POINT-FREE ULTRAMETRIC SPACES AND THE CATEGORY OF FUZZY SUBSETS

Giangiacomo Gerla
DMI
University of Salerno
Via S. Allende 84081
Baronissi (SA), Italy
ggerla@unisa.it

Cristina Coppola
DMA
University Federico II,
Via Cintia, 80126 Napoli,
Italy,
cristina.coppola@dma.unina.it

Tiziana Pacelli
DMI
University of Salerno
Via S. Allende 84081
Baronissi (SA), Italy
tpacelli@unisa.it

Abstract. Some attempts to establish a link between point-free geometry and the categorical approach to fuzzy set theory is exposed. In fact, it is possible to find functors between the category of fuzzy sets as defined by Höhle in [4] and a category whose objects are the pointless ultrametric spaces.

Keywords: Point-free geometry, fuzzy sets, similarity, category, ultrametrics.

1. Introduction.

The aim of point-free geometry is to give an axiomatic basis to geometry in which the notion of *point* is not assumed as a primitive. The first example in such a direction was furnished by Whitehead's researches [8,9] where the primitives are the *regions* and the *inclusion* relation. Later, Whitehead proposed the topological notion of *connection* instead of the inclusion [10]. More recently, in [2, 3], Gerla proposed a system of axioms in which *regions, inclusion, distance* and *diameter* are assumed as primitives.

In this note we expose some attempts to establish a link between point-free geometry and the categorical approach to fuzzy sets theory as proposed by Höhle in [6]. More precisely, Section 2 is devoted to give some preliminary notions. In Section 3, starting from the definition of *pointless metric spaces*, we introduce the *pointless ultrametric spaces*. In Section 4, we define the *semi-metric spaces* and the *semi-similarities*, and we verify the relations between these two structures. In particular, we focus on *semi-ultrametric spaces* and the semi-similarity with the *Gödel t-norm*, called *G-semi-similarity*. In Section 5, we give a characterization of *G-semi-similarities* by the notion of *semi-equivalence*. In Section 6, we examine the relations existing between *pointless metric spaces* and *semi-metric spaces*, and, in particular, between *pointless ultrametric* and *semi-ultrametric* spaces. Besides, we verify the connection of *G-semi-similarities* with *pointless ultrametric spaces.* In Section 7, we organize the class of *pointless ultrametric spaces* into a category and we define two functors to relate such a category with Höhle's category. Finally, in Section 8, we exhibit an example of *G-semi-similarity* . Some final remarks are given in Section 9.

2. Preliminaries.

We introduce some basic notions in multi-valued logic.

Definition 2.1. A *continuous triangular norm* (*t-norm*)**,** is a continuous binary operation $\otimes$ on $[0,1]$ such that, for all $x, x_1, x_2, y_1, y_2 \in [0, 1]$

- $\otimes$ is commutative,
- $\otimes$ is associative,
- $\otimes$ is isotone in both arguments, i.e.,
  $$x_1 \leq x_2 \Rightarrow x_1 \otimes x \leq x_2 \otimes x,$$
  $$y_1 \leq y_2 \Rightarrow x \otimes y_1 \leq x \otimes y_2,$$
- $1 \otimes x = x = x \otimes 1$ and $0 \otimes x = 0 = x \otimes 0.$

The most important continuous *t*-norms are *minimum* (or *Gödel t-norm*), *product* and *Lukasiewicz conjunction* defined by setting $a \otimes b = max(0, a+b-1)$.

Definition 2.2. Let $\otimes$ be a continuous *t*–norm. The *residuation* is the operation $\rightarrow_{\otimes}$ defined by

$$a \rightarrow_{\otimes} b = Sup\{x : a \otimes x \leq b\}$$

It is immediate that

$$a\otimes x\leq b \Leftrightarrow x\leq a\rightarrow_{\otimes} b.$$

As an example, if $\otimes$ is the *Gödel t*-norm then the residuation operation is the *Gödel implication* $\rightarrow_G$:

$$a \rightarrow_G b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } b < a. \end{cases} \quad (2.1)$$

Definition 2.3. A continuous *t*–norm $\otimes$ is called *Archimedean* if, for any $x, y \in [0,1]$, $y \neq 0$, an integer n exists such that $x^{(n)}< y$, where $x^{(n)}$ is defined by $x^{(1)} = x$ and $x^{(n+1)} = x^{(n)} \otimes x$.

The Archimedean *t*-norms are important, because they are characterized by some functions.

Definition 2.4. An *additive generator* of a *t*-norm $\otimes$ is a continuous, strictly decreasing function f: $[0,1]\rightarrow[0,\infty]$, such that $f(1)=0$ and

$$x\otimes y = f^{[-1]} (f(x)+f(y)),$$

where $f^{[-1]}$: $[0,\infty]\rightarrow[0,1]$ is defined by

$$f^{[-1]}(y)= \begin{cases} f^{-1}(y) & \text{if } y\in f([0,1]), \\ 0 & \text{otherwise} \end{cases}$$

and it is called *pseudoinverse* of f.

Proposition 2.1. *A function* $\otimes$: $[0,1]^2\rightarrow[0,1]$ *is a continuous Archimedean t-norm iff it has an additive generator.*

The product *t*-norm and Lukasiewicz *t*-norm are examples of Archimedean *t*-norms and their additive generators are $f_p(x) = -lnx$ and $f_L(x)=1- x$, respectively. The *Gödel t-norm* is not Archimedean.

Observe that, if an additive generator exists for a *t*-norm $\otimes$, then

$$x\rightarrow_{\otimes} y = f^{[-1]} (f(y)-f (x)).$$

Now, let S be a nonempty set. We call *fuzzy subset of S* any map $s: S \rightarrow [0,1]$. For any $x\in S$, the value $s(x)$ is interpreted as the membership degree of x to s. Given $\lambda\in [0,1]$, the *closed λ-cut* of s is the subset $C(s,\lambda) = \{x\in S : s(x)\geq\lambda\}$ of S. A *fuzzy relation* in S is a fuzzy subset of $S\times S$, i.e., a map $r: S\times S\rightarrow[0, 1]$. We are interested to the following properties for a fuzzy relation r :

(i) $r(x,x) = 1$ (reflexivity)

(ii) $r(x,y) \otimes r(y,z)\leq r(x,z)$ (transitivity)

(iii) $r(x,y)= r(y,x)=1 \Rightarrow x=y$ (antisymmetry)

(iv) $r(x,y) = r(y,x)$ (symmetry)

for every $x, y, z \in S$.

3. Pointless ultrametric spaces.

In order to give a metric approach to point-free geometry, Gerla in [2] defines the notion of *pointless metric space*, briefly *pm-space*. A *pm-space* is a structure $(R, \leq, \delta, |\,|)$, where

- $(R,\leq)$ is an ordered set,
- δ: $R\times R\rightarrow[0,\infty)$ is an order-reversing map,
- $|\,|$:$R \rightarrow[0, \infty]$ is an order-preserving map

and, for every $x, y, z \in R$:

(a1) $\delta(x, x) = 0$

(a2) $\delta(x, y) = \delta(y, x)$

(a3) $\delta(x, y) \leq \delta(x, z) + \delta(z, y) + |z|$.

The elements in R are called *regions*, the number $\delta(x, y)$ the *distance* between x and y, $|x|$ the *diameter* of x.

In this paper we are interested to a particular class of *pm-spaces* which is related with the notion of ultrametric space.

Recall that in literature a *pseudo-ultrametric space* is defined as a structure (M, d) such that:

- $d(x, x) = 0$,
- $d(x, y) = d(y, x)$,
- $d(x, y) \leq d(x, z)\vee d(z, y)$

where $\vee$ is the maximum. Since

$$\delta(x, z)\vee\delta(z, y) \leq \delta(x, z) + \delta(z, y),$$

any pseudo-ultrametric space is a pseudo-metric space. This definition suggests the following one in the framework of point-free geometry.

Definition 3.1. A *pointless ultrametric space,* briefly *pu-space,* is a *pm*-space $\mathcal{R}=(R, \leq, \delta, |\,|)$ such that

(A3) $\delta(x, y) \leq \delta(x, z)\vee\delta(z, y)\vee| z|$.

Observe that, since

$\delta(x, z)\vee\delta(z, y)\vee |z| \leq \delta(x, z) + \delta(z, y) + |z|$,

(A3) entails (a3). A class of basic examples of *pm*-spaces and *pu-spaces* is obtained by starting from a metric space.

Proposition 3.1. *Let* (M, d) *be a pseudo-metric space and let C be a nonempty class of bounded and nonempty subsets of M. Define* δ *and* $|\,|$ *by setting*

$$\delta(X, Y) = inf\{d(x, y) : x\in X, y\in Y\} \quad (3.1)$$

$$|X| = sup\{d(x, y) : x, y \in X\}. \quad (3.2)$$

Then $(C, \subseteq, \delta, |\,|)$ *is a pm-space. If* (M,d) *is a* *pseudo-ultrametric space, then* $(C, \subseteq, \delta, |\ |)$ *is a pu-space.*

Proof. (a1) and (a2) are immediate. To prove (a3), let X, Y and Z be subsets of M, $x \in X$, $y \in Y$, z and $z' \in Z$; then

$$\delta(X, Y) \leq d(x, y) \leq d(x, z) + d(z, z') + d(z', y) \leq d(x, z) + d(z', y) + |Z|.$$

Consequently,

$$\delta(X, Y) \leq \delta(X, Z) + \delta(Z, Y) + |Z|.$$

Assume that (M, d) is a pseudo-ultrametric space. Then

$$\delta(X, Y) \leq d(x, y) \leq d(x, z) \vee d(z, z') \vee d(z', y) \leq d(x, z) \vee d(z', y) \vee |Z|,$$

and therefore $(C, \subseteq, \delta, |\ |)$ is a *pu*-space.

We call *canonical* the so obtained spaces.

4. Semi-metrics and semi-similarities.

We introduce a new class of structures satisfying symmetry and a triangular inequality, but not reflexivity.

Definition 4.1. A *semi-metric space,* briefly *sm-space*, is a structure (R, d) where R is a set whose elements are called *regions* and $d: R\times R \rightarrow [0,\infty]$ is a function we call *semi-distance*, such that, for any $x, y, z \in R$:

(b1) $d(x, y) = d(y, x)$,

(b2) $d(x, y) \leq d(x, z) + d(z, y)$.

Given a semi-distance d, we define a *diameter* by setting:

$$|x|_d = d(x, x) . \tag{4.1}$$

Observe that by setting $y = x$ and $z = y$ in (b2), we obtain that $d(x,x) \leq d(x,y)+d(y,x)$ and therefore, by (b1), that $d(x,x) \leq 2d(x,y)$. Likewise we have that $d(y, y) \leq 2d(x, y)$ and therefore it results

$$d(x, y) \geq \frac{|x|_d}{2} \vee \frac{|y|_d}{2}. \tag{4.2}$$

So we can have $d(x, y) = 0$ only in the case both x and y have zero diameter. In literature it is possible to find a duality between the notion of metric and the notion of similarity (see for example Hájek [5]). Likewise we can give the next definition as a dual concept of semi-distance.

Definition 4.2. Let $\otimes$ be a t-norm. A *semi-similarity* is a fuzzy relation E on R such that

(e1) $E(x, y) = E(y, x)$ (symmetry)

(e2) $E(x, z) \otimes E(z, y) \leq E(x, y)$ (transitivity)

for every $x, y, z \in R$. A *similarity* is a semi-similarity such that

(e3) $E(x,x) = 1$.

$E(x,y)$ is regarded as truth-value of a statement like $x =_R y$. Semi-similarities are used to give a general approach to fuzzy sets theory based on the notion of category (see also M. Fourman and D.S. Scott [1]). Semi-similarities are strictly related with *sm-spaces*. We examine two cases regarding Definition 4.2**:** the case of Archimedean t-norms and the case of the *Gödel t-norm*. In the first one we use, as in Gerla ([4]), the notion of additive generator which characterizes the Archimedean t-norms.

Proposition 4.1. *Let* $f : [0,1] \rightarrow [0,\infty]$ *be an additive generator of an Archimedean t-norm* $\otimes$ *and let d be a semi-distance on a set R. Then the fuzzy-relation* $E_f(d)$ *defined by*

$$E_f(d)(x,y) = f^{[-1]}(d(x,y)) \tag{4.3}$$

is a semi-similarity with the t-norm $\otimes$. *Conversely, let E be a semi-similarity on R with the t-norm* $\otimes$, *then the structure* $(R, d_f(E))$ *where* $d_f(E)$ *is defined by*

$$d_f(E)(x,y) = f(E(x,y)), \tag{4.4}$$

is a sm-space.

If the t-norm is the *Gödel t*-norm, in Definition 4.2, the transitivity is given by

(e2*) $E(x, z) \wedge E(z, y) \leq E(x, y)$.

In such a case, a semi-similarity is called *G-semi-similarity* and, setting $y = x$ in (e2*), we obtain that

$$E(x, z) \wedge E(z, x) \leq E(x, x)$$

and therefore that $E(x, z) \leq E(x, x)$. Then

$$E(x, z) \leq E(x, x) \wedge E(z, z).$$

Observe that, since the *Gödel t-norm* is not Archimedean, the Proposition 4.1 doesn't hold for it. But, in this case, we consider a subclass of *sm-spaces,* shrinking the codomain of the semi-distance and adding an axiom.

Definition 4.3. A *semi-ultrametric space,* briefly *su-space*, is an *sm-space* (R,d), where the semi-distance is a function $d : R\times R \rightarrow [0,1]$, such that, for any $x, y, z \in R$:

(B2) $d(x, y) \leq d(x, z) \vee d(z, y)$.

Obviously, (B2) entails (b2). Observe that by setting $y = x$ and $z = y$ in (B2), we obtain that $d(x,x) \leq d(x,y) \vee d(y,z)$ and therefore, by (b1), that $d(x, x) \leq d(x, y)$. Likewise we have that $d(y, y) \leq d(x, y)$ and therefore it results

$$d(x, y) \geq |x|_d \vee |y|_d. \qquad (4.5)$$

Now we are able to describe the relation between the G-semi-similarities and the *sm*-spaces.

Proposition 4.2. *Let d be a semi-ultrametric on a set R, then the fuzzy-relation E_d defined by*

$$E_d(x, y) = 1 - d(x, y) \qquad (4.6)$$

is a G-semi-similarity. Conversely, let E be a G-semi-similarity on R, then the structure (R, d_E), defined by

$$d_E(x,y) = 1 - E(x,y) \qquad (4.7)$$

is a su-space.

Proof. Define E_d by (4.6). Then (e1) is immediate. To prove (e2*) observe that

$$E_d(x, y) \wedge E_d(y, z) = (1 - d(x, y)) \wedge (1 - d(y, z))$$
$$= 1 - (d(x, y) \vee d(y, z)) \leq 1 - d(x, z)$$
$$= E_d(x, z).$$

Now define d_E by (4.7). Then (b1) is immediate. To prove (B2) it is sufficient to observe that

$$d(x, y) = 1 - E(x, y) \leq (1 - E(x, z)) \vee (1 - E(z, y))$$
$$= d(x, z) \vee d(z, y) .$$

5. Characterization of the *G*-semi-similarities.

We can characterize G-semi-similarities in terms of related cuts.

Definition 5.1. Let S be a nonempty set. A (classical) relation R on S is called *semi-equivalence* provided that is symmetric and transitive.

Let denote by $D_R = \{x \in S$ / there is an element $y \in S : (x,y) \in R\}$ the domain of R. Then, if R is a semi-equivalence relation, it results that if $x \in D_R$, then $(x,x) \in R$. Then R is reflexive in D_R. Therefore, every semi-equivalence relation R on S is an equivalence relation on its domain D_R. Viceversa, if R is an equivalence relation on D_R and if it is symmetric on S, then R is a semi-equivalence relation on S.

Definition 5.2. A family $(R_\lambda)_{\lambda \in [0,1]}$ of semi-equivalence relations on a set S is called *order-reversing* if it results that

- $R_\beta \subseteq R_\alpha$ for every $\alpha \leq \beta$, $\alpha, \beta \in [0,1]$;
- $R_0 = S \times S$.

Proposition 5.1. *A fuzzy relation E is a G-semi-similarity if and only if the cuts of E define an order-reversing family $(C(E,\lambda))_{\lambda \in [0,1]}$ of semi-equivalences.*

Also, any order-reversing family of semi-equivalence relations defines a G-semi-similarity.

Proposition 5.2. *Let $(R_\lambda)_{\lambda \in [0,1]}$ be an order-reversing family of semi-equivalence relations. Then the fuzzy relation E defined by setting*

$$E(x, y) = Sup\{\lambda : (x, y) \in R_\lambda\} \qquad (5.1)$$

is a G-semi-similarity.

Proof. Condition (e1) is immediate by the symmetry of R_λ. To prove (e2*), let us consider

$E(x, z) = Sup\{\lambda : (x, z) \in R_\lambda\} = \mu$
$E(z, y) = Sup\{\lambda : (z, y) \in R_\lambda\} = \xi$
$E(x, y) = Sup\{\lambda : (x, y) \in R_\lambda\} = \eta$.

Suppose $\mu \leq \xi$ (likewise $\xi \leq \mu$). Since $(R_\lambda)_{\lambda \in [0,1]}$ is an order-reversing family of relations, it results $R_\xi \subseteq R_\mu$. Therefore we have $(x, z) \in R_\mu$ and $(z, y) \in R_\mu$ and then, by transitivity, $(x, y) \in R_\mu$. But $\eta = Sup\{\lambda : (x, y) \in R_\lambda\}$, then $\eta \geq \mu$ and, since $\mu \wedge \xi = \mu$, the condition (e2*)

$$E(x, z) \wedge E(z, y) \leq E(x, y)$$

is verified.

6. A connection between *pm*-spaces and *sm*-spaces.

In order to establish a connection between *pm*-spaces and *sm*-spaces, we observe that in defining *pm*-spaces we can consider the inclusion relation as a derived notion. In fact, as proved in [2], the following holds true:

Proposition 6.1. *Let $(R, \delta, |\ |)$ be a structure satisfying* (a1), (a2) *and* (a3) *and define $\leq$ by setting $x \leq y$ provided that*

$$|x| \leq |y| \text{ and } \delta(x,z) \geq \delta(y,z) \text{ for any } z \in R.$$

Then $(R, \delta, |\ |)$ is a pm-space.

In accordance with such a proposition, in the following we denote by $(R, \delta, |\ |)$ a *pm*-space whose order relation is defined as in Proposition 6.1. It is possible to associate any *pm*-space with a *sm*-space.

Proposition 6.2. *Let* $(R,\delta,|\ |)$ *be a pm-space and define* d_δ *by setting, for any* $x, y \in R$,

$$d_\delta(x, y) = \delta(x, y)+\frac{|x|}{2}+\frac{|y|}{2}. \qquad (6.1)$$

Then the structure (R, d_δ) *is a sm-space whose diameter coincides with* $|\ |$.

Proof. (b1) and the equality $|\ |_d = |\ |$ are trivial. Besides,

$$d_\delta(x, y) = \delta(x,y)+\frac{|x|}{2}+\frac{|y|}{2}$$
$$\le \delta(x, z)+\delta(z, y) +|z|+\frac{|x|}{2}+\frac{|y|}{2}$$
$$= (\delta(x,z)+\frac{|x|}{2}+\frac{|z|}{2}) + (\delta(z, y)+\frac{|z|}{2}+\frac{|y|}{2})$$
$$= d_\delta(x, z)+ d_\delta(z, y).$$

Conversely, we can associate any *sm-space* with a *pm-space.*

Proposition 6.3. *Let* (R, d) *be a sm-space and define* δ_d *by setting* $\delta_d(x,y)= d(x,y) - \frac{|x|_d}{2} - \frac{|y|_d}{2}$ *if* $d(x,y) \ge \frac{|x|_d}{2}+\frac{|y|_d}{2}$ *and* $\delta_d(x,y)=0$ *otherwise. Then the structure* $(R,\delta_d, |\ |_d)$, *where* $|\ |_d$ *is defined by* (4.1), *is a pm-space .*

Proof. Axioms (a1) and (a2) are immediate. If $d(x,y)< \frac{|x|_d}{2}+\frac{|y|_d}{2}$ (a3) is trivial. Otherwise, by (b2),

$$\delta_d(x,y)= d(x,y) - \frac{|x|_d}{2} - \frac{|y|_d}{2} \le$$
$$\le d(x,z)+d(z,y) - \frac{|x|_d}{2} - \frac{|y|_d}{2} +$$
$$+ \left(|z|_d - \frac{|z|_d}{2} - \frac{|z|_d}{2}\right) =$$
$$= \delta_d(x,z)+ \delta_d(z,y) + |z|_d .$$

Observe that the definitions of d_δ and δ_d in Proposition 6.2 and in Proposition 6.3 are not the unique possible ways to associate a *pm*-space with a *sm*-space and viceversa. For example, it is possible to associate any *pu*-space $(R,\delta, |\ |)$ with a *su*-space (R, d_δ) by setting

$$d_\delta(x, y) = \delta(x, y)\vee |x|\vee|y|, \qquad (6.2)$$

for any $x, y \in R$

Proposition 6.4. *Let* $(R,\delta,|\ |)$ *be a pu-space, then the structure* (R, d_δ) *defined by* (6.2) *is a su-space whose diameter coincides with* $|\ |$.

Proof. (b1) and the equality $|\ |_d = |\ |$ are trivial. Besides,

$$d_\delta(x, z) = \delta(x,z)\vee |x| \vee |z|$$
$$\le \delta(x,y)\vee\delta(y,z)\vee|y|\vee|x|\vee|z|=$$
$$= (\delta(x,y)\vee|x|\vee|y|)\vee(\delta(y, z)\vee|y|\vee|z|)=$$
$$= d_\delta(x, y)\vee d_\delta(y, z).$$

Conversely, we can associate any *su*-space (R, d) with a *pu*-space $(R,\delta_d, |\ |_d)$ by setting, for any $x,y\in R$,

$$\delta_d(x,y)=\begin{cases} d(x,y) & \text{if } d(x,y)=|x|_d \vee |y|_d \\ 0 & \text{if } d(x,y)>|x|_d \vee |y|_d \end{cases} \qquad (6.3)$$

Proposition 6.5. *Let* (R, d) *be a su-space, then the structure* $(R,\delta_d, |\ |_d)$ *defined by* (6.3) *and* (4.1) *is a pu-space .*

Proof. Axioms (a1) and (a2) are immediate. To prove that

$$\delta_d(x, z)\vee\delta_d(z, y)\vee|z|_d \ge \delta_d(x, y),$$

assume that $d(x,z)\ge d(z,y)$. Now, in the case $\delta_d(x,y) = 0$ and in the case $|z|_d \ge d(x,y)$ such an inequality is trivial. So, it is not restrictive to assume that $d(x,y) >|x|_d\vee|y|_d$, and therefore that $\delta_d(x, y) = d(x, y)$ and that $d(x,y)>|z|_d$,. In such a case, by (B2),

$$d(x,z) = d(x,z)\vee d(z,y)$$
$$\ge d(x,y) >|x|_d\vee|z|_d$$

and therefore

$$\delta_d(x, z)\vee \delta_d(x, z)\ge \delta_d(x,z) =$$
$$= d(x,z)\vee d(z,y) \ge d(x,y) = \delta_d(x, y).$$

Likewise we proceed in the case $d(x,z)\le d(z,y)$.

Besides, we can also set

$$\delta_d(x,y)=\begin{cases} d(x,y) & \text{if } d(x,y)>|x|_d \vee |y|_d \\ 0 & \text{if } x = y \end{cases} \qquad (6.4)$$

instead of (6.3).

In accordance with Proposition 4.2, Proposition 6.4 and Proposition 6.5 any connection between *su*-spaces and *pu*-spaces is also a connection between semi-similarities and *pu*-spaces.

Proposition 6.6. *Let E be a G-semi-similarity, define* $| \ |_E : R \to [0,1]$ *by setting*

$$|x|_E = 1 - E(x, x) \tag{6.5}$$

and $\delta_E : R \times R \to [0, 1]$ *by*

$$\delta_E(x,y) = \begin{cases} 0 & \text{if } E(x,y) = E(x,x) \wedge E(y,y) \\ 1 - E(x,y) & \text{if } E(x,y) < E(x,x) \wedge E(y,y) \end{cases} \tag{6.6}$$

for every $x, y \in R$. *Then* $(R, \delta_E, | \ |_E)$ *is a pu-space. Conversely, let* $(R, \delta, | \ |)$ *be a pu-space and define* $E_{\delta, | |} : R \times R \to [0,1]$ *by setting*

$$E_{\delta, | |}(x,y) = 1 - (\delta(x,y) \vee |x| \vee |y|) \tag{6.7}$$

Then $E_{\delta, | |}$ *is a G-semi-similarity.*

This last proposition is useful to describe, in the next section, the link of point-free geometry with fuzzy sets theory by a categorical point of view.

7. The categories of semi-similarities and of *pu*-spaces.

In order to organize the class of semi-similarities into a category, we refer to the category *M*-SET* described by Hőhle in [4]. Namely, while Hőhle defines this category for any *GL-monoid*, we are interested only with the particular *GL*-monoid in $[0,1]$ defined by the *Gödel t-norm*. In such a case we have the following simplified definition.

Definition 7.1. The *category of the G-semi-similarities* is the category ***GSS*** such that:

- the objects are structures (R,E) in which E is a G-semi-similarity;
- a morphism from (R, E) to (R',E') is a map $f : R \to R'$ satisfying the axioms

(M1) $E'(f(x), f(x)) \leq E(x, x)$

(M2) $E(x, y) \leq E'(f(x), f(y))$

for every $x, y \in X$.

Observe that from (M2) we have that $E(x,x) \leq E'(f(x),f(x))$ and therefore, by (M1),

$$E(x, x) = E'(f(x), f(x)).$$

The second category we consider is defined by the class of *pu*-spaces.

Definition 7.2. The category ***PU*** *of the pu-spaces* is the category such that

- the objects are *pu*-spaces;
- a morphism from $(R, \delta, | \ |)$ to $(R', \delta', | \ |')$ is a map $f: R \to R'$ such that

(1) $\delta(x, y) \geq \delta'(f(x),f(y))$

(2) $|x| \geq |f(x)|'$

In both the categories the *composition* is the usual composition of maps and the *identities* are the identical maps. Proposition 6.6 enables us to associate any G-semi-similarity with a *pu*-space $(R, \delta_E, | \ |_E)$. This suggests the definition of a suitable functor from ***GSS*** to ***PU***.

Proposition 7.1. *Define the map F from* ***GSS*** *to* ***PU*** *by setting*

- $F((R, E)) = (R, \delta_E, | \ |_E)$
- $F(f) = f$.

Then F is a functor from ***GSS*** *to* ***PU***.

Proof. We have only to prove that if f is a morphism from (R,E) to (R', E'), then f is a morphism from $(R, \delta_E, | \ |_E)$ to $(R', \delta_{E'}, | \ |_{E'})$. Indeed, it is immediate that

$$|f(x)|_{E'} = 1 - E'(f(x),f(x)) = 1 - E(x,x) = |x|_E.$$

To prove that

$$\delta_E(x,y) \geq \delta_{E'}(f(x),f(y)) \tag{7.1}$$

it is not restrictive to assume that $\delta_{E'}(f(x),f(y)) \neq 0$ and therefore that

$$E'(f(x),f(y)) < E'(f(x), f(x)) \wedge E'(f(y), f(y)).$$

and $\delta_{E'}(f(x),f(y)) = 1 - E'(f(x),f(y))$. In such a case, since

$$\begin{aligned} E(x,y) &\leq E'(f(x),f(y)) \\ &< E'(f(x), f(x)) \wedge E'(f(y), f(y)) \\ &= E(x,x) \wedge E(y,y), \end{aligned}$$

we have that $\delta_E(x,y) = 1 - E(x,y)$. So, (7.1) is a trivial consequence of (M2).

Observe that in proving that F is a functor we obtain that

$$|f(x)|_{E'} = |x|_E. \tag{7.2}$$

On the other hand, it is easy to find a morphism h in ***PU*** such that $|f(x)|_{E'} < |x|_E$ for a suitable region. Then, the proposed functor is faithful, but not full. We can consider the subcategory ***PU***** of ***PU*** obtained by considering only the morphisms f satisfying (7.2). Proposition 6.6 suggests a definition of a functor from ***PU***** to ***GSS***.

Proposition 7.2. *Define the map F' from* ***PU**** *to* ***GSS*** *by setting*

- $F'((R, \delta, | \ |)) = (R, E_{\delta, | |})$
- $F(f) = f$.

Then F' is a functor from ***PU***** *to* ***GSS***.

Proof. Let $(R,\ \delta,\ |\ |)$ and $(R',\ \delta',\ |\ |')$ be two *pu*-spaces, $(R,\ E)$ and $(R',\ E')$ the structures, where the semi-similarities E and E' are defined by (6.7), and f a morphism from $(R,\ \delta, |\ |)$ to $(R', \delta', |\ |')$. Then

$$E'(f(x),f(x)) = 1-|f(x)|' = 1-|x| = E(x,x).$$

Moreover,

$$E(x,y) = 1-(\delta(x,y)\vee|x|\vee|y|)$$
$$\leq 1-(\delta'(f(x),f(y))\vee|f(x)|'\vee|f(y)|')$$
$$= E'(f(x),f(y)).$$

8. A class of examples.

Let X and Y be two nonempty sets and denote by $F(X,\ Y)$ the class of partial functions from X to Y. If $f \in F(X,\ Y)$ we denote by D_f the domain of f and by U_f the complement of D_f, i.e. the set of elements in which f is not defined. Let f, g be elements of $F\ (X,\ Y)$, then the *equalizer* of f and g , is defined by

$$eq(f,\ g) = \{x \in X : x \in D_f \cap D_g\ , f(x)=g(x)\}.$$

The *contrast* between f and g is defined as the complement of the equalizer, i.e.

$$contr(f,g) = -\ eq(f,\ g).$$

Observe that

$$contr(f,g) = C_{fg} \cup U_f \cup U_g$$

where

$$C_{fg} = \{x \in X : x \in D_f \cap D_g \text{ and } f(x) \neq g(x)\}. \quad (8.1)$$

In other words, $contr(f,g)$ contains the elements on which f and g *"contrast"* and the elements in which either f or g is not defined. In particular $contr(f,f) = U_f$.

Definition 8.1. Consider a map $irl{:}X\rightarrow[0,1]$ we call *fuzzy subset of irrelevant elements*. Then the *irrelevancy degree of a set S,* is

$$Irl(S)= Inf\{irl(x) : x \in S\}. \quad (8.2)$$

We interpret $irl(x)$ as the *"degree of irrelevancy"* of an element x and $Irl(S)$ as a measure of the degree of validity of the claim "all the elements in S are irrelevant". Trivially, we have that for any pair S_1, S_2 of subsets of X,

$$Irl(S_1 \cup S_2) = Irl(S_1) \wedge Irl(S_2).$$

Proposition 8.1. *Let X be a nonempty class of partial functions and set*

$$E(f,\ g) = Irl(contr(f,g)). \quad (8.3)$$

Then E is a G- semi-similarity.

Proof. (e1) is immediate. To prove (e2*), observe that for every $f,\ g,\ h \in$ X,

$$C_{fg} \subseteq C_{fh} \cup C_{hg} \cup U_h$$

and therefore,

$$contr(f,g) \subseteq contr(f,h) \cup contr(h,g).$$

This entails

$$E(f,g) \geq E(f,h) \wedge E(h,g).$$

We interpret $E(f,g)$ as a measure of the truth degree of the claim "in all the relevant elements f and g are defined and coincide". Observe that

$$E(f,f) = Irl(U_f)$$

and therefore $E(f,f)$ is the valuation of the claim that f is defined in all the relevant elements. In particular, if f is total, then $E(f,f)$ is equal to 1, if f is totally undefined, i.e. $U_f = X$, then $E(f,\ f) = 0$.

9. Conclusions and future works.

This note is a first attempt to establish a link between point-free geometry and fuzzy set theory. In spite of some promising results, the proposed functors are not yet satisfactory. Also, it is an open question to give a geometric interpretation of the objects of the category of the fuzzy sets as suggested by the obtained results. Future works will be addressed to this aims.

References

[1] Fourman M., Scott D.S., *Sheaves and logic*, in "Applications of Sheaves", Lecture Notes in Mathematics 753, Springer-Verlag (1979), 302-401.

[2] Gerla G., *Pointless Metric Space*, Journal of Symbolic Logic 55, 1, (1990), 207-219.

[3] Gerla G., *Pointless Geometries*, in "Handbook of Incidence Geometries", F. Buekenhouted (1994) Elsevier Science, 1015-1031.

[4] Gerla G., *Representation Theorems for Fuzzy Orders and Quasi-metrics,* to appear on "Soft Computing".

[5] Hájek P., *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[6] Höhle U., *Presheaves over GL-monoid.* in "Nonclassical Logics and Their Applications to Fuzzy Subsets" (U. Höhle, E. P. Klement, eds), Kluwer, (1995), 127-157 .

[7] Novak V., Perfilieva I., Mockor J., *Mathematical Principles of Fuzzy Logic,* Kluwer Academic Publishers, London, 1999.

[8] Whitehead A. N., *An inquiry concerning the Principles of Natural Knowledge,* Univ. Press, Cambridge, 1919.
[9] Whitehead A. N., *The Concept of Nature,* Univ. Press, Cambridge, 1920.
[10] Whitehead A. N., *Process and Reality,* McMillian, N.Y., 1929.