lond. **A 173** (1939) 211 [INSPIRE].

[58] G. Velo and D. Zwanziger, *Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential*, Phys. Rev. **186** (1969) 1337 [INSPIRE].

[59] G. Velo and D. Zwanziger, *Noncausality and other defects of interaction lagrangians for particles with spin one and higher*, Phys. Rev. **188** (1969) 2218 [INSPIRE].

[60] G. Velo, *Anomalous behaviour of a massive spin two charged particle in an external electromagnetic field*, Nucl. Phys. **B 43** (1972) 389 [INSPIRE].

[61] M. Porrati, *Massive spin* 5/2 *fields coupled to gravity: Tree level unitarity versus the equivalence principle*, Phys. Lett. **B 304** (1993) 77 [gr-qc/9301012] [INSPIRE].

[62] R. Doria, J. Helayel-Neto and S. Mokhtari, *The Minimal coupling of charged spin* 3/2 *fields to an extended Abelian gauge model*, Commun. Theor. Phys. **21** (1994) 121 [INSPIRE].

[63] A. Cucchieri, M. Porrati and S. Deser, *Tree level unitarity constraints on the gravitational couplings of higher spin massive fields*, Phys. Rev. **D 51** (1995) 4543 [hep-th/9408073] [INSPIRE].

[64] I. Giannakis, J.T. Liu and M. Porrati, *Massive higher spin states in string theory and the principle of equivalence*, Phys. Rev. **D 59** (1999) 104013 [hep-th/9809142] [INSPIRE].

[65] S. Klishevich, *Massive fields of arbitrary half integer spin in constant electromagnetic field*, Int. J. Mod. Phys. **A 15** (2000) 609 [hep-th/9811030] [INSPIRE].

[66] S. Deser, V. Pascalutsa and A. Waldron, *Massive spin* 3/2 *electrodynamics*, Phys. Rev. **D 62** (2000) 105031 [hep-th/0003011] [INSPIRE].

[67] S. Deser and A. Waldron, *Inconsistencies of massive charged gravitating higher spins*, Nucl. Phys. **B 631** (2002) 369 [hep-th/0112182] [INSPIRE].

[68] M. Porrati and R. Rahman, *Intrinsic Cutoff and Acausality for Massive Spin 2 Fields Coupled to Electromagnetism*, Nucl. Phys. **B 801** (2008) 174 [arXiv:0801.2581] [INSPIRE].

JHEP07(2012)041

[69] M. Porrati, *Universal Limits on Massless High-Spin Particles*, *Phys. Rev.* **D 78** (2008) 065016 [arXiv:0804.4672] [INSPIRE].

[70] M. Porrati and R. Rahman, *A Model Independent Ultraviolet Cutoff for Theories with Charged Massive Higher Spin Fields*, *Nucl. Phys.* **B 814** (2009) 370 [arXiv:0812.4254] [INSPIRE].

[71] M. Porrati and R. Rahman, *Causal Propagation of a Charged Spin* 3/2 *Field in an External Electromagnetic Background*, *Phys. Rev.* **D 80** (2009) 025009 [arXiv:0906.1432] [INSPIRE].

[72] Y. Zinoviev, *On electromagnetic interactions for massive mixed symmetry field*, *JHEP* **03** (2011) 082 [arXiv:1012.2706] [INSPIRE].

[73] Y. Zinoviev, *Gravitational cubic interactions for a massive mixed symmetry gauge field*, *Class. Quant. Grav.* **29** (2012) 015013 [arXiv:1107.3222] [INSPIRE].

[74] R. Rahman, *Helicity-1/2 Mode as a Probe of Interactions of Massive Rarita-Schwinger Field*, arXiv:1111.3366 [INSPIRE].

[75] S. Deser and A. Waldron, *Partially Massless Spin 2 Electrodynamics*, *Phys. Rev.* **D 74** (2006) 084036 [hep-th/0609113] [INSPIRE].

[76] P.C. Argyres and C.R. Nappi, *Massive spin-2 bosonic string states in an electromagnetic background*, *Phys. Lett.* **B 224** (1989) 89 [INSPIRE].

[77] M. Porrati and R. Rahman, *Notes on a Cure for Higher-Spin Acausality*, *Phys. Rev.* **D 84** (2011) 045013 [arXiv:1103.6027] [INSPIRE].

[78] M. Porrati, R. Rahman and A. Sagnotti, *String Theory and The Velo-Zwanziger Problem*, *Nucl. Phys.* **B 846** (2011) 250 [arXiv:1011.6411] [INSPIRE].

[79] M. Taronna, *Higher Spins and String Interactions*, arXiv:1005.3061 [INSPIRE].

[80] A. Sagnotti and M. Taronna, *String Lessons for Higher-Spin Interactions*, *Nucl. Phys.* **B 842** (2011) 299 [arXiv:1006.5242] [INSPIRE].

[81] M. Bianchi, L. Lopez and R. Richter, *On stable higher spin states in Heterotic String Theories*, *JHEP* **03** (2011) 051 [arXiv:1010.1177] [INSPIRE].

[82] D. Polyakov, *Higher Spins and Open Strings: Quartic Interactions*, *Phys. Rev.* **D 83** (2011) 046005 [arXiv:1011.0353] [INSPIRE].

[83] O. Schlotterer, *Higher Spin Scattering in Superstring Theory*, *Nucl. Phys.* **B 849** (2011) 433 [arXiv:1011.1235] [INSPIRE].

[84] M. Bianchi and P. Teresi, *Scattering higher spins off D-branes*, *JHEP* **01** (2012) 161 [arXiv:1108.1071] [INSPIRE].

[85] S. Lee and D. Polyakov, *String Amplitudes and Frame-like Formalism for Higher Spins*, *Phys. Rev.* **D 85** (2012) 106014 [arXiv:1203.0909] [INSPIRE].

[86] Y. Zinoviev, *On massive spin 2 interactions*, *Nucl. Phys.* **B 770** (2007) 83 [hep-th/0609170] [INSPIRE].

[87] Y. Zinoviev, *On spin 2 electromagnetic interactions*, *Mod. Phys. Lett.* **A 24** (2009) 17 [arXiv:0806.4030] [INSPIRE].

[88] R. Metsaev, *Cubic interaction vertices of massive and massless higher spin fields*, *Nucl. Phys.* **B 759** (2006) 147 [hep-th/0512342] [INSPIRE].

[89] R. Metsaev, *Cubic interaction vertices for fermionic and bosonic arbitrary spin fields*, Nucl. Phys. **B 859** (2012) 13 [arXiv:0712.3526] [INSPIRE].

[90] R. Manvelyan, K. Mkrtchyan and W. Rühl, *General trilinear interaction for arbitrary even higher spin gauge fields*, Nucl. Phys. **B 836** (2010) 204 [arXiv:1003.2877] [INSPIRE].

[91] M. Taronna, *Higher-Spin Interactions: four-point functions and beyond*, JHEP **04** (2012) 029 [arXiv:1107.5843] [INSPIRE].

[92] E. Joung and M. Taronna, *Cubic interactions of massless higher spins in (A)dS: metric-like approach*, Nucl. Phys. **B 861** (2012) 145 [arXiv:1110.5918] [INSPIRE].

[93] M. Vasiliev, *Cubic Vertices for Symmetric Higher-Spin Gauge Fields in* $(A)dS_d$, Nucl. Phys. **B 862** (2012) 341 [arXiv:1108.5921] [INSPIRE].

[94] C. Aragone, S. Deser and Z. Yang, *Massive higher spin from dimensional reduction of gauge fields*, Annals Phys. **179** (1987) 76 [INSPIRE].

[95] S. Rindani, D. Sahdev and M. Sivakumar, *Dimensional reduction of symmetric higher spin actions. 1. Bosons*, Mod. Phys. Lett. **A 4** (1989) 265 [INSPIRE].

[96] S. Rindani, M. Sivakumar and D. Sahdev, *Dimensional reduction of symmetric higher spin actions. 2: Fermions*, Mod. Phys. Lett. **A 4** (1989) 275 [INSPIRE].

[97] M. Bianchi, P.J. Heslop and F. Riccioni, *More on La Grande Bouffe*, JHEP **08** (2005) 088 [hep-th/0504156] [INSPIRE].

[98] C. Fronsdal, *Massless Fields with Integer Spin*, Phys. Rev. **D 18** (1978) 3624 [INSPIRE].

[99] J. Fang and C. Fronsdal, *Massless Fields with Half Integral Spin*, Phys. Rev. **D 18** (1978) 3630 [INSPIRE].

[100] R. Metsaev, *CFT adapted gauge invariant formulation of arbitrary spin fields in AdS and modified de Donder gauge*, Phys. Lett. **B 671** (2009) 128 [arXiv:0808.3945] [INSPIRE].

[101] D. Francia and A. Sagnotti, *Minimal local Lagrangians for higher-spin geometry*, Phys. Lett. **B 624** (2005) 93 [hep-th/0507144] [INSPIRE].

[102] D. Francia, J. Mourad and A. Sagnotti, *Current Exchanges and Unconstrained Higher Spins*, Nucl. Phys. **B 773** (2007) 203 [hep-th/0701163] [INSPIRE].

[103] A. Gover, A. Shaukat and A. Waldron, *Tractors, Mass and Weyl Invariance*, Nucl. Phys. **B 812** (2009) 424 [arXiv:0810.2867] [INSPIRE].

[104] C. Fronsdal, *Singletons and Massless, Integral Spin Fields on de Sitter Space (Elementary Particles in a Curved Space. 7.*, Phys. Rev. **D 20** (1979) 848 [INSPIRE].

[105] R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, Phys. Lett. **B 354** (1995) 78 [INSPIRE].

[106] R. Metsaev, *Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space*, Lect. Notes Phys. **524** (1997) 331 [hep-th/9810231] [INSPIRE].

[107] I. Bars, *Gauge Symmetry in Phase Space, Consequences for Physics and Spacetime*, Int. J. Mod. Phys. **A 25** (2010) 5235 [arXiv:1004.0688] [INSPIRE].

[108] H. van Dam and M. Veltman, *Massive and massless Yang-Mills and gravitational fields*, Nucl. Phys. **B 22** (1970) 397 [INSPIRE].

[109] V. Zakharov, *Linearized gravitation theory and the graviton mass*, JETP Lett. **12** (1970) 312 [INSPIRE].

JHEP07(2012)041

[110] M. Porrati, *No van Dam-Veltman-Zakharov discontinuity in AdS space*, *Phys. Lett.* **B 498** (2001) 92 [hep-th/0011152] [INSPIRE].

[111] I.I. Kogan, S. Mouslopoulos and A. Papazoglou, *The $m \to 0$ limit for massive graviton in dS_4 and AdS_4: How to circumvent the van Dam-Veltman-Zakharov discontinuity*, *Phys. Lett.* **B 503** (2001) 173 [hep-th/0011138] [INSPIRE].

[112] M. Porrati, *Fully covariant van Dam-Veltman-Zakharov discontinuity and absence thereof*, *Phys. Lett.* **B 534** (2002) 209 [hep-th/0203014] [INSPIRE].

[113] I. Buchbinder, A. Galajinsky and V. Krykhtin, *Quartet unconstrained formulation for massless higher spin fields*, *Nucl. Phys.* **B 779** (2007) 155 [hep-th/0702161] [INSPIRE].

[114] W. Siegel, *All free conformal representations in all dimensions*, *Int. J. Mod. Phys.* **A 4** (1989) 2015 [INSPIRE].

[115] J. Dixmier, *Représentations intégrables du groupe de de Sitter*, *B. Soc. Math. Fr.* **89** (1961) 9 [http://archive.numdam.org/article/BSMF_1961__89__9_0.pdf].

[116] E. Thieleker, *The unitary representations of the generalized Lorentz groups*, *Trans. Amer. Math. Soc.* **199** (1974) 327 [http://www.ams.org/journals/tran/1974-199-00/S0002-9947-1974-0379754-8/S0002-9947-1974-0379754-8.pdf].

[117] E. Joung and J. Mourad, *Boundary action of free AdS higher-spin gauge fields and the holographic correspondence*, arXiv:1112.5620 [INSPIRE].

[118] A.A. Tseytlin, *On limits of superstring in $AdS_5 \times S^5$*, *Theor. Math. Phys.* **133** (2002) 1376 [hep-th/0201112] [INSPIRE].

[119] G. Bonelli, *On the covariant quantization of tensionless bosonic strings in AdS space-time*, *JHEP* **11** (2003) 028 [hep-th/0309222] [INSPIRE].

[120] A. Sagnotti and M. Tsulaia, *On higher spins and the tensionless limit of string theory*, *Nucl. Phys.* **B 682** (2004) 83 [hep-th/0311257] [INSPIRE].

[121] P. Dempster and M. Tsulaia, *On the Structure of Quartic Vertices for Massless Higher Spin Fields on Minkowski Background*, arXiv:1203.5597 [INSPIRE].

JHEP07(2012)041