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THÈSE

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À mes parents
et à mes grands-parents

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Contents

Introduction (en français)

Ce manuscrit contient trois résultats principaux :

0.1 Conjecture du soufflet au niveau des flexions infinitésimales

Un polyèdre (plus précisement, une surface polyèdrale) est appelé *flexible* si sa forme spatiale peut être changée continûement seulement par suite des changements de ses angles dièdres, c'est à dire, si chaque face reste congruente à elle-même pendant la déformation. Cette déformation est dite *une flexion continue* du polyèdre.

En 1897 Raoul Bricard a décrit tous les octaèdres flexibles dans $\mathbb{R}^3$. La méthode moderne de construction des octaèdres de Bricard a été proposée par Henri Lebesgue [Leb67]. Les octaèdres de Bricard sont les premiers exemples des polyèdres flexibles (avec des auto-intersections). En 1976 Robert Connelly [Con] a construit le premier polyèdre flexible plongé dans $\mathbb{R}^3$.

La conjecture du soufflet qui déclare que chaque polyèdre flexible conserve son volume orienté pendant la flexion continue, est une question très connue dans la théorie de rigidité des polyèdres. En 1996 Idjad Sabitov [Sab96] a donné une réponse affirmative à la conjecture du soufflet dans l'espace euclidien à trois dimensions. Une preuve améliorée de ce résultat se trouve dans le papier [CSW97] de Robert Connelly, Idjad Sabitov et Anke Walz. En 1997 Victor Alexandrov [Ale97] a construit un polyèdre flexible dans l'espace sphérique à trois dimensions qui change son volume pendant la flexion continue. La question si la conjecture du soufflet est vrai dans l'espace hyperbolique à trois dimensions est encore ouverte.

Une déformation d'une surface polyèdrale $\mathcal{S}$ est une famille des surfaces $\mathcal{S}(t)$, $t \in (-1, 1)$, qui dépend analytiquement du paramètre t, conserve la structure combinatoire de $\mathcal{S}$ et telle que $\mathcal{S}(0) = \mathcal{S}$. Une déformation d'une surface polyèdrale $\mathcal{S}$ avec les faces triangulaires est dite sa *flexion infinitésimale* si les longueurs de toutes les arêtes de $\mathcal{S}(t)$ sont stationnaires à $t = 0$. Une flexion infinitésimale est dite *nontriviale* s'il y a deux sommets de $\mathcal{S}(t)$ qui ne sont pas connectés par une arête de $\mathcal{S}(t)$ et tels que la distance spatiale entre eux n'est pas stationnaire. Un polyèdre est dit *infinitésimalement flexible* s'il possède une flexion infinitésimale nontriviale.

Pour attaquer la conjecture du soufflet, déjà en 1980 Idjad Sabitov a proposé de considérer la conjecture des soufflets au niveau des flexions infinitésimales. En gros, nous pouvons formuler la question de Sabitov comme suit : est-ce que c'est vrai que le volume de chaque polyèdre infinitésimalement flexible est stationnaire sous la flexion infinitésimale ? Dans [Ale89] et [Ale97] Victor Alexandrov répond par la négative à cette question pour les polyèdres infinitésimalement flexibles dans les espaces euclidien et sphérique à trois dimensions.

Dans le chapitre 1 de la thèse je donne une réponse négative à la conjecture du soufflet au niveau des flexions infinitésimales dans l'espace hyperbolique à trois dimensions [Slu11] :

Théorème 0.1. *Dans l'espace hyperbolique à trois dimensions il y a un polyèdre sans auto-intersections, homéomorphe à une sphère et il y a une flexion infinitésimale tels que le volume du* *polyèdre n'est pas stationnaire sous la flexion infinitésimale.*

J'ai présenté ce résultat à la conférence scientifique internationale "Les inégalités sur des volumes" à Banff, Alberta, Canada en mars 2010.

0.2 Condition de flexibilité d'une suspension dans $\mathbb{H}^3$

Une *suspension* est un polyèdre avec deux sommets spéciaux (appelés les pôle nord et sud) qui n'ont pas d'arête commune, et tels que tous les autres sommets du polyèdre (appelés les sommets de l'équateur) sont joints par arêtes avec les deux pôles, et les arêtes qui joignent des sommets de l'équateur forment un cycle.

Les octaèdres de Bricard [Leb67] sont des exemples des suspensions flexibles. En 2002 Hellmuth Stachel [Sta06] a démontré la flexibilité des analogues des octaèdres de Bricard dans l'espace hyperbolique.

En 1975 Robert Connelly [Con75] a démontré qu'une combinaison des longueurs des arêtes de l'équateur d'une suspension flexible dans $\mathbb{R}^3$ est égale à zéro (dans cette combinaison, chaque longueur est prise soit positive soit négative). En 2001 Sergey Mikhalev [Mik01] a redémontré le résultat susmentionné de Connelly par des méthodes algébriques. De plus, Mikhalev a démontré que pour chaque quadrilatère spatial formé par des arêtes d'une suspension flexible qui contient ses deux pôles il y a une combinaison des longueurs (prises soit positives soit négatives) des arêtes du quadrilatère qui est égale à zéro.

À la suite de Robert Connelly et Sergey Mikhalev, j'ai démontré le résultat suivant [Slu13] dont la preuve est donnée aussi dans le chapitre 2 de ce maniscrit :

Théorème 0.2. *Soit $\mathcal{P}$ une suspension flexible non dégénérée dans l'espace hyperbolique à trois dimensions avec les pôles S et N et avec les sommets de l'équateur P_j, $j = 1, ..., W$. Alors*

$$\sum_{j=1}^{W} \sigma_{j,j+1}|P_jP_{j+1}| = 0,$$

où $\sigma_{j,j+1} \in \{+1, -1\}$, $|P_jP_{j+1}|$ est la longueur de l'arête P_jP_{j+1}, $j = 1, ..., W$, et, par définition, $P_WP_{W+1} \stackrel{\text{def}}{=} P_WP_1$, $\sigma_{W,W+1} \stackrel{\text{def}}{=} \sigma_{W,1}$.

Dans [Slu13] je vérifie également le théorème 0.2 pour les octaèdres de Bricard-Stachel dans l'espace hyperbolique à trois dimensions.

J'ai présenté ce résultat à la conférence scientifique internationale "La quatrième rencontre géométrique" dediée à la centenaire de A. D. Alexandrov à Saint-Pétersbourg, Russie, en août 2012.

0.3 Métriques polyèdrales sur les bords de variétés quasi-Fuchsiennes convexes

Tout d'abord je rappelle deux résultats très connus dans la géométrie métrique. Le premier est dû à Alexandr Alexandrov et Alexei Pogorelov [Pog73] :

Théorème 0.3. *Soit h une métrique C^∞-régulière sur la sphère S^2 à courbure strictement supérieure à -1, il existe alors une immersion isométrique de (S^2, h) dans $\mathbb{H}^3$, unique aux isométries de $\mathbb{H}^3$ près. De plus, cette immersion borde un convexe de $\mathbb{H}^3$.*

Le deuxième est dû à Mikhael Gromov [Gro86] :

Théorème 0.4. *Soit S une surface compacte de genre supérieur ou égal à 2, munie d'une métrique h C^∞-régulière à courbure supérieure à -1. Il existe alors un groupe fuchsien Γ agissant sur $\mathbb{H}^3$ tel que (S, h) se plonge isométriquement dans $\mathbb{H}^3/\Gamma$.*

Une variété de dimension trois hyperbolique compacte M à bord ∂M est dite *strictement convexe* [Lab92] si deux points quelconques de M peuvent être joints par une géodésique minimisante incluse dans l'intérieur de M. Cette condition entraîne que la courbure intrinsèque de ∂M est supérieure à -1 (ici hyperbolique signifie courbure constante -1).

En 1992 François Labourie [Lab92] a obtenu le résultat suivant qui peut être conçu comme la généralisation des théorèmes 0.3 et 0.4 :

Théorème 0.5. *Soit M une variété compacte à bord (différente du tore plein) et qui admette une structure de variété hyperbolique strictement convexe. Soit h une métrique C^∞-régulière sur ∂M à courbure strictement plus grande que -1, il existe alors une métrique hyperbolique convexe g sur M qui induise h sur ∂M :*

$$g\mid_{\partial M} = h.$$

Une variété hyperbolique M est dite *quasi-Fuchsienne* si l'ensemble limite Λ_M sur le bord à l'infini du revêtement universel $\tilde{M}$ de M est une courbe de Jordan.

Récemment j'ai obtenu l'extension suivante du théorème 0.5 :

Théorème 0.6. *Soit M une variété compacte à bord de genre supérieur ou égal à 2 et qui admette une structure de variété quasi-Fuchsienne strictement convexe. Soit h une métrique hyperbolique à singularités coniques d'angle inférieur à 2π sur ∂M, il existe alors une métrique hyperbolique g sur M à bord convexe, pour laquelle la métrique induite sur le bord est h.*

Les chapitres 3 et 4 de ma thèse contiennent la preuve de ce résultat.

Rappelons un résultat classique sur les polyèdres convexes dû à Alexandr Alexandrov [Ale06] :

Théorème 0.7. *Soit h une métrique sur la sphère S^2 à courbure sectionnelle constante K avec des singularités coniques tels que l'angle total autour chaque point singulier de h plus petit que 2π. Il existe alors un polyèdre convexe muni de métrique h dans l'espace à trois dimensions R_K à courbure constante K, $K \in \mathbb{R}$, unique aux isométries de R_K près. Ici, nous incluons les polygones convexes doublement couverts dans l'ensemble des polyèdres convexes.*

Le théorème 0.6 peut être conçu aussi comme un analogue du théorème 0.7 pour les variétés hyperboliques convexes à bord polyèdral.

En 2002 Jean-Marc Schlenker [Sch06] a démontré l'unicité de la métrique g dans le théorème 0.5. Ainsi, il a obtenu

Théorème 0.8. *Soit M une variété compacte connectée à bord (différente du tore plein) et qui admette une métrique hyperbolique complète convexe co-compacte. Soit g une métrique hyperbolique de M telle que ∂M est C^∞-régulier et strictement convexe. Alors la métrique induite I sur ∂M a la courbure intrinsèque $K > -1$. Chaque métrique C^∞-régulière sur ∂M avec $K > -1$ est induite sur ∂M pour un choix unique de g.*

Il est alors naturel de conjecturer que la métrique g dans l'énoncé du théorème 0.6 est unique. Les méthodes que j'utilise dans la démonstration du théorème 0.6 ne me permettent pas pour l'instant d'attaquer ce problème.

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Chapter 1

An infinitesimally nonrigid polyhedron with nonstationary volume in hyperbolic 3-space

The Bellows Conjecture states that every flexible polyhedron preserves its oriented volume during the flex. In 1996 I. Kh. Sabitov [Sab96] gave an affirmative answer to the Bellows Conjecture in Euclidean 3-space. An improved demonstration of this result is given in the paper [CSW97] by R. Connelly, I. Kh. Sabitov, and A. Walz. In 1997 V. A. Alexandrov [Ale97] has built a flexible polyhedron in spherical 3-space which changes its volume during the flex. The question whether the Bellows Conjecture holds true in hyperbolic 3-space is still open.

In the note of the editor of the Russian translation of [Con80] I. Kh. Sabitov proposed to consider the Bellows Conjecture at the level of infinitesimal flexes. We say that a polyhedral surface is non-trivial if none of its vertices lies in the interior of a piece of the surface contained in a plane. We can now formulate I. Kh. Sabitov's question as follows: is it true that, for every infinitesimally non-rigid non-trivial polyhedron, the volume it bounds is stationary under its infinitesimal flex? In case the answer to I. Kh. Sabitov's question were positive, we would automatically validate the Bellows Conjecture for the flexible polyhedra.

Having constructed a non-trivial counterexample in [Ale89], V. A. Alexandrov gave a negative answer to I. Kh. Sabitov's question for infinitesimally nonrigid polyhedra in Euclidean 3-space. An example of a flexible polyhedron in spherical 3-space, constructed in [Ale97], which changes its volume during the flex, yields that the answer to this question is also negative for infinitesimally nonrigid polyhedra in spherical 3-space. In this Chapter we prove

Theorem 1.1. *In hyperbolic 3-space there is a non-trivial, non-self intersecting polyhedral surface, homeomorphic to a sphere, that has an infinitesimal flex such that the volume it bounds is not stationary under the flex.*

This result is published in [Slu11].

The polyhedron mentioned in Theorem 1.1 is built explicitely. It's similar to a polyhedron in Euclidean 3-space which was first constructed by A. D. Alexandrov and S. M. Vladimirova [AV62] and later studied by A. D. Milka [Mil02]. Another example of an infinitesimally nonrigid polyhedron in Euclidean 3-space (an octahedron of a special type) was described by H. Gluck in [Glu75].

1.1 Constructing $\mathcal{S}$

Throughout this chapter we call a polyhedral surface a polyhedron.

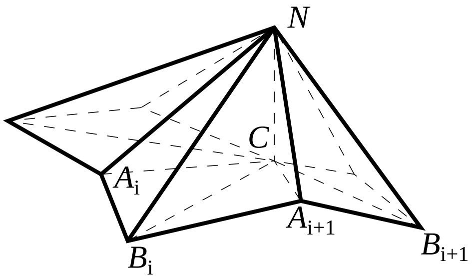

Figure 1.1: The lateral surface of $\mathcal{P}$.

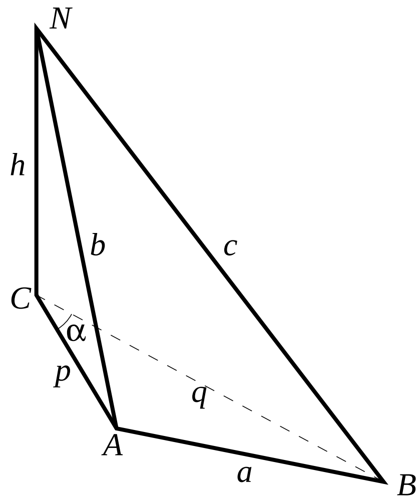

Figure 1.2: The tetrahedron $\mathcal{T}$.

Consider a regular pyramid $\mathcal{P}$ in hyperbolic 3-space with a regular concave star with n petals as the base. We denote vertices of the star by A_i, B_i, $i = 1, ..., n$, and we note that the orthogonal projection of the vertex N of $\mathcal{P}$ onto its base coincides with the center C of the star, see Fig. 1.1. We reflect $\mathcal{P}$ in the plane that contains its base and denote by $\mathcal{S}$ a suspension which consists of both initial and reflected pyramids with their common base. We denote by S the vertex of $\mathcal{S}$ symmetric to N with respect to the plane containing the base of $\mathcal{P}$. A cycle formed by the edges of the base of $\mathcal{P}$ is called the equator of the suspension $\mathcal{S}$.

Note that the lengths of all edges of the equator of $\mathcal{S}$ are equal to each other by construction. Moreover, the lengths of all edges SA_i, NA_i, $i = 1, ..., n$, are equal to each other, and also the lengths of all edges NB_i, SB_i, $i = 1, ..., n$, are equal to each other too.

By construction, $\mathcal{S}$ possesses multiple symmetries and the spatial body bounded by $\mathcal{S}$ consists of identical tetrahedral "bricks". Consider one of these tetrahedra, see Fig. 1.2. Denote its surface by $\mathcal{T}$, and its vertices by N, A, B, C. Note that $\angle ACN = \angle BCN = \pi/2$ by construction. Let's use the following notation for the lengths of the edges and for the plane angles of $\mathcal{T}$: $|CN| = h$, $|CA| = p$, $|CB| = q$, $|AB| = a$, $|NA| = b$, $|NB| = c$, $\angle ACB = \alpha$, $\angle CAN = \beta$, $\angle BAN = \gamma$, $\angle CAB = \delta$, $\angle CBN = \varphi$, $\angle CBA = \psi$, $\angle ABN = \theta$, $\angle ANB = \lambda$, $\angle CNA = \mu$, $\angle CNB = \nu$. Denote the dihedral angles of $\mathcal{T}$ at the edge AB by $\angle AB$, at the edge NA by $\angle NA$, and at the edge NB by $\angle NB$.

By construction, the dihedral angle of $\mathcal{T}$ at the edge CN is equal to α, the dihedral angles of $\mathcal{S}$ at the edges of its equator are equal to $2\angle AB$, at the edges NA_i and SA_i , $i = 1, ..., n$, are equal to $2\angle NA$, and at the edges NB_i and SB_i , $i = 1, ..., n$, are equal to $2\angle NB$.

Further we show that the suspension $\mathcal{S}$ constructed above can be taken as a polyhedron whose existence is proclaimed by Theorem 1.1.

1.2 A condition for infinitesimal nonrigidity

A deformation of a polyhedral surface $\mathcal{S}$ is a family of surfaces $\mathcal{S}(t)$, $t \in (-1, 1)$, which depends analytically on the parameter t, preserves the combinatorial structure of $\mathcal{S}$, and is such that $\mathcal{S}(0) = \mathcal{S}$.

A deformation of a polyhedral surface $\mathcal{S}$ with triangular faces is called its infinitesimal flex if the lengths of all edges of $\mathcal{S}(t)$ are stationary at $t = 0$.

An infinitesimal flex is called nontrivial if there exist two vertices of $\mathcal{S}(t)$ which are not connected by an edge of $\mathcal{S}(t)$ and are such that the spatial distance between them is not stationary.

A polyhedron is called infinitesimally nonrigid if it possesses a nontrivial infinitesimal flex.

Determine a deformation of the suspension $\mathcal{S}$ constructed in the previous section as follows. The point C is fixed. At the moment t, the point N goes to the point $N(t)$ lying on the ray $\overrightarrow{CN}$ at the distance from C determined by the formula

$$h(t) = h + tu, \tag{1.1}$$

where u is a real number which has a meaning of velocity and which will be specified below. The point S goes to the point $S(t)$ lying on the ray $\overrightarrow{CS}$ at the distance from C determined by the formula (1.1). The point A_i, $i = 1, ..., n$, goes to the point $A_i(t)$ lying on the ray $\overrightarrow{CA_i}$ at the distance from C determined by the formula $p(t) = p + tv$, where v is a real number which has a meaning of velocity. The point B_i, $i = 1, ..., n$, goes to the point $B_i(t)$ lying on the ray $\overrightarrow{CB_i}$ at the distance from C determined by the formula $q(t) = q + tw$, where w is a real number which has a meaning of velocity and which will be specified below.

In order to determine the movements of other points of the suspension $\mathcal{S}(t)$ let's use the statement of Ceva's theorem in hyperbolic space [Pra04]:

Theorem 1.2. *Given a triangle $\triangle ABC$ and points $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ that lie on sides BC, CA, and AB of $\triangle ABC$. Then the segments $A\tilde{A}$, $B\tilde{B}$, and $C\tilde{C}$ intersect at one point if and only if one of the following equivalent relations holds:*

$$\frac{\sin\angle AC\tilde{C}}{\sin\angle\tilde{C}CB}\frac{\sin\angle BA\tilde{A}}{\sin\angle\tilde{A}AC}\frac{\sin\angle CB\tilde{B}}{\sin\angle\tilde{B}BA} = 1;$$

$$\frac{\sinh A\tilde{C}}{\sinh\tilde{C}B}\frac{\sinh B\tilde{A}}{\sinh\tilde{A}C}\frac{\sinh C\tilde{B}}{\sinh\tilde{B}A} = 1. \tag{1.2}$$

In terms of the statement of Ceva's Theorem 1.2, let's take the point $P(t)$ of the segment $A(t)B(t)$ for which the equality

$$\frac{\sinh A(t)P(t)}{\sinh P(t)B(t)} = \frac{\sinh AP}{\sinh PB}$$

holds true, as a new position of any point P of the edge AB at the moment t.

To determine the movement of an internal point Q of the face $\triangle ABC$, at first we construct points $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, as the intersections of the edges BC, CA, and AB with the rays AQ, BQ, and CQ, and then determine their positions $\tilde{A}(t)$, $\tilde{B}(t)$, and $\tilde{C}(t)$ at the moment t by the method described above. By Ceva's Theorem 1.2, the segments $A(t)\tilde{A}(t)$, $B(t)\tilde{B}(t)$, and $C(t)\tilde{C}(t)$ intersect at one point (the relation (1.2) remains true at every moment t). Consider this point of intersection as a new position $Q(t)$ of the point Q at the moment t.

The deformation of $\mathcal{S}$ described above, naturally produces a deformation of the tetrahedron $\mathcal{T}$ which we denote by $\mathcal{T}(t)$. The lengths of all edges as well as the values of all plane and dihedral angles of $\mathcal{T}$ are functions in t and their notation naturally follow from the notation for the corresponding entities of $\mathcal{T}$. For example, we denote the length of the edge $N(t)A(t)$ by $b(t)$, the value of the plane angle $\angle CA(t)N(t)$ by $\beta(t)$, and the value of the dihedral angle of $\mathcal{T}(t)$ at the edge $N(t)A(t)$ by $\angle N(t)A(t)$, etc.

Let's find a relation between u, v, and w implying that the deformation $\mathcal{S}(t)$ is an infinitesimal flex. We only need to study the deformation of the face ABN in $\mathcal{T}$ because all faces of $\mathcal{S}$ move in the same way.

Apply the Pythagorean Theorem for hyperbolic space [AVS93] to the triangle $\triangle N(t)CA(t)$:

$$\cosh b(t) = \cosh(h + tu)\cosh(p + tv) \tag{1.3}$$

and to the triangle $\triangle N(t)CB(t)$:

$$\cosh c(t) = \cosh(h + tu)\cosh(q + tw) \tag{1.4}$$

of $\mathcal{T}(t)$.

Using the Cosine Law for hyperbolic space [AVS93] applied to the triangle $\triangle A(t)CB(t)$, and taking it into account that the angle α remains constant during the deformation (and is equal to $\frac{\pi}{n}$), we get:

$$\cosh a(t) = \cosh(p + tv)\cosh(q + tw) - \sinh(p + tv)\sinh(q + tw)\cos\alpha. \tag{1.5}$$

Further it will be useful for us to study stationarity of the function $f(t) = \cosh l(t)$ instead of stationarity of the length $l(t)$ of any edge of $\mathcal{S}(t)$, because $f'(0) = l'(0)\sinh l(0)$ and $l(0) > 0$, and thus $f'(0) = 0$ if and only if $l'(0) = 0$.

Let's differentiate (1.3): $(\cosh b(t))' = u\sinh(h + tu)\cosh(p + tv) + v\cosh(h + tu)\sinh(p + tv)$. Thus, stationarity of the length $b(t)$ of the edge $N(t)A(t)$ is equivalent to the condition $(\cosh b(t))'|_{t=0} = u\sinh h\cosh p + v\cosh h\sinh p = 0$, or

$$v = -\frac{\tanh h}{\tanh p}u. \tag{1.6}$$

Similarly, stationarity of the length $c(t)$ of the edge $N(t)B(t)$ is equivalent to the condition

$$w = -\frac{\tanh h}{\tanh q}u. \tag{1.7}$$

Differentiating (1.5), we find the condition for stationarity of the length $a(t)$ of the edge $A(t)B(t)$:

$$(\cosh a(t))'|_{t=0} = v\sinh p\cosh q + w\cosh p\sinh q - \cos\alpha\{v\cosh p\sinh q + w\sinh p\cosh q\} = 0. \tag{1.8}$$

Substituting (1.6) and (1.7) into (1.8), we get:

$$u\tanh h\Big[\cos\alpha\Big\{\frac{\cosh p\sinh q}{\tanh p} + \frac{\sinh p\cosh q}{\tanh q}\Big\} - \frac{\sinh p\cosh q}{\tanh p} - \frac{\cosh p\sinh q}{\tanh q}\Big] = 0.$$

Thus, the deformation under consideration of $\mathcal{S}$ is an infinitesimal flex if and only if (1.6), (1.7) and

$$\cos\alpha\Big\{\frac{\cosh p\sinh q}{\tanh p} + \frac{\sinh p\cosh q}{\tanh q}\Big\} = 2\cosh p\cosh q$$

hold true. Hence, $\mathcal{S}$ allows the infinitesimal flex of the form described in the beginning of this section if and only if p, q, and α satisfy the following relation:

$$\frac{\tanh p}{\tanh q} = \frac{1 \pm \sin\alpha}{\cos\alpha}. \tag{1.9}$$

The so-constructed infinitesimal flex is nontrivial because the distance between the poles $N(t)$ and $S(t)$ is not stationary.

As Professor Robert Connelly remarked, there is a natural correspondence between infinitesimal flexes of a polyhedron (or framework) in Euclidean space and infinitesimal flexes of a polyhedron (or framework) in hyperbolic space (and in spherical space as well). One way to see this is through the Pogorelov correspondence and another way is by coning. The consequence is that the infinitesimal flex of the polyhedron $\mathcal{S}$ can also be checked in Euclidean space (say, when $\mathcal{S}$ is placed in the Kleinian (projective) model of hyperbolic space, $\mathcal{S}$ can be considered as an Euclidean polyhedron as well), but the parameters of the flex of $\mathcal{S}$ must be recalculated properly for Euclidean space. There is an interesting projective approach to the study of infinitesimal flexes and other rigidity problems in some of Walter Whiteley's papers, for example, in [CW82].

1.3 Calculating metric elements of $\mathcal{T}(t)$

Let's obtain formulae for the dihedral angles $\angle A(t)B(t)$, $\angle N(t)A(t)$, and $\angle N(t)B(t)$ of the tetrahedron $\mathcal{T}(t)$, which will be used in a proof of Theorem 1.1.

First we calculate the sines and cosines of the plane angles of $\mathcal{T}(t)$.

Apply the Cosine Law for hyperbolic space to the triangle $\triangle CA(t)N(t)$ to calculate the cosine of the angle $\beta(t)$: $\cosh(h+tu) = \cosh(p+tv)\cosh b(t) - \sinh(p+tv)\sinh b(t)\cos\beta(t)$. Thus, taking into account (1.3) and formulae of hyperbolic trigonometry, we get:

$$\cos\beta(t) = \frac{\sinh(p+tv)\cosh(h+tu)}{\sinh b(t)} = \frac{\sinh(p+tv)\cosh(h+tu)}{\sqrt{\cosh^2(h+tu)\cosh^2(p+tv)-1}}. \tag{1.10}$$

(Here and below $\sqrt{s}$ stands for a branch of the square root that takes a positive real value for a positive real s.) To calculate the sine of $\beta(t)$ we apply the Sine Law for hyperbolic space [AVS93] to $\triangle CA(t)N(t)$:

$$\frac{\sin\beta(t)}{\sinh(h+tu)} = \frac{\sin\pi/2}{\sinh b(t)} = \frac{1}{\sqrt{\cosh^2(h+tu)\cosh^2(p+tv)-1}},$$

and therefore,

$$\sin\beta(t) = \frac{\sinh(h+tu)}{\sinh b(t)} = \frac{\sinh(h+tu)}{\sqrt{\cosh^2(h+tu)\cosh^2(p+tv)-1}}. \tag{1.11}$$

Similarly, we obtain the formulae for the cosine and sine of the angle $\varphi(t)$ in $\triangle CB(t)N(t)$:

$$\cos\varphi(t) = \frac{\sinh(q+tw)\cosh(h+tu)}{\sinh c(t)} = \frac{\sinh(q+tw)\cosh(h+tu)}{\sqrt{\cosh^2(h+tu)\cosh^2(q+tw)-1}}, \tag{1.12}$$

$$\sin\varphi(t) = \frac{\sinh(h+tu)}{\sinh c(t)} = \frac{\sinh(h+tu)}{\sqrt{\cosh^2(h+tu)\cosh^2(q+tw)-1}}, \tag{1.13}$$

for the cosine and sine of the angle $\mu(t)$ in $\triangle CA(t)N(t)$:

$$\cos\mu(t) = \frac{\sinh(h+tu)\cosh(p+tv)}{\sinh b(t)} = \frac{\sinh(h+tu)\cosh(p+tv)}{\sqrt{\cosh^2(h+tu)\cosh^2(p+tv)-1}}, \tag{1.14}$$

$$\sin\mu(t) = \frac{\sinh(p+tv)}{\sinh b(t)} = \frac{\sinh(p+tv)}{\sqrt{\cosh^2(h+tu)\cosh^2(p+tv)-1}}, \tag{1.15}$$

and for the cosine and sine of the angle $\nu(t)$ in $\triangle CB(t)N(t)$:

$$\cos\nu(t) = \frac{\sinh(h+tu)\cosh(q+tw)}{\sinh c(t)} = \frac{\sinh(h+tu)\cosh(q+tw)}{\sqrt{\cosh^2(h+tu)\cosh^2(q+tw)-1}}, \tag{1.16}$$

$$\sin\nu(t) = \frac{\sinh(q+tw)}{\sinh c(t)} = \frac{\sinh(q+tw)}{\sqrt{\cosh^2(h+tu)\cosh^2(q+tw)-1}}. \tag{1.17}$$

The Cosine Law for hyperbolic space applied twice to the triangle $\triangle A(t)CB(t)$ leads us to the formulae:

$$\cos\delta(t) = \frac{\cosh(p+tv)\cosh a(t) - \cosh(q+tw)}{\sinh(p+tv)\sinh a(t)}, \tag{1.18}$$

$$\cos\psi(t) = \frac{\cosh(q+tw)\cosh a(t) - \cosh(p+tv)}{\sinh(q+tw)\sinh a(t)}. \tag{1.19}$$

From the Sine Law for hyperbolic space applied to $\triangle A(t)CB(t)$, it follows that:

$$\frac{\sin\delta(t)}{\sinh(q+tw)} = \frac{\sin\alpha}{\sinh a(t)} = \frac{\sin\psi(t)}{\sinh(p+tv)},$$

and thus the formulae

$$\sin\delta(t) = \frac{\sin\alpha\sinh(q+tw)}{\sinh a(t)}, \tag{1.20}$$

$$\sin\psi(t) = \frac{\sin\alpha\sinh(p+tv)}{\sinh a(t)} \tag{1.21}$$

hold true.

The Cosine Law for hyperbolic space applied three times to the triangle $\triangle A(t)N(t)B(t)$ leads us to the formulae:

$$\cos\theta(t) = \frac{\cosh a(t)\cosh c(t) - \cosh b(t)}{\sinh a(t)\sinh c(t)}, \tag{1.22}$$

$$\cos\gamma(t) = \frac{\cosh a(t)\cosh b(t) - \cosh c(t)}{\sinh a(t)\sinh b(t)}, \tag{1.23}$$

$$\cos\lambda(t) = \frac{\cosh b(t)\cosh c(t) - \cosh a(t)}{\sinh b(t)\sinh c(t)}. \tag{1.24}$$

Taking into account (1.3)–(1.5), we calculate $\sinh a(t)$, $\sinh b(t)$, and $\sinh c(t)$ from (1.10)–(1.24):

$$\sinh a(t) = \sqrt{\cosh^2 a(t) - 1} = \sqrt{(\cosh(p+tv)\cosh(q+tw) - \sinh(p+tv)\sinh(q+tw)\cos\alpha)^2 - 1},$$

$$\sinh b(t) = \sqrt{\cosh^2 b(t) - 1} = \sqrt{(\cosh(h+tu)\cosh(p+tv))^2 - 1},$$

$$\sinh c(t) = \sqrt{\cosh^2 c(t) - 1} = \sqrt{(\cosh(h+tu)\cosh(q+tw))^2 - 1}.$$

The fact that the values of the angles in a hyperbolic triangle are greater than 0 and less than π yields that the sines of the angles of a hyperbolic triangle are nonnegative. Hence, $\sin\theta(t) = \sqrt{1-\cos^2\theta(t)}$, $\sin\gamma(t) = \sqrt{1-\cos^2\gamma(t)}$, $\sin\lambda(t) = \sqrt{1-\cos^2\lambda(t)}$.

Consider the unit sphere Σ centered at the vertex $A(t)$ of $\mathcal{T}(t)$. Denote the points of the intersection of Σ and the rays $\overrightarrow{A(t)C}$, $\overrightarrow{A(t)N(t)}$, and $\overrightarrow{A(t)B(t)}$ by $C_A(t)$, $N_A(t)$, and $B_A(t)$ correspondingly. They determine a triangle $\triangle C_A(t)N_A(t)B_A(t)$ which consists of the points of the intersection of Σ and the rays emitted from $A(t)$ and passing through the points of the face $\triangle CB(t)N(t)$ of $\mathcal{T}(t)$. By construction, the angle of the spherical triangle $\triangle C_A(t)N_A(t)B_A(t)$ at the vertex $C_A(t)$ is equal to $\pi/2$, the angle at $N_A(t)$ is equal to $\angle N(t)A(t)$, the angle at $B_A(t)$ is equal to $\angle A(t)B(t)$, the length of the side $C_A(t)N_A(t)$ is equal to $\beta(t)$, the length of $N_A(t)B_A(t)$ is equal to $\gamma(t)$, and the length of $C_A(t)B_A(t)$ is equal to $\delta(t)$.

Similarly, we build a spherical triangle $\triangle C_B(t)N_B(t)A_B(t)$. Its angle at the vertex $C_B(t)$ is equal to $\pi/2$, the angle at $N_B(t)$ is equal to $\angle N(t)B(t)$, the angle at $A_B(t)$ is equal to $\angle A(t)B(t)$, the length of the side $C_B(t)N_B(t)$ is equal to $\varphi(t)$, the length of $N_B(t)A_B(t)$ is equal to $\theta(t)$, and the length of $C_B(t)A_B(t)$ is equal to $\psi(t)$.

Applying the Cosine Law for spherical space [AVS93] twice to $\triangle C_A(t)N_A(t)B_A(t)$, we obtain the formulae:

$$\cos\angle A(t)B(t) = \frac{\cos\beta(t) - \cos\gamma(t)\cos\delta(t)}{\sin\gamma(t)\sin\delta(t)},$$

$$\cos\angle N(t)A(t) = \frac{\cos\delta(t) - \cos\gamma(t)\cos\beta(t)}{\sin\gamma(t)\sin\beta(t)}.$$

Again, applying the Cosine Law for spherical space to $\triangle C_B(t)N_B(t)A_B(t)$, we get:

$$\cos\angle N(t)B(t) = \frac{\cos\psi(t) - \cos\varphi(t)\cos\theta(t)}{\sin\varphi(t)\sin\theta(t)}.$$

Now apply the Sine Law for spherical space [AVS93] to $\triangle C_A(t)N_A(t)B_A(t)$:

$$\frac{\sin\angle N(t)A(t)}{\sin\delta(t)} = \frac{\sin\angle A(t)B(t)}{\sin\beta(t)} = \frac{\sin\pi/2}{\sin\gamma(t)}.$$

Hence,

$$\sin\angle A(t)B(t) = \frac{\sin\beta(t)}{\sin\gamma(t)} \quad \text{and} \quad \sin\angle N(t)A(t) = \frac{\sin\delta(t)}{\sin\gamma(t)}.$$

Again, apply the Sine Law for spherical space to $\triangle C_B(t)N_B(t)A_B(t)$:

$$\frac{\sin\angle N(t)A(t)}{\sin\nu(t)} = \frac{\sin\angle N(t)B(t)}{\sin\mu(t)} = \frac{\sin\alpha}{\sin\lambda(t)}.$$

Thus,

$$\sin\angle N(t)B(t) = \sin\alpha\frac{\sin\mu(t)}{\sin\lambda(t)}.$$

In the proof of Theorem 1.1 given below we use also the following three evident relations:

$$\frac{d\angle N(t)A(t)}{dt} = -\frac{\frac{d}{dt}(\cos\angle N(t)A(t))}{\sin\angle N(t)A(t)},$$

$$\frac{d\angle N(t)B(t)}{dt} = -\frac{\frac{d}{dt}(\cos\angle N(t)B(t))}{\sin\angle N(t)B(t)},$$

and

$$\frac{d\angle A(t)B(t)}{dt} = -\frac{\frac{d}{dt}(\cos\angle A(t)B(t))}{\sin\angle A(t)B(t)}.$$

1.4 Proof of Theorem 1.1

Remind that, according to the Schläfli formula for polyhedra in hyperbolic 3-space [AVS93] of the curvature -1, the equality

$$dV = -\frac{1}{2}\sum_e l_e d\theta_e \tag{1.25}$$

holds true, where dV stands for the variation of the volume of the polyhedron, l_e stands for the length of an edge e of the polyhedron, $d\theta_e$ stands for the variation of the dihedral angle of the polyhedron attached to the edge e, and summation is taken over all edges e of the polyhedron.

Show that the polyhedron $\mathcal{S}(0)$ from the family of suspensions $\mathcal{S}(t)$, $t \in (-1, 1)$, constructed in Section 1.1, with parameters of the tetrahedron $\mathcal{T}$

$$p = \operatorname{artanh}\frac{1}{2}, \quad q = \operatorname{artanh}\frac{\sqrt{3}}{2}, \quad h = \operatorname{artanh}\frac{1}{2}, \quad \alpha = \frac{\pi}{6} \quad \text{(i. e. } n = 6) \tag{1.26}$$

and the velocities of deformation

$$u = \frac{\sqrt{3}}{4}, \quad v = -\frac{\sqrt{3}}{4}, \quad w = -\frac{1}{4}, \tag{1.27}$$

can be taken as a polyhedron whose existence is asserted in Theorem 1.1.

The suspension $\mathcal{S}(0)$ is not infinitesimally rigid because p, q, and α from (1.26) satisfy (1.9).

Let's verify that the nontrivial infinitesimal flex from Section 1.2 with the coefficients (1.27) can be taken as an infinitesimal flex whose existence is stated in Theorem 1.1.

Using the Schläfli formula (1.25) and taking into account notation and remarks of Section 1.1, we see that the variation of the volume of $\mathcal{S}(t)$ at $t = 0$ can be written as follows:

$$dV_{\mathcal{S}(0)} = -12\Big(a(0)\frac{d\angle A(t)B(t)}{dt}(0) + b(0)\frac{d\angle N(t)A(t)}{dt}(0) + c(0)\frac{d\angle N(t)B(t)}{dt}(0)\Big)dt. \tag{1.28}$$

Substituting the values of parameters from (1.26) and (1.27) into the formulae of Sections 1.2 and 1.3, we sequentially find the hyperbolic sines and cosines of the lengths of the edges and the variations of the dihedral angles of the tetrahedron $\mathcal{T}(t)$ at $t = 0$:

$$\cosh a(t) = \cosh\Big(-\operatorname{artanh}\frac{1}{2}\Big)\cosh\Big(-\operatorname{artanh}\frac{\sqrt{3}}{2}\Big) - \frac{\sqrt{3}}{2}\sinh\Big(-\operatorname{artanh}\frac{1}{2}\Big)\sinh\Big(-\operatorname{artanh}\frac{\sqrt{3}}{2}\Big),$$

$$\cosh b(0) = \cosh\Big(-\operatorname{artanh}\frac{1}{2}\Big)\cosh\Big(\operatorname{artanh}\frac{1}{2}\Big), \quad \cosh c(t) = \cosh\Big(\operatorname{artanh}\frac{1}{2}\Big)\cosh\Big(-\operatorname{artanh}\frac{\sqrt{3}}{2}\Big),$$

$$\frac{d\angle A(t)B(t)}{dt}(0) = \frac{\sqrt{13}}{4}, \quad \frac{d\angle N(t)A(t)}{dt}(0) = \frac{\sqrt{7}}{4}, \quad \frac{d\angle N(t)B(t)}{dt}(0) = -\frac{\sqrt{13}}{4},$$

and thus, by (1.28),

$$\frac{dV_{\mathcal{S}(0)}}{dt} = -12\Big[\frac{\sqrt{13}}{4}\operatorname{arcosh}\Big(\cosh\Big(-\operatorname{artanh}\frac{1}{2}\Big)\cosh\Big(-\operatorname{artanh}\frac{\sqrt{3}}{2}\Big) -$$

$$-\frac{\sqrt{3}}{2}\sinh\Big(-\operatorname{artanh}\frac{1}{2}\Big)\sinh\Big(-\operatorname{artanh}\frac{\sqrt{3}}{2}\Big)\Big) + \frac{\sqrt{7}}{4}\operatorname{arcosh}\Big(\cosh\Big(-\operatorname{artanh}\frac{1}{2}\Big)\cosh\Big(\operatorname{artanh}\frac{1}{2}\Big)\Big) -$$

$$-\frac{\sqrt{13}}{4}\operatorname{arcosh}\Big(\cosh\Big(\operatorname{artanh}\frac{1}{2}\Big)\cosh\Big(-\operatorname{artanh}\frac{\sqrt{3}}{2}\Big)\Big)\Big] =$$

$$-3\Big[\sqrt{7}\,\mathrm{arcosh}\,\frac{4}{3}+\sqrt{13}\Big(\mathrm{arcosh}\,\frac{5}{2\sqrt{3}}-\mathrm{arcosh}\,\frac{4}{\sqrt{3}}\Big)\Big]=-3\Big[\sqrt{7}\ln\frac{4+\sqrt{7}}{3}+\sqrt{13}\ln\frac{7-\sqrt{13}}{6}\Big]<$$

$$<-\frac{3\sqrt{7}}{8}\Big[8\ln\frac{4+\sqrt{7}}{3}+11\ln\frac{7-\sqrt{13}}{6}\Big]=-\frac{3\sqrt{7}}{8}\ln\Big[\Big(\frac{4+\sqrt{7}}{3}\Big)^{8}\Big(\frac{7-\sqrt{13}}{6}\Big)^{11}\Big]<0. \quad \square$$

Another polyhedra (say, hyperbolic analogues of H. Gluck's infinitesimally nonrigid octahedra [Glu75]) could probably also serve as an example for Theorem 1.1, but we don't verify it here.

1.5 Concluding remarks

Using notation of Section 1.4, we determine the integral mean curvature of a polyhedron $\mathcal{S}(t)$ in 3-space as follows:

$$M(\mathcal{S}(t))=\frac{1}{2}\sum_{e} l_e(t)(\pi-\theta_e(t)).$$

R. Alexander [Ale85] proved that the integral mean curvature of any polyhedron in Euclidean 3-space is stationary under every its infinitesimal flex.

The lengths of the edges of the suspension $\mathcal{S}(t)$ are stationary under the infinitesimal flex of $\mathcal{S}(t)$ from Section 1.2. Hence, the variation of the integral mean curvature of $\mathcal{S}(t)$ at $t=0$ is equal to the variation of the volume $dV_{\mathcal{S}(0)}$. Therefore, the proof of Theorem 1.1 automatically implies that the variation of the integral mean curvature for the infinitesimal flex of $\mathcal{S}(t)$ constructed above is not equal to zero. Thus, the integral mean curvature of an infinitesimally nonrigid polyhedron is not always stationary in hyperbolic space as well as in the spherical space but is always stationary in Euclidean space.

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Chapter 2

A necessary flexibility condition of a nondegenerate suspension in hyperbolic 3-space

A polyhedron (more precisely, a polyhedral surface) is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e., if every face remains congruent to itself during the flex.

In 1897 R. Bricard [Bri97] described all flexible octahedra in Euclidean 3-space. The Bricard's octahedra were the first examples of flexible polyhedra (with self-intersections). BricardTs octahedra are special cases of Euclidean flexible suspensions. In 1974 R. Connelly [Con75] proved that some combination of the lengths of all edges of the equator of a flexible suspension in Euclidean 3-space is equal to zero (each length is taken either positive or negative in this combination). The method applied by R. Connelly, is to reduce the problem to the study of an analytic function of complex variable in neighborhoods of its branch points.

In 2001 S. N. Mikhalev [Mik01] reproved the above-mentioned result of R. Connelly by algebraic methods. Moreover, S. N. Mikhalev proved that for every spatial quadrilateral formed by edges of a flexible suspension and containing its both poles there is a combination of the lengths (taken either positive or negative) of the edges of the quadrilateral, which is equal to zero.

The aim of this work is to prove a similar result for the equator of a flexible suspension in hyperbolic 3-space, applying the method of Connelly [Con75].

2.1 Formulating the flexibility condition

Let $\mathcal{K}$ be a simplicial complex. A *polyhedron* (a *polyhedral surface*) in hyperbolic 3-space is a continuous map from $\mathcal{K}$ to $\mathbb{H}^3$, which sends every k-dimensional simplex of $\mathcal{K}$ into a subset of a k-dimensional plane of hyperbolic space ($k \leq 2$). Images of topological 2-simplices are called faces, images of topological 1-simplices are called edges and images of topological 0-simplices are called vertices of the polyhedron. Note that in our definition an image of a simplex can be degenerate (for instance, a face can lie on a straight the hyperbolic line, and an edge can be reduced to one point), and faces can intersect in their interior points. If $v_1, ..., v_W$ are the vertices of $\mathcal{K}$, and if $\mathcal{P} : \mathcal{K} \to \mathbb{H}^3$ is a polyhedron, then $\mathcal{P}$ is determined by W points $P_1, ..., P_W \in \mathbb{H}^3$, where $P_j \stackrel{\text{def}}{=} \mathcal{P}(v_j)$, $j = 1, ..., W$.

If $\mathcal{P} : \mathcal{K} \to \mathbb{H}^3$ and $\mathcal{Q} : \mathcal{K} \to \mathbb{H}^3$ are two polyhedra, then we say $\mathcal{P}$ and $\mathcal{Q}$ are *congruent* if there exists a motion $\mathcal{A} : \mathbb{H}^3 \to \mathbb{H}^3$ such that $\mathcal{Q} = \mathcal{A} \circ \mathcal{P}$ (i.e. the isometric mapping $\mathcal{A}$ sends every vertex of $\mathcal{P}$ into a corresponding vertex of $\mathcal{Q}$: $Q_j = \mathcal{A}(P_j)$, or in other words

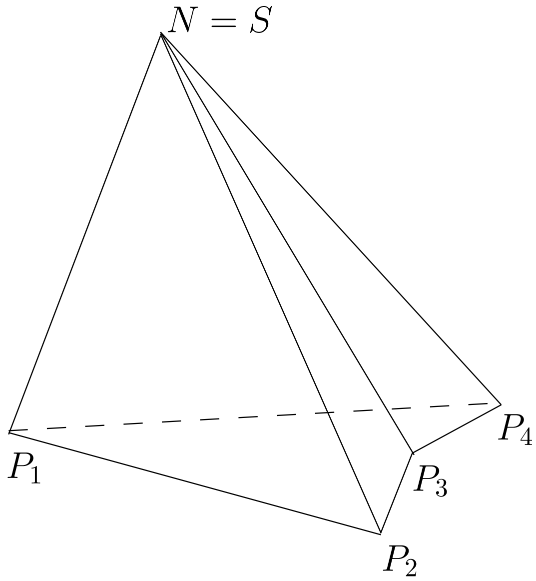

Figure 2.1: A double covered cap.

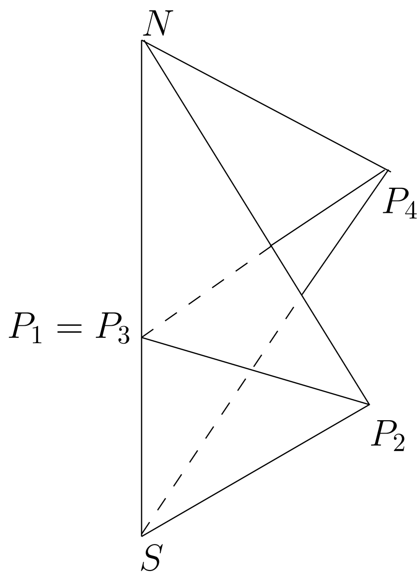

Figure 2.2: A suspension with a wing.

$\mathcal{Q}(v_j) = \mathcal{A}(\mathcal{P}(v_j))$, $j = 1, ..., W$). We say $\mathcal{P}$ and $\mathcal{Q}$ are *isometric* (*in the intrinsic metric*) if each edge of $\mathcal{P}$ has the same length as the corresponding edge of $\mathcal{Q}$, i.e. if $\langle v_j, v_k \rangle$ is a 1-simplex of $\mathcal{K}$ then $\mathrm{d}_{\mathbb{H}^3}(Q_j, Q_k) = \mathrm{d}_{\mathbb{H}^3}(P_j, P_k)$, where $\mathrm{d}_{\mathbb{H}^3}(\cdot, \cdot)$ stands for the distance in hyperbolic space $\mathbb{H}^3$.

A polyhedron $\mathcal{P}$ is *flexible* if, for some continuous one parameter family of polyhedra $\mathcal{P}_t : \mathcal{K} \to \mathbb{H}^3$, $0 \le t \le 1$, the following three conditions hold true: (1) $\mathcal{P}_0 = \mathcal{P}$; (2) each $\mathcal{P}_t$ is isometric to $\mathcal{P}_0$; (3) some $\mathcal{P}_t$ is not congruent to $\mathcal{P}_0$.

Let $\mathcal{K}$ be defined as follows: $\mathcal{K}$ has vertices $v_0, v_1, ..., v_V, v_{V+1}$, where $v_1, ..., v_V$ form a cycle (v_j adjacent to v_{j+1}, $j = 1, ..., V-1$, and v_V adjacent to v_1), and v_0 and v_{V+1} are each adjacent to all of $v_1, ..., v_V$. Each polyhedron $\mathcal{P}$ based on $\mathcal{K}$ is called a *suspension*. Call $N \stackrel{\mathrm{def}}{=} \mathcal{P}(v_0)$ the north pole, and $S \stackrel{\mathrm{def}}{=} \mathcal{P}(v_{V+1})$ the south pole, and $P_j \stackrel{\mathrm{def}}{=} \mathcal{P}(v_j)$, $j = 1, ..., V$ vertices of the *equator* $\mathcal{P}$.

Assume that a suspension $\mathcal{P}$ is flexible. If we suppose the segment NS to be an extra edge, then $\mathcal{P}$ becomes a set of V tetrahedra glued cyclically along their common edge NS. We call a suspension *nondegenerate* if none of these tetrahedra lies in a hyperbolic 2-plane. Note that a nondegenerate suspension $\mathcal{P}$ does not flex if the distance between N and S remains constant. Therefore, as in the Euclidean case [Con75] we assume that the length of NS is variable during the flex of $\mathcal{P}$. Examples of degenerate suspensions are a double covered cap — a suspension with coinciding poles (see Fig. 2.1), and a suspension with a wing — a suspension whose vertices N, S, P_{i-1}, and P_{i+1} lie on a straight line for some i (see Fig. 2.2). In this chapter we will not study the degenerate flexible suspensions.

In this Chapter we prove

Theorem 2.1. *Let $\mathcal{P}$ be a nondegenerate flexible suspension in hyperbolic 3-space with the poles S and N, and with the vertices of the equator P_j, $j = 1, ..., V$. Then for some set of signs $\sigma_{j,j+1} \in \{+1, -1\}$, $j = 1, ..., V$, the combination of the lengths $e_{j,j+1}$ of all edges P_jP_{j+1} of the*

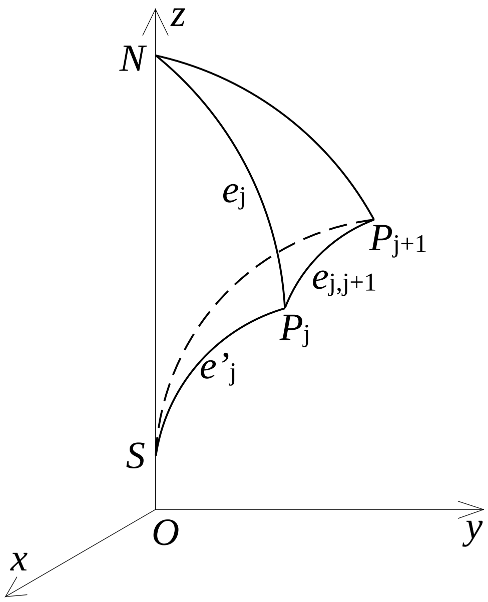

Figure 2.3: A fragment of the lateral surface of $\mathcal{P}$.

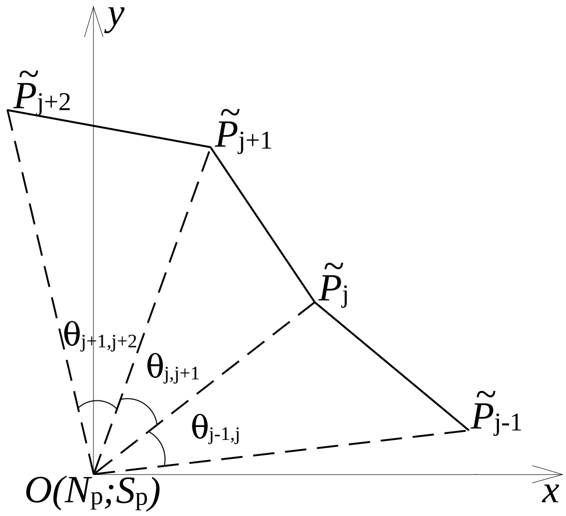

Figure 2.4: A projection of $\mathcal{P}$ on Oxy .

equator of $\mathcal{P}$ taken with the corresponding signs $\sigma_{j,j+1}$ is equal to zero, i.e.

$$\sum_{j=1}^{V} \sigma_{j,j+1} e_{j,j+1} = 0. \tag{2.1}$$

(*Here and below, by definition, it is considered that $P_{V+1} \stackrel{\text{def}}{=} P_1$, $P_V P_{V+1} \stackrel{\text{def}}{=} P_V P_1$, $\sigma_{V,V+1} \stackrel{\text{def}}{=} \sigma_{V,1}$, and $e_{V,V+1} \stackrel{\text{def}}{=} e_{V,1}$.*)

This result is published in [Slu13].

2.2 Connelly's equation of flexibility of a suspension

R. Connelly in [Con75] obtained an equation of flexibility of a nondegenerate suspension in Euclidean 3-space. Following him, in this section we will obtain an equation of flexibility of a nondegenerate suspension in hyperbolic 3-space.

Let us place a nondegenerate suspension $\mathcal{P}$ into the Poincaré upper half-space model [And05] of hyperbolic 3-space $\mathbb{H}^3$ in such a way that the poles N and S of $\mathcal{P}$ lie on the axis Oz of the Cartesian coordinate system of the Poincaré model (see Fig. 2.3). Let S has the coordinates $(0, 0, z_S)$, N has the coordinates $(0, 0, z_N)$, and P_j has the coordinates (x_j, y_j, z_j), $j = 1, ..., V$. Also we denote the length of the edge NP_j by e_j, and the length of SP_j by e'_j, $j = 1, ..., V$.

Consider a Euclidean orthogonal projection $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ in the plane Oxy (see Fig. 2.4). Also $\widetilde{\mathcal{P}}$ is a the hyperbolic projection of $\mathcal{P}$ on Oxy from the only point at infinity of $\mathbb{H}^3$ which does not lie on Oxy. This projection sends poles N and S of $\mathcal{P}$ to the origin O $(0, 0)$ in the plane Oxy, P_j to the point $\widetilde{P}_j$ (x_j, y_j), edges NP_j and SP_j to the Euclidean segment $O\widetilde{P}_j$, and the egde $P_j P_{j+1}$

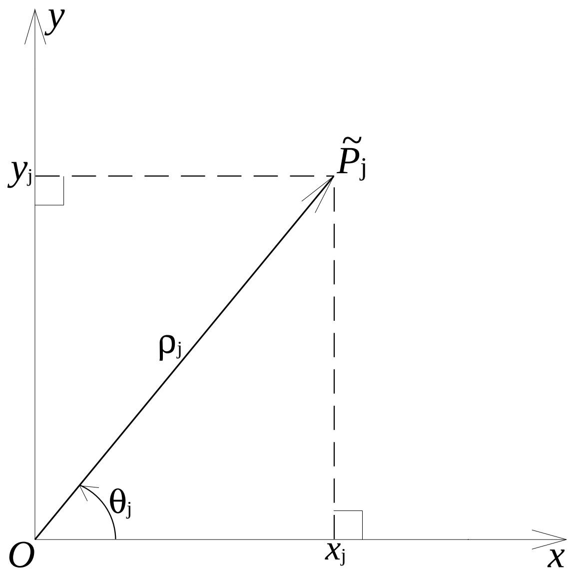

Figure 2.5: The coordinates of $\widetilde{P}_j$.

of the equator of $\mathcal{P}$ to the Euclidean segment $\widetilde{P}_j\widetilde{P}_{j+1}$, $j = 1, ..., V$ (here and below $\widetilde{P}_{V+1} \stackrel{\text{def}}{=} \widetilde{P}_1$, $x_{V+1} \stackrel{\text{def}}{=} x_1$, $y_{V+1} \stackrel{\text{def}}{=} y_1$, $z_{V+1} \stackrel{\text{def}}{=} z_1$).

Polar coordinates (ρ_j, θ_j) of $\widetilde{P}_j$, $j = 1, ..., V$, are related to its Cartesian coordinates by the formulas (see Fig. 2.5):

$$\rho_j = \sqrt{x_j^2 + y_j^2}, \quad \sin\theta_j = \frac{y_j}{\rho_j} = \frac{y_j}{\sqrt{x_j^2 + y_j^2}}, \quad \cos\theta_j = \frac{x_j}{\rho_j} = \frac{x_j}{\sqrt{x_j^2 + y_j^2}}. \tag{2.2}$$

Note that by construction, the dihedral angle $\theta_{j,j+1}$ of the tetrahedron NSP_jP_{j+1} at the edge NS is equal to the flat angle $\angle\widetilde{P}_jO\widetilde{P}_{j+1}$, $j = 1, ..., V$, and

$$\theta_{j,j+1} = \theta_{j+1} - \theta_j. \tag{2.3}$$

Note as well that the value of $\theta_{j,j+1}$ can be negative. Applying the trigonometric ratio of the difference of two angles and (2.3), we get:

$$\cos\theta_{j,j+1} = \cos\theta_{j+1}\cos\theta_j + \sin\theta_{j+1}\sin\theta_j, \quad \sin\theta_{j,j+1} = \sin\theta_{j+1}\cos\theta_j - \cos\theta_{j+1}\sin\theta_j. \tag{2.4}$$

Taking into account (2.2) we reduce (2.4) to

$$\cos\theta_{j,j+1} = \frac{x_jx_{j+1} + y_jy_{j+1}}{\sqrt{x_{j+1}^2 + y_{j+1}^2}\sqrt{x_j^2 + y_j^2}}, \quad \sin\theta_{j,j+1} = \frac{x_jy_{j+1} - y_jx_{j+1}}{\sqrt{x_{j+1}^2 + y_{j+1}^2}\sqrt{x_j^2 + y_j^2}}.$$

Then, according to Euler's formula,

$$e^{i\theta_{j,j+1}} = \cos\theta_{j,j+1} + i\sin\theta_{j,j+1} = \frac{(x_jx_{j+1} + y_jy_{j+1}) + i(x_jy_{j+1} - y_jx_{j+1})}{\sqrt{x_{j+1}^2 + y_{j+1}^2}\sqrt{x_j^2 + y_j^2}}. \tag{2.5}$$

Following R. Connelly [Con75], we remark that the sum of the dihedral angles $\theta_{j,j+1}$ of all tetrahedra NSP_jP_{j+1}, $j = 1, ..., V$, at the edge NS is constant and a multiple of 2π (here and below $\theta_{V,V+1} \stackrel{\text{def}}{=} \theta_{V,1}$, $\theta_{V+1} \stackrel{\text{def}}{=} \theta_1$, $\rho_{V+1} \stackrel{\text{def}}{=} \rho_1$), i.e.

$$\sum_{j=1}^{V} \theta_{j,j+1} = 2\pi m \quad \text{for some integer } m, \tag{2.6}$$

and remains so during the deformation of the suspension, when the values of the angles $\theta_{j,j+1}$, $j = 1, ..., V$, vary continuously.

We rewrite the equation of flexibility (2.6) in a convenient form:

$$\prod_{j=1}^{V} e^{i\theta_{j,j+1}} = 1. \tag{2.7}$$

Thus, taking into account (2.5), we see that coordinates of vertices of $\mathcal{P}$ are related as follows:

$$\prod_{j=1}^{V} \frac{(x_j x_{j+1} + y_j y_{j+1}) + i(x_j y_{j+1} - y_j x_{j+1})}{x_j^2 + y_j^2} = 1, \tag{2.8}$$

or in other notation

$$\prod_{j=1}^{V} F_{j,j+1} = \prod_{j=1}^{V} \frac{G_{j,j+1}}{\rho_j \rho_{j+1}} = \prod_{j=1}^{V} \frac{G_{j,j+1}}{\rho_j^2} = 1, \tag{2.9}$$

where $G_{j,m} = (x_j x_m + y_j y_m) + i(x_j y_m - y_j x_m)$, $F_{j,m} = \frac{G_{j,m}}{\rho_j \rho_m}$, $j, m = 1, ..., V$, and $G_{V,V+1} \stackrel{\text{def}}{=} G_{V,1}$, $F_{V,V+1} \stackrel{\text{def}}{=} F_{V,1}$.

When studying the deformation $\mathcal{P}_t$ of the suspension $\mathcal{P}$, all objects and values related to $\mathcal{P}_t$ naturally succeed from the notation for the corresponding entities related to $\mathcal{P}$. For example, the coordinate $x_j(t)$ of the point $P_j(t)$ of the deformation $\mathcal{P}_t$ corresponds to the coordinate x_j of the point P_j of the suspension $\mathcal{P}$, the dihedral angle $\theta_{j,j+1}(t)$ of the tetrahedron $N(t)S(t)P_j(t)P_{j+1}(t)$ at the edge $N(t)S(t)$ corresponds to the dihedral angle $\theta_{j,j+1}$ of the tetrahedron NSP_jP_{j+1} at the edge NS, etc.

2.3 The equation of flexibility of a suspension in terms of the lengths of its edges

In this section we are going to express the equation of flexibility of a suspension (2.8) in terms of the lengths of edges of $\mathcal{P}$. Recall that the lengths of the edges of $\mathcal{P}$ remain constant during the flex. To this purpose we need to demonstrate the truth of two following statements. The first of them can be verified by direct calculation (see also Fig. 2.6).

Lemma 2.2. *Given a Poincaré upper half-plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$). Then the distance between the points A (ρ_0, z_A) and B (ρ_0, z_B), having the same first coordinate ρ_0, is calculated by the formula*

$$\mathrm{d}_{\mathbb{H}^2}(A, B) = \left| \ln \frac{z_B}{z_A} \right|. \tag{2.10}$$

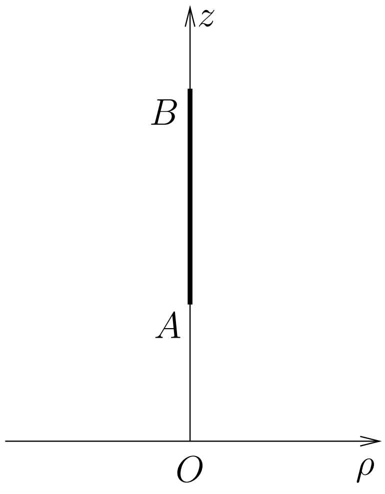

Figure 2.6: Points in a plane from the lemma 2.2.

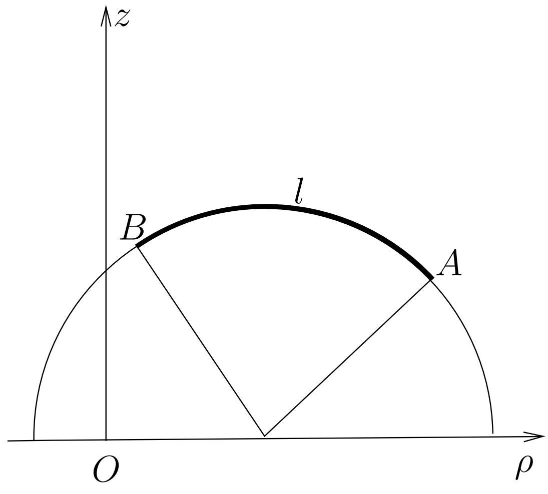

Figure 2.7: Points in a plane from the lemma 2.3.

Lemma 2.3. *Given a Poincaré upper half-plane* $\mathbb{H}^2$ *with the coordinates* (ρ, z) *(i.e., with the metric given by the formula* $ds^2 = \frac{d\rho^2+dz^2}{z^2}$*). Then the distance* $l \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(A, B)$ *between the points* A (ρ_A, z_A) *and* B (ρ_B, z_B) *is related to their coordinates by the formula*

$$(\rho_B - \rho_A)^2 + z_A^2 + z_B^2 = 2 z_A z_B \cosh l. \tag{2.11}$$

Proof. According to the part (2) of the Corollary $A.5.8$ [BP03], the distance between the points with the coordinates (x, t) and (y, s) in the Poincaré upper half-space model $\mathbb{R}^n \times \mathbb{R}^+$ of hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ is calculated by the formula

$$\mathrm{d}_{\mathbb{H}^{n+1}}((x,t),(y,s)) = 2\,\mathrm{artanh}\left(\frac{\|x-y\|^2+(t-s)^2}{\|x-y\|^2+(t+s)^2}\right)^{1/2}, \tag{2.12}$$

where the symbol $\|\cdot\|$ stands for the standard Euclidean norm in $\mathbb{R}^n$.

By (2.12) the distance between the points A and B (see Fig. 2.7) is calculated by the formula

$$l = 2\,\mathrm{artanh}\left(\frac{(\rho_A-\rho_B)^2+(z_A-z_B)^2}{(\rho_A-\rho_B)^2+(z_A+z_B)^2}\right)^{1/2}, \tag{2.13}$$

where $n = 1$, $(x, t) = (\rho_A, z_A)$ and $(y, s) = (\rho_B, z_B)$.

After a series of transformations of the formula (2.13) we get:

$$(\rho_A-\rho_B)^2\left(\cosh^2\frac{l}{2}-\sinh^2\frac{l}{2}\right)+(z_A^2+z_B^2)\left(\cosh^2\frac{l}{2}-\sinh^2\frac{l}{2}\right) = 2z_Az_B\left(\cosh^2\frac{l}{2}+\sinh^2\frac{l}{2}\right). \tag{2.14}$$

By two identities of the hyperbolic geometry, $\cosh^2\frac{l}{2} - \sinh^2\frac{l}{2} = 1$ and $\cosh l = \cosh^2\frac{l}{2} + \sinh^2\frac{l}{2}$, (2.14) reduces to (2.11). □

Let us express $G_{j,j+1}$ and ρ_j^2 in terms of the length of edges of $\mathcal{P}$.

We assume that the coordinates of the south pole S are $(0, 0, 1)$. Let $t \stackrel{\text{def}}{=} e^{\mathrm{d}_{\mathbb{H}^3}(N,S)}$, where $\mathrm{d}_{\mathbb{H}^3}(N, S)$ is the distance between the poles N and S of $\mathcal{P}$. Without loss of generality, we assume that $z_N \geq z_S$. Then, by Lemma 2.2, the coordinates of N are $(0, 0, t)$.

Applying Lemma 2.3 to the points S and P_j lying in the hyperbolic plane SNP_j, by the formula (2.11) we get:

$$\rho_j^2 + z_j^2 + 1 = 2z_j \cosh e'_j. \tag{2.15}$$

Now we apply Lemma 2.3 to the vertices N and P_j:

$$\rho_j^2 + z_j^2 + t^2 = 2tz_j \cosh e_j. \tag{2.16}$$

Subtracting (2.15) from (2.16), under the assumption that $t\cosh e_j \neq \cosh e'_j$, we get:

$$z_j = \frac{t^2-1}{2(t\cosh e_j - \cosh e'_j)}. \tag{2.17}$$

Also, taking into account (2.15) and (2.17), we obtain:

$$\rho_j^2 = 2z_j\cosh e'_j - z_j^2 - 1 = \frac{(t^2-1)\cosh e'_j}{(t\cosh e_j - \cosh e'_j)} - \frac{(t^2-1)^2}{4(t\cosh e_j - \cosh e'_j)^2} - 1. \tag{2.18}$$

Let $\rho_{j,j+1}$ denote the Euclidean distance between the points $\widetilde{P}_j$ and $\widetilde{P}_{j+1}$, $j = 1, ..., V$ (here and below $\rho_{V,V+1} \stackrel{\text{def}}{=} \rho_{V,1}$). Applying Lemma 2.3 to the vertices P_j and P_{j+1}, we get:

$$\rho_{j,j+1}^2 = 2z_j z_{j+1}\cosh e_{j,j+1} - z_j^2 - z_{j+1}^2. \tag{2.19}$$

By the Pythagorean theorem $\rho_{j,j+1}$ is related to the Cartesian coordinates of $\widetilde{P}_j$ and $\widetilde{P}_{j+1}$ by the formula

$$\rho_{j,j+1} = \sqrt{(x_{j+1}-x_j)^2 + (y_{j+1}-y_j)^2}. \tag{2.20}$$

By (2.2) the equation (2.20) reduces to:

$$\rho_{j,j+1}^2 = (x_j^2+y_j^2) + (x_{j+1}^2+y_{j+1}^2) - 2(x_jx_{j+1}+y_jy_{j+1}) = \rho_j^2 + \rho_{j+1}^2 - 2(x_jx_{j+1}+y_jy_{j+1}).$$

Thus, taking into account (2.18) and (2.19), the expression $x_jx_{j+1}+y_jy_{j+1}$, which is a part of $G_{j,j+1}$ from (2.9), is related to the lengths of edges of $\mathcal{P}$ by the formula

$$x_jx_{j+1}+y_jy_{j+1} = \frac{\rho_j^2+\rho_{j+1}^2-\rho_{j,j+1}^2}{2} = z_j\cosh e'_j + z_{j+1}\cosh e'_{j+1} - z_jz_{j+1}\cosh e_{j,j+1} - 1. \tag{2.21}$$

Substituting (2.17) in (2.21) we get:

$$x_jx_{j+1}+y_jy_{j+1} = \frac{1}{2}\Bigg(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j-\cosh e'_j)} + \frac{(t^2-1)\cosh e'_{j+1}}{(t\cosh e_{j+1}-\cosh e'_{j+1})} - $$

$$- \frac{(t^2-1)^2\cosh e_{j,j+1}}{2(t\cosh e_j-\cosh e'_j)(t\cosh e_{j+1}-\cosh e'_{j+1})} - 2\Bigg). \tag{2.22}$$

Let us now express $x_jy_{j+1}-y_jx_{j+1}$, which is also a part of $G_{j,j+1}$, in terms of the length of edges of $\mathcal{P}$.

According to (2.5) we know that

$$\cos\theta_{j,j+1} = \frac{x_jx_{j+1}+y_jy_{j+1}}{\rho_j\rho_{j+1}} \quad \text{and} \quad \sin\theta_{j,j+1} = \frac{x_jy_{j+1}-y_jx_{j+1}}{\rho_j\rho_{j+1}}. \tag{2.23}$$

Note that by definition (2.2), $\rho_j > 0$, $j = 1, ..., V$.

By the Pythagorean trigonometric identity, the formula

$$\sin\theta_{j,j+1} = \sigma_{j,j+1}\sqrt{1-\cos^2\theta_{j,j+1}} \tag{2.24}$$

holds true, where $\sigma_{j,j+1} = 1$ if $\sin\theta_{j,j+1} \geq 0$, and $\sigma_{j,j+1} = -1$ if $\sin\theta_{j,j+1} < 0$ (remind that $\theta_{j,j+1}$ is determined in (2.3)). Then (2.23) and (2.24) imply

$$x_jy_{j+1} - y_jx_{j+1} = \rho_j\rho_{j+1}\sin\theta_{j,j+1} = \sigma_{j,j+1}\rho_j\rho_{j+1}\sqrt{1-\cos^2\theta_{j,j+1}} =$$

$$= \sigma_{j,j+1}\rho_j\rho_{j+1}\sqrt{1-\frac{(x_jx_{j+1}+y_jy_{j+1})^2}{\rho_j^2\rho_{j+1}^2}} = \sigma_{j,j+1}\sqrt{\rho_j^2\rho_{j+1}^2-(x_jx_{j+1}+y_jy_{j+1})^2}. \tag{2.25}$$

Substituting (2.18) and (2.22) in (2.25) we get

$$x_jy_{j+1} - y_jx_{j+1} = \sigma_{j,j+1}\Bigg[\bigg(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j - \cosh e'_j)} - \frac{(t^2-1)^2}{4(t\cosh e_j-\cosh e'_j)^2} - 1\bigg)\times$$

$$\times\bigg(\frac{(t^2-1)\cosh e'_{j+1}}{(t\cosh e_{j+1} - \cosh e'_{j+1})} - \frac{(t^2-1)^2}{4(t\cosh e_{j+1}-\cosh e'_{j+1})^2} - 1\bigg) - \frac{1}{4}\bigg(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j - \cosh e'_j)}+$$

$$+\frac{(t^2-1)\cosh e'_{j+1}}{(t\cosh e_{j+1} - \cosh e'_{j+1})} - \frac{(t^2-1)^2\cosh e_{j,j+1}}{2(t\cosh e_j - \cosh e'_j)(t\cosh e_{j+1} - \cosh e'_{j+1})} - 2\bigg)^2\Bigg]^{\frac{1}{2}}. \tag{2.26}$$

Substituting (2.18), (2.22), and (2.26) in (2.8) we obtain the equation of flexibility of a suspension in terms of the lengths of edges of $\mathcal{P}$.

2.4 Proof of Theorem 2.1

Assume that a nondegenerate suspension $\mathcal{P}$ flexes. Then, as we have already mentioned in the section 2.1, the distance l_{NS} between the poles of $\mathcal{P}$ changes during the flex. Let $t \stackrel{\text{def}}{=} e^{l_{NS}}$ be the parameter of the flex of $\mathcal{P}$. The identity (2.9) holds true at every moment t of the flex, as the values of the expressions $F_{j,j+1}$, $G_{j,j+1}$, ρ_j^2, $j = 1, ..., V$, which make part (2.9), vary as t changes. Here the functions $G_{j,j+1}(t) = [x_jx_{j+1} + y_jy_{j+1}](t) + i[x_jy_{j+1} - y_jx_{j+1}](t)$ and $\rho_j^2(t)$, $j = 1, ..., V$, are determined in (2.18), (2.22) and (2.26).

Assume now that for some $j \in \{1, ..., V\}$ the dihedral angle $\theta_{j,j+1}(t)$ remains constant (the value of $\theta_{j,j+1}(t)$ can also be equal to zero) as t changes. In this case the length of the edge $N(t)S(t)$ of the tetrahedron $N(t)S(t)P_j(t)P_{j+1}(t)$ must be constant as well (all other edges of the tetrahedron are also the edges of $\mathcal{P}_t$, therefore there lengths are fixed), i.e. the value of t does not change. As we mentioned in the section 2.1, in this case $\mathcal{P}$ can not be flexible. Thus we have the contradiction. Therefore, the values of the angles $\theta_{j,j+1}(t)$, $j = 1, ..., V$, change continuously during the flex. Hence, there exists such an interval (t_1, t_2) that for all $t \in (t_1, t_2)$ it is true that $\theta_{j,j+1}(t) \neq 0$ for every $j \in \{1, ..., V\}$.

We extend both sides of the equation of flexibility (2.9) as functions in t on the whole complex plane $\mathbb{C}$. By Theorem on the uniqueness of the analytic function [Bit84], the expression (2.9) remains valid.

Recall that a function $\omega = f(z)$ of a single complex variable z is called algebraic, if there is a polynomial $p(\omega, z)$ in two variables which does not vanish identically and such that $p(f(z), z) \equiv 0$. It is known that an analytic function of a single complex variable is an algebraic function if and only if it has a finite number of branches and at most algebraic singularities [Ahl78, Theorem 4, p. 306]. Thus, the analytic functions $F_{j,j+1}(t)$, $j=1,...,V$, which are also algebraic, have a finite number of branch points. Without loss of generality we can assume that none of these points lies in the interval (t_1,t_2). For every $F_{j,j+1}(t)$, $j=1,...,V$, we choose a single-valued branch $(F_{j,j+1}(t),D)$, where $D\subset\mathbb{C}$ is an unbounded domain containing (t_1,t_2). Let $\mathcal{W}\subset D$ be a path connecting $t_0\in(t_1,t_2)$ and ∞, such that t_0 is a unique real point of $\mathcal{W}$. Let us calculate the limit of $F_{j,j+1}(t)$ as $t\to\infty$ along $\mathcal{W}$.

Taking into account (2.18) we get

$$\lim_{t\to\infty}\frac{\rho_j^2(t)}{t^2}=\lim_{t\to\infty}\Big[\frac{1}{t^2}\Big(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j-\cosh e'_j)}-\frac{(t^2-1)^2}{4(t\cosh e_j-\cosh e'_j)^2}-1\Big)\Big]=-\frac{1}{4\cosh^2 e_j}. \tag{2.27}$$

Similarly, from (2.22) we derive that

$$\lim_{t\to\infty}\frac{(x_jx_{j+1}+y_jy_{j+1})(t)}{t^2}=-\frac{\cosh e_{j,j+1}}{4\cosh e_j\cosh e_{j+1}}. \tag{2.28}$$

Also from (2.25) and taking into account (2.27) and (2.28) we have:

$$\lim_{t\to\infty}\frac{(x_jy_{j+1}-y_jx_{j+1})^2(t)}{t^4}=\lim_{t\to\infty}\left[\frac{\rho_j^2(t)\rho_{j+1}^2(t)-(x_jx_{j+1}+y_jy_{j+1})^2(t)}{t^4}\right]=$$

$$=\frac{1}{16\cosh^2 e_j\cosh^2 e_{j+1}}-\frac{\cosh^2 e_{j,j+1}}{16\cosh^2 e_j\cosh^2 e_{j+1}}=\frac{1-\cosh^2 e_{j,j+1}}{16\cosh^2 e_j\cosh^2 e_{j+1}}.$$

Hence,

$$\lim_{t\to\infty}\frac{(x_jy_{j+1}-y_jx_{j+1})(t)}{t^2}=i\sigma_{j,j+1}\frac{\sqrt{\cosh^2 e_{j,j+1}-1}}{4\cosh e_j\cosh e_{j+1}}, \tag{2.29}$$

where $\sigma_{j,j+1}\in\{+1,-1\}$ is determined by the single-valued branch $(F_{j,j+1}(t),D)$ and by the path $\mathcal{W}$.

By definition of $G_{j,j+1}(t)$ and according to (2.28) and (2.29), we get:

$$\lim_{t\to\infty}\frac{G_{j,j+1}(t)}{t^2}=-\frac{\cosh e_{j,j+1}+\sigma_{j,j+1}\sqrt{\cosh^2 e_{j,j+1}-1}}{4\cosh e_j\cosh e_{j+1}}. \tag{2.30}$$

By (2.30) and (2.27), the limit of the left-hand side of (2.9) at $t\to\infty$

$$\lim_{t\to\infty}\prod_{j=1}^{V}F_{j,j+1}(t)=\lim_{t\to\infty}\prod_{j=1}^{V}\frac{F_{j,j+1}(t)/t^2}{\rho_j^2(t)/t^2}=\prod_{j=1}^{V}\Big(\cosh e_{j,j+1}+\sigma_{j,j+1}\sqrt{\cosh^2 e_{j,j+1}-1}\Big),$$

and (2.9) at $t\to\infty$ transforms to

$$\prod_{j=1}^{V}\Big(\cosh e_{j,j+1}+\sigma_{j,j+1}\sqrt{\cosh^2 e_{j,j+1}-1}\Big)=1. \tag{2.31}$$

By the following trigonometric identity of the hyperbolic geometry, $\cosh^2 x-\sinh^2 x=1$, and because $e_{j,j+1}>0$, we have

$$\sqrt{\cosh^2 e_{j,j+1}-1}=\sqrt{\sinh^2 e_{j,j+1}}=\sinh e_{j,j+1}. \tag{2.32}$$

By (2.32) the equation (2.31) transforms to

$$\prod_{j=1}^{V} \big(\cosh e_{j,j+1} + \sigma_{j,j+1} \sinh e_{j,j+1} \big) = 1. \tag{2.33}$$

By $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, we have

$$\cosh e_{j,j+1} + \sigma_{j,j+1} \sinh e_{j,j+1} = \left\{ \begin{array}{ll} e^{e_{j,j+1}} & \text{for} \quad \sigma_{j,j+1} = 1, \\ e^{-e_{j,j+1}} & \text{for} \quad \sigma_{j,j+1} = -1. \end{array} \right. = e^{\sigma_{j,j+1} e_{j,j+1}}. \tag{2.34}$$

Substituting (2.34) in (2.33) and taking the logarithm of the resulting equation, we get (2.1) □.

The study of the behavior of the equation of flexibility (2.9) in neighborhoods of other branch points of the left-hand side of (2.9) did not give us interesting results: either we were obtaining trivial identities like $1 = 1$ (for instance, as $t \to \pm 1$), or the limit of the left-hand side of the equation of flexibility was too complicated to distinguish interesting patterns there.

2.5 Verification of the necessary flexibility condition of a nondegenerate suspension for the Bricard-Stachel octahedra in hyperbolic 3-space

In 2002 H. Stachel [Sta06] proved the flexibility of the analogues of the Bricard's octahedra in hyperbolic 3-space. Let us verify the validity of the necessary flexibility condition of a nondegenerate suspension for the Bricard-Stachel octahedra in hyperbolic 3-space.

We define an *octahedron* $\mathcal{O}$ as the suspension $NABCDS$ with the poles N and S, and with the vertices of the equator A, B, C, and D. Note that we can consider vertices A and C as the poles of $\mathcal{O}$ (in this case the quadrilateral $NDSB$ serves as the equator of $\mathcal{O}$). Also we can consider vertices B and D as the poles of $\mathcal{O}$ (in this case the quadrilateral $NCSA$ serves as the equator of $\mathcal{O}$).

2.5.1 Bricard-Stachel octahedra of types 1 and 2

The procedure of construction of the Bricard-Stachel octahedra of types 1 and 2 in hyperbolic 3-space is the same as for the Bricard's octahedra of types 1 and 2 in Euclidean 3-space [Sta06], [Ale10].

Any *Bricard-Stachel octahedron of type 1* in $\mathbb{H}^3$ can be constructed in the following way. Consider a disk-homeomorphic piece-wise linear surface $\mathcal{S}$ in $\mathbb{H}^3$ composed of four triangles ABN, BCN, CDN, and DAN such that $\mathrm{d}_{\mathbb{H}^3}(A,B) = \mathrm{d}_{\mathbb{H}^3}(C,D)$ and $\mathrm{d}_{\mathbb{H}^3}(B,C) = \mathrm{d}_{\mathbb{H}^3}(D,A)$. It is known that a spatial quadrilateral $ABCD$ which opposite sides have the same lengths, is symmetric with respect to a line $\mathcal{L}$ passing through the middle points of its diagonals AC and BD (see Fig. 2.8; for a more precise analogy with the Euclidean case, in this Figure as well as in the following Figures we draw polyhedra in the Kleinian model of hyperbolic space where lines and planes are intersections of Euclidean lines and planes with a fixed unit ball). Glue together $\mathcal{S}$ and its symmetric image with respect to L along $ABCD$. Denote by S the symmetric image of N under the symmetry with respect to L (see Fig. 2.9). The resulting polyhedral surface $NABCDS$ with self-intersections is flexible (because $\mathcal{S}$ is flexible) and combinatorially it is an octahedron (according to the definition given above). We will call it a Bricard-Stachel octahedron of type 1. By construction it follows that $\mathrm{d}_{\mathbb{H}^3}(A,N) = \mathrm{d}_{\mathbb{H}^3}(C,S)$, $\mathrm{d}_{\mathbb{H}^3}(B,N) = \mathrm{d}_{\mathbb{H}^3}(D,S)$, $\mathrm{d}_{\mathbb{H}^3}(C,N) = \mathrm{d}_{\mathbb{H}^3}(A,S)$, and $\mathrm{d}_{\mathbb{H}^3}(D,N) = \mathrm{d}_{\mathbb{H}^3}(B,S)$.

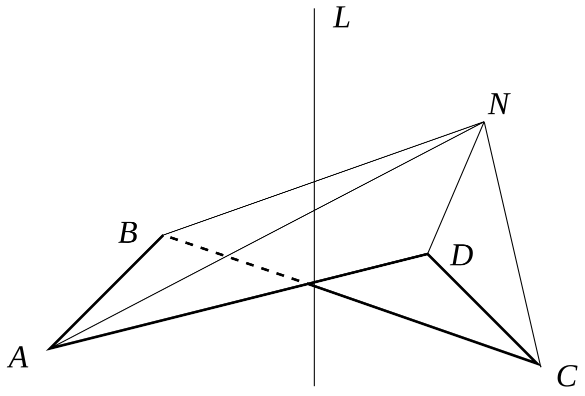

Figure 2.8: The construction of the Bricard-Stachel octahedron of type 1. Step 1.

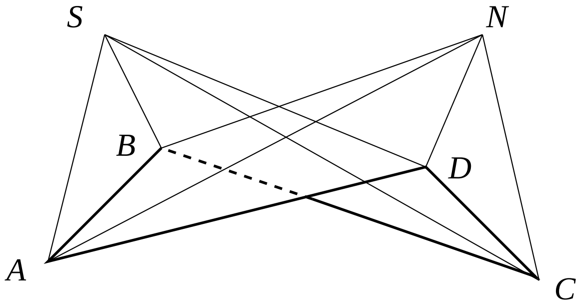

Figure 2.9: The construction of the Bricard-Stachel octahedron of type 1. Step 2.

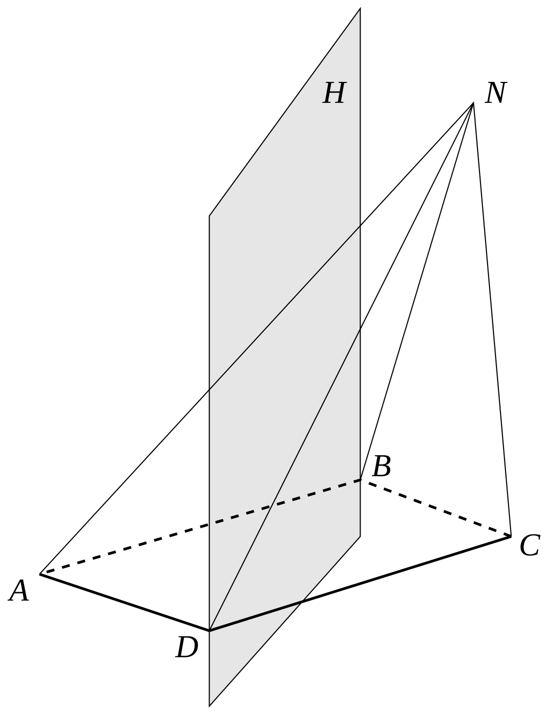

Figure 2.10: The construction of the Bricard-Stachel octahedron of type 2. Step 1.

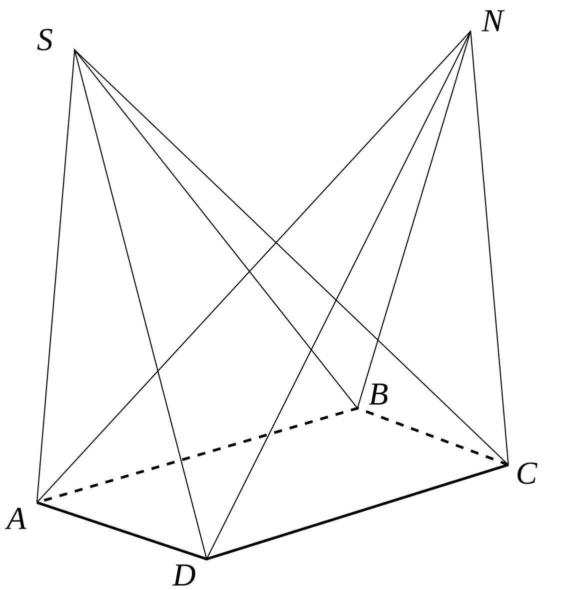

Figure 2.11: The construction of the Bricard-Stachel octahedron of type 2. Step 2.

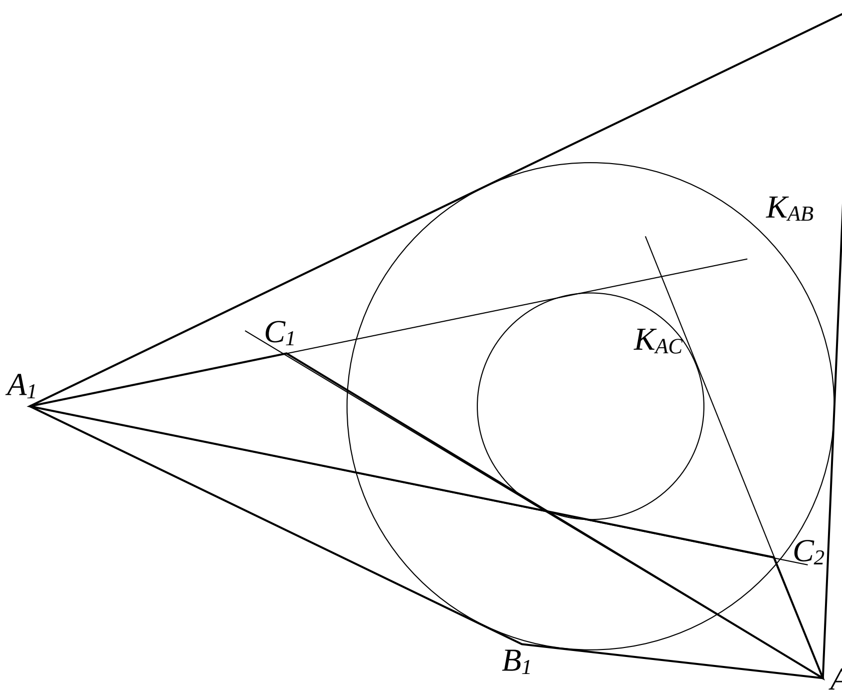

Figure 2.12: The construction of the Bricard-Stachel octahedron of type 3 based on circles. Step 1.

Any *Bricard-Stachel octahedron of type 2* in $\mathbb{H}^3$ can be constructed as follows. Consider a disk-homeomorphic piece-wise linear surface $\mathcal{S}$ in $\mathbb{H}^3$ composed of four triangles ABN, BCN, CDN, and DAN such that $\mathrm{d}_{\mathbb{H}^3}(A,B) = \mathrm{d}_{\mathbb{H}^3}(B,C)$ and $\mathrm{d}_{\mathbb{H}^3}(C,D) = \mathrm{d}_{\mathbb{H}^3}(D,A)$. It is known that a spatial quadrilateral $ABCD$ which neighbor sides at the vertices B and D have the same lengths, is symmetric with respect to a plane H which dissects the dihedral angle between the half-planes ABD and CBD (see Fig. 2.10). Glue together $\mathcal{S}$ and its symmetric image with respect to H along $ABCD$. Denote by S the symmetric image of N under the symmetry with respect to H (see Fig. 2.9). The resulting polyhedral surface $NABCDS$ with self-intersections is flexible (because $\mathcal{S}$ is flexible) and combinatorially it is an octahedron. We will call it a Bricard-Stachel octahedron of type 2. By construction it follows that $\mathrm{d}_{\mathbb{H}^3}(A,N) = \mathrm{d}_{\mathbb{H}^3}(C,S)$, $\mathrm{d}_{\mathbb{H}^3}(C,N) = \mathrm{d}_{\mathbb{H}^3}(A,S)$, $\mathrm{d}_{\mathbb{H}^3}(B,N) = \mathrm{d}_{\mathbb{H}^3}(B,S)$, and $\mathrm{d}_{\mathbb{H}^3}(D,N) = \mathrm{d}_{\mathbb{H}^3}(D,S)$.

It remains to note that for every considered octahedron each of three its equators has two pairs of edges of the same lengths. Hence, Theorem 2.1 is valid for the Bricard-Stachel octahedra of types 1 and 2.

2.5.2 Bricard-Stachel octahedra of type 3

There are three subtypes of the Bricard-Stachel octahedra of type 3 in hyperbolic space [Sta06] which construction is based on circles, horocycles or hypercircles correspondingly. The procedure of construction is common for all subtypes of the Bricard-Stachel octahedra of type 3 and it is the same as for the Bricard's octahedra of type 3 in Euclidean space.

Any *Bricard-Stachel octahedron of type 3* in $\mathbb{H}^3$ can be constructed in the following way. Let K_{AC} and K_{AB} be two different circles (horocycles, hypercircles) in $\mathbb{H}^2$ with the common center M and let A_1, A_2 be two different finite points outside K_{AC} and K_{AB}. In addition,

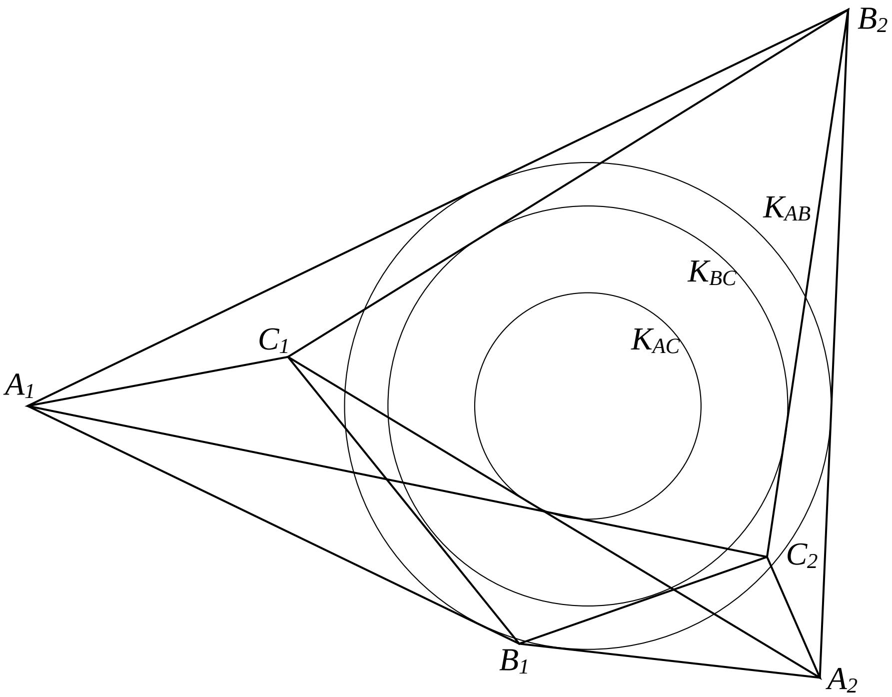

Figure 2.13: The construction of the Bricard-Stachel octahedron of type 3 based on circles. Step 2.

suppose that K_{AC}, K_{AB}, A_1 and A_2 are taken in such a way that the straight lines tangent to K_{AB} and passing through A_1 and A_2 intersect pairwise in finite points of $\mathbb{H}^2$ and form a quadrilateral $A_1B_1A_2B_2$ tangent to K_{AB}; moreover, that the straight lines tangent to K_{AC} and passing through A_1 and A_2 intersect pairwise in finite points of $\mathbb{H}^2$ and form a quadrilateral $A_1C_1A_2C_2$ tangent to K_{AC} (see Fig. 2.12; for clarity, we placed circles K_{AB} and K_{AC} so that their common center coincides with the center of the Kleinian model of hyperbolic space. In this case K_{AB} and K_{AC} are Euclidean circles as well). A polyhedron $\mathcal{O}$ with the vertices A_i, B_j, C_k, with the edges A_iB_j, A_iC_k, B_jC_k, and with the faces $\triangle A_iB_jC_k$, $i, j, k \in \{1, 2\}$, is an octahedron in the sense of the definition given above (see Fig. 2.13). The following pairs of vertices can serve as the poles of $\mathcal{O}$: (A_1, A_2) with the corresponding equator $B_1C_1B_2C_2$, (B_1, B_2) with the equator $A_1C_1A_2C_2$, and (C_1, C_2) with the equator $A_1B_1A_2B_2$. Suppose in addition that $\mathcal{O}$ does not have symmetries. We will call such octahedron $\mathcal{O}$ a Bricard-Stachel octahedron of type 3.

According to H. Stachel [Sta06], $\mathcal{O}$ flexes continuously in $\mathbb{H}^3$. Moreover, $\mathcal{O}$ admits two flat positions during the flex (we constructed $\mathcal{O}$ in one of its flat positions). Hence, for every equator of $\mathcal{O}$, $A_1B_1A_2B_2$, $B_1C_1B_2C_2$, and $A_1C_1A_2C_2$, all straight lines containing a side of the equator are tangent to some circle (horocycle, hypercircle) at least in one flat position of $\mathcal{O}$. Using this fact, we will prove that Theorem 2.1 is valid for the Bricard-Stachel octahedra of type 3. We have to consider three possible cases: when an equator of $\mathcal{O}$ is tangent to a circle, to a horocycle, or to a hypercircle in $\mathbb{H}^2$. Here we study the most common situation when any three vertices of an equator of a flexible octahedron in its flat position do not lie on a straight line.

An equator of a Bricard-Stachel octahedron of type 3 is tangent to a circle in $\mathbb{H}^2$

Let M be the center of the circle K_{AB} with the radius R in $\mathbb{H}^2$ and let all straight lines containing a side of the quadrilateral $A_1B_1A_2B_2$ are tangent to K_{AB}. Let us draw the segments MP_1, MP_2, MP_3, MP_4 connecting M with the straight lines A_1B_2, A_2B_2, A_2B_1, A_1B_1 and perpendicular to the corresponding lines. By construction, $\mathrm{d}_{\mathbb{H}^2}(M,P_1) = \mathrm{d}_{\mathbb{H}^2}(M,P_2) = \mathrm{d}_{\mathbb{H}^2}(M,P_3) = \mathrm{d}_{\mathbb{H}^2}(M,P_4) = R$.

By the Pythagorean theorem for hyperbolic space [AVS93] applied to $\triangle A_1MP_1$ and $\triangle A_1MP_4$, we obtain: $\cosh \mathrm{d}_{\mathbb{H}^2}(A_1,P_1) = \cosh \mathrm{d}_{\mathbb{H}^2}(A_1,P_4) = \cosh \mathrm{d}_{\mathbb{H}^2}(A_1,M)/\cosh R$. Then $a \stackrel{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(A_1,P_1) = \mathrm{d}_{\mathbb{H}^2}(A_1,P_4)$. Similarly we get: $b \stackrel{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(B_2,P_1) = \mathrm{d}_{\mathbb{H}^2}(B_2,P_2)$, $c \stackrel{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(A_2,P_2) = \mathrm{d}_{\mathbb{H}^2}(A_2,P_3)$, and $d \stackrel{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(B_1,P_3) = \mathrm{d}_{\mathbb{H}^2}(B_1,P_4)$.

If the circle K_{AB} is inscribed in the quadrilateral $A_1B_1A_2B_2$ (see Fig. 2.12), then $\mathrm{d}_{\mathbb{H}^2}(A_1,B_2) = a+b$, $\mathrm{d}_{\mathbb{H}^2}(A_2,B_2) = b+c$, $\mathrm{d}_{\mathbb{H}^2}(A_2,B_1) = c+d$, $\mathrm{d}_{\mathbb{H}^2}(A_1,B_1) = a+d$, and the identity

$$\mathrm{d}_{\mathbb{H}^2}(A_1,B_2) - \mathrm{d}_{\mathbb{H}^2}(A_2,B_2) + \mathrm{d}_{\mathbb{H}^2}(A_1,B_1) - \mathrm{d}_{\mathbb{H}^2}(A_1,B_1) = 0 \tag{2.35}$$

holds true.

If the circle K_{AB} is tangent to the quadrilateral $A_1B_1A_2B_2$ externally (this case corresponds to the quadrilateral $A_1C_1A_2C_2$ and to the circle K_{AC} in the Fig. 2.12), then $\mathrm{d}_{\mathbb{H}^2}(A_1,B_2) = a-b$, $\mathrm{d}_{\mathbb{H}^2}(A_2,B_2) = b+c$, $\mathrm{d}_{\mathbb{H}^2}(A_2,B_1) = c-d$, $\mathrm{d}_{\mathbb{H}^2}(A_1,B_1) = a+d$, and the identity

$$\mathrm{d}_{\mathbb{H}^2}(A_1,B_2) + \mathrm{d}_{\mathbb{H}^2}(A_2,B_2) - \mathrm{d}_{\mathbb{H}^2}(A_1,B_1) - \mathrm{d}_{\mathbb{H}^2}(A_1,B_1) = 0 \tag{2.36}$$

holds true.

By (2.35) and (2.36), the theorem 2.1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a circle in at least one of its flat positions.

An equator of a Bricard-Stachel octahedron of type 3 is tangent to a horocycle in $\mathbb{H}^2$

Let us consider the Poincaré upper half-plane model of the hyperbolic plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2+dz^2}{z^2}$). Without loss of generality we can assume that the center of the horocycle tangent to the equator of a Bricard-Stachel octahedron $\mathcal{O}$ of type 3, coincides with the (unique) point ∞ at infinity of $\mathbb{H}^2$ which does not lie on the Euclidean line $z = 0$. We denote the family of such horocycles by $K = \{\rho = R | R > 0\}$. Let $K_R \in K$ and let $A_1 = (\rho_{A_1}, z_{A_1})$ and $A_2 = (\rho_{A_2}, z_{A_2})$ be two opposite vertices of $\mathcal{O}$, such that the straight line (in $\mathbb{H}^2$) passing through A_1 and A_2 is not tangent to K_R. All the vertices of $\mathcal{O}$ are located outside K_R, hence $z_{A_1} < R$ and $z_{A_2} < R$. We will construct all possible quadrangles tangent to K_R with the opposite vertices A_1 and A_2, i.e., all quadrangles that can serve as equators of $\mathcal{O}$. Then we will verify the validity of the theorem 2.1 for such quadrangles.

Let $T = (\rho_T, z_T)$ be a point in $\mathbb{H}^2$ and let Λ be a straight line in $\mathbb{H}^2$ passing through T which is realized in the Poincaré upper half-plane as the Euclidean demi-circle with the radius $\sqrt{(\rho_T - \rho_{T,\Lambda})^2 + z_T^2}$ and with the center $O_\Lambda^T = (\rho_{T,\Lambda}, 0)$. Then the angle $\varphi_T^\Lambda \stackrel{\mathrm{def}}{=} \angle T O_\Lambda^T \rho \in (0,\pi)$ determines uniquely a position of T on Λ.

Remark 2.4. *For every finite point $T = (\rho_T, z_T)$, $z_T < R$, there exist precisely two straight lines Λ_l^T and Λ_r^T tangent to the horocycle K_R and containing T. They are realized in the Poincaré upper half-plane as the Euclidean demi-circles with the radius R and with the centers $O_l^T = (\rho_{T,l}, 0)$ and $O_r^T = (\rho_{T,r}, 0)$, $\rho_{T,l} \le \rho_T \le \rho_{T,r}$. The angles $\varphi_T^l \stackrel{\mathrm{def}}{=} \angle T O_l^T \rho$ and $\varphi_T^r \stackrel{\mathrm{def}}{=} \angle T O_r^T \rho$ serve as the coordinates of T on Λ_l^T and Λ_r^T correspondingly. Then, by construction, we get: $\varphi_T^r = \pi - \varphi_T^l$. Hence,*

$$\cos \varphi_T^r = -\cos \varphi_T^l. \tag{2.37}$$

According to the remark 2.4, there are two straight lines, $\Lambda_l^{A_1}$, and $\Lambda_r^{A_1}$, passing through A_1 and tangent to K_R, which are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the radius R and with the centers $O_l^{A_1} = (\rho_{A_1,l}, 0)$, $O_r^{A_1} = (\rho_{A_1,r}, 0)$, $\rho_{A_1,l} \leq \rho_{A_1} \leq \rho_{A_1,r}$. The angles $\varphi_{A_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_l^{A_1} \rho$, $\varphi_{A_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_r^{A_1} \rho$ serve as the coordinates of A_1 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$ correspondingly. Moreover,

$$\cos \varphi_{A_1}^{\Lambda_r^{A_1}} = -\cos \varphi_{A_1}^{\Lambda_l^{A_1}}. \tag{2.38}$$

Similarly, there are two straight lines, $\Lambda_l^{A_2}$, and $\Lambda_r^{A_2}$, passing through A_2 and tangent to K_R, which are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the radius R and with the centers $O_l^{A_2} = (\rho_{A_2,l}, 0)$, $O_r^{A_2} = (\rho_{A_2,r}, 0)$, $\rho_{A_2,l} \leq \rho_{A_2} \leq \rho_{A_2,r}$. The angles $\varphi_{A_2}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_l^{A_2} \rho$, $\varphi_{A_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_r^{A_2} \rho$ serve as the coordinates of A_2 on $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{A_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{A_2}^{\Lambda_l^{A_2}}. \tag{2.39}$$

Suppose that $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point B_1. Then the angles $\varphi_{B_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_1} \rho$, $\varphi_{B_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_2} \rho$ serve as the coordinates of B_1 on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{B_1}^{\Lambda_l^{A_2}} = -\cos \varphi_{B_1}^{\Lambda_l^{A_1}}. \tag{2.40}$$

Also suppose that $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point B_2. Then the angles $\varphi_{B_2}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_1} \rho$, $\varphi_{B_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_2} \rho$ serve as the coordinates of B_2 on $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{B_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{B_2}^{\Lambda_r^{A_1}}. \tag{2.41}$$

Let the straight lines $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point C_1. Then the angles $\varphi_{C_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle C_1 O_r^{A_1} \rho$, $\varphi_{C_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle C_1 O_l^{A_2} \rho$ serve as the coordinates of C_1 on $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{C_1}^{\Lambda_l^{A_2}} = -\cos \varphi_{C_1}^{\Lambda_r^{A_1}}. \tag{2.42}$$

Also, let the straight lines $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point C_2. Then the angles $\varphi_{C_2}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle C_2 O_l^{A_1} \rho$, $\varphi_{C_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle C_2 O_r^{A_2} \rho$ serve as the coordinates of C_2 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{C_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{C_2}^{\Lambda_l^{A_1}}. \tag{2.43}$$

By construction, the quadrangles $A_1 B_1 A_2 B_2$ and $A_1 C_1 A_2 C_2$ are tangent to K_R, and the points A_1, A_2 are opposite vertices of each of these quadrangles. In order to verify the validity of Theorem 2.1 for the flexible octahedra with the equator $A_1 B_1 A_2 B_2$ or $A_1 C_1 A_2 C_2$ we need to prove the following easy statement.

Lemma 2.5. *Given a Poincaré upper half-plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$). Let A and B be points on the straight line Λ realized in $\mathbb{H}^2$ as the Euclidean demi-circle with the raduis R and with the center $O_\Lambda = (\rho_{O_\Lambda}, 0)$, and let the angles $\varphi_A \stackrel{\text{def}}{=} \angle A O_\Lambda \rho$, $\varphi_B \stackrel{\text{def}}{=} \angle B O_\Lambda \rho$ serve as the coordinates of A and B correspondingly* *on* Λ*. Also we assume that* $0 < \varphi_A \leq \phi_B < \pi$*. Then the distance between* A *and* B *is calculated as follows:*

$$\mathrm{d}_{\mathbb{H}^2}(A, B) = \frac{1}{2} \ln \Big[\Big(\frac{1 + \cos\varphi_A}{1 + \cos\varphi_B} \Big) \Big(\frac{1 - \cos\varphi_B}{1 - \cos\varphi_A} \Big) \Big]. \tag{2.44}$$

Proof. The hyperbolic segment Λ_{AB} connecting the points A and B is specified parametrically by the formulas $\Lambda_{AB}(t) : (\rho(\varphi), z(\varphi))$, $\varphi \in [\varphi_A, \varphi_B]$, where $\rho(\varphi) = \rho_{O_\Lambda} + R\cos\varphi$, $z(\varphi) = R\sin\varphi$, $A = \Lambda_{AB}(\varphi_A)$, $B = \Lambda_{AB}(\varphi_B)$. The direct calculation shows that the lengths of Λ_{AB} is equal to the right-hand side of (2.44). □

By Lemma 2.5, the lengths of the edges of the quadrilateral $A_1B_1A_2B_2$ are calculated as follows:

$$\mathrm{d}_{\mathbb{H}^2}(A_1, B_1) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{A_1}^{\Lambda_l^{A_1}}}{1 + \cos\varphi_{B_1}^{\Lambda_l^{A_1}}} \right) \left(\frac{1 - \cos\varphi_{B_1}^{\Lambda_l^{A_1}}}{1 - \cos\varphi_{A_1}^{\Lambda_l^{A_1}}} \right) \right], \tag{2.45}$$

$$\mathrm{d}_{\mathbb{H}^2}(A_2, B_1) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{A_2}^{\Lambda_l^{A_2}}}{1 + \cos\varphi_{B_1}^{\Lambda_l^{A_2}}} \right) \left(\frac{1 - \cos\varphi_{B_1}^{\Lambda_l^{A_2}}}{1 - \cos\varphi_{A_2}^{\Lambda_l^{A_2}}} \right) \right], \tag{2.46}$$

$$\mathrm{d}_{\mathbb{H}^2}(B_2, A_1) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{B_2}^{\Lambda_r^{A_1}}}{1 + \cos\varphi_{A_1}^{\Lambda_r^{A_1}}} \right) \left(\frac{1 - \cos\varphi_{A_1}^{\Lambda_r^{A_1}}}{1 - \cos\varphi_{B_2}^{\Lambda_r^{A_1}}} \right) \right], \tag{2.47}$$

$$\mathrm{d}_{\mathbb{H}^2}(B_2, A_2) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{B_2}^{\Lambda_r^{A_2}}}{1 + \cos\varphi_{A_2}^{\Lambda_r^{A_2}}} \right) \left(\frac{1 - \cos\varphi_{A_2}^{\Lambda_r^{A_2}}}{1 - \cos\varphi_{B_2}^{\Lambda_r^{A_2}}} \right) \right]. \tag{2.48}$$

Then, by (2.38)—(2.41), we get:

$$\mathrm{d}_{\mathbb{H}^2}(A_1, B_1) + \mathrm{d}_{\mathbb{H}^2}(A_2, B_1) - \mathrm{d}_{\mathbb{H}^2}(B_2, A_1) - \mathrm{d}_{\mathbb{H}^2}(B_2, A_2) = 0. \tag{2.49}$$

By Lemma 2.5, the lengths of the edges of the quadrilateral $A_1C_1A_2C_2$ are calculated as follows:

$$\mathrm{d}_{\mathbb{H}^2}(C_1, A_1) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{C_1}^{\Lambda_r^{A_1}}}{1 + \cos\varphi_{A_1}^{\Lambda_r^{A_1}}} \right) \left(\frac{1 - \cos\varphi_{A_1}^{\Lambda_r^{A_1}}}{1 - \cos\varphi_{C_1}^{\Lambda_r^{A_1}}} \right) \right], \tag{2.50}$$

$$\mathrm{d}_{\mathbb{H}^2}(C_2, A_1) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{C_2}^{\Lambda_l^{A_1}}}{1 + \cos\varphi_{A_1}^{\Lambda_l^{A_1}}} \right) \left(\frac{1 - \cos\varphi_{A_1}^{\Lambda_l^{A_1}}}{1 - \cos\varphi_{C_2}^{\Lambda_l^{A_1}}} \right) \right], \tag{2.51}$$

$$\mathrm{d}_{\mathbb{H}^2}(A_2, C_1) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{A_2}^{\Lambda_l^{A_2}}}{1 + \cos\varphi_{C_1}^{\Lambda_l^{A_2}}} \right) \left(\frac{1 - \cos\varphi_{C_1}^{\Lambda_l^{A_2}}}{1 - \cos\varphi_{A_2}^{\Lambda_l^{A_2}}} \right) \right], \tag{2.52}$$

$$\mathrm{d}_{\mathbb{H}^2}(A_2, C_2) = \frac{1}{2} \ln \left[\left(\frac{1 + \cos\varphi_{A_2}^{\Lambda_r^{A_2}}}{1 + \cos\varphi_{C_2}^{\Lambda_r^{A_2}}} \right) \left(\frac{1 - \cos\varphi_{C_2}^{\Lambda_r^{A_2}}}{1 - \cos\varphi_{A_2}^{\Lambda_r^{A_2}}} \right) \right]. \tag{2.53}$$

By (2.38), (2.39), (2.42), and (2.43), it is easy to verify that

$$\mathrm{d}_{\mathbb{H}^2}(C_2, A_1) + \mathrm{d}_{\mathbb{H}^2}(C_1, A_1) - \mathrm{d}_{\mathbb{H}^2}(A_2, C_1) - \mathrm{d}_{\mathbb{H}^2}(A_2, C_2) = 0. \tag{2.54}$$

According to (2.49) and (2.54), the theorem 2.1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a horocycle in at least one of its flat positions.

An equator of a Bricard-Stachel octahedron of type 3 is tangent to a hypercircle in $\mathbb{H}^2$

Let us consider the Poincaré upper half-plane model of the hyperbolic plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2+dz^2}{z^2}$). Without loss of generality we can assume that the hypercircle tangent to the equator of a Bricard-Stachel octahedron $\mathcal{O}$ of type 3, passes through the (unique) point ∞ at infinity of $\mathbb{H}^2$ which does not lie on the Euclidean line $z = 0$, and through the point $O = (0, 0)$ at infinity of $\mathbb{H}^2$. Every such hypercircle is specified by the equation $z = \rho \tan\alpha$ for some $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. By the symmetry of $\mathbb{H}^2$ with respect to the straight line $\rho = 0$, it is sufficient to consider the family of hypercircles $K = \{z = \rho\tan\alpha | \alpha \in (0, \frac{\pi}{2})\}$. Let $K_\alpha \in K$. We will construct all possible quadrangles tangent to K_α such that none of their vertices belongs to K_α, i.e., all quadrangles that can serve as equators of $\mathcal{O}$. Then we will verify the validity of the theorem 2.1 for such quadrangles.

Let us study the quadrangles based on the straight lines $\Lambda_l^{A_1}$, $\Lambda_r^{A_1}$, $\Lambda_l^{A_2}$, $\Lambda_r^{A_2}$, tangent to K_α, which are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the centers $O_l^{A_1} = (\rho_{A_1,l}, 0)$, $O_r^{A_1} = (\rho_{A_1,r}, 0)$, $O_l^{A_2} = (\rho_{A_2,l}, 0)$, $O_r^{A_2} = (\rho_{A_2,r}, 0)$. Also, let $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$ intersect at a point A_1, $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$ intersect at a point A_2. Assume that A_1 and A_2 are two opposite vertices of $\mathcal{O}$, and that the inequalities $0 < \rho_{A_1,l} < \rho_{A_1,r}$, $0 < \rho_{A_2,l} < \rho_{A_2,r}$ hold true.

Remark 2.6. *Let $T = (\rho_T, z_T)$ be a point in $\mathbb{H}^2$, which serves as the intersection of straight lines Λ_l^T and Λ_r^T tangent to a hypercircle K_α, and let Λ_l^T and Λ_r^T are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the centers $O_l^T = (\rho_{T,l}, 0)$, $O_r^T = (\rho_{T,r}, 0)$ ($\rho_{T,l} < \rho_{T,r}$). Then, by Remark 2.4, the angles $\varphi_T^l \stackrel{\text{def}}{=} \angle TO_l^T\rho$ and $\phi_T^r \stackrel{\text{def}}{=} \angle TO_r^T\rho$ determine uniquely the positions of T on Λ_l^T and Λ_r^T correspondingly. Moreover,*

$$\cos\varphi_T^l = \frac{\rho_{T,r}}{\rho_{T,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_T^r = \frac{\rho_{T,l}}{\rho_{T,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.55}$$

Proof. Λ_l^T and Λ_r^T are tangent to K_α. Hence, the radii R_l and R_r of the demi-circles realizing Λ_l^T and Λ_r^T in $\mathbb{H}^2$ are determined by the formulas

$$R_l = \rho_{T,l}\sin\alpha \quad \text{and} \quad R_r = \rho_{T,r}\sin\alpha. \tag{2.56}$$

Let T_∞ be a point with coordinates $(\rho_T, 0)$. Applying the Euclidean Pythagorean theorem to $\triangle TT_\infty O_r^T$ and simplifying the obtained expression, we get:

$$\rho_T^2 + z_T^2 = 2\rho_T\rho_{T,l} - \rho_{T,l}^2\cos^2\alpha. \tag{2.57}$$

Similarly, from $\triangle TT_\infty O_l^T$ we get that

$$\rho_T^2 + z_T^2 = 2\rho_T\rho_{T,r} - \rho_{T,r}^2\cos^2\alpha. \tag{2.58}$$

Subtracting (2.57) from (2.58), we easily deduce:

$$\rho_T = \frac{\rho_{T,r} + \rho_{T,l}}{2}\cos^2\alpha. \tag{2.59}$$

From the definitions of the cosines of φ_T^l and φ_T^r ($\cos\varphi_T^l = (\rho_T - \rho_{T,l})/R_l$ and $\cos\varphi_T^r = (\rho_T - \rho_{T,r})/R_r$), taking into account (2.56) and (2.59), we obtain (2.55). □

By Remark 2.6, the angles $\varphi_{A_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_l^{A_1}\rho$ and $\varphi_{A_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_r^{A_1}\rho$ determine uniquely the positions of A_1 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$ correspondingly. Moreover,

$$\cos\varphi_{A_1}^{\Lambda_l^{A_1}} = \frac{\rho_{A_1,r}}{\rho_{A_1,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{A_1}^{\Lambda_r^{A_1}} = \frac{\rho_{A_1,l}}{\rho_{A_1,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.60}$$

Similarly, the angles $\varphi_{A_2}^{\Lambda_l^{A_2}} \overset{\text{def}}{=} \angle A_2 O_l^{A_2} \rho$ and $\varphi_{A_2}^{\Lambda_r^{A_2}} \overset{\text{def}}{=} \angle A_2 O_r^{A_2} \rho$ serve as the coordinates of A_2 on $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{A_2}^{\Lambda_l^{A_2}} = \frac{\rho_{A_2,r}}{\rho_{A_2,l}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{A_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_2,l}}{\rho_{A_2,r}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.61}$$

Suppose that the straight lines $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point B_1. Then the angles $\varphi_{B_1}^{\Lambda_l^{A_1}} \overset{\text{def}}{=} \angle B_1 O_l^{A_1} \rho$ and $\varphi_{B_1}^{\Lambda_l^{A_2}} \overset{\text{def}}{=} \angle B_1 O_l^{A_2} \rho$ serve as the coordinates of B_1 on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{B_1}^{\Lambda_l^{A_1}} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{B_1}^{\Lambda_l^{A_2}} = \frac{\rho_{A_1,l}}{\rho_{A_2,l}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.62}$$

Suppose also that $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point B_2. Then the angles $\varphi_{B_2}^{\Lambda_r^{A_1}} \overset{\text{def}}{=} \angle B_2 O_r^{A_1} \rho$ and $\varphi_{B_2}^{\Lambda_r^{A_2}} \overset{\text{def}}{=} \angle B_2 O_r^{A_2} \rho$ serve as the coordinates of B_2 on $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{B_2}^{\Lambda_r^{A_1}} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{B_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_1,r}}{\rho_{A_2,r}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.63}$$

Suppose that $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point C_1. Then the angles $\varphi_{C_1}^{\Lambda_r^{A_1}} \overset{\text{def}}{=} \angle C_1 O_r^{A_1} \rho$ and $\varphi_{C_1}^{\Lambda_l^{A_2}} \overset{\text{def}}{=} \angle C_1 O_l^{A_2} \rho$ serve as the coordinates of C_1 on $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{C_1}^{\Lambda_l^{A_2}} = \frac{\rho_{A_1,r}}{\rho_{A_2,l}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{C_1}^{\Lambda_r^{A_1}} = \frac{\rho_{A_2,l}}{\rho_{A_1,r}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.64}$$

Suppose also that $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point C_2. Then the angles $\varphi_{C_2}^{\Lambda_l^{A_1}} \overset{\text{def}}{=} \angle C_2 O_l^{A_1} \rho$ and $\varphi_{C_2}^{\Lambda_r^{A_2}} \overset{\text{def}}{=} \angle C_2 O_r^{A_2} \rho$ serve as the coordinates of C_2 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{C_2}^{\Lambda_l^{A_1}} = \frac{\rho_{A_2,r}}{\rho_{A_1,l}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{C_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_1,l}}{\rho_{A_2,r}} \frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{2.65}$$

As in the case of the quadrangles tangent to a horocycle in $\mathbb{H}^2$, the lengths of the edges of $A_1B_1A_2B_2$ are expressed in (2.45)—(2.48), and the lengths of the edges of $A_1C_1A_2C_2$ are calculated in (2.50)—(2.53). Taking into account (2.60)—(2.65), it is easy to state the validity of (2.49) and (2.54).

According to (2.49) and (2.54), the theorem 2.1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a hypercircle in at least one of its flat positions.

The case when three vertices of an equator of a flexible octahedron in its flat position lie on a straight line, is similar. The case when all four vertices of an equator lie on a straight line, is trivial.

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Chapter 3

Construction of a compact convex quasi-Fuchsian manifold with a prescribed hyperbolic polyhedral metric on the boundary

The problem of existence and uniqueness of an isometric realization of a surface with a prescribed metric in a given ambient space is classical in the metric geometry. Initially stated in the Euclidean case, it can be posed for surfaces in other spaces, in particular, in hyperbolic 3-space $\mathbb{H}^3$.

One of the first fundamental results in this theory is due to A. D. Alexandrov. It concerns the realization of polyhedral surfaces in the spaces of constant curvature.

As in [Ale06], R_K stands for spherical 3-space of curvarure K in the case $K > 0$; R_K stands for hyperbolic 3-space of curvarure K when $K < 0$; and in the case $K = 0$, R_K denotes Euclidean 3-space.

Then the result of A. D. Alexandrov reads as follows:

Theorem 3.1. *Let h be a metric of a constant sectional curvature K with cone singularities on a sphere S^2 such that the total angle around every singular point of h do not exceed 2π. Then there exists a closed convex polyhedron in R_K equipped with the metric h which is unique up to the isometries of R_K. Here we include the doubly covered convex polygons, which are plane in R_K, in the set of convex polyhedra.*

Later, A. D. Alexandrov and A. V. Pogorelov proved the following statement in $\mathbb{H}^3$ [Pog73]:

Theorem 3.2. *Let h be a C^∞-regular metric of a sectional curvature which is strictly greater than -1 on a sphere S^2. Then there exists an isometric immersion of the sphere (S^2, h) into hyperbolic 3-space $\mathbb{H}^3$ which is unique up to the isometries of $\mathbb{H}^3$. Moreover, this immersion bounds a convex domain in $\mathbb{H}^3$.*

Definition. [MT98, p. 30], [Ota96, p. 11] A discrete finitely generated subgroup $\Gamma_F \subset PSL_2(\mathbb{R})$ without torsion and such that the quotient $\mathbb{H}^2/\Gamma_F$ has a finite volume, is called a *Fuchsian group.*

Given a hyperbolic plane $\mathcal{P}$ in $\mathbb{H}^3$ and a Fuchsian group $\Gamma_{\mathcal{P}} \subset PSL_2(\mathbb{R})$ acting on $\mathcal{P}$, we can canonically extend the action of the group $\Gamma_{\mathcal{P}}$ on the whole space $\mathbb{H}^3$.

Here we recall another result on the above-mentioned problem considered for a special type of hyperbolic manifolds, namely, for Fuchsian manifolds, which is due to M. Gromov [Gro86]:

Theorem 3.3. *Let S be a compact surface of genus greater than or equal to 2, equipped with a C^∞-regular metric h of a sectional curvature which is greater than -1 everywhere. Then there exists a Fuchsian group Γ_F acting on $\mathbb{H}^3$, such that the surface (S, h) is isometrically embedded in $\mathbb{H}^3/\Gamma_F$.*

Remark 3.4. *The hyperbolic manifold $\mathbb{H}^3/\Gamma_F$ from the statement of Theorem 3.3 is called Fuchsian. Note also that the limit set $\Lambda(\Gamma_F) \subset \partial_\infty \mathbb{H}^3$ of a Fuchsian group Γ_F is a geodesic circle in projective space $\mathbb{CP}^1$ regarded as the boundary at infinity $\partial_\infty \mathbb{H}^3$ of the Poincaré ball model of hyperbolic 3-space $\mathbb{H}^3$.*

Definition. [Lab92] A compact hyperbolic manifold M is said to be *strictly convex* if any two points in M can be joined with a minimizing geodesic which lies inside the interior of M. This condition implies that the intrinsic curvature of ∂M is greater than -1 everywhere (the term "hyperbolic" means for us "of a constant curvature equal to -1 everywhere").

In 1992 F. Labourie [Lab92] obtained the following result which can be considered as a generalization of Theorems 3.2 and 3.3:

Theorem 3.5. *Let M be a compact manifold with boundary (different from the solid torus) which admits a structure of a strictly convex hyperbolic manifold. Let h be a C^∞-regular metric on ∂M of a sectional curvature which is strictly greater than -1 everywhere. Then there exists a convex hyperbolic metric g on M which induces h on ∂M:*

$$g \mid_{\partial M} = h.$$

Definition. [MT98, p. 120] A *quasi-Fuchsian space* is the quasiconformal deformation space $QH(\Gamma_F)$ of a Fuchsian group $\Gamma_F \subset PSL_2(\mathbb{R})$.

In other words, the quasi-Fuchsian manifold $QH(\Gamma_F)$ is a quotient $\mathbb{H}^3/\Gamma_{qF}$ of $\mathbb{H}^3$ by a discrete finitely generated group $\Gamma_{qF} \subset PSL_2(\mathbb{R})$ of hyperbolic isometries of $\mathbb{H}^3$ such that the limit set $\Lambda(\Gamma) \subset \partial_\infty \mathbb{H}^3$ of Γ is a Jordan curve which can be obtained from the circle $\Lambda(\Gamma_F) \subset \partial_\infty \mathbb{H}^3$ by a quasiconformal deformation of $\partial_\infty \mathbb{H}^3$.

In geometric terms, a quasi-Fuchsian manifold is a complete hyperbolic manifold homeomorphic to $\mathcal{S} \times \mathbb{R}$, where $\mathcal{S}$ is a closed connected surface of genus at least 2, which contains a convex compact subset.

Our main goal is to prove the following extension of Theorem 3.5:

Theorem 3.6. *Let $\mathcal{M}$ be a compact connected 3-manifold with boundary of the type $\mathcal{S} \times [-1, 1]$ where $\mathcal{S}$ is a closed connected surface of genus at least 2. Let h be a hyperbolic metric with cone singularities of angle less than 2π on $\partial\mathcal{M}$ such that every singular point of h possesses a neighborhood in $\partial\mathcal{M}$ which does not contain other singular points of h. Then there exists a hyperbolic metric g in $\mathcal{M}$ with a convex boundary $\partial\mathcal{M}$ such that the metric induced on $\partial\mathcal{M}$ is h.*

Theorem 3.6 can also be considered as an analogue of Theorem 3.1 for the convex hyperbolic manifolds with polyhedral boundary.

Definition. [CEG06]*A pleated surface* in a hyperbolic 3-manifold $\mathcal{M}$ is a complete hyperbolic surface $\mathcal{S}$ together with an isometric map $f : \mathcal{S} \to \mathcal{M}$ such that every $s \in \mathcal{S}$ is in the interior of some geodesic arc which is mapped by f to a geodesic arc in $\mathcal{M}$.

A pleated surface resembles a polyhedron in the sense that it has flat faces that meet along edges. Unlike a polyhedron, a pleated surface has no corners, but it may have infinitely many edges that form a lamination.

Remark 3.7. *The surfaces serving as the connected components of the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$ from the statement of Theorem 3.6, which are equipped by assumption with hyperbolic polyhedral metrics, do not necessarily have to be polyhedra embedded in $\mathcal{M}$: these surfaces can be partially pleated.*

Definition. [MS09] Let $\mathcal{M}$ be the interior of a compact manifold with boundary. A complete hyperbolic metric g on $\mathcal{M}$ is convex co-compact if $\mathcal{M}$ contains a compact subset $\mathcal{K}$ which is convex: any geodesic segment c in $(\mathcal{M}, g)$ with endpoints in $\mathcal{K}$ is contained in $\mathcal{K}$.

In 2002 J.-M. Schlenker [Sch06] proved uniqueness of the metric g in Theorem 3.5. Thus, he obtained

Theorem 3.8. *Let M be a compact connected 3-manifold with boundary (different from the solid torus) which admits a complete hyperbolic convex co-compact metric. Let g be a hyperbolic metric on M such that ∂M is C^∞-regular and strictly convex. Then the induced metric I on ∂M has curvature $K > -1$. Each C^∞-regular metric on ∂M with $K > -1$ is induced on ∂M for a unique choice of g.*

It would be natural to conjecture that the metric g in the statement of Theorem 3.6 is unique. The methods used in the demonstration of Theorem 3.6 do not presently allow to attack this problem.

At last, recalling that the convex quasi-Fuchsian manifolds are special cases of the convex co-compact manifolds, we can guess that Theorem 3.6 remains valid in the case when $\mathcal{M}$ is a convex co-compact manifold. It would be interesting to verify this hypothesis in the future.

3.1 Proof of Theorem 3.6

A compact connected 3-manifold $\mathcal{M}$ of the type $\mathcal{S}\times[-1,1]$ from the statement of Theorem 3.6, where $\mathcal{S}$ is a closed connected surface of genus at least 2, can be regarded as a convex compact 3-dimensional domain of an unbounded quasi-Fuchsian manifold $\mathcal{M}^\circ = \mathbb{H}^3/\Gamma_{QF}$ where Γ_{QF} stands for a quasi-Fuchsian group of isometries of hyperbolic space $\mathbb{H}^3$. Note that the boundary $\partial\mathcal{M}$ of such domain $\mathcal{M}$ consists of two distinct locally convex compact 2-surfaces in $\mathcal{M}^\circ$. Thus, the metric h from the statement of Theorem 3.6 is a pair of hyperbolic metrics with cone singularities of angle less than 2π (or, in other words, a pair of hyperbolic polyhedral metrics) of compact connected surfaces of the same with $\mathcal{M}$ genus, and our aim is to find such quasi-Fuchsian subgroup Γ_{QF} of isometries of hyperbolic space $\mathbb{H}^3$ and such convex compact domain $\mathcal{M} \subset \mathcal{M}^\circ$ that the induced metric of its boundary $\partial\mathcal{M}$ coincides with h.

The main idea of the proof of Theorem 3.6 is

(1) to approximate the metric h with singularities by a sequence $\{h_n\}_{n\in\mathbb{N}}$ of C^∞-regular metrics for which the Labourie-Schlenker Theorem 3.8 is applicable, and therefore, there are such quasi-Fuchsian groups Γ_n of isometries of $\mathbb{H}^3$ and such convex compact domains $\mathcal{M}_n$ in the quasi-Fuchsian manifolds $\mathcal{M}_n^\circ = \mathbb{H}^3/\Gamma_n$ that the induced metrics of the boundaries $\partial\mathcal{M}_n$ of the sets $\mathcal{M}_n$ are exactly h_n, $n \in \mathbb{N}$;

(2) to find a sequence of positive integers $n_k \xrightarrow[k\to\infty]{} \infty$ such that the subsequences of groups $\{\Gamma_{n_k}\}_{k\in\mathbb{N}}$ and of domains $\{\mathcal{M}_{n_k}\}_{k\in\mathbb{N}}$ converge (the types of convergence will be precised later);

(3) and to show that the induced metric on the boundary of the limit domain $\mathcal{M}$ coincides with h.

For convenience, let us introduce new notation of some entities that we considered before: we redefine the domain $\mathcal{M}$ and the quasi-Fuchsian manifold $\mathcal{M}^\circ$ by the symbols $\mathcal{M}_\infty$ and $\mathcal{M}_\infty^\circ$, correspondingly. Also, let us denote the connected components of the boundary $\partial\mathcal{M}_\infty$ of the limit domain $\mathcal{M}_\infty$ by $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$, and the induced metrics on the surfaces $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$ by h_∞^+ and h_∞^-, respectively. Therefore, to define the metric h from the statement of Theorem 3.6 means to give a pair of hyperbolic polyhedral metrics h_∞^+ and h_∞^-.

3.1.1 Construction of sequences of metrics converging to the prescribed metrics

In this Subsection, we obtain two preliminary results.

Lemma 3.9. *Let $\mathcal{S}$ be a surface with a hyperbolic polyhedral metric h (i.e. of the sectional curvature -1 everywhere except at a discrete set of points with conic singularities of angles less than 2π). Then there is a sequence of C^∞-regular metrics $\{h_n\}_{n\in\mathbb{N}}$ with sectional curvatures greater than or equal to -1 everywhere, converging to the metric h.*

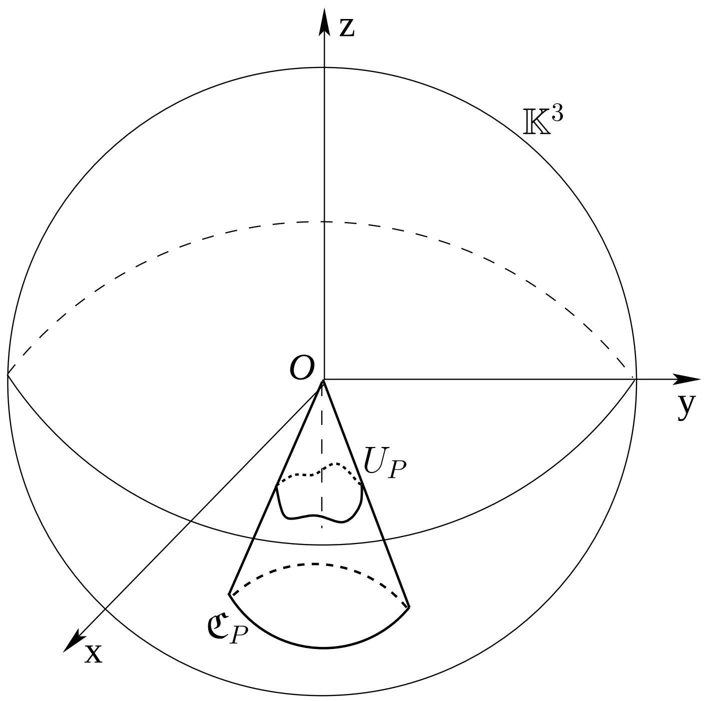

Figure 3.1: The circular cone $\mathfrak{C}_P$ in the Kleinian model $\mathbb{K}^3$ of hyperbolic space $\mathbb{H}^3$.

Proof. Consider a singular point $P \in \mathcal{S}$ of a hyperbolic polyhedral metric h together with a neighborhood $U_P \subset \mathcal{S}$ which does not contain other singular points of h. The domain U_P equipped with the restriction $h|_{U_P}$ of the metric h is isometric to a piece of a circular cone $\mathfrak{C}_P$ in hyperbolic space $\mathbb{H}^3$, where the point P corresponds to the apex of $\mathfrak{C}_P$.

The Kleinian model $\mathbb{K}^3$ of hyperbolic space $\mathbb{H}^3$ can be viewed as the unitary ball centered at the origin O of the Cartesian coordinate system $Oxyz$ in Euclidean 3-space $\mathbb{R}^3$. Recall that the hyperbolic geodesics in $\mathbb{K}^3$ are Euclidean segments. Thus, a hyperbolic cone in the projective model $\mathbb{K}^3$ of $\mathbb{H}^3$ is a Euclidean cone in $\mathbb{R}^3$. Let us place the cone $\mathfrak{C}_P$ into the Kleinian model $\mathbb{K}^3$ so that the apex of $\mathfrak{C}_P$ is identified with the origin O of the Cartesian coordinates $Oxyz$ and the axis of symmetry of $\mathfrak{C}_P$ coincides with the axis Oz (see Fig. 3.1). Then the cone $\mathfrak{C}_P$ can be represented as the surface of revolution around the axis Oz of the graph of a function of the type

$$z = f_\mu(x) \stackrel{\text{def}}{=} \mu|x|, \quad x \in \mathbb{R},$$

where the parameter μ is a negative real number.

Recall the following classical result due to S. L. Sobolev:

Theorem 3.10 (Theorem in §2.4 of Chapter I [Sob63], p. 13)**.** *For every function $\phi \in L_p$ there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ of C^∞-regular functions converging strongly to ϕ.*

In the demonstration of Theorem 3.10 given in [Sob63] the regular approximations ϕ_k, $k \in \mathbb{N}$, of the function ϕ are constructed by convolution of ϕ with applications of the type

$$\omega_r(x) = \begin{cases} \frac{1}{c_r} e^{\frac{x^2}{x^2-r^2}}, & x \in [-r,r], \\ 0, & x \in \mathbb{R} \setminus [-r,r], \end{cases} \quad \text{where the constant} \quad c_r = \int_{-r}^{r} e^{\frac{t^2}{t^2-r^2}}\,dt, \tag{3.1}$$

and the parameter r is a positive real number.

Since we need to consider only a small part of the cone $\mathfrak{C}_P$ which is placed inside the Euclidean unitary ball centered at O (the interesting part corresponds to the neighborhood U_P of the point $P \in \mathcal{S}$), it suffices to assume that the function $f_\mu(x)$ is defined in the segment $[-1,1]$. Hence, being a continuous function with a compact support, f_μ belongs to Lebesgue space L_p for any p. Therefore, choosing a monotonically decreasing sequence of small positive real numbers $r_k \xrightarrow[k\to\infty]{} 0$ and convoluting f_μ with the applications ω_{r_k}, $k \in \mathbb{N}$, we construct a sequence of convex even functions $\{z = f_\mu^k(x) \overset{\text{def}}{=} f_\mu * \omega_{r_k}(x)\}_{k\in\mathbb{N}}$ converging to f_μ. By Theorem 3.10, the functions f_μ^k are C^∞-regular, $k \in \mathbb{N}$.

Let us study the graphs of the functions f_μ^k, $k \in \mathbb{N}$.

The first generalized derivative $Df_\mu(x)$ of the application $f_\mu(x)$ can be characterized by the following representative:

$$Df_\mu(x) = \begin{cases} -\mu, & x \in [-\infty, 0[, \\ 0, & x = 0, \\ \mu, & x \in]0, \infty]. \end{cases} \tag{3.2}$$

Note that it can be expressed through the Heaviside function

$$H(x) = \begin{cases} 0, & x \in [-\infty, 0[, \\ \frac{1}{2}, & x = 0, \\ 1, & x \in]0, \infty], \end{cases}$$

as follows:

$$Df_\mu(x) = 2\mu H(x) - \mu. \tag{3.3}$$

By a property of the convolution, the first generalized derivative $Df_\mu^k(x)$ of the application $f_\mu^k(x)$ is related with $Df_\mu(x)$ as follows:

$$Df_\mu^k(x) = D[f_\mu * \omega_{r_k}](x) = (Df_\mu) * \omega_{r_k}(x).$$

Also, according to (3.1), the function $\omega_{r_k}(x-t)$ of the variable t is zero outside the segment $[x - r_k, x + r_k]$. Moreover, by (3.2), for any $x > r_k$ we have that $Df_\mu(t) = \mu$ for all $t \in [x - r_k, x + r_k]$. Thus, for any $x > r_k$

$$Df_\mu^k(x) = (Df_\mu) * \omega_{r_k}(x) = \int_{-\infty}^{\infty} \omega_{r_k}(x-t) Df_\mu(t)dt = \int_{x-r_k}^{x+r_k} \omega_{r_k}(x-t) Df_\mu(t)dt$$

$$= \int_{x-r_k}^{x+r_k} \omega_{r_k}(x-t)\mu dt \overset{\tau=x-t}{=} \mu \int_{r_k}^{-r_k} \omega_{r_k}(\tau)[-d\tau] = \mu \int_{-r_k}^{r_k} \omega_{r_k}(\tau)d\tau = \mu,$$

which implies that the function $f_\mu^k(x)$ is a linear application on the half-line $[r_k, \infty[$ of the type $f_\mu^k(x) = \mu x + c_+$, where c_+ is a real constant. Similarly we obtain that $f_\mu^k(x)$ is a linear application on the half-line $]-\infty, -r_k]$ of the type $f_\mu^k(x) = -\mu x + c_-$, where $c_- \in \mathbb{R}$. By symmetry we get that $c_+ = c_-$. Since the functions f_μ^k, $k \in \mathbb{N}$, approximate $f_\mu = \mu|x|$, we can put $c_+ = c_- = 0$. We have just showed that the graphs of the maps f_μ^k, $k \in \mathbb{N}$, coincide with the graph of f_μ outside small neighborhoods of $(0,0) \in \mathbb{R}^2$.

Let us now study the convexity of the functions f_μ^k, $k \in \mathbb{N}$. By the formula (3.3), the second derivative $D^2 f_\mu(x)$ of the application $f_\mu(x)$ regarded as a generalized function is equal to $2\mu\delta(x)$, where $\delta(x)$ stands for the Dirac delta function (remind that $DH(x) = \delta(x)$). Also, by construction, the applications ω_{r_k}, $k \in \mathbb{N}$, are even functions. Hence, the generalized second derivative $D^2 f_\mu^k$ of f_μ^k can be calculated as follows:

$$D^2 f_\mu^k(x) = D^2[f_\mu * \omega_{r_k}](x) = (D^2 f_\mu) * \omega_{r_k}(x) = 2\mu\delta * \omega_{r_k}(x) = 2\mu\omega_{r_k}(-x) = 2\mu\omega_{r_k}(x).$$

Recall that the constant μ is negative. Taking into account (3.1), we conclude that $D^2 f_\mu^k(x) \leq 0$ for all $x \in \mathbb{R}$. Thus, for any $k \in \mathbb{N}$ the function f_μ^k is concave everywhere on $\mathbb{R}$, and the graph of f_μ^k smoothes out the angle formed by the graph of f_μ at the point $(0,0) \in \mathbb{R}^2$.

Rotating the graphs of the functions f_μ^k, $k \in \mathbb{N}$, on the plane Oxz around the axis Oz, we obtain a sequence of convex C^∞-regular surfaces $\{\mathfrak{C}_P^k\}_{k\in\mathbb{N}}$ which converges to the cone $\mathfrak{C}_P$. Again, the surfaces $\mathfrak{C}_P^k$ smooth out the conic singularity of $\mathfrak{C}_P$ at its apex, and they coincide with $\mathfrak{C}_P$ outside small neighborhoods of $O \in \mathbb{K}^3$, $k \in \mathbb{N}$.

As the notions of convexity are equivalent in Euclidean space $\mathbb{R}^3$ and in the Kleinian model $\mathbb{K}^3$ of hyperbolic space $\mathbb{H}^3$, the sets $\mathfrak{C}_P^k$, $k \in \mathbb{N}$, regarded as surfaces in $\mathbb{H}^3$, are convex. Therefore, the Gaussian curvature of the surfaces $\mathfrak{C}_P^k \subset \mathbb{H}^3$ is greater than or equal to -1 everywhere, $k \in \mathbb{N}$. Denote by $h_k|_{U_P}$ the induced metrics of the surfaces $\mathfrak{C}_P^k \subset \mathbb{H}^3$, $k \in \mathbb{N}$, restricted on the sets corresponding to the neighborhood U_P of the point $P \in \mathcal{S}$. By construction, the sequence of C^∞-smooth metrics $\{h_k|_{U_P}\}_{k\in\mathbb{N}}$ converges to $h|_{U_P}$ as $k \to \infty$, and moreover, these metrics coincide with $h|_{U_P}$ near the boundary of U_P on $\mathcal{S}$. Thus, replacing the metric $h|_{U_P}$ as a part of the metric h on the surface $\mathcal{S}$ by the metrics $h_k|_{U_P}$, $k \in \mathbb{N}$, we obtain a sequence of metrics $\{h_k\}_{k\in\mathbb{N}}$ on $\mathcal{S}$ converging to h as $k \to \infty$.

The procedure described above should be applied simultaneously to all singular points of the metric h. □

Lemma 3.11. *Consider a regular metric surface $(\mathcal{S}, h)$, where $\mathcal{S}$ stands for a 2-dimensional surface, h is a metric provided on $\mathcal{S}$, and $K_h(x)$ denotes the sectional curvature of $(\mathcal{S}, h)$ at a point $x \in \mathcal{S}$. If we consider another metric surface $(\mathcal{S}, g)$, where the metric $g = \lambda h$ is a multiple of h and $\lambda > 0$ is a positive constant, then the sectional curvature $K_g(x)$ of $(\mathcal{S}, g)$ at a point $x \in \mathcal{S}$ is related to $K_h(x)$ as follows:*

$$K_g(x) = \frac{1}{\lambda} K_h(x). \tag{3.4}$$

Proof. First, according to Theorem 2.51 [GHL04, p. 70], the consistence of the connection ∇ with the metric h means that for any vector vields U, V, and W on $\mathcal{S}$ the following relation holds:

$$U.h(V, W) = h(\nabla_U V, W) + h(V, \nabla_U W). \tag{3.5}$$

Multiplying (3.5) by λ and recalling that $g = \lambda h$, we easily get:

$$U.g(V, W) = g(\nabla_U V, W) + g(V, \nabla_U W).$$

Hence, the connection ∇ is also consistent with the metric g.

Then, by Definition 3.7 [GHL04, p. 107], we remark that the *curvature tensor* $R_x(u,v)w$ defined for an arbitrary point $x \in \mathcal{S}$ and for any vectors u, v, and w of the tangent space $T_x\mathcal{S}$ depends only on the connection ∇ consistent with the metrics h and g. Thus, the curvature tensor $R_x(u,v)w$ is common for both metrics h and g on the surface $\mathcal{S}$.

At last, according to Definition 3.3 [GHL04, p. 109], the *sectional curvature* $K_h(x)$ of the surface $(\mathcal{S}, h)$ at a point $x \in \mathcal{S}$ can be expressed as follows:

$$K_h(x) = \frac{h_x(R_x(u,v)u,v)}{h_x(u,u)h_x(v,v) - (h_x(u,v))^2} \tag{3.6}$$

and does not depend on the choice of an orthogonal basis $\{u, v\}$ of the tangent space $T_x\mathcal{S}$. Similarly, the sectional curvature $K_g(x)$ of the surface $(\mathcal{S}, g)$ at a point $x \in \mathcal{S}$ is defined by the formula:

$$K_g(x) = \frac{g_x(R_x(u,v)u,v)}{g_x(u,u)g_x(v,v) - (g_x(u,v))^2}, \tag{3.7}$$

where $\{u, v\}$ is an orthogonal basis of the tangent space $T_x\mathcal{S}$.

Comparing the relations (3.6) and (3.7) and taking into account that $g = \lambda h$, we obtain (3.4). □

3.1.2 Convergence of convex surfaces in a compact domain in $\mathbb{H}^3$

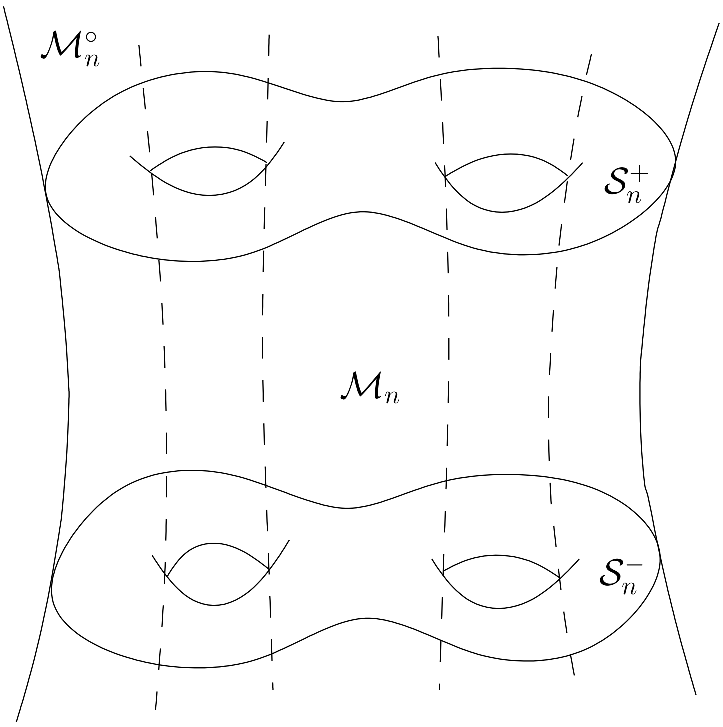

Figure 3.2: The surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ in the quasi-Fuchsian manifold $\mathcal{M}_n^\circ$.

Let h_∞^+ and h_∞^- be two polyhedral hyperbolic metrics on a closed compact surface $\mathcal{S}$ of genus g. By Lemma 3.9, there are two sequences of C^∞-smooth metrics $\{\hbar_n^+\}_{n\in\mathbb{N}}$ and $\{\hbar_n^-\}_{n\in\mathbb{N}}$ on $\mathcal{S}$, with sectional curvature ≥ -1 everywhere, converging to h_∞^+ and h_∞^- as $n \to \infty$. In order to be able to apply the Labourie-Schlenker Theorem 3.8, let us chose a monotonically decreasing sequence of real numbers $\lambda_n \xrightarrow[n\to\infty]{} 1$ and let us define the metrics $h_n^+ \stackrel{\text{def}}{=} \lambda_n \hbar_n^+$ and $h_n^- \stackrel{\text{def}}{=} \lambda_n \hbar_n^-$ on $\mathcal{S}$, $n \in \mathbb{N}$. Thus, by Lemma 3.11, the sectional curvatures of the metrics h_n^+ and h_n^- is strictly greater than -1 everywhere on $\mathcal{S}$, and, by construction, the sequences of C^∞-smooth metrics $\{h_n^+\}_{n\in\mathbb{N}}$ and $\{h_n^-\}_{n\in\mathbb{N}}$ converge to h_∞^+ and h_∞^- as $n \to \infty$. Therefore, by the Labourie-Schlenker Theorem 3.8, for each $n \in \mathbb{N}$ there is a unique compact convex domain $\mathcal{M}_n$ of a quasi-Fuchsian manifold $\mathcal{M}_n^\circ$ with hyperbolic metric g_n such that the induced metrics of the components $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ of the boundary $\partial\mathcal{M}_n \stackrel{\text{def}}{=} \mathcal{S}_n^+ \cup \mathcal{S}_n^-$ are equal to h_n^+ and h_n^- (see also Fig. 3.2). It means that, for each $n \in \mathbb{N}$ there exist isometric embeddings $f_{\mathcal{S}_n^+} : (\mathcal{S}, h_n^+) \to \mathcal{M}_n^\circ$ and $f_{\mathcal{S}_n^-} : (\mathcal{S}, h_n^-) \to \mathcal{M}_n^\circ$ such that $f_{\mathcal{S}_n^+}(\mathcal{S}) = \mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $f_{\mathcal{S}_n^-}(\mathcal{S}) = \mathcal{S}_n^- \subset \mathcal{M}_n^\circ$.

As $\mathcal{M}_n^\circ$ can be retracted by deformation on $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, we conclude that their fundamental groups are homomorphic:

$$\pi_1(\mathcal{S}_n^+) \simeq \pi_1(\mathcal{M}_n^\circ) \simeq \pi_1(\mathcal{S}_n^-).$$

Also, by construction,

$$\pi_1(\mathcal{S}_n^+) \simeq \pi_1(\mathcal{S}) \simeq \pi_1(\mathcal{S}_n^-).$$

Hence, for all $n \in \mathbb{N}$

$$\pi_1(\mathcal{M}_n^\circ) \simeq \pi_1(\mathcal{S}). \tag{3.8}$$

Since the manifolds $\mathcal{M}_n^\circ$, $n \in \mathbb{N}$, are hyperbolic, their universal coverings $\widetilde{\mathcal{M}_n^\circ}$ are actually copies of hyperbolic 3-space $\mathbb{H}^3$. Moreover, as each $\mathcal{M}_n^\circ$ is quasi-Fuchsian, there exists a holonomy representation $\rho_n : \pi_1(\mathcal{M}_n^\circ) \to \mathcal{I}(\widetilde{\mathcal{M}_n^\circ})(= \mathcal{I}(\mathbb{H}^3))$ of the fundamental group of $\mathcal{M}_n^\circ$ in the group of isometries of the universal covering $\widetilde{\mathcal{M}_n^\circ}(= \mathbb{H}^3)$ such that $\mathcal{M}_n^\circ = \widetilde{\mathcal{M}_n^\circ}/[\rho_n(\pi_1(\mathcal{M}_n^\circ))] = \mathbb{H}^3/[\rho_n(\pi_1(\mathcal{M}_n^\circ))]$ and the limit set $\Lambda_{\rho_n} \subset \partial_\infty\mathbb{H}^3$ of $\rho_n(\pi_1(\mathcal{M}_n^\circ))$ is homotopic to a circle. By (3.8), we can also speak about the holonomy representation $\rho_n^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\widetilde{\mathcal{M}_n^\circ})(= \mathcal{I}(\mathbb{H}^3))$ of the fundamental group of $\mathcal{S}$ in the group of isometries of the universal covering $\widetilde{\mathcal{M}_n^\circ}(= \mathbb{H}^3)$ such that $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S})) = \rho_n(\pi_1(\mathcal{M}_n^\circ))$. Thus we have that $\mathcal{M}_n^\circ = \widetilde{\mathcal{M}_n^\circ}/[\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))] = \mathbb{H}^3/[\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))]$ and the limit set $\Lambda_{\rho_n^{\mathcal{S}}}$ of $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ is just Λ_{ρ_n}, $n \in \mathbb{N}$. We also suppose that $\pi_1(\mathcal{S})$ is generated by the elements $\{\gamma_1, ..., \gamma_l\}$.

Inside $\widetilde{\mathcal{M}_n^\circ}(= \mathbb{H}^3)$, $n \in \mathbb{N}$, we can find a convex set $\widetilde{\mathcal{M}}_n$ serving as a universal covering of the domain $\mathcal{M}_n \subset \mathcal{M}_n^\circ$, i.e. such that $\mathcal{M}_n = \widetilde{\mathcal{M}}_n/[\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))]$, and a pair of convex surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$ serving as universal coverings of the surfaces $\mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $\mathcal{S}_n^- \subset \mathcal{M}_n^\circ$ (see Fig. 3.3), i.e. such that $\mathcal{S}_n^+ = \widetilde{\mathcal{S}}_n^+/[\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))]$ and $\mathcal{S}_n^- = \widetilde{\mathcal{S}}_n^-/[\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))]$. By construction, $\partial\widetilde{\mathcal{M}}_n = \widetilde{\mathcal{S}}_n^+ \cup \widetilde{\mathcal{S}}_n^-$ and the boundaries at infinity $\partial_\infty\widetilde{\mathcal{M}}_n = \partial_\infty\widetilde{\mathcal{S}}_n^+ = \partial_\infty\widetilde{\mathcal{S}}_n^- = \Lambda_{\rho_n^{\mathcal{S}}}$. Denote by $p_n : \widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ the projection of $\widetilde{\mathcal{M}}_n$ on $\mathcal{M}_n$, $n \in \mathbb{N}$. By construction, $\mathcal{S}_n^+ = p_n(\widetilde{\mathcal{S}}_n^+)$ and $\mathcal{S}_n^- = p_n(\widetilde{\mathcal{S}}_n^-)$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we lift the metric g_n of the manifold $\mathcal{M}_n$ to the metric $\tilde{g}_n$ of the universal covering $\widetilde{\mathcal{M}}_n$ in such a way that for any $\gamma \in \pi_1(\mathcal{S})$ and for $x \in \mathcal{M}_n$ and $\tilde{x} \in \widetilde{\mathcal{M}}_n$ satisfying the relation $x = p_n(\tilde{x})$, we have $\tilde{g}_n(\tilde{x}) = {p_n}^* g_n(x)$, i.e. the metric $\tilde{g}_n(\tilde{x}) \in T_{\tilde{x}}^*\widetilde{\mathcal{M}}_n$ is a pull-back of the metric $g_n(x) \in T_x^*\mathcal{M}_n$. We have already remarked that, since g_n is hyperbolic, $\tilde{g}_n$ is hyperbolic too. Denote by $\tilde{h}_n^+$ the restriction of the metric $\tilde{g}_n$ on the surface $\widetilde{\mathcal{S}}_n^+$ and by $\tilde{h}_n^-$ the restriction of the metric $\tilde{g}_n$ on the surface $\widetilde{\mathcal{S}}_n^-$, $n \in \mathbb{N}$. By construction, the metric $\tilde{h}_n^+$ is the lift of h_n^+ from the surface $\mathcal{S}_n^+$ to its universal covering $\widetilde{\mathcal{S}}_n^+$ and the metric $\tilde{h}_n^-$ is the lift of h_n^- from $\mathcal{S}_n^-$ to $\widetilde{\mathcal{S}}_n^-$, $n \in \mathbb{N}$.

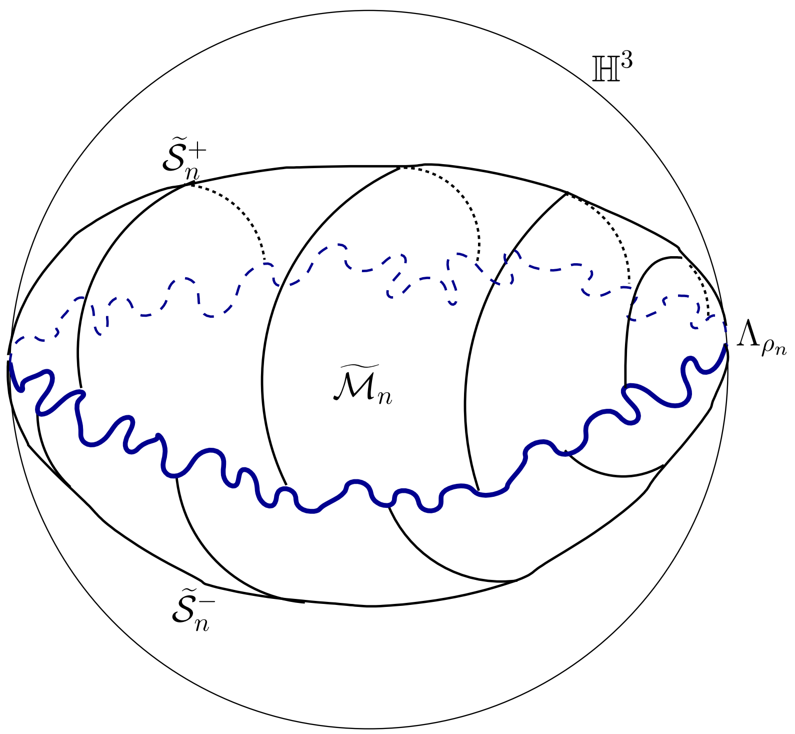

Figure 3.3: The universal coverings $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$ in the Kleinian model $\mathbb{K}^3$ of hyperbolic space $\mathbb{H}^3$.

Definition. The *diameter* δ of a set S with a metric h is the following quantity: $\delta \stackrel{\text{def}}{=} \sup\{\mathrm{d}_h(u,v) | u,v \in S\}$ where $\mathrm{d}_h(u,v)$ stands for the distance between points u and v in the metric h.

Lemma 3.12. *There exists a positive constant $\delta_{\mathcal{S}} < \infty$ which bounds from above the diameters δ_n^+ and δ_n^- of the surfaces $(\mathcal{S}, h_n^+)$ and $(\mathcal{S}, h_n^-)$ for all $n \in \mathbb{N}$.*

Proof. First, note that the procedure of construction of the metrics $\hbar_n^+$ and $\hbar_n^-$, $n \in \mathbb{N}$, by smoothing the hyperbolic polyhedral metrics h_∞^+ and h_∞^- with the help of convolution, as it was described in the proof of Lemma 3.9, does not increase the distance between any two points on the surface $\mathcal{S}$. On the other hand, multiplying the metrics $\hbar_n^+$ and $\hbar_n^-$ by the real number $\lambda_n > 1$, and thus, obtaining the metrics h_n^+ and h_n^-, $n \in \mathbb{N}$, we increase all distances on $\mathcal{S}$ by $\sqrt{\lambda_n}$.

Recall that the sequence of numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ is decreasing. Hence, $\lambda_1 \geq \lambda_n$ for every $n \in \mathbb{N}$. Therefore, the distances on $\mathcal{S}$ measured in the metric $h_\lambda^+ \stackrel{\text{def}}{=} \lambda_1 h_\infty^+$ are not smaller than the corresponding distances measured in the metrics h_n^+ for all $n \in \mathbb{N}$. Similarly, the distances on $\mathcal{S}$ measured in the metric $h_\lambda^- \stackrel{\text{def}}{=} \lambda_1 h_\infty^-$ are not smaller than the corresponding distances measured in the metrics h_n^- for all $n \in \mathbb{N}$.

Since $\mathcal{S}$ is compact, the diameters δ_λ^+ and δ_λ^- of the surfaces $(\mathcal{S}, h_\lambda^+)$ and $(\mathcal{S}, h_\lambda^-)$ are finite numbers. We can pose $\delta_{\mathcal{S}} = \max(\delta_\lambda^+, \delta_\lambda^-)$. ☐

Lemma 3.13. *There exists a positive constant $\delta_{\mathcal{M}} < \infty$ such that for each $n \in \mathbb{N}$ and for every pair of points $u \in \mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $v \in \mathcal{S}_n^- \subset \mathcal{M}_n^\circ$ the distance $\mathrm{d}_{g_n}(u,v)$ between u and v in the* *manifold* $\mathcal{M}_n^\circ$ *is less than* $\delta_{\mathcal{M}}$.

Proof. By Theorem 4.1 in Chapter 4 of this thesis, the distances $\sigma_n^{\mathcal{S}}$ between the surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, $n \in \mathbb{N}$, are uniformly bounded by a constant $\sigma_{\mathcal{S}}$. Also, by Lemma 3.12, the diameters of $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ are both bounded by a constant $\delta_{\mathcal{S}}$ which does not depend on n. Hence, our assertion is valid if we take $\delta_{\mathcal{M}}$ to be equal to $\sigma_{\mathcal{S}} + 2\delta_{\mathcal{S}}$. ☐

Professor Gregory McShane remarked that the existence of a constant $\delta_{\mathcal{M}} > 0$ which serves as an common upper bound for the distances between the boundary components $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ of the domains $\mathcal{M}_n$, $n \in \mathbb{N}$ does not guarantee that the diameters of $\mathcal{M}_n$ are uniformly bounded from above.

Indeed, Jeffrey Brock in his PhD thesis (see also [Bro01]) studied the following example.

Given a pair of homeomorphic Riemann surfaces X and Y of finite type and a "partial pseudo Anosov" mapping class ϕ, by the Ahlfors-Bers simultaneous uniformization theorem there is a sequence of quasi-Fuchsian manifolds $\{Q(\phi^n X, Y)\}_{n=1}^{\infty}$. The diameters of each of the boundary components of the convex hull of $Q(\phi^n X, Y)$ is uniformly bounded in n and so is the distance between the two boundary components but the diameter of the convex hull of $Q(\phi^n X, Y)$ goes to infinity because of a "cusp growing there" as $n \to \infty$.

However, the diameters of the domains $\mathcal{M}_n$, $n \in \mathbb{N}$ do not play role in the demonstration of Theorem 3.6; only the distances between the surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, $n \in \mathbb{N}$, are of importance here.

Let us now return to the proof of Theorem 3.6.

Let us fix an arbitrary point $x \in \mathcal{S}$, which is not, however, a point of singularity for the metrics h_∞^+ and h_∞^- on $\mathcal{S}$, and let us denote $x_n^+ \overset{\text{def}}{=} f_{\mathcal{S}_n^+}(x) \in \mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $x_n^- \overset{\text{def}}{=} f_{\mathcal{S}_n^-}(x) \in \mathcal{S}_n^- \subset \mathcal{M}_n^\circ$, $n \in \mathbb{N}$. Denote also the distance between the points x_n^+ and x_n^- in $\mathcal{M}_n^\circ$ by σ_n^x, $n \in \mathbb{N}$. By Lemma 3.13, $\sigma_n^x < \delta_{\mathcal{M}}$ for all $n \in \mathbb{N}$.

Let us consider two copies $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ of the universal covering of the surface $\mathcal{S}$ with the projections $p^+ : \widetilde{\mathcal{S}}^+ \to \mathcal{S}$ and $p^- : \widetilde{\mathcal{S}}^- \to \mathcal{S}$ and let us fix some points $\tilde{x}^+ \in \widetilde{\mathcal{S}}^+$ and $\tilde{x}^- \in \widetilde{\mathcal{S}}^-$ such that $p^+(\tilde{x}^+) = x$ and $p^-(\tilde{x}^-) = x$. Without loss of generality we may think that the fundamental group $\pi_1(\mathcal{S})$ acts on $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ in the sense that $\mathcal{S} \simeq \widetilde{\mathcal{S}}^+/\pi_1(\mathcal{S})$ and $\mathcal{S} \simeq \widetilde{\mathcal{S}}^-/\pi_1(\mathcal{S})$. For every $n \in \mathbb{N}$ we fix an arbitrary pair of points $\tilde{x}_n^+ \in \widetilde{\mathcal{S}}_n^+ \subset \widetilde{\mathcal{M}}_n^\circ (= \mathbb{H}^3)$ and $\tilde{x}_n^- \in \widetilde{\mathcal{S}}_n^- \subset \widetilde{\mathcal{M}}_n^\circ$ verifying the conditions $p_n(\tilde{x}_n^+) = x_n^+$ and $p_n(\tilde{x}_n^-) = x_n^-$, and such that the distance in $\mathcal{M}_n^\circ$ between $\tilde{x}_n^+$ and $\tilde{x}_n^-$ is equal to σ_n^x. The functions $f_{\mathcal{S}_n^+} : \mathcal{S} \to \mathcal{S}_n^+$ and $f_{\mathcal{S}_n^-} : \mathcal{S} \to \mathcal{S}_n^-$ defined above induce the canonical bijective developing maps $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{S}}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \widetilde{\mathcal{S}}_n^-$ with the properties $\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+) = \tilde{x}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\tilde{x}^-) = \tilde{x}_n^-$ and such that for any $\gamma \in \pi_1(\mathcal{S})$ it is true that $\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma.\tilde{x}^+) = \rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\gamma.\tilde{x}^-) = \rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^-$, $n \in \mathbb{N}$.

Remark 3.14. *The above-mentioned property of developing maps holds for any points* $\tilde{y}^+ \in \widetilde{\mathcal{S}}^+$, $\tilde{y}^- \in \widetilde{\mathcal{S}}^-$ *and for every* $\gamma \in \pi_1(\mathcal{S})$:

$$\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma.\tilde{y}^+) = \rho_n^{\mathcal{S}}(\gamma).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}^+) \quad \text{and} \quad \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\gamma.\tilde{y}^-) = \rho_n^{\mathcal{S}}(\gamma).\tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\tilde{y}^-), \quad n \in \mathbb{N}.$$

Let the metrics $\tilde{h}_\lambda^+$ and $\tilde{h}_\lambda^-$ on the universal coverings $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ of the surface $\mathcal{S}$ be the pull-backs of the metrics h_λ^+ and h_λ^- on $\mathcal{S}$ defined in the proof of Lemma 3.12. We are now able to construct the Dirichlet domains $\Delta^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widetilde{\mathcal{S}}^-$ of $\mathcal{S}$ with respect to the metrics h_λ^+ and h_λ^- based in the points $\tilde{x}^+ \in \widetilde{\mathcal{S}}^+$ and $\tilde{x}^- \in \widetilde{\mathcal{S}}^-$, respectively. In what follows we will work with the fundamental domains $\Delta^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widetilde{\mathcal{S}}^-$ of $\mathcal{S}$.

Lemma 3.15. *For each $n \in \mathbb{N}$ the domains $\Delta_n^+ \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\Delta^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\Delta_n^- \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\Delta^-) \subset \widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are included in the hyperbolic balls $B(\tilde{x}_n^+, \delta_{\mathcal{S}})$ and $B(\tilde{x}_n^-, \delta_{\mathcal{S}})$ of radius $\delta_{\mathcal{S}}$ centered at the points $\tilde{x}_n^+$ and $\tilde{x}_n^-$ correspondingly.*

Proof. It suffices to prove this statement for the domain Δ_n^+.

Assume that the surface $\widetilde{\mathcal{S}}^+$ is equipped with the metric $\tilde{h}_\lambda^+$. It follows from the definition of the Dirichlet domain that the distance from any point $x \in \Delta^+ \subset \widetilde{\mathcal{S}}^+$ to the center $\tilde{x}^+$ of Δ^+ is not greater than the diameter of the surface $(\mathcal{S}, h_\lambda^+)$ which is less than or equal to $\delta_{\mathcal{S}}$ (see the proof of Lemma 3.12). Recall that the developing map $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{S}}_n^+$ can be viewed as the identical application from one copy of the surface $\widetilde{\mathcal{S}}^+$ equipped with the metric $\tilde{h}_\lambda^+$ to another copy of $\widetilde{\mathcal{S}}^+$ equipped with the metric $\tilde{h}_n^+$. Also, by the construction made in the proof of Lemma 3.12, all distances on the surface $\mathcal{S}$ measured in the metric h_n^+ do not exceed the corresponding distances on $\mathcal{S}$ in the metric h_λ^+. Hence, this property is valid for the pull-backs $\tilde{h}_n^+$ and $\tilde{h}_\lambda^+$ on $\widetilde{\mathcal{S}}^+$ of the metrics $\tilde{h}_n^+$ and h_λ^+ on $\mathcal{S}$. Therefore, the distance from any point $v \in \Delta_n^+ = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\Delta^+) \subset \widetilde{\mathcal{S}}_n^+$ to the center $\tilde{x}_n^+ = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+)$ of Δ_n^+ is not greater than $\delta_{\mathcal{S}}$.

To complete the proof we remark that for any couple of points $v_1, v_2 \in \widetilde{\mathcal{S}}_n^+$ the distance between them in the hyperbolic metric of 3-space $\mathbb{H}^3$ does not exceed the distance between v_1 and v_2 in the induced metric $\tilde{h}_n^+$ on the 2-surface $\widetilde{\mathcal{S}}_n^+$: $\mathrm{d}_{\mathbb{H}^3}(v_1, v_2) \leq \mathrm{d}_{\tilde{h}_n^+}(v_1, v_2)$. □

Denote by $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ the union of Δ^+ with all "neighbor" fundamental domains of $\mathcal{S}$ of the form $\gamma.\Delta^+$ for all $\gamma \in \pi_1(\mathcal{S})$ such that $\operatorname{cl}\Delta^+ \cap \operatorname{cl}\gamma.\Delta^+ \neq \emptyset$. Similarly we define the set $\widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$.

Lemma 3.16. *For each $n \in \mathbb{N}$ the domains $\widehat{\Delta}_n^+ \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\widehat{\Delta}_n^- \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\widehat{\Delta}^-) \subset \widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are included in the hyperbolic balls $B(\tilde{x}_n^+, 3\delta_{\mathcal{S}})$ and $B(\tilde{x}_n^-, 3\delta_{\mathcal{S}})$ of radius $3\delta_{\mathcal{S}}$ centered at the points $\tilde{x}_n^+$ and $\tilde{x}_n^-$ correspondingly.*

Proof. It suffices to prove this statement for the domain $\widehat{\Delta}_n^+$.

First, by Lemma 3.15, the domain Δ_n^+ is inscribed in the ball $B(\tilde{x}_n^+, \delta_{\mathcal{S}})$. Similarly, for each $\gamma \in \pi_1(\mathcal{S})$ the domain $\rho_n^{\mathcal{S}}(\gamma).\Delta_n^+$ (isometric to Δ_n^+) is inscribed in the ball $B(\rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^+, \delta_{\mathcal{S}})$. Note that $\widehat{\Delta}_n^+$ is the union of Δ_n^+ with the domains of the form $\rho_n^{\mathcal{S}}(\gamma).\Delta_n^+$ such that $\operatorname{cl}\Delta_n^+ \cap \operatorname{cl}\rho_n^{\mathcal{S}}(\gamma).\Delta_n^+ \neq \emptyset$, where $\gamma \in \pi_1(\mathcal{S})$. Thus, the set $\widehat{\Delta}_n^+$ is contained in the union $\mathcal{U}_B$ of the ball $B(\tilde{x}_n^+, \delta_{\mathcal{S}})$ and all balls of the type $B(\rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^+, \delta_{\mathcal{S}})$ such that $B(\rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^+, \delta_{\mathcal{S}}) \cap B(\tilde{x}_n^+, \delta_{\mathcal{S}}) \neq \emptyset$. Clearly, $\mathcal{U}_B$ lies entirely inside the ball $B(\tilde{x}_n^-, 3\delta_{\mathcal{S}})$. □

The following statement is an immediate corollary of Lemmas 3.13 and 3.16.

Lemma 3.17. *For each $n \in \mathbb{N}$ the domains $\widehat{\Delta}_n^+ \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\widehat{\Delta}_n^- \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\widehat{\Delta}^-) \subset \widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are both included in the hyperbolic balls $B(\tilde{x}_n^+, 3\delta_{\mathcal{S}} + \delta_{\mathcal{M}})$ and $B(\tilde{x}_n^-, 3\delta_{\mathcal{S}} + \delta_{\mathcal{M}})$ of radius $3\delta_{\mathcal{S}} + \delta_{\mathcal{M}}$ centered at the points $\tilde{x}_n^+$ and $\tilde{x}_n^-$.*

It is high time to identify the universal coverings $\widetilde{\mathcal{M}}_n^\circ$ (which are copies of $\mathbb{H}^3$) by supposing that the points $\tilde{x}_n^+$ coincide for all $n \in \mathbb{N}$. Let us temporarily forget the 3-dimensional domains $\widetilde{\mathcal{M}}_n$ of hyperbolic space $\mathbb{H}^3$ in order to concentrate our attention on the study of properties of the sequences of surfaces $\{\widetilde{\mathcal{S}}_n^+\}_{n\in\mathbb{N}}$ and $\{\widetilde{\mathcal{S}}_n^-\}_{n\in\mathbb{N}}$.

Recall the statement of the classical Arzelà-Ascoli Theorem.

Theorem 3.18 (Theorem 7.5.7 in [Die60], p. 137). *Suppose F is a Banach space and E a compact metric space. In order that a subset H of the Banach space $\mathcal{C}_F(E)$ of continuous functions from*

E to F be relatively compact, necessary and sufficient conditions are that H be equicontinuous and that, for each $x \in E$ the set H_x of all $f(x)$ such that $f \in H$ be relatively compact in F.

We will apply it in the following

Lemma 3.19. *There exist subsequences of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_{n_k}} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_{n_k}} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ that converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}^+_\infty} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widehat{\Delta}^- \to \mathbb{H}^3$ correspondingly.*

Proof. It suffices to find a converging subsequence of the sequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$. To this purpose we will apply the Arzelà-Ascoli Theorem 3.18.

Let us equip the domain $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ with the restriction $\tilde{h}^+_\lambda \mid_{\widehat{\Delta}^+}$ of the metric $\tilde{h}^+_\lambda$. Consider the domain $(\widehat{\Delta}^+, \tilde{h}^+_\lambda \mid_{\widehat{\Delta}^+})$ as a compact metric space E from the statement of Theorem 3.18; hyperbolic space $\mathbb{H}^3$ as a Banach space F; the sequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ in the space of continuous functions from $(\widehat{\Delta}^+, \tilde{h}^+_\lambda \mid_{\widehat{\Delta}^+})$ to $\mathbb{H}^3$ as the set $H \subset \mathcal{C}_F(E)$.

By Lemma 3.17, the images $\widehat{\Delta}^+_n = \tilde{f}_{\widetilde{\mathcal{S}}^+_n}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}^+_n \subset \mathbb{H}^3$ of the maps $\tilde{f}_{\widetilde{\mathcal{S}}^+_n}$, $n \in \mathbb{N}$, are all included in the ball $B(\tilde{x}^+_n, 3\delta_{\mathcal{S}} + \delta_{\mathcal{M}})$ (recall that we identified all points $\tilde{x}^+_n \in \mathbb{H}^3$, $n \in \mathbb{N}$). Thus, for each $x \in E$ the set H_x is relatively compact in F.

As it was already done in the proof of Lemma 3.15, we consider every developing map $\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widehat{\Delta}^+ \to \widetilde{\mathcal{S}}^+_n$ as the inclusion of the domain $\widehat{\Delta}^+$ equipped with the metric $\tilde{h}^+_\lambda \mid_{\widehat{\Delta}^+}$ to the surface $\widetilde{\mathcal{S}}^+$ with the metric $\tilde{h}^+_n$, $n \in \mathbb{N}$. So, for any $\varepsilon > 0$ if we pose $\delta := \varepsilon$ then for every pair of points $x, y \in \widehat{\Delta}^+$ such that $\mathrm{d}_{\tilde{h}^+_\lambda}(x,y) < \delta$ it is true that $\mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}^+_n}(x), \tilde{f}_{\widetilde{\mathcal{S}}^+_n}(y)) \le \mathrm{d}_{\tilde{h}^+_n}(\tilde{f}_{\widetilde{\mathcal{S}}^+_n}(x), \tilde{f}_{\widetilde{\mathcal{S}}^+_n}(y)) < \varepsilon$ (recall that, by construction, distances measured in the metric $\tilde{h}^+_\lambda$ are not smaller than the corresponding distances measured in the metric $\tilde{h}^+_n$), $n \in \mathbb{N}$. Thus, the functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ are equicontinuous.

Therefore, by the Arzelà-Ascoli Theorem 3.18, there exists a subsequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_{n_k}} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ that converges to some continuous function $\tilde{f}_{\widetilde{\mathcal{S}}^+_\infty} : \widehat{\Delta}^+ \to \mathbb{H}^3$. Similarly we obtain that there exists a subsequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_{n_k}} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ that converges to some continuous function $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widehat{\Delta}^- \to \mathbb{H}^3$. ☐

Assumption 3.20. *Further we assume that the sequences of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_n} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}^+_\infty} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widehat{\Delta}^- \to \mathbb{H}^3$.*

3.1.3 Convergence of the holonomy representations $\{\rho^{\mathcal{S}}_n\}_{n\in\mathbb{N}}$ and of the developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_n} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$

Now we need to derive several properties of the holonomy representations $\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))$, $n \in \mathbb{N}$.

Lemma 3.21. *Given two points $y^1, y^2 \in \mathbb{H}^3$ together with orthogonal bases $\{e^1, e^2, e^3\}$ and $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\}$ of the tangent spaces $T_{y^1}\mathbb{H}^3$ and $T_{y^2}\mathbb{H}^3$, there is a unique isometry $\vartheta \in \mathcal{I}(\mathbb{H}^3)$ such that $y^2 = \vartheta.y^1$ and $\hat{e}^i = d_{y^1}\vartheta(e^i)$, $i = 1, ..., 3$.*

Proof. Following Chapter 1, § 1.5 in [AVS93, p. 13] let us recall the construction of the hyperboloid model $\mathbb{I}^3$ of hyperbolic space $\mathbb{H}^3$. Denoting the coordinates in space $\mathbb{R}^4$ by x_0, x_1, x_2, x_3, we introduce the Minkowski scalar product in $\mathbb{R}^4$ by the formula

$$(x, y)_M = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{3.9}$$

which turns $\mathbb{R}^4$ into a pseudo-Euclidean vector space, denoted by $\mathbb{R}^{3,1}$.

A basis $\{u^0, u^1, u^2, u^3\} \subset \mathbb{R}^{3,1}$ is said to be *orthonormal* if $(u^0, u^0)_M = -1$, $(u^i, u^i)_M = 1$ for $i \neq 0$, and $(u^i, u^j)_M = 0$ for $i \neq j$. For example, the standard basis

$$\{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\} = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\} \subset \mathbb{R}^{3,1} \tag{3.10}$$

is orthonormal.

Each pseudo-orthogonal (i.e. preserving the above scalar product) transformation of $\mathbb{R}^{3,1}$ takes an open cone of time-like vectors

$$\mathfrak{C} = \{x \in \mathbb{R}^{3,1} : (x, x)_M < 0\}$$

consisting of two connected components

$$\mathfrak{C}^+ = \{x \in \mathfrak{C} : x_0 > 0\}, \quad \mathfrak{C}^- = \{x \in \mathfrak{C} : x_0 < 0\}$$

onto itself. Denote by $O(3,1)$ the group of all pseudo-orthogonal transformations of space $\mathbb{R}^{3,1}$, and by $O'(3,1)$ its subgroup of index 2 consisting of those pseudo orthogonal transformations which map each connected component of the cone $\mathfrak{C}$ onto itself.

Using notation developed in § A.1 [BP03, p. 1] we remind that the manifold

$$\mathbb{I}^3 = \{x \in \mathbb{R}^{3,1} : (x, x)_M = -1, x_0 > 0\}$$

with the metric induced by the pseudo-Euclidean metric (3.9) is called the hyperboloid model $\mathbb{I}^3$ of hyperbolic space $\mathbb{H}^3$, and the restrictions of the elements of $O'(3,1)$ on $\mathbb{I}^3$ form the group $\mathcal{I}(\mathbb{H}^3)$ of all isometries of $\mathbb{H}^3$.

Again, by Chapter 1, § 1.5 in [AVS93, p. 13], for any $x \in \mathbb{I}^3$ we can naturally identify the tangent space $T_x\mathbb{I}^3$ with the orthogonal complement of the vector x in space $\mathbb{R}^{3,1}$, which is a 3-dimensional Euclidean space (with respect to the same scalar product). If $\{u^1, u^2, u^3\}$ is an orthonormal basis in it, then $\{x, u^1, u^2, u^3\}$ is an orthonormal basis in the space $\mathbb{R}^{3,1}$.

Obviously, the vector ϵ^0 of the standard basis (3.10) $\mathbb{R}^{3,1}$ lies in $\mathbb{I}^3$ and the vectors $\{\epsilon^1, \epsilon^2, \epsilon^3\}$ defined in (3.10) form an orthonormal basis of the tangent space $T_{\epsilon^0}\mathbb{I}^3$. Also, according to a fact mentioned in the previous paragraph, the sets of four vectors $\{y^1, e^1, e^2, e^3\} \subset \mathbb{R}^{3,1}$ and $\{y^2, \hat{e}^1, \hat{e}^2, \hat{e}^3\} \subset \mathbb{R}^{3,1}$ from the statement of Lemma 3.21 are orthonormal bases of $\mathbb{R}^{3,1}$. Define the linear transformations ϑ_1 and ϑ_2 of $\mathbb{R}^{3,1}$ determined by their 4×4-real matrices $M_1^\vartheta \stackrel{\text{def}}{=} (y^1, e^1, e^2, e^3)$ and $M_2^\vartheta \stackrel{\text{def}}{=} (y^2, \hat{e}^1, \hat{e}^2, \hat{e}^3)$ with the columns consisting of the coordinates of the corresponding vectors in the standard basis of $\mathbb{R}^{3,1}$. A direct calculation shows the transformations ϑ_1 and ϑ_2 send the standard base to the orthonormal bases $\{y^1, e^1, e^2, e^3\}$ and $\{y^2, \hat{e}^1, \hat{e}^2, \hat{e}^3\}$ of $\mathbb{R}^{3,1}$, respectively. Moreover, we know that the vectors ϵ^0, y^1, and y^2 belong to the upper cone $\mathfrak{C}^+$. Hence, ϑ_1 and ϑ_2 are elements of the group $O'(3,1)$, and we can take the transformation ϑ from the statement of Lemma 3.21 to be equal to $\vartheta_2[\vartheta_1]^{-1}$. □

Definition. Given a sequence of hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n\in\mathbb{N}}$ determined by points $y_n^1, y_n^2 \in \mathbb{H}^3$ and orthogonal bases $\{e_n^1, e_n^2, e_n^3\}$, $\{\hat{e}_n^1, \hat{e}_n^2, \hat{e}_n^3\}$ of the tangent spaces $T_{y_n^1}\mathbb{H}^3$ and $T_{y_n^2}\mathbb{H}^3$, we say that the isometries $\{\vartheta_n\}_{n\in\mathbb{N}}$ *converge* to an isometry $\vartheta_\infty \in \mathcal{I}(\mathbb{H}^3)$ *in the sense* *of Lemma* 3.21 if the sequences of base points $\{y_n^1\}_{n\in\mathbb{N}}$, $\{y_n^2\}_{n\in\mathbb{N}}$ converge to points $y_\infty^1, y_\infty^2 \in \mathbb{H}^3$ and the sequences of orthogonal bases $\{e_n^1, e_n^2, e_n^3\}_{n\in\mathbb{N}}$, $\{\hat{e}_n^1, \hat{e}_n^2, \hat{e}_n^3\}_{n\in\mathbb{N}}$ converge to orthogonal bases $\{e_\infty^1, e_\infty^2, e_\infty^3\}$, $\{\hat{e}_\infty^1, \hat{e}_\infty^2, \hat{e}_\infty^3\}$ of the tangent spaces $T_{y_\infty^1}\mathbb{H}^3$ and $T_{y_\infty^2}\mathbb{H}^3$, and the above-mentioned limits define uniquely the isometry ϑ_∞. Denote a convergence of isometries in the sense of Lemma 3.21 by $\vartheta_n \Rightarrow \vartheta_\infty$ as $n \to \infty$.

Definition. We say that hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n\in\mathbb{N}}$ *converge* to an isometry $\vartheta_\infty \in \mathcal{I}(\mathbb{H}^3)$ *in a "weak" sense* if for any point $y \in \mathbb{H}^3$ the sequence $\{\vartheta_n.y\}_{n\in\mathbb{N}}$ converges to the point $\vartheta_\infty.y \in \mathbb{H}^3$ as $n \to \infty$. Denote a "weak" convergence of isometries by $\vartheta_n \xrightarrow[n\to\infty]{} \vartheta_\infty$.

Lemma 3.22. *Given a collection of hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n=1}^\infty$, $\vartheta_n \Rightarrow \vartheta_\infty$ as $n \to \infty$ if and only if $\vartheta_n \xrightarrow[n\to\infty]{} \vartheta_\infty$.*

Proof. A hyperbolic isometry $\vartheta : \mathbb{H}^3 \to \mathbb{H}^3$ which sends any $y \in \mathbb{H}^3$ to the point $\vartheta.y \in \mathbb{H}^3$ can be interpreted as a linear transformation of Minkowski space $\mathbb{R}^{3,1}$ as it was mentioned in the proof of Lemma 3.21. Therefore, $\vartheta(y)$ depends continuously on $y \in \mathbb{H}^3$.

Suppose that $\vartheta_n \Rightarrow \vartheta_\infty$ as $n \to \infty$. By construction, a transformation $\vartheta \in \mathcal{I}(\mathbb{H}^3)$ from Lemma 3.21 depends continuously on the parameters $y^1, y^2 \in \mathbb{H}^3$, $\{e^1, e^2, e^3\} \subset T_{y^1}\mathbb{H}^3$, and $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\} \subset T_{y^2}\mathbb{H}^3$. Hence, for any point $y \in \mathbb{H}^3$ the sequence $\{\vartheta_n.y\}_{n\in\mathbb{N}}$ converges to the point $\vartheta_\infty.y \in \mathbb{H}^3$ as $n \to \infty$, which means that the convergence of the isometries $\{\vartheta_n\}_{n\in\mathbb{N}}$ in the sense of Lemma 3.21 implies also the "weak" convergence of these isometries to ϑ_∞.

Suppose now that $\vartheta_n \xrightarrow[n\to\infty]{} \vartheta_\infty$. Being a linear transformation of Minkowski space $\mathbb{R}^{3,1}$, the hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n=1}^\infty$ are represented in the standard basis of $\mathbb{R}^{3,1}$ by the 4×4-real matrices $M^{\vartheta_n} \stackrel{\text{def}}{=} (\vartheta_n^0, \vartheta_n^1, \vartheta_n^2, \vartheta_n^3)$, where ϑ_n^k, $k = 0, 1, 2, 3$, are the columns of M_n^ϑ.

Let $P_0 \stackrel{\text{def}}{=} (1, 0, 0, 0)^T \in \mathbb{I}^3 \subset \mathbb{R}^{3,1}$. The "weak" convergence of the isometries $\{\vartheta_n\}_{n\in\mathbb{N}}$ at the point P_0 means that $M^{\vartheta_n}.P_0 \xrightarrow[n\to\infty]{} M^{\vartheta_\infty}.P_0$, i.e.

$$\vartheta_n^0 \xrightarrow[n\to\infty]{} \vartheta_\infty^0. \tag{3.11}$$

Let $P_1 \stackrel{\text{def}}{=} (\sqrt{2}, 1, 0, 0)^T \in \mathbb{I}^3 \subset \mathbb{R}^{3,1}$. The "weak" convergence of the isometries $\{\vartheta_n\}_{n\in\mathbb{N}}$ at the point P_1 means that $M^{\vartheta_n}.P_1 \xrightarrow[n\to\infty]{} M^{\vartheta_\infty}.P_1$, i.e. $\sqrt{2}\vartheta_n^0 + \vartheta_n^1 \xrightarrow[n\to\infty]{} \sqrt{2}\vartheta_\infty^0 + \vartheta_\infty^0$. Taking into account (3.11), we obtain that $\vartheta_n^1 \xrightarrow[n\to\infty]{} \vartheta_\infty^1$. Similarly we get that $\vartheta_n^2 \xrightarrow[n\to\infty]{} \vartheta_\infty^2$ and $\vartheta_n^3 \xrightarrow[n\to\infty]{} \vartheta_\infty^3$. Thus, the "weak" convergence of the isometries $\{\vartheta_n\}_{n\in\mathbb{N}}$ to ϑ_∞ as $n \to \infty$ implies also their convergence in the sense of Lemma 3.21. ☐

Lemma 3.23. *For each $n \in \mathbb{N}$ let a pair of surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ (which are the images of developing maps $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{S}}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \widetilde{\mathcal{S}}_n^-$) be invariant under the actions of a quasi-Fuchsian group $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$. Suppose in addition that the restrictions of the developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ on the domains $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ and $\widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$ defined in Subsection 3.1.2 converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widehat{\Delta}^- \to \mathbb{H}^3$. Then there is a sequence of positive integers $n_k \xrightarrow[k\to\infty]{} \infty$ such that the morphisms $\{\rho_{n_k}^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)\}_{k\in\mathbb{N}}$ converge to a morphism $\rho_\infty^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)$ in the sense of Lemma 3.21, i.e. for every $\gamma \in \pi_1(\mathcal{S})$ there exists a hyperbolic isometry which we denote by $\rho_\infty^{\mathcal{S}}(\gamma)$ such that $\rho_{n_k}^{\mathcal{S}}(\gamma) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma)$ as $k \to \infty$.*

Proof. First, we prove that there is a sequence of positive integers $n_k \xrightarrow[k\to\infty]{} \infty$ such that for any generator γ_i of the group $\pi_1(\mathcal{S})$ together with its inverse element $\gamma_i^{-1} \in \pi_1(\mathcal{S})$, $i = 1, ..., l$, the subsequences of isometries $\rho_{n_k}^{\mathcal{S}}(\gamma_i) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma_i)$ and $\rho_{n_k}^{\mathcal{S}}(\gamma_i^{-1}) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma_i^{-1})$ converge as $k \to \infty$.

Indeed, since for any $i = 1, ..., l$ points $\tilde{x}^+$, $\gamma_i.\tilde{x}^+$, and $\gamma_i^{-1}.\tilde{x}^+$ lie inside $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ by construction, and because of convergence of the developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ to a continuous function $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$, we know that the sequences of points $\tilde{x}_n^+ (= \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+)) \xrightarrow[n\to\infty]{} \tilde{x}_\infty^+ (= \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{x}^+))$, $\rho_n^{\mathcal{S}}(\gamma_i).\tilde{x}_n^+ (= \rho_n^{\mathcal{S}}(\gamma_i).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma_i.\tilde{x}^+)) \xrightarrow[n\to\infty]{} \rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{x}_\infty^+ (= \rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\gamma_i.\tilde{x}^+))$, and $[\rho_n^{\mathcal{S}}(\gamma_i)]^{-1}.\tilde{x}_n^+ (= \rho_n^{\mathcal{S}}(\gamma_i^{-1}).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma_i^{-1}.\tilde{x}^+)) \xrightarrow[n\to\infty]{} [\rho_\infty^{\mathcal{S}}(\gamma_i)]^{-1}.\tilde{x}_\infty^+ (= \rho_\infty^{\mathcal{S}}(\gamma_i^{-1}).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\gamma_i^{-1}.\tilde{x}^+))$ converge in $\mathbb{H}^3$.

Also we know that for each $n \in \mathbb{N}$ and for every $i = 1, ..., l$, the differential $d_{\tilde{x}_n^+}\rho_n^{\mathcal{S}}(\gamma_i)$ sends an orthonormal base $\{e_1^{n,i}, e_2^{n,i}, e_3^{n,i}\}$ of the tangent space $T_{\tilde{x}_n^+}\mathbb{H}^3$ to an orthonormal base $\{\hat{e}_1^{n,i}, \hat{e}_2^{n,i}, \hat{e}_3^{n,i}\}$ of $T_{\rho_n^{\mathcal{S}}(\gamma_i).\tilde{x}_n^+}\mathbb{H}^3$ (recall that, by constructions all the points $\tilde{x}_n^+$, $n \in \mathbb{N}$ coincide). Since the subsequences $\{e_j^{n,i}\}_{n\in\mathbb{N}}$, $\{\hat{e}_j^{n,i}\}_{n\in\mathbb{N}}$, $j = 1, 2, 3$, $i = 1, ..., l$, of unitary vectors are bounded, there exists a sequence of positive integers $n_k \xrightarrow[k\to\infty]{} \infty$ such that the pairs of subsequences of orthonormal bases $\{e_1^{n_k,i}, e_2^{n_k,i}, e_3^{n_k,i}\}_{k\in\mathbb{N}}$ and $\{\hat{e}_1^{n_k,i}, \hat{e}_2^{n_k,i}, \hat{e}_3^{n_k,i}\}_{k\in\mathbb{N}}$ converge all together ($i = 1, ..., l$) ensemble to orthonormal bases $\{e_1^{\infty,i}, e_2^{\infty,i}, e_3^{\infty,i}\}$ and $\{\hat{e}_1^{\infty,i}, \hat{e}_2^{\infty,i}, \hat{e}_3^{\infty,i}\}$. Hence, by Lemma 3.21, there exists a hyperbolic isometry that we denote by $\rho_\infty^{\mathcal{S}}(\gamma_i)$ which sends the point $\tilde{x}_\infty^+$ to the point $\rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{x}_\infty^+$ defined above, and which differential $d_{\tilde{x}_\infty^+}\rho_\infty^{\mathcal{S}}(\gamma_i)$ sends an orthonormal base $\{e_1^{\infty,i}, e_2^{\infty,i}, e_3^{\infty,i}\}$ of the tangent space $T_{\tilde{x}_\infty^+}\mathbb{H}^3$ to an orthonormal base $\{\hat{e}_1^{\infty,i}, \hat{e}_2^{\infty,i}, \hat{e}_3^{\infty,i}\}$ of $T_{\rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{x}_\infty^+}\mathbb{H}^3$ such that $\rho_{n_k}^{\mathcal{S}}(\gamma_i) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma_i)$ as $k \to \infty$.

Secondly, we derive that for any element $\gamma \in \pi_1(\mathcal{S})$ the subsequences of isometries $\rho_{n_k}^{\mathcal{S}}(\gamma) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma)$ converges as $k \to \infty$. Indeed, every $\gamma \in \pi_1(\mathcal{S})$ can be decomposed in a product of generators of $\pi_1(\mathcal{S})$ together with their inverse elements, for which the demanded convergence has already been shown. ☐

Assumption 3.24. *Further we assume that the sequence of holonomy representations $\{\rho_n^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)\}_{n\in\mathbb{N}}$ (where the groups $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$ are quasi-Fuchsian) converges to a holonomy representation $\rho_\infty^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)$ (where $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ is a discrete group of isometries of $\mathbb{H}^3$) in the sense of Lemma 3.21 as $n \to \infty$.*

Let us now prove the following property of the functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widehat{\Delta}^- \to \mathbb{H}^3$ with respect to the group of isometries $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of space $\mathbb{H}^3$.

Remark 3.25. *If for a pair of points $\tilde{y}_1^+, \tilde{y}_2^+ \in \widehat{\Delta}^+$ there exists a transformation $\gamma^+ \in \pi_1(\mathcal{S})$ such that $\tilde{y}_2^+ = \gamma^+.\tilde{y}_1^+$, then the following equality holds:*

$$\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_2^+) = \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+). \tag{3.12}$$

Similarly, if for a pair of points $\tilde{y}_1^-, \tilde{y}_2^- \in \widehat{\Delta}^-$ there exists a transformation $\gamma^- \in \pi_1(\mathcal{S})$ such that $\tilde{y}_2^- = \gamma^-.\tilde{y}_1^-$, then

$$\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}_2^-) = \rho_\infty^{\mathcal{S}}(\gamma^-).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}_1^-).$$

Proof. It suffices to prove the formula (3.12).

By Remark 3.14, the relation

$$\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_2^+) = \rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+) \tag{3.13}$$

is valid for all $n \in \mathbb{N}$.

By Assumption 3.20, the sequence $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_2^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ converges to the point $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_2^+) \in \mathbb{H}^3$. Hence, taking into account the formula (3.13) we see that in order to prove the equality (3.12) we need to demonstrate the convergence of the sequence $\{\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ to the point $\rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)$, i.e., fixing $\varepsilon > 0$, we ought to find such $n_0 \in \mathbb{N}$ that

$$\forall n > n_0 \quad \text{the inequality} \quad \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) < \varepsilon \quad \text{holds.} \tag{3.14}$$

First, by the above-mentioned Assumption 3.20, the sequence $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ converges to the point $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+) \in \mathbb{H}^3$. Therefore,

$$\exists n_1 \in \mathbb{N} : \forall n > n_1 \quad \text{the inequality} \quad \mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) < \frac{\varepsilon}{2} \quad \text{is valid.} \tag{3.15}$$

Also, by Assumption 3.24, $\rho_n^{\mathcal{S}}(\gamma^+) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma^+)$ as $n \to \infty$. Hence, by Lemma 3.22, the sequence of points $\{\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ converges to the point $\rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+) \in \mathbb{H}^3$, i.e.

$$\exists n_2 \in \mathbb{N} : \forall n > n_2 \quad \text{the inequality} \quad \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) < \frac{\varepsilon}{2} \quad \text{is true.} \tag{3.16}$$

Applying the triangle inequality, we get:

$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_\infty(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) \leq$$
$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) + \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)). \tag{3.17}$$

The fact that $\rho_n^{\mathcal{S}}(\gamma^+)$ is an isometry of $\mathbb{H}^3$ implies the equality:

$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) = \mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)). \tag{3.18}$$

Therefore, substituting (3.18) in (3.17), we obtain:

$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) \leq$$
$$\mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) + \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)). \tag{3.19}$$

Hence, by (3.19), (3.15), and (3.16), we conclude that it is sufficient to pose $n_0 = \max(n_1, n_2)$ to satisfy the condition (3.14). □

Now we are able to extend the functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widehat{\Delta}^- \to \mathbb{H}^3$ to the whole domains $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$. Let us do it as follows: for arbitrary points $\tilde{y}^+ \in \widetilde{\mathcal{S}}^+$ and $\tilde{y}^- \in \widetilde{\mathcal{S}}^-$ we find such points $\tilde{y}_\Delta^+$ and $\tilde{y}_\Delta^-$ in the fundamental domains $\Delta^+ \subset \widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$ of the surface $\mathcal{S}$ and such elements $\gamma^+, \gamma^- \in \pi_1(\mathcal{S})$ that $\tilde{y}^+ = \gamma^+.\tilde{y}_\Delta^+$ and $\tilde{y}^- = \gamma^-.\tilde{y}_\Delta^-$, then we define $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}^+) \overset{\text{def}}{=} \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_\Delta^+)$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}^-) \overset{\text{def}}{=} \rho_\infty^{\mathcal{S}}(\gamma^-).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}_\Delta^-)$. By construction, the surfaces $\widetilde{\mathcal{S}}_\infty^+ \overset{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\widetilde{\mathcal{S}}^+)$ and $\widetilde{\mathcal{S}}_\infty^- \overset{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\widetilde{\mathcal{S}}^-)$ are invariant under the actions of the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$.

Repeating almost literally the demonstration of Remark 3.25, we can prove

Lemma 3.26. *The sequences of developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3$.*

Finally, we show

Remark 3.27. *The boundaries at infinity* $\partial_\infty \widetilde{\mathcal{S}}_\infty^+ \subset \partial_\infty \mathbb{H}^3$ *and* $\partial_\infty \widetilde{\mathcal{S}}_\infty^- \subset \partial_\infty \mathbb{H}^3$ *of the surfaces* $\widetilde{\mathcal{S}}_\infty^+$ *and* $\widetilde{\mathcal{S}}_\infty^-$ *coincide with the limit set* $\Lambda_{\rho_\infty^{\mathcal{S}}}$ *of the group* $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$*. Moreover, the group* $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ *of isometries of* $\mathbb{H}^3$ *from Lemma 3.23 is quasi-Fuchsian.*

Proof. By Lemma 3.26, the sequences of surfaces $\{\widetilde{\mathcal{S}}_n^+\}_{n\in\mathbb{N}}$ and $\{\widetilde{\mathcal{S}}_n^-\}_{n\in\mathbb{N}}$ bounding the convex connected hyperbolic domains $\{\widetilde{\mathcal{M}}_n\}_{n\in\mathbb{N}}$ converge to the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ in $\mathbb{H}^3$. Hence, the sets $\{\widetilde{\mathcal{M}}_n\}_{n\in\mathbb{N}}$ converge to a convex connected hyperbolic domain $\widetilde{\mathcal{M}}_\infty$. Moreover, the boundaries at infinity $\{\partial_\infty \widetilde{\mathcal{S}}_n^+\}_{n\in\mathbb{N}}$ and $\{\partial_\infty \widetilde{\mathcal{S}}_n^-\}_{n\in\mathbb{N}}$ converge to the curves $\partial_\infty \widetilde{\mathcal{S}}_\infty^+ \subset \partial_\infty \mathbb{H}^3$ and $\partial_\infty \widetilde{\mathcal{S}}_\infty^- \subset \partial_\infty \mathbb{H}^3$. Indeed, our surfaces in the Poincaré disc model of $\mathbb{H}^3$ considered as Euclidean surfaces inside a unitary ball converge together with their boundaries.

Recall that, by the Labourie-Schlenker Theorem 3.8, for each $n \in \mathbb{N}$ the curves $\partial_\infty \widetilde{\mathcal{S}}_n^+$ and $\partial_\infty \widetilde{\mathcal{S}}_n^-$ coincide with the limit set $\Lambda_{\rho_n^{\mathcal{S}}}$ of the quasi-Fuchsian holonomy representations $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ which is homotopic to a circle in $\partial_\infty \mathbb{H}^3$. On the other hand, by Assumption 3.24, $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S})) \rightrightarrows \rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ as $n \to \infty$, which implies that the sequence of the limit sets $\{\Lambda_{\rho_n^{\mathcal{S}}}\}_{n\in\mathbb{N}}$ converges to the limit set $\Lambda_{\rho_\infty^{\mathcal{S}}}$ (see, for instance, [Mat04, p. 323]).

Thus, the boundaries at infinity $\partial_\infty \widetilde{\mathcal{S}}_\infty^+$ and $\partial_\infty \widetilde{\mathcal{S}}_\infty^-$ of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ coincide with the limit set $\Lambda_{\rho_\infty^{\mathcal{S}}}$ of the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$. Furthermore, we conclude that the boundary $\partial\widetilde{\mathcal{M}}_\infty$ of the domain $\widetilde{\mathcal{M}}_\infty$ consists of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$, and the boundary at infinity $\partial_\infty\widetilde{\mathcal{M}}_\infty$ of $\widetilde{\mathcal{M}}_\infty$ also coincides with $\Lambda_{\rho_\infty^{\mathcal{S}}}$.

Since the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ are topological discs embedded in $\mathbb{H}^3$, their common boundary at infinity is homotopic to a circle. Therefore, by definition, the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ is quasi-Fuchsian. ☐

Note that the domain $\widetilde{\mathcal{M}}_\infty$ which appeared during the demonstration of Remark 3.27, is invariant under the actions of the quasi-Fuchsian group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$.

3.1.4 Adaptation of a classical theorem of A. D. Alexandrov to the hyperbolic case

Recall a classical result due to A. D. Alexandrov:

Theorem 3.28 (Theorem 1 in Sec. 1 of Chapter III [Ale06], p. 91)**.** *If a sequence of closed convex surfaces* $\mathcal{F}_n$ *converges to a closed convex surface* $\mathcal{F}$ *and if two sequences of points* X_n *and* Y_n *on* $\mathcal{F}_n$ *converge to two points* X *and* Y *of* $\mathcal{F}$*, respectively, then the distances between the points* X_n *and* Y_n *measured on the surfaces* $\mathcal{F}_n$ *converge to the distance between the points* X *and* Y *measured on* $\mathcal{F}$*, i.e.,* $\mathrm{d}_{\mathcal{F}}(X,Y) = \lim_{n\to\infty} \mathrm{d}_{\mathcal{F}_n}(X_n, Y_n)$.

A. D. Alexandrov demonstrated this theorem in Euclidean 3-space. Slightly modifying his proof, here we show the validity of Theorem 3.28 in hyperbolic space $\mathbb{H}^3$. We will largely use this result in Subsection 3.1.5.

First we remark that the proof of Theorem 3.28 in the Euclidean case is based on the two following lemmas which hold true in all Hadamard spaces (i.e. in the hyperbolic space as well), and it uses the mentioned below properties of the arc length in any complete metric space:

Lemma 3.29 (Lemma 2 in Sec. 1 of Chapter III [Ale06], p. 93)**.** *If a curve* L *lies outside a closed convex surface* $\mathcal{F}$*, then the length of this curve is not less than the distance on* $\mathcal{F}$ *between the projections of its endpoints to the surface* $\mathcal{F}$*. In particular, if the ends* A *and* B *of the curve* L *lie on* $\mathcal{F}$*, then the length of the curve* L *is not less than the length of the shortest arc* AB *on the surface* $\mathcal{F}$*.*

Lemma 3.30 (Lemma 3 in Sec. 1 of Chapter III [Ale06], p. 93)**.** *If a sequence of closed convex surfaces $\mathcal{F}_n$ converges to a nondegenerate surface $\mathcal{F}$ and if points X_n and Y_n converge to the same point X on $\mathcal{F}$, then the distance between X_n and Y_n on $\mathcal{F}_n$ converges to zero:* $\lim_{n\to\infty} \mathrm{d}_{\mathcal{F}_n}(X_n, Y_n) = 0$.

Property 3.31 (Theorem 3 in Sec. 2 of Chapter II [Ale06], p. 66)**.** *There is a shortest arc of every two points on a manifold with complete intrinsic metric.*

Property 3.32 (Theorem 4 in Sec. 1 of Chapter II [Ale06], p. 59)**.** *We can choose a convergent subsequence from each infinite set of curves in a compact domain of length not exceeding a given one.*

Property 3.33 (Theorem 5 in Sec. 1 of Chapter II [Ale06], p. 59)**.** *If curves L_n converge to a curve L, then the length of L is not greater than the lower limit of the lengths of L_n.*

However, there is a place in the proof of Theorem 3.28 which uses some particular properties of Euclidean space, specifically, of the Euclidean homothety. In the following statement we formulate what is shown there:

Lemma 3.34. *If a sequence of closed convex surfaces $\mathcal{F}_n$ converges to a nondegenerate closed convex surface $\mathcal{F}$ and if two sequences of points X_n and Y_n on $\mathcal{F}_n$ converge to two points X and Y of $\mathcal{F}$, respectively, then*

$$\limsup_{n\to\infty} \mathrm{d}_{\mathcal{F}_n}(X_n, Y_n) \leq \mathrm{d}_{\mathcal{F}}(X, Y). \tag{3.20}$$

Proof of Lemma 3.34 in the Euclidean case [Ale06, pp. 95–96]. Take a point O inside the surface $\mathcal{F}$ and perform the homothety transform with the center at O of the surfaces $\mathcal{F}_n$ so that all these surfaces turn out to be inside $\mathcal{F}$. Note that if the initial surface $\mathcal{F}_n$ lies inside $\mathcal{F}$ then we do not need to apply the homothety, so we pose the coefficient of homothety $\lambda_n = 1$; otherwise we perform the scaling back homothety transform with $\lambda_n < 1$. Since the surfaces $\mathcal{F}_n$ converge to $\mathcal{F}$, the coefficients λ_n can be taken closer and closer to 1 as n increases and $\lambda_n \to 1$ as $n \to \infty$. The surfaces and points, which are obtained from the surfaces $\mathcal{F}_n$ and the points X_n and Y_n as a result of this transformation, will be denoted by $\lambda_n\mathcal{F}_n$, $\lambda_n X_n$, and $\lambda_n Y_n$. Since $\lambda_n \to 1$ and the points X_n and Y_n tend to X and Y, the points $\lambda_n X_n$ and $\lambda_n Y_n$ also converge to X and Y, respectively.

Let X'_n and Y'_n be the projections of the points X and Y to the surfaces $\lambda_n\mathcal{F}_n$. By Lemma 3.29,

$$\mathrm{d}_{\lambda_n\mathcal{F}_n}(X'_n, Y'_n) \leq \mathrm{d}_{\mathcal{F}}(X, Y). \tag{3.21}$$

Obviously, the points X'_n converge to X as $n \to \infty$, and at the same time, the points $\lambda_n X_n$ also converge to X. Therefore, by Lemma 3.30,

$$\mathrm{d}_{\lambda_n\mathcal{F}_n}(\lambda_n X_n, X'_n) \to 0, \tag{3.22}$$

and, by the same arguments,

$$\mathrm{d}_{\lambda_n\mathcal{F}_n}(Y'_n, \lambda_n Y_n) \to 0. \tag{3.23}$$

By the "triangle inequality",

$$\mathrm{d}_{\lambda_n\mathcal{F}_n}(\lambda_n X_n, \lambda_n Y_n) \leq \mathrm{d}_{\lambda_n\mathcal{F}_n}(\lambda_n X_n, X'_n) + \mathrm{d}_{\lambda_n\mathcal{F}_n}(X'_n, Y'_n) + \mathrm{d}_{\lambda_n\mathcal{F}_n}(Y'_n, \lambda_n Y_n). \tag{3.24}$$

Using the inequality (3.21) and the relations (3.22) and (3.23) and passing to the limit in (3.24) as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \mathrm{d}_{\lambda_n\mathcal{F}_n}(\lambda_n X_n, \lambda_n Y_n) \leq \mathrm{d}_{\mathcal{F}}(X, Y). \tag{3.25}$$

But under the homothety with coefficient λ_n, all distances change by λ_n times, and, therefore,

$$\mathrm{d}_{\lambda_n \mathcal{F}_n}(\lambda_n X_n, \lambda_n Y_n) = \lambda_n \mathrm{d}_{\mathcal{F}_n}(X_n, Y_n); \tag{3.26}$$

since $\lambda_n \to 1$, the formula (3.25) implies (3.20). □

Let us adapt the proof of Lemma 3.34 for hyperbolic 3-space.

Modification of the proof of Lemma 3.34 for the hyperbolic case. Further we will use the notation developed in the proof of the Euclidean version of Lemma 3.34. Considering the surfaces $\mathcal{F} \subset \mathbb{H}^3$ and $\mathcal{F}_n \subset \mathbb{H}^3$ ($n \in \mathbb{N}$) in the projective model $\mathbb{K}^3$ of hyperbolic space $\mathbb{H}^3$ as surfaces of Euclidean space $\mathbb{R}^3$ and supposing in addition that the center $O_{\mathbb{K}}$ of the Kleinian model $\mathbb{K}^3$ lies inside the surface $\mathcal{F}$, as previously, let us perform the Euclidean homothety transforms with the center at $O_{\mathbb{K}}$ of the surfaces $\mathcal{F}_n$ so that all resulting surfaces $\lambda_n \mathcal{F}_n$ turn out to be inside $\mathcal{F}$ (here λ_n are the Euclidean homothety coefficients, $n \in \mathbb{N}$). Below we will call *Euclidean homothety transform* any transformation of hyperbolic space $\mathbb{H}^3$ which corresponds to a homothety transformation of Euclidean space $\mathbb{R}^3$ when we identify $\mathbb{R}^3$ with the projective model $\mathbb{K}^3$ of $\mathbb{H}^3$. We already know that in the Euclidean case the distances between corresponding pairs of points $X_n, Y_n \in \mathcal{F}_n$ and $\lambda_n X_n, \lambda_n Y_n \in \lambda_n \mathcal{F}_n$ in the induced metrics of the surfaces $\mathcal{F}_n$ and $\lambda_n \mathcal{F}_n$ satisfy the relation (3.26). Let us now find a similar condition in the case when $\mathcal{F}_n$ and $\lambda_n \mathcal{F}_n$ are regarded as surfaces of hyperbolic space $\mathbb{H}^3$.

All closed convex surfaces $\mathcal{F}_n$ together with their limit surface $\mathcal{F}$ can be included into a sufficiently large ball $\mathcal{B} \subset \mathbb{H}^3$ centered at $O_{\mathbb{K}}$. Let us put $\mathcal{B}$ into the Kleinian model $\mathbb{K}^3$ of $\mathbb{H}^3$ and let $\rho_{\mathcal{B}} < 1$ stands for the Euclidean radius of $\mathcal{B}$ in $\mathbb{K}^3$.

An Euclidean homothety transform τ centered at $O_{\mathbb{K}} \in \mathbb{K}^3$ with a coefficient $\lambda \leq 1$ sends any point Z inside $\mathcal{B}$ to the point λZ. Denote by $\rho (< \rho_{\mathcal{B}})$ the length of the Euclidean radius-vector connecting the points $O_{\mathbb{K}}$ and Z in the projective model $\mathbb{K}^3$ of $\mathbb{H}^3$. The differential $d\tau$ of the hyperbolic transformation τ sends any vector $v_Z \in T_Z\mathbb{H}^3$ codirectional with the geodesic L_Z which contains the points $O_{\mathbb{K}}$, Z, and λZ, to the vector $v_{\lambda Z} \in T_{\lambda Z}\mathbb{H}^3$ also codirectional with L_Z. A direct calculation shows that the norms of the vectors v_Z and $v_{\lambda Z}$ are related as follows:

$$\|v_{\lambda Z}\| = \frac{\lambda(1-\rho^2)}{1-\lambda^2\rho^2}\|v_Z\|. \tag{3.27}$$

It is easy to verify that for $\lambda \leq 1$ the function $f_\lambda(\rho) \stackrel{\text{def}}{=} \frac{\lambda(1-\rho^2)}{1-\lambda^2\rho^2}$ in ρ is monotonically decreasing in the segment $[0, \rho_{\mathcal{B}}]$. Together with (3.27), this fact implies:

$$\|v_{\lambda Z}\| \geq \frac{\lambda(1-{\rho_{\mathcal{B}}}^2)}{1-\lambda^2{\rho_{\mathcal{B}}}^2}\|v_Z\|. \tag{3.28}$$

Similarly, the differential $d\tau$ sends any vector $v_Z^\perp \in T_Z\mathbb{H}^3$ perpendicular to the geodesic L_Z, to the vector $v_{\lambda Z}^\perp \in T_{\lambda Z}\mathbb{H}^3$ also perpendicular to L_Z. A direct calculation shows that the norms of the vectors $v_Z^\perp$ and $v_{\lambda Z}^\perp$ are related as follows:

$$\|v_{\lambda Z}^\perp\| = \frac{\lambda\sqrt{1-\rho^2}}{\sqrt{1-\lambda^2\rho^2}}\|v_Z^\perp\|. \tag{3.29}$$

It is easy to verify that for $\lambda \leq 1$ the function $g_\lambda(\rho) \stackrel{\text{def}}{=} \frac{\lambda\sqrt{1-\rho^2}}{\sqrt{1-\lambda^2\rho^2}}$ in ρ is monotonically decreasing in the segment $[0, \rho_{\mathcal{B}}]$. Together with (3.29), it implies:

$$\|v_{\lambda Z}^\perp\| \geq \frac{\lambda\sqrt{1-{\rho_{\mathcal{B}}}^2}}{\sqrt{1-\lambda^2{\rho_{\mathcal{B}}}^2}}\|v_Z^\perp\|. \tag{3.30}$$

Any vector $u \in T_Z\mathbb{H}^3$ can be decomposed as the sum of two vectors $u = v + v^\perp$, $v, v^\perp \in T_Z\mathbb{H}^3$, such that the vector v is codirectional with the geodesic L_Z, and the vector $v^\perp$ is perpendicular to L_Z. Hence, (**??**) and (3.30) imply that the norms of the vectors $u \in T_Z\mathbb{H}^3$ and $u_\lambda \overset{\text{def}}{=} d\tau(Z).u \in T_{\lambda Z}\mathbb{H}^3$ satisfy the following inequality:

$$\|u_\lambda\| \geq \min\left\{\frac{\lambda(1-\rho_{\mathcal{B}}{}^2)}{1-\lambda^2\rho_{\mathcal{B}}{}^2}, \frac{\lambda\sqrt{1-\rho_{\mathcal{B}}{}^2}}{\sqrt{1-\lambda^2\rho_{\mathcal{B}}{}^2}}\right\}\|u\| = \frac{\lambda(1-\rho_{\mathcal{B}}{}^2)}{1-\lambda^2\rho_{\mathcal{B}}{}^2}\|u\| \tag{3.31}$$

as $0 < \lambda \leq 1$.

Recall that the length of a curve $c : [0,1] \to \mathbb{H}^3$ which is C^1-smooth almost everywhere is given by the formula $l(c) \overset{\text{def}}{=} \int_0^1 \|c'(t)\|dt$ where $c'(t) \in T_{c(t)}\mathbb{H}^3$ for almost all $t \in [0,1]$. Suppose in addition that the curve c lies in the interior of the ball $\mathcal{B}$, apply the Euclidean homothety transform τ to c, and denote the resulting curve by c_λ. Hence, taking into account the inequality (3.1.4), we see that the lengths of the curves c and c_λ are related as follows:

$$l(c_\lambda) \geq \frac{\lambda(1-\rho_{\mathcal{B}}{}^2)}{1-\lambda^2\rho_{\mathcal{B}}{}^2} l(c).$$

Thus, returning to the consideration of the distances between the pairs of points $X_n, Y_n \in \mathcal{F}_n$ and $\lambda_n X_n, \lambda_n Y_n \in \lambda_n\mathcal{F}_n$ in the induced metrics of the surfaces $\mathcal{F}_n$ and $\lambda_n\mathcal{F}_n$, we conclude that in the hyperbolic case the inequality

$$\mathrm{d}_{\lambda_n\mathcal{F}_n}(\lambda_n X_n, \lambda_n Y_n) \geq \frac{\lambda_n(1-\rho_{\mathcal{B}}{}^2)}{1-\lambda_n^2\rho_{\mathcal{B}}{}^2}\mathrm{d}_{\mathcal{F}_n}(X_n, Y_n) \tag{3.32}$$

holds. Substituting (3.32) in the formula (3.25) which is valid in both Euclidean and hyperbolic situations, we get:

$$\limsup_{n\to\infty} \frac{\lambda_n(1-\rho_{\mathcal{B}}{}^2)}{1-\lambda_n^2\rho_{\mathcal{B}}{}^2}\mathrm{d}_{\mathcal{F}_n}(X_n, Y_n) \leq \mathrm{d}_{\mathcal{F}}(X, Y). \tag{3.33}$$

Since the expression $\frac{\lambda_n(1-\rho_{\mathcal{B}}{}^2)}{1-\lambda_n^2\rho_{\mathcal{B}}{}^2}$ tends to 1 as the numbers λ_n approach to 1, the formula (3.33) implies (3.20). □

We have just adapted to the hyperbolic situation the only place in the proof of Theorem 3.28 largely depending on properties of Euclidean space. Therefore, Theorem 3.28 remains valid in hyperbolic 3-space.

When the present work was already written, the author found that A. D. Alexandrov proved the hyperbolic version of Theorem 3.28 using different methods long ago in 1945 (see his paper [Ale45, Theorem 3] in Russian).

3.1.5 Induced metrics of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$

Return to consideration of the family of convex domains $\{\widetilde{\mathcal{M}}_n\}_{n=1}^\infty$ with the boundaries $\partial\widetilde{\mathcal{M}}_n = \widetilde{\mathcal{S}}_n^+ \cup \widetilde{\mathcal{S}}_n^-$ (see Subsections 3.1.2 and 3.1.3) in hyperbolic space $\mathbb{H}^3$. Assume in addition that the marked points $\tilde{x}_n^+ \in \widetilde{\mathcal{S}}_n^+$, $n = 1, ..., \infty$, are all identified with an arbitrary point $O_{\mathbb{H}} \in \mathbb{K}^3$.

Consider a ball $\hat{\mathcal{B}} \subset \mathbb{H}^3$ centered at $O_{\mathbb{H}}$ of a sufficiently big hyperbolic radius $\hat{\rho}$ (it will be enough to put $\hat{\rho} = 9\delta_{\mathcal{S}} + \delta_{\mathcal{M}}$, where the constants $\delta_{\mathcal{S}}$ and $\delta_{\mathcal{M}}$ are defined in Lemmas 3.12 and 3.13). Define the convex compact hyperbolic sets $\mathcal{M}_n^{\mathcal{B}} \overset{\text{def}}{=} \widetilde{\mathcal{M}}_n \cap \hat{\mathcal{B}}$, and denote by $\hat{\mathcal{S}}_n^+ \overset{\text{def}}{=} \partial\mathcal{M}_n^{\mathcal{B}} \cap \widetilde{\mathcal{S}}_n^+$ and $\hat{\mathcal{S}}_n^- \overset{\text{def}}{=} \partial\mathcal{M}_n^{\mathcal{B}} \cap \widetilde{\mathcal{S}}_n^-$ the intersections of the boundary $\partial\mathcal{M}_n^{\mathcal{B}}$ of the domain $\mathcal{M}_n^{\mathcal{B}}$ with the surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$, $n = 1, ..., \infty$. By construction, the sets $\widehat{\Delta}_n^+$ and $\widehat{\Delta}_n^-$ defined in Lemma 3.16 are subsets of $\hat{\mathcal{S}}_n^+$ and $\hat{\mathcal{S}}_n^-$ correspondingly, $n = 1, ..., \infty$.

Remark 3.35. *The ball $\hat{\mathcal{B}}$ is taken big enough in order to provide the following property: for an arbitrary pair of points $A^+, B^+ \in \widehat{\Delta}_n^+$ there exists a path $\zeta^+ \subset \widehat{\Delta}_n^+$ connecting A^+ and B^+ which is shorter than any path $\xi^+ \subset \partial\mathcal{M}_n^{\mathcal{B}}$ connecting A^+ and B^+ and such that $\xi^+ \cap (\partial\mathcal{M}_n^{\mathcal{B}} \setminus \hat{\mathcal{S}}_n^+) \neq \emptyset$. Similarly, for points $A^-, B^- \in \widehat{\Delta}_n^-$ there exists a path $\zeta^- \subset \widehat{\Delta}_n^-$ connecting A^- and B^- which is shorter than any path $\xi^- \subset \partial\mathcal{M}_n^{\mathcal{B}}$ connecting A^- and B^- and such that $\xi^- \cap (\partial\mathcal{M}_n^{\mathcal{B}} \setminus \hat{\mathcal{S}}_n^-) \neq \emptyset$. For this purpose, radius $\hat{\rho} = 9\delta_{\mathcal{S}} + \delta_{\mathcal{M}}$ of the ball $\hat{\mathcal{B}}$ is sufficient although not optimal.*

Recall that, by Lemma 3.26, the sequences of developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3$, and the images of the maps $\tilde{f}_{\widetilde{\mathcal{S}}_n^+}$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-}$ are convex surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^+$ respectively, $n = 1, ..., \infty$. Therefore, by construction, the surfaces $\{\widehat{\Delta}_n^+\}_{n\in\mathbb{N}}$ and $\{\widehat{\Delta}_n^-\}_{n\in\mathbb{N}}$ converge to $\widehat{\Delta}_\infty^+$ and $\widehat{\Delta}_\infty^-$, and moreover, the sequence of closed convex nondegenerate surfaces $\{\partial\mathcal{M}_n^{\mathcal{B}}\}_{n\in\mathbb{N}}$ converges to the closed convex nondegenerate surface $\partial\mathcal{M}_\infty^{\mathcal{B}}$ in $\mathbb{H}^3$. Applying the hyperbolic version of Theorem 3.28 to the family of surfaces $\{\partial\mathcal{M}_n^{\mathcal{B}}\}_{n\in\mathbb{N}}$ which converges to $\partial\mathcal{M}_\infty^{\mathcal{B}}$ we conclude that the sequence of induced metrics on $\partial\mathcal{M}_n^{\mathcal{B}}$ tends to the induced metric on $\partial\mathcal{M}_\infty^{\mathcal{B}}$ as $n \to \infty$. In particular, given any two sequences of points A_n^+ and B_n^+ in $\widehat{\Delta}_n^+ \subset \partial\mathcal{M}_n^{\mathcal{B}}$ converging to two points A_∞^+ and B_∞^+ in $\widehat{\Delta}_\infty^+ \subset \partial\mathcal{M}_n^{\mathcal{B}}$, respectively, the distances between the points A_n^+ and B_n^+ measured on the surfaces $\partial\mathcal{M}_n^{\mathcal{B}}$ converge to the distance between the points A_∞^+ and B_∞^+ measured on $\partial\mathcal{M}_\infty^{\mathcal{B}}$, i.e.

$$\mathrm{d}_{\partial\mathcal{M}_\infty^{\mathcal{B}}}(A_\infty^+, B_\infty^+) = \lim_{n\to\infty}\mathrm{d}_{\partial\mathcal{M}_n^{\mathcal{B}}}(A_n^+, B_n^+). \tag{3.34}$$

By Remark 3.35, the distance between the points A_n^+ and B_n^+ measured on $\partial\mathcal{M}_n^{\mathcal{B}}$ is equal to the distance between these points measured on $\hat{\mathcal{S}}_n^+$; also, by construction, $\hat{\mathcal{S}}_n^+$ is a convex subset of the surface $\widetilde{\mathcal{S}}_n^+$ with the induced metric $\tilde{h}_n^+$, therefore

$$\mathrm{d}_{\partial\mathcal{M}_n^{\mathcal{B}}}(A_n^+, B_n^+) = \mathrm{d}_{\tilde{h}_n^+}(A_n^+, B_n^+), \tag{3.35}$$

$n = 1, ..., \infty$. Substituting (3.35) in (3.34), we get:

$$\mathrm{d}_{\tilde{h}_\infty^+}(A_\infty^+, B_\infty^+) = \lim_{n\to\infty}\mathrm{d}_{\tilde{h}_n^+}(A_n^+, B_n^+).$$

Hence, the sequence of the induced metrics $\tilde{h}_n^+$ of the surfaces $\widetilde{\mathcal{S}}_n^+$ restricted on the sets $\widehat{\Delta}_n^+$ converges to the induced metric $\tilde{h}_\infty^+$ of the surface $\widetilde{\mathcal{S}}_\infty^+$ restricted on $\widehat{\Delta}_\infty^+$ as $n \to \infty$. By analogy, the sequence of the induced metrics $\{\tilde{h}_n^-|_{\widehat{\Delta}_n^-}\}_{n\in\mathbb{N}}$ converges to the induced metric $\tilde{h}_\infty^-|_{\widehat{\Delta}_\infty^-}$.

In Subsections 3.1.2 and 3.1.3 we constructed the surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$ to be invariant under the actions of the discrete group $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$ for each $n = 1, ..., \infty$. Hence, the induced metrics $\tilde{h}_n^+$ and $\tilde{h}_n^-$ on the surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$, respectively, are periodic with respect to the group $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$, $n = 1, ..., \infty$. We have just proved that the metrics $\tilde{h}_n^+$ and $\tilde{h}_n^-$ converge to $\tilde{h}_\infty^+$ and $\tilde{h}_\infty^-$, correspondingly, in the neighborhoods $\widehat{\Delta}_n^+ \subset \widetilde{\mathcal{S}}_n^+$ and $\widehat{\Delta}_n^- \subset \widetilde{\mathcal{S}}_n^-$ of the fundamental domains $\Delta_n^+ \subset \widetilde{\mathcal{S}}_n^+$ and $\Delta_n^- \subset \widetilde{\mathcal{S}}_n^-$ of the surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$. Since, by Assumption 3.24 and Remark 3.27, the sequence of quasi-Fuchsian groups $\{\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))\}_{n\in\mathbb{N}}$ converges to a quasi-Fuchsian group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$, we now conclude that the metrics $\tilde{h}_n^+$ and $\tilde{h}_n^-$ converge to $\tilde{h}_\infty^+$ and $\tilde{h}_\infty^-$ everywhere on $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$ as $n \to \infty$.

To complete the proof of Theorem 3.6 let us consider the convex compact hyperbolic domain $\mathcal{M}_\infty \stackrel{\text{def}}{=} \widetilde{\mathcal{M}}_\infty/[\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))]$ with the boundary

$$\partial\mathcal{M}_\infty \stackrel{\text{def}}{=} \mathcal{S}_\infty^+ \cup \mathcal{S}_\infty^- \stackrel{\text{def}}{=} \left(\widetilde{\mathcal{S}}_\infty^+/[\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))]\right) \bigcup \left(\widetilde{\mathcal{S}}_\infty^-/[\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))]\right)$$

in the unbounded hyperbolic manifold $\mathcal{M}_\infty^\circ \overset{\text{def}}{=} \mathbb{H}^3/[\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))]$. The metric $\tilde{h}_\infty^+$ on the universal covering $\widetilde{\mathcal{S}}_\infty^+$ of the boundary component $\mathcal{S}_\infty^+$ of the domain $\mathcal{M}_\infty$ induces the metric $\breve{h}_\infty^+$ on the compact surface $\mathcal{S}_\infty^+$. We have recently showed that the pull-backs $\tilde{h}_n^+$ of the metrics h_n^+ (see Subsection 3.1.2) converge to the pull-back $\tilde{h}_\infty^+$ of the metric $\breve{h}_\infty^+$. Hence, the sequence of metrics $\{h_n^+\}_{n\in\mathbb{N}}$ tends to the metric $\breve{h}_\infty^+$ as $n \to \infty$. But in the very beginning of Subsection 3.1.2 the C^∞-smooth metrics $\{h_n^+\}_{n\in\mathbb{N}}$ were constructed in order to approximate the polyhedral metric h_∞^+. Therefore, the induced metric $\breve{h}_\infty^+$ on $\mathcal{S}_\infty^+$ coincides with the prescribed metric h_∞^+. Similarly we obtain that the metric on the surface $\mathcal{S}_\infty^-$ is exactly h_∞^-.

We sum up that the convex hyperbolic bounded domain $\mathcal{M}_\infty$ with the boundary $\partial\mathcal{M}_\infty = \mathcal{S}_\infty^+ \cup \mathcal{S}_\infty^-$ in the quasi-Fuchsian manifold $\mathcal{M}_\infty^\circ$ was constructed in such a way that the induced metrics of the boundary components $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$ coincide with the prescribed polyhedral metrics h_∞^+ and h_∞^-. Theorem 3.6 is proved. $\square$

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Chapter 4

Distance between boundary components of a convex compact domain in a quasi-Fuchsian manifold

Consider a sequence of convex bounded domains $\mathcal{M}_n$ with the upper boundaries $\mathcal{S}_n^+$ and the lower boundaries $\mathcal{S}_n^-$ in quasi-Fuchsian manifolds $\mathcal{M}_n^\circ$, such that for all n the convex regular metric surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ with the induced metrics h_n^+ and h_n^-, respectively, are topologically the same surface $\mathcal{S}$.

Definition. The *distance $d(\mathcal{K},\mathcal{L})$ between subsets $\mathcal{K}$ and $\mathcal{L}$ of a set $\mathcal{N}$* is defined as follows: $d(\mathcal{K},\mathcal{L}) \stackrel{\text{def}}{=} \inf\{\mathrm{d}_{\mathcal{N}}(u,v) | u \in \mathcal{K}, v \in \mathcal{L}\}$, where $\mathrm{d}_{\mathcal{N}}(u,v)$ stands for the distance between points u and v in $\mathcal{N}$.

In this chapter, we prove the following result which is essentially used in the demonstration of Theorem 3.6 from the previous chapter:

Theorem 4.1. *Let the metrics h_n^+ tend to some metric h_∞^+ (correspondingly, h_n^- tend to h_∞^-) as n goes to ∞. Then there is a common upper bound for the distances between $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ in $\mathcal{M}_n^\circ$ which does not depend on n.*

The proof of Theorem 4.1 is essentially based on

Theorem 4.2. *Given a convex bounded domain $\mathcal{M}$ with the upper boundary $\mathcal{S}^+$ and the lower boundary $\mathcal{S}^-$ in a quasi-Fuchsian manifold $\mathcal{M}^\circ$. If the metric surface $\mathcal{S}^+$ possesses two homotopically different nontrivial closed simple intersecting curves c_1^+ and c_2^+ of the lengths l_1^+ and l_2^+, and $\mathcal{S}^-$ possesses two homotopically different nontrivial closed simple intersecting curves c_1^- and c_2^- of the lengths l_1^- and l_2^- such that c_1^+ and c_1^-, as well as c_2^+ and c_2^-, are homotopically equivalent pairs of curves in $\mathcal{M}$, then the distance $d(\mathcal{S}^+,\mathcal{S}^-)$ between $\mathcal{S}^+$ and $\mathcal{S}^-$ is bounded from above by the constant*

$$d(\mathcal{S}^+,\mathcal{S}^-) < \max\left\{ \left(l_1^+ + l_1^- + \ln\frac{2l_1^+}{l_1^-}\right), \left(l_1^+ + l_1^- + \ln\frac{2l_1^-}{l_1^+}\right), \left(l_2^+ + l_2^- + \ln\frac{2l_2^+}{l_2^-}\right), \left(l_2^+ + l_2^- + \ln\frac{2l_2^-}{l_2^+}\right),\right.$$

$$2\operatorname{arcosh}\left[\cosh l_1^+ \cosh\left(l_1^+ + \operatorname{arcosh}\frac{e^{l_1^+}(l_1^+)^2}{\varepsilon_3^2}\right)\right], 2\operatorname{arcosh}\left[\cosh l_1^- \cosh\left(l_1^- + \operatorname{arcosh}\frac{e^{l_1^-}(l_1^-)^2}{\varepsilon_3^2}\right)\right],$$

$$\left. 2\operatorname{arcosh}\left[\cosh l_2^+ \cosh\left(l_2^+ + \operatorname{arcosh}\frac{e^{l_2^+}(l_2^+)^2}{\varepsilon_3^2}\right)\right], 2\operatorname{arcosh}\left[\cosh l_2^- \cosh\left(l_2^- + \operatorname{arcosh}\frac{e^{l_2^-}(l_2^-)^2}{\varepsilon_3^2}\right)\right]\right\},$$

where the symbol ε_3 stands for the Margulis constant of hyperbolic space $\mathbb{H}^3$ (this constant will be defined shortly).

This result is of independent interest as well. Note that we do not require the regularity of surface metrics in Theorems 4.1 and 4.2.

Let us show how Theorem 4.2 implies Theorem 4.1.

Proof of Theorem 4.1.

Consider two homotopically different nontrivial closed curves c_1 and c_2 on the surface $\mathcal{S}$ such that they intersect each other but do not intersect with the singular points of the metrics h^+_∞ and h^-_∞ on $\mathcal{S}$. Since the sequence of metrics $\{h^+_n\}_{n\in\mathbb{N}}$ converges to the metric h^+_∞, the lengths $l_1^{+,n}$ of the curve $c_1 \in \mathcal{S}$ measured in the metrics h^+_n, $n \in \mathbb{N}$, tend to the length $l_1^{+,\infty} > 0$ of c_1 measured in the metric h^+_∞ as $n \to \infty$. The converging sequence of the positive real numbers $\{l_1^{+,n}\}_{n\in\mathbb{N}}$ is bounded from below by a real number $\omega_1^+ > 0$ and from above by a real number $\Omega_1^+ > 0$. Similarly, the lengths $l_1^{-,n}$ of the curve $c_1 \in \mathcal{S}$ measured in the metrics h^-_n, $n \in \mathbb{N}$, are bounded from below by some $\omega_1^- > 0$ and from above by some $\Omega_1^- > 0$; the lengths $l_2^{+,n}$ of the curve $c_2 \in \mathcal{S}$ measured in the metrics h^+_n, $n \in \mathbb{N}$, are bounded from below by some $\omega_2^+ > 0$ and from above by some $\Omega_2^+ > 0$; and the lengths $l_2^{-,n}$ of the curve $c_2 \in \mathcal{S}$ measured in the metrics h^-_n, $n \in \mathbb{N}$, are bounded from below by some $\omega_2^- > 0$ and from above by some $\Omega_2^- > 0$.

By Theorem 4.2, the distance $d(\mathcal{S}^+_n, \mathcal{S}^-_n)$ between the surfaces $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$ in the quasi-Fuchsian manifold $\mathcal{M}^\circ_n$ is uniformly bounded from above for any $n \in \mathbb{N}$:

$$
\begin{aligned}
d(\mathcal{S}^+_n, \mathcal{S}^-_n) < \max\Bigg\{ & \left(\Omega_1^+ + \Omega_1^- + \ln\frac{2\Omega_1^+}{\omega_1^-}\right), \left(\Omega_1^+ + \Omega_1^- + \ln\frac{2\Omega_1^-}{\omega_1^+}\right), \left(\Omega_2^+ + \Omega_2^- + \ln\frac{2\Omega_2^+}{\omega_2^-}\right), \\
& \left(\Omega_2^+ + \Omega_2^- + \ln\frac{2\Omega_2^-}{\omega_2^+}\right), 2\operatorname{arcosh}\left[\cosh\Omega_1^+ \cosh\left(\Omega_1^+ + \operatorname{arcosh}\frac{e^{\Omega_1^+}(\Omega_1^+)^2}{\varepsilon_3^2}\right)\right], \\
& 2\operatorname{arcosh}\left[\cosh\Omega_1^- \cosh\left(\Omega_1^- + \operatorname{arcosh}\frac{e^{\Omega_1^-}(\Omega_1^-)^2}{\varepsilon_3^2}\right)\right], \\
& 2\operatorname{arcosh}\left[\cosh\Omega_2^+ \cosh\left(\Omega_2^+ + \operatorname{arcosh}\frac{e^{\Omega_2^+}(\Omega_2^+)^2}{\varepsilon_3^2}\right)\right], \\
& 2\operatorname{arcosh}\left[\cosh\Omega_2^- \cosh\left(\Omega_2^- + \operatorname{arcosh}\frac{e^{\Omega_2^-}(\Omega_2^-)^2}{\varepsilon_3^2}\right)\right]\Bigg\}.
\end{aligned}
$$

□

Our aim now is to demonstrate Theorem 4.2. We will widely use the Margulis lemma to prove this fact. In the most general case the Margulis lemma reads as follows [BP03, Theorem D.1.1, p. 134]:

General Margulis Lemma. *For every $m \in \mathbb{N}$ there exists a constant $\varepsilon_m \geq 0$ such that for any properly discontinuous subgroup Γ of the group $\mathcal{I}(\mathbb{H}^m)$ of isometries of $\mathbb{H}^m$ and for any $x \in \mathbb{H}^m$, the group $\Gamma_{\varepsilon_m}(x)$ generated by the set $F_{\varepsilon_m}(x) = \{\gamma \in \Gamma : \mathrm{d}_{\mathbb{H}^m}(x, \gamma(x)) \leq \varepsilon_m\}$ is almost-nilpotent, where $\mathrm{d}_{\mathbb{H}^m}(\cdot,\cdot)$ stands for the distance in hyperbolic space $\mathbb{H}^m$.*

If we restrict the General Margulis Lemma to the case of the quasifuchsian isometries of hyperbolic 3-space $\mathbb{H}^3$ which is interesting to us, then the lemma can be rewritten in this way [Ota03, Theorem B, p. 100]:

Margulis Lemma. *There is a universal constant $\varepsilon_3 > 0$ such that for any properly discontinuous subgroup Γ of the group $\mathcal{I}(\mathbb{H}^3)$ of isometries of $\mathbb{H}^3$ if two closed simple intersecting curves $\tilde\gamma_1$ and $\tilde\gamma_2$ of the manifold $\mathbb{H}^3/\Gamma$ have lengths less than ε_3, then $\tilde\gamma_1$ and $\tilde\gamma_2$ are homotopically equivalent in $\mathbb{H}^3/\Gamma$.*

Hence, the main idea of the proof of Theorem 4.2 is to find a pair of closed simple intersecting curves inside $\mathcal{M}$ of lengths less than the Margulis constant ε_3 and such that they are not homotopically equivalent once the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$ is big enough. Then, by the Margulis lemma, the curves under consideration ought to be homotopically equivalent, which leads us to a contradiction. Let us now give a more detailed plan of the proof of Theorem 4.2:

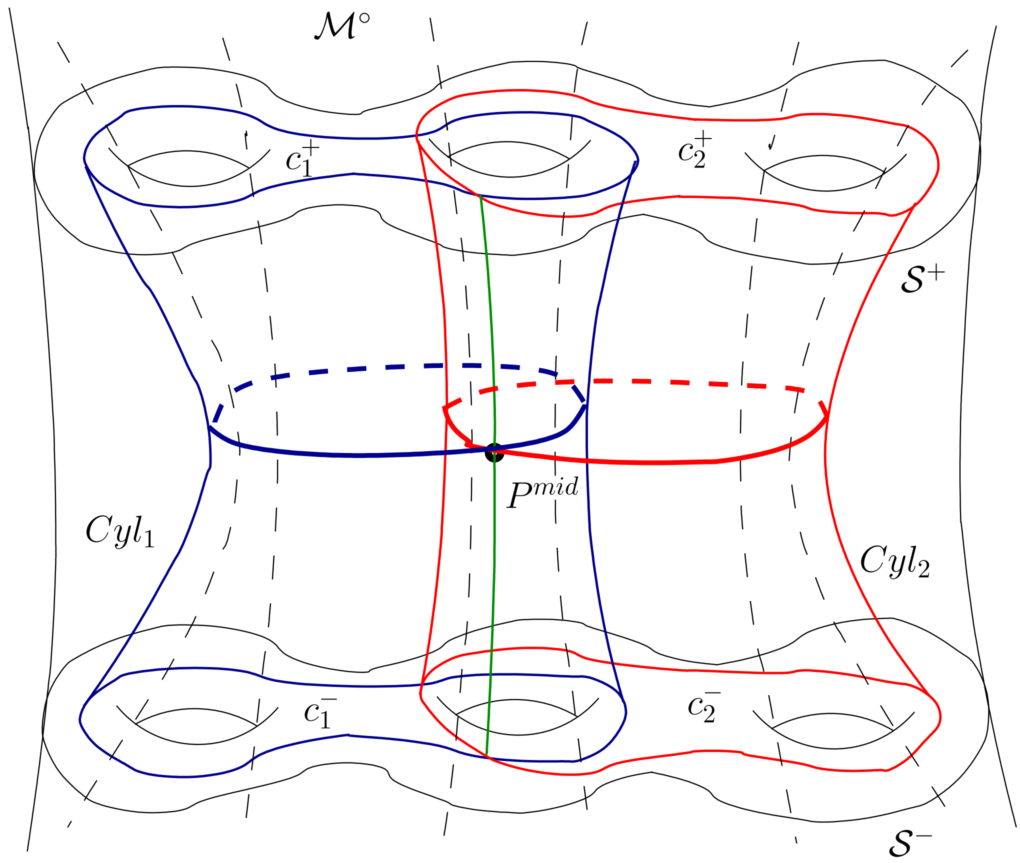

Figure 4.1: The cylinders Cyl_1 and Cyl_2 in the manifold $\mathcal{M}^\circ$.

- Suppose that the curves c_1^+ and c_2^+ intersect at a point P^+ (this point is not necessarily unique), and the curves c_1^- and c_2^- intersect at a point P^-. We will construct cylinders Cyl_1 and Cyl_2 in $\mathcal{M}$ that realize homotopies between c_1^+ and c_1^- and between c_2^+ and c_2^- correspondingly. Then the intersection of Cyl_1 and Cyl_2 contains a (curved) line with ends P^+ and P^-. Denote the midpoint of this line by P^{mid}.
- We will find a constant based on l_1^+, l_1^-, l_2^+, l_2^-, and ε_3, and we will construct curves on Cyl_1 and Cyl_2 (see Fig. 4.1) passing through P^{mid} such that if the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$ is greater than the constant mentioned above then both constructed curves are shorter than ε_3.

4.1 Construction of the cylinders Cyl_1 and Cyl_2

We consider a quasifuchsian manifold $\mathcal{M}^\circ$. By definition, it means that $\mathcal{M}^\circ$ is a quotient $\mathbb{H}^3/\Gamma^\circ$ where Γ° is a quasifuchsian subgroup of the group $\mathcal{I}(\mathbb{H}^3)$ of isometries of hyperbolic 3-space. Note that Γ° is homomorphic to the fundamental group $\pi_1(\mathcal{M}^\circ)$.

Denote by γ_1 the closed geodesic of $\mathcal{M}^\circ$ homotopically equivalent to c_1^+ and c_1^-. Similarly, denote by γ_2 the closed geodesic of $\mathcal{M}^\circ$ homotopically equivalent to c_2^+ and c_2^-. By abuse of notation, we denote by γ_1 and γ_2 the elements of $\pi_1(\mathcal{M}^\circ)$ corresponding to the closed geodesics under consideration. The universal covering of the domain $\mathcal{M} \subset \mathcal{M}^\circ$ is a convex simply connected subset $\widetilde{\mathcal{M}}$ of $\mathbb{H}^3$. Denote by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ the isometries of $\mathbb{H}^3$ corresponding to the elements γ_1 and γ_2 of $\pi_1(\mathcal{M}^\circ)$.

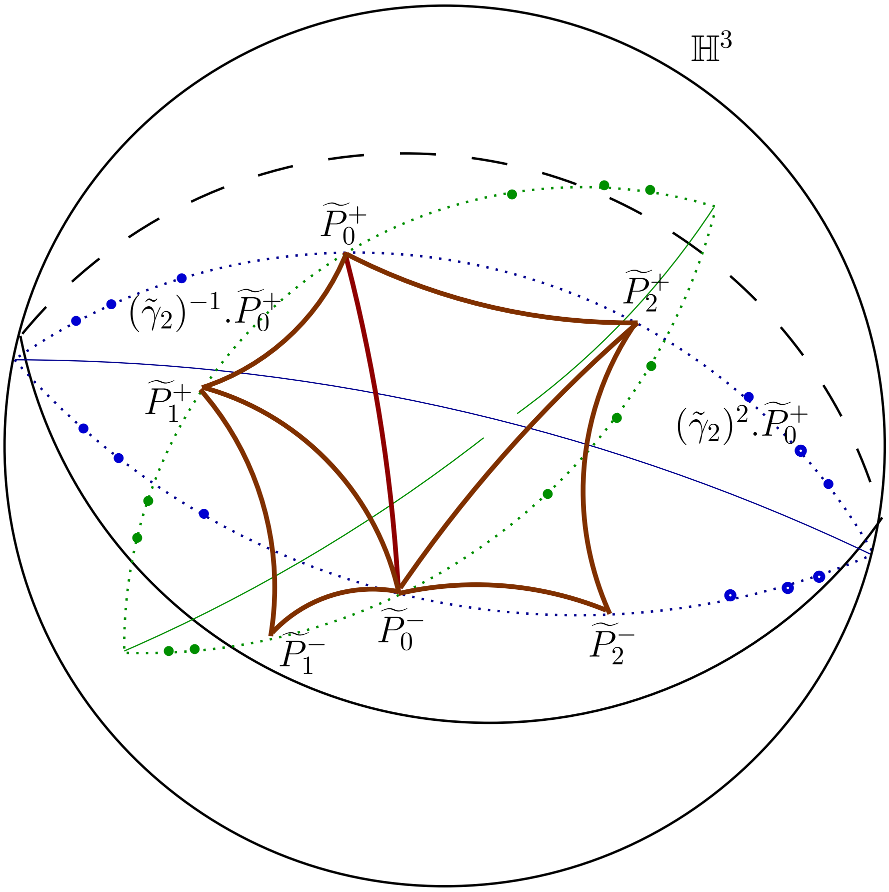

Figure 4.2: Construction of fundamental domains of the cylinders Cyl_1 and Cyl_2 in the Poincaré model of $\mathbb{H}^3$.

Let us now consider any single point $\widetilde{P}_0^+ \in \mathbb{H}^3$ serving as a pre-image of $P^+ \in c_1^+ \cap c_2^+$ in the universal covering $\widetilde{\mathcal{M}}$. Among all the points in the pre-image of $P^- \in c_1^- \cap c_2^-$ in $\widetilde{\mathcal{M}}$, we choose $\widetilde{P}_0^- \in \mathbb{H}^3$ to be the closest to $\widetilde{P}_0^+$ (in case there are several points realizing the minimal distance to $\widetilde{P}_0^+$, we choose one of them arbitrarily). Denote $\widetilde{P}_1^+ \overset{\text{def}}{=} \tilde{\gamma}_1.\widetilde{P}_0^+$, $\widetilde{P}_1^- \overset{\text{def}}{=} \tilde{\gamma}_1.\widetilde{P}_0^-$, $\widetilde{P}_2^+ \overset{\text{def}}{=} \tilde{\gamma}_2.\widetilde{P}_0^+$, $\widetilde{P}_2^- \overset{\text{def}}{=} \tilde{\gamma}_2.\widetilde{P}_0^-$ (recall that for every point $T \in \mathbb{H}^3$ and for every $\tilde{\gamma} \in \mathcal{I}(\mathbb{H}^3)$ the symbol $\tilde{\gamma}.T$ stands for the image of T under the isometry $\tilde{\gamma}$). Then we set the unions of flat hyperbolic triangles $\triangle\widetilde{P}_0^+\widetilde{P}_0^-\widetilde{P}_1^+ \cup \triangle\widetilde{P}_1^+\widetilde{P}_1^-\widetilde{P}_0^-$ and $\triangle\widetilde{P}_0^+\widetilde{P}_0^-\widetilde{P}_2^+ \cup \triangle\widetilde{P}_2^+\widetilde{P}_2^-\widetilde{P}_0^-$ in $\mathbb{H}^3$ to be fundamental domains of the cylinders Cyl_1 and Cyl_2 (see Fig. 4.2).

The fundamental domain $\tilde{c}_1^+ \subset \mathbb{H}^3$ of the curve c_1^+ has the same length l_1^+ as c_1^+. We can choose $\tilde{c}_1^+$ to connect $\widetilde{P}_0^+$ and $\widetilde{P}_1^+$. Hence, the length of the straight (hyperbolic) segment $\widetilde{P}_0^+\widetilde{P}_1^+$ is less than or equal to l_1^+. Similarly, $d_{\mathbb{H}^3}(\widetilde{P}_0^-, \widetilde{P}_1^-) \leq l_1^-$, $d_{\mathbb{H}^3}(\widetilde{P}_0^+, \widetilde{P}_2^+) \leq l_2^+$, and $d_{\mathbb{H}^3}(\widetilde{P}_0^-, \widetilde{P}_2^-) \leq l_2^-$. Also, by construction, the midpoints $\widetilde{P}_0^{mid}$, $\widetilde{P}_1^{mid}$, and $\widetilde{P}_2^{mid}$ of the segments $\widetilde{P}_0^+\widetilde{P}_0^-$, $\widetilde{P}_1^+\widetilde{P}_1^-$, and $\widetilde{P}_2^+\widetilde{P}_2^-$ serve as pre-images of the midpoint P^{mid} of the segment P^+P^- lying in the intersection $Cyl_1 \cap Cyl_2$.

Evidently, Cyl_1 and Cyl_2 can be prolonged to realize homotopies between the pairs of closed curves (c_1^+, c_1^-) and (c_2^+, c_2^-) as it was announced in our plan, but it will not be needed further.

Let us study properties of the cylinders constructed alike Cyl_1 and Cyl_2.

4.2 Properties of the cylinders of the type Cyl

Definition. A cylinder Cyl_0 is said to be *of the type Cyl* if and only if Cyl_0 possesses

1) a fundamental domain $FD(Cyl_0) \stackrel{\text{def}}{=} \triangle\widetilde{R}^+\widetilde{R}^-\widetilde{Q}^+ \cup \triangle\widetilde{Q}^+\widetilde{Q}^-\widetilde{R}^-$ constructed of two totally geodesic triangles in $\mathbb{H}^3$ such that $d_{\mathbb{H}^3}(\widetilde{Q}^+, \widetilde{Q}^-) = d_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}^-)$, and
2) the hyperbolic isometry $\tilde{\gamma} \in \mathcal{I}(\mathbb{H}^3)$ sending the geodesic segment $\widetilde{R}^+\widetilde{R}^-$ to the geodesic segment $\widetilde{Q}^+\widetilde{Q}^-$ and such that for every point $\widetilde{R}_\sharp^- \in \{\tilde{\gamma}_\sharp.\widetilde{R}^- | \tilde{\gamma}_\sharp \in \langle\tilde{\gamma}\rangle\}$ the inequality $d_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}^-) \leq d_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}_\sharp^-)$ holds true (here and below the symbol $\langle\tilde{\gamma}\rangle$ stands for the group generated by the element $\tilde{\gamma}$). Note that $\widetilde{Q}^- \in \{\tilde{\gamma}_\sharp.\widetilde{R}^- | \tilde{\gamma}_\sharp \in \langle\tilde{\gamma}\rangle\}$ by construction.

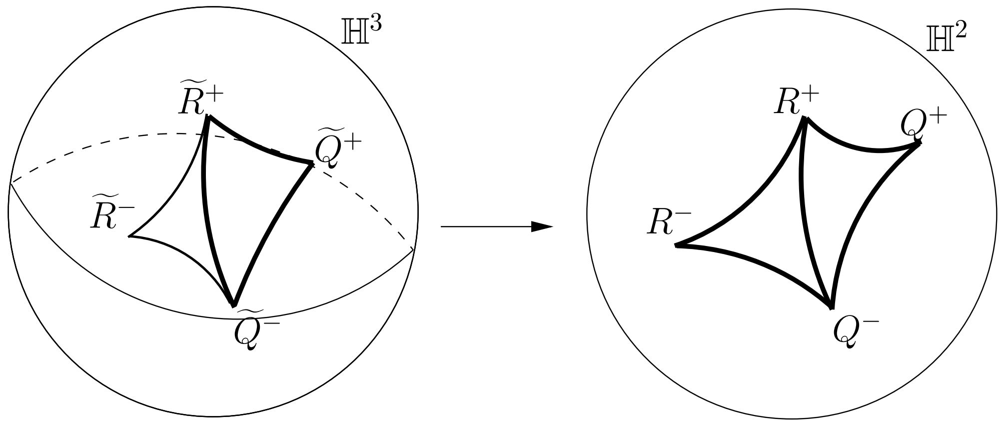

Figure 4.3: The quadrilaterals $\widetilde{R}^+\widetilde{R}^-\widetilde{Q}^+\widetilde{Q}^-$ in $\mathbb{H}^3$ and $R^+R^-Q^+Q^-$ in $\mathbb{H}^2$.

Remark that the metric of Cyl_0 induced from the ambient space is hyperbolic. Let us flatten $FD(Cyl_0)$ and obtain a hyperbolic quadrilateral $R^+R^-Q^+Q^- \subset \mathbb{H}^2$ isometric to $FD(Cyl_0)$ such that the vertices with tildes in $\mathbb{H}^3$ correspond to the vertices of the same name but without tildes in $\mathbb{H}^2$ (see Fig. 4.3).

The quadrilateral $R^+R^-Q^+Q^-$ serves as a fundamental domain of Cyl_0 in its universal covering in $\mathbb{H}^2$. Denote by χ_R and χ_Q the hyperbolic straight lines in $\mathbb{H}^2$ containing the segments R^+R^- and Q^+Q^- correspondingly. Remark that the connected domain of $\mathbb{H}^2$ between χ_R and χ_Q is actually a fundamental domain of the unbounded hyperbolic cylinder Cyl_0° containing Cyl_0. We will call it $FD(Cyl_0^\circ)$. Indeed, the fundamental group $\pi_1(Cyl_0^\circ) = \mathbb{Z}$. Hence, Cyl_0° possesses a closed geodesic χ° and there is a hyperbolic straight line χ in $\mathbb{H}^2$ serving as a lift of χ° and related to the isometry $\bar{\chi}$ of $\mathbb{H}^2$ such that $Cyl_0^\circ = \mathbb{H}^2/\langle\bar{\chi}\rangle$. We show the existence of such geodesic χ in the following

Lemma 4.3. *Consider two nonintersecting geodesics χ_R and χ_Q in $\mathbb{H}^2$ which are not asymptotic, with marked points $R \in \chi_R$ and $Q \in \chi_Q$. There is a unique hyperbolic straight line χ in $\mathbb{H}^2$ such that the angles of intersection of χ with χ_R and χ_Q are equal, and moreover, if we denote $R' \overset{\text{def}}{=} \chi_R \cap \chi$ and $Q' \overset{\text{def}}{=} \chi_Q \cap \chi$, then $\mathrm{d}_{\mathbb{H}^2}(R, R') = \mathrm{d}_{\mathbb{H}^2}(Q, Q')$ and the points R and Q lie in the same half-plane with respect to χ.*

Proof. Let us consider the Beltrami-Klein model $\mathbb{K}^2$ of the hyperbolic plane $\mathbb{H}^2$. Recall that $\mathbb{K}^2$ is a unit disc in the Euclidean plane $\mathbb{R}^2$ and all geodesics of $\mathbb{K}^2$ are restrictions of Euclidean straight lines on this disc. Without loss of generality the geodesics $\chi_R \subset \mathbb{K}^2$ and $\chi_Q \subset \mathbb{K}^2$ can be taken symmetric with respect to the axis Ox of the cartesian coordinate system on $\mathbb{R}^2$, both at an arbitrary distance ζ from Ox. Let χ_R lie in the upper half-space of $\mathbb{R}^2$ with respect to Ox and χ_Q lie in the lower half-space of $\mathbb{R}^2$ with respect to Ox. At last we fix arbitrary points $R \in \chi_R$ and $Q \in \chi_Q$.

By construction, every geodesic in $\mathbb{K}^2$ passing through the origin O of the cartesian coordinate system on $\mathbb{R}^2$ either intersects χ_R and χ_Q at the same angle or does not intersect them. Let us consider a family Φ_τ of such geodesics $R_\tau Q_\tau$ lying between the straight lines OR and OQ where $R_\tau \in \chi_R$, $Q_\tau \in \chi_Q$, τ stands for the hyperbolic distance between R and R_τ, and the line $OQ \in \Phi_\tau$ corresponds to the value $\hat{\tau}$ of the parameter τ.

Note that

– R and Q lie in the same half-plane with respect to any $R_\tau Q_\tau \in \Phi_\tau$.
– As τ grows up monotonically from 0 to $\hat{\tau}$, the distance $\mathrm{d}_{\mathbb{H}^2}(Q, Q_\tau)$ decreases monotonically from $\mathrm{d}_{\mathbb{H}^2}(Q, Q_{\hat{\tau}})$ to 0. Hence, there exists a unique $\tau_0 \in [0, \hat{\tau}]$ such that $\mathrm{d}_{\mathbb{H}^2}(R, R_{\tau_0}) = \mathrm{d}_{\mathbb{H}^2}(Q, Q_{\tau_0})$.

We choose χ to be $R_{\tau_0} Q_{\tau_0} \in \Phi_\tau$. χ is unique since τ_0 is unique. ☐

Remark 4.4. *Let $Set(R^-) \overset{\text{def}}{=} \{\bar{\chi}_\sharp.R^- | \bar{\chi}_\sharp \in \langle\bar{\chi}\rangle\}$ (by construction, $Q^- \in Set(R^-)$). Then for every point $R^-_\sharp \in Set(R^-)$ the inequality $\mathrm{d}_{\mathbb{H}^2}(R^+, R^-) \le \mathrm{d}_{\mathbb{H}^2}(R^+, R^-_\sharp)$ holds true.*

Proof. By construction, $\mathrm{d}_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}^-) = \mathrm{d}_{\mathbb{H}^2}(R^+, R^-)$, and the surfaces $\langle\bar{\chi}\rangle.R^+R^-Q^+Q^- \subset \mathbb{H}^2$ (which is the union $\bigcup_{\bar{\chi}_\sharp \in \langle\bar{\chi}\rangle} \bar{\chi}_\sharp.R^+R^-Q^+Q^-$ of the quadrilaterals $\bar{\chi}_\sharp.R^+R^-Q^+Q^-$ isometric to $R^+R^-Q^+Q^-$) and $\langle\bar{\chi}\rangle.FD(Cyl_0) \subset \mathbb{H}^3$ are isometric in their intrinsic metrics. Evidently, for any points $\widetilde{T}_1$ and $\widetilde{T}_2$ in $\langle\bar{\chi}\rangle.FD(Cyl_0)$ it is true that $\mathrm{d}_{\mathbb{H}^3}(\widetilde{T}_1, \widetilde{T}_2) \le \mathrm{d}^{int}_{\langle\bar{\chi}\rangle.FD(Cyl_0)}(\widetilde{T}_1, \widetilde{T}_2)$, where $\mathrm{d}^{int}_{\langle\bar{\chi}\rangle.FD(Cyl_0)}(\cdot, \cdot)$ stands for the intrinsic metric of $\langle\bar{\chi}\rangle.FD(Cyl_0)$. At last, the part 2) of the definition of a cylinder Cyl_0 of the type Cyl allows us to conclude that Remark 4.4 is valid. ☐

Remark 4.5. *Let $R'Q'$ be a segment of the geodesic $\chi \subset \mathbb{H}^2$ between χ_R and χ_Q serving as a fundamental domain of $\chi^\circ \subset Cyl^\circ_0$ on χ (here $R' \in \chi_R$ and $Q' \in \chi_Q$). Then either $R'Q' \subset R^+R^-Q^+Q^-$ or $R'Q' \cap R^+R^-Q^+Q^- = \emptyset$.*

Proof. Recall that the points R^+ and Q^+ are pre-images in $\mathbb{H}^2$ of the same point on Cyl_0, and one can be obtained from another by applying an isometry of $\mathbb{H}^2$ which is an element of the group $\langle\bar{\chi}\rangle$ preserving the straight hyperbolic line χ. Hence, R^+ and Q^+ lie in one half-plane of $\mathbb{H}^2$ with respect to χ and, by consequence, the segment R^+Q^+ does not intersect χ. Similarly, $R^-Q^- \cap \chi = \emptyset$.

We conclude that if R^+Q^+ and R^-Q^- lie in the same half-plane of $\mathbb{H}^2$ with respect to χ then $R'Q' \cap R^+R^-Q^+Q^- = \emptyset$. Otherwise, if R^+Q^+ and R^-Q^- lie in different half-planes with respect to χ, then $R'Q' \subset R^+R^-Q^+Q^-$. ☐

4.3 h-neighborhood of a geodesic in $\mathbb{H}^2$

In this section, we study hyperbolic quadrilaterals of one special type and half-neighborhoods of geodesics containing one of the sides of our quadrilaterals which are inscribed in and circumscribed about these quadrilaterals. Properties of these objects will be largely used in obtaining bounds on a possible size of cylinders of the type Cyl.

The object of our interest is a quadrilateral $OO'PP' \subset \mathbb{H}^2$ with the sides $d_{\mathbb{H}^2}(O, O') = l$, $d_{\mathbb{H}^2}(P, P') = l'$, and $d_{\mathbb{H}^2}(O, P) = d_{\mathbb{H}^2}(O', P') = h'$, such that the edges OP and $O'P'$ are perpendicular to OO'. Draw a curve γ_h at a distance $h < h'$ from the geodesic containing OO' such that γ_h intersects OP and $O'P'$ at points T and T' correspondingly. Denote a segment of γ_h between OP and $O'P'$ by $\widehat{TT'}$, and the hyperbolic length of $\widehat{TT'}$ by l_h.

A direct calculation shows that

Remark 4.6. *The following relation holds true:*

$$l_h = l \cosh h.$$

Remark 4.7. *If $h = h'$ then T and T' coincide with P and P', $\widehat{TT'}$ intersects $OO'PP'$ as a solid body only at its ends P and P', and, evidently, $l_{h'} > l'$ (any path connecting two points can not be shorter then a geodesic segment between them).*

Remark 4.8. *Suppose that $h' > l'$. If $h \leq h' - l'$ then $\widehat{TT'} \subset OO'PP'$ and $l_h < l'$.*

Proof. Consider hyperbolic balls $B_{l'}(P)$ and $B_{l'}(P')$ of the radius l' with the centers P and P'. These balls contain the segment PP'. Also, $B_{l'}(P)$ and $B_{l'}(P')$ are perpendicular to OP and $O'P'$ correspondingly. By construction, $\widehat{TT'}$ is perpendicular to OP and $O'P'$ as well. Moreover, $\widehat{TT'}$ is a convex curve. Hence, $\widehat{TT'}$ lies outside the interior of $B_{l'}(P)$ and $B_{l'}(P')$ for $h \leq h' - l'$. It means that the geodesic segment PP' does not intersect $\widehat{TT'}$, and $\widehat{TT'} \subset OO'PP'$.

Denote by $OO'\widehat{TT'}$ the convex domain in $\mathbb{H}^2$ bounded by the segments OT, OO', $O'T'$ and the curve $\widehat{TT'}$. By construction, the orthogonal projection of PP' onto $OO'\widehat{TT'}$ is $\widehat{TT'}$. Since the orthogonal projection on the boundary of a convex hyperbolic domain is contracting [BGS85, p. 9] (see also [CEG06, II.1.3.4, p. 124]), we get $l_h < l'$. □

4.4 Fundamental domains of Cyl_1 and Cyl_2 in $\mathbb{H}^2$

Following the construction of a fundamental domain of a cylinder of the type Cyl in $\mathbb{H}^2$ from Section 4.2, we define for the cylinder Cyl_1 its fundamental domain $P_0^+P_0^-P_1^+P_1^- \subset \mathbb{H}_1^2$, where $\mathbb{H}_1^2$ is just a copy of the hyperbolic plane $\mathbb{H}^2$. We denote by χ_{P_0} and χ_{P_1} the hyperbolic straight lines in $\mathbb{H}_1^2$ containing the segments $P_0^+P_0^-$ and $P_1^+P_1^-$ correspondingly. Following the content of Section 4.3, we find the hyperbolic segment $O_0O_1 \subset \mathbb{H}_1^2$ corresponding to the element γ_1 of the fundamental group $\pi_1(\mathcal{M}^\circ)$ (see Section 4.1) with the points $O_0 \in \chi_{P_0}$ and $O_1 \in \chi_{P_1}$.

Similarly, we define the quadrilateral $P_0^+P_0^-P_2^+P_2^- \subset \mathbb{H}_2^2$ to be a fundamental domain of the cylinder Cyl_2, where $\mathbb{H}_2^2$ is another copy of $\mathbb{H}^2$. Denote by χ_{P_0} and χ_{P_2} the geodesics in $\mathbb{H}_2^2$ containing $P_0^+P_0^-$ and $P_2^+P_2^-$ correspondingly. We also find the hyperbolic segment $O_0O_2 \subset \mathbb{H}_2^2$ corresponding to $\gamma_2 \in \pi_1(\mathcal{M}^\circ)$ with the points $O_0 \in \chi_{P_0}$ and $O_2 \in \chi_{P_2}$.

An attentive reader has already remarked the following abuse of notation: the geodesic χ_{P_0} with the points P_0^+, P_0^-, and O_0 on it lie both in $\mathbb{H}_1^2$ and $\mathbb{H}_2^2$ as if these copies $\mathbb{H}_1^2$ and $\mathbb{H}_2^2$ of the hyperbolic plane intersect at χ_{P_0}. It is very logic since the segment $P_0^+P_0^- \subset \chi_{P_0}$ corresponds to the segment P^+P^- in the intersection of the cylinders Cyl_1 and Cyl_2 related to $\mathbb{H}_1^2$ and $\mathbb{H}_2^2$.

We are now prepared to prove Theorem 4.2. In order to do this, according to Remark 4.5 we must consider two separate situations.

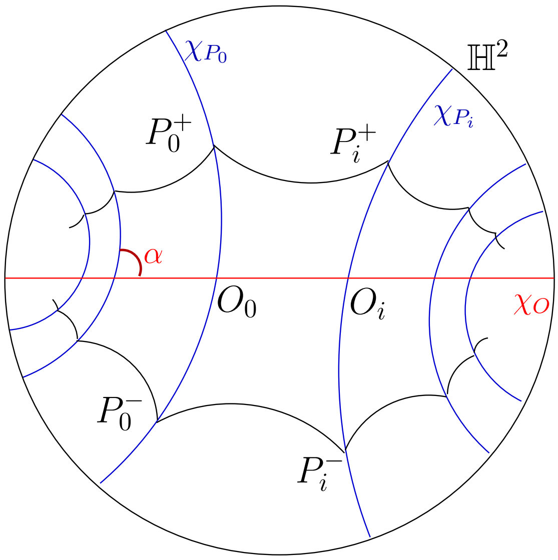

Figure 4.4: The quadrilateral $P_0^+P_0^-P_i^+P_i^-$, $i = 1, 2$, in Situation 1.

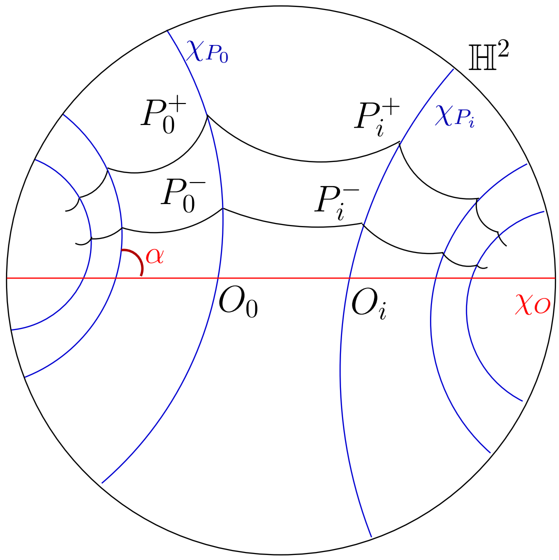

Figure 4.5: The quadrilateral $P_0^+P_0^-P_i^+P_i^-$, $i = 1, 2$, in Situation 2.

Situation 1. If for both cylinders Cyl_1 and Cyl_2 their fundamental domains $P_0^+P_0^-P_1^+P_1^- \subset \mathbb{H}_1^2$ and $P_0^+P_0^-P_2^+P_2^- \subset \mathbb{H}_1^2$ contain the segments O_0O_1 and O_0O_2 correspondingly (see Fig. 4.4), then the distance between the surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$ from the statement of Theorem 4.2 is bounded from above due to the Margulis lemma.
Indeed, recall that P^{mid} is the midpoint of the segment $P^+P^- \subset Cyl_1 \cap Cyl_2$, then the midpoints P_0^{mid}, P_1^{mid}, and P_2^{mid} of the segments $P_0^+P_0^- \subset \chi_{P_0}$, $P_1^+P_1^- \subset \chi_{P_1}$, and $P_2^+P_2^- \subset \chi_{P_2}$ are the pre-images of P^{mid} in $P_0^+P_0^-P_1^+P_1^- \subset \mathbb{H}_1^2$ or $P_0^+P_0^-P_2^+P_2^- \subset \mathbb{H}_2^2$. Following the content of Section 4.3, we construct the paths $\widehat{P_0^{mid}P_1^{mid}} \subset \mathbb{H}_1^2$ and $\widehat{P_0^{mid}P_2^{mid}} \subset \mathbb{H}_2^2$ connecting P_0^{mid} with P_1^{mid} and P_2^{mid}, and lying at the distance $\mathrm{d}_{\mathbb{H}^2}(P_0^{mid}, O_0)$ from O_0O_1 and O_0O_2. We will demonstrate that, once the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$ (consequently, the hyperbolic length of P^+P^-) is bigger then a constant depending on l_1^+, l_1^-, l_2^+, and l_2^- (see Section 4.1 for definitions), then two intersecting homotopically different curves in $\mathcal{M}$ with fundamental domains $\widehat{P_0^{mid}P_1^{mid}} \subset \mathbb{H}_1^2$ and $\widehat{P_0^{mid}P_2^{mid}} \subset \mathbb{H}_2^2$ have the lengths less than the Margulis constant ε_3, which is impossible.

Situation 2. If for at least one of the cylinders Cyl_1 or Cyl_2 the corresponding segment O_0O_1 or O_0O_2 does not intersect $P_0^+P_0^-P_1^+P_1^-$ or $P_0^+P_0^-P_2^+P_2^-$ (see Fig. 4.5), then we will prove that the hyperbolic length of the segment $P^+P^- \subset Cyl_1 \cap Cyl_2$ (and, hence, the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$) is necessarily bounded by a constant depending on either l_1^+ and l_1^-, or l_2^+ and l_2^-.

First, we will prove Theorem 4.2 supposing in addition that the segments O_0O_1 and O_0O_2 are orthogonal to the pairs of geodesics (χ_{P_0}, χ_{P_1}) and (χ_{P_0}, χ_{P_2}) correspondingly. We will call it the *orthogonality* condition. A reader may check that if the hyperbolic isometries $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of $\mathbb{H}^3$ (see the beginning of Section 4.1 for definitions) do not have rotational components then the orthogonality condition is satisfied.

4.5 Consideration of Situation 1 in case the *orthogonality* condition holds true

Here we prove

Lemma 4.9. *Let a cylinder of the type Cyl contain a closed geodesic and possess a fundamental domain $R^+R^-Q^+Q^- \subset \mathbb{H}^2$ satisfying the orthogonality condition. Define by l^+_{RQ} and l^-_{RQ} the lengths of the sides R^+Q^+ and R^-Q^-. There is a constant*

$$H^{ort}_{int} = 2\max\left\{\left(l^+_{RQ} + \operatorname{arcosh}\frac{e^{l^+_{RQ}}(l^+_{RQ})^2}{\varepsilon_3^2}\right), \left(l^-_{RQ} + \operatorname{arcosh}\frac{e^{l^-_{RQ}}(l^-_{RQ})^2}{\varepsilon_3^2}\right)\right\}$$

such that if the length of the sides R^+R^- and Q^+Q^- is greater than H^{ort}_{int} then there is a path in $R^+R^-Q^+Q^-$ connecting the midpoints of R^+R^- and Q^+Q^- with the length which is smaller than the Margulis constant ε_3.

Consider the cylinder Cyl_0 of the type Cyl with a fundamental domain $R^+R^-Q^+Q^- \subset \mathbb{H}^2$. Here the orthogonality condition means that there are points $O_R \in R^+R^-$ and $O_Q \in Q^+Q^-$ such that $d_{\mathbb{H}^2}(R^+, O_R) = d_{\mathbb{H}^2}(Q^+, O_Q)$ (and $d_{\mathbb{H}^2}(R^-, O_R) = d_{\mathbb{H}^2}(Q^-, O_Q)$), and the segment $O_RO_Q \subset R^+R^-Q^+Q^-$ is orthogonal to R^+R^- and Q^+Q^-. Denote the midpoints of R^+R^- and Q^+Q^- by R^{mid} and Q^{mid}, the midpoints of O_RR^+ and O_QQ^+ by T^+_R and T^+_Q, the midpoints of O_RR^- and O_QQ^- by T^-_R and T^-_Q; the lengths of O_RR^{mid} and O_RO_Q by h_{mid} and l_O, the lengths of $O_RT^+_R$, O_RR^+, and R^+Q^+ by h^+_T, h^+_{RQ}, and l^+_{RQ}, the lengths of $O_RT^-_R$, O_RR^-, and R^-Q^- by h^-_T, h^-_{RQ}, and l^-_{RQ}. By construction, $R^{mid} \in T^-_RT^+_R$, $Q^{mid} \in T^-_QT^+_Q$,

$$h^+_T = \frac{h^+_{RQ}}{2}, \quad \text{and} \quad h^-_T = \frac{h^-_{RQ}}{2}.$$

Also, the length h of the segments R^+R^- and Q^+Q^- can be expressed as follows:

$$h = h^+_{RQ} + h^-_{RQ}. \tag{4.1}$$

Since the orthogonal projection on a geodesic segment is contracting, we have

$$l_O \leq l^+_{RQ} \quad \text{and} \quad l_O \leq l^-_{RQ}.$$

Let us construct the paths $\widehat{R^{mid}Q^{mid}}$, $\widehat{T^+_RT^+_Q}$ and $\widehat{T^-_RT^-_Q}$ at the distances h_{mid}, h^+_T and h^+_T from the segment O_RO_Q as in Section 4.3. In case $h^+_{RQ} \geq h^-_{RQ}$ we have that $R^{mid} \in O_RT^+_R$, $Q^{mid} \in O_QT^+_Q$, and

$$h_{mid} \leq h^+_T. \tag{4.2}$$

According to Remark 4.6 and by (4.2), $\widehat{R^{mid}Q^{mid}}$ is shorter than $\widehat{T^+_RT^+_Q}$ and if we find a condition on h^+_{RQ} that guarantees the length of $\widehat{T^+_RT^+_Q}$ to be less than the Margulis constant ε_3, then the length of $\widehat{R^{mid}Q^{mid}}$ is less than ε_3 as well. Similarly, $\widehat{R^{mid}Q^{mid}}$ is shorter than $\widehat{T^-_RT^-_Q}$ when $h^+_{RQ} < h^-_{RQ}$ and a condition on h^-_{RQ} providing the length of $\widehat{T^-_RT^-_Q}$ to be less than ε_3, guarantees that the length of $\widehat{R^{mid}Q^{mid}}$ is also less than ε_3.

We need the following

Lemma 4.10. *Let us consider a quadrilateral O_RO_QRQ as in Section 4.3 with the fixed length l_{RQ} of the edge RQ. There is a constant*

$$h_{int}^{ort} = l_{RQ} + \operatorname{arcosh} \frac{e^{l_{RQ}} l_{RQ}^2}{\varepsilon_3^2}.$$

such that if the length h_{RQ} of the sides O_RR and O_QQ is greater than h_{int}^{ort} then the length of the path $\widehat{T_RT_Q}$ at the distance $h_T \stackrel{\text{def}}{=} h_{RQ}/2$ from O_RO_Q connecting the midpoints T_R and T_Q of O_RR and O_QQ is smaller than the Margulis constant ε_3.

Proof. Denote by l_O the length of O_RO_Q. Once l_{RQ} is fixed, suppose that h_{RQ} can be arbitrarily big, in particular, bigger than l_{RQ}.

There are points $T'_R \in O_RR$ and $T'_Q \in O_QQ$ at the distance h'_T from O_R and O_Q correspondingly, such that the length of the path $\widehat{T'_RT'_Q}$ as in Section 4.3 is equal to ε_3. By Remark 4.6,

$$l_O \cosh h'_T = \varepsilon_3. \tag{4.3}$$

Indeed, if T'_R and T'_Q do not exist then

$$l_O > \varepsilon_3. \tag{4.4}$$

By Remarks 4.6 and 4.8 applied to the quadrilateral O_RO_QRQ,

$$l_O \cosh(h_{RQ} - l_{RQ}) < l_{RQ}. \tag{4.5}$$

Mixing (4.4) and (4.5), we get

$$\varepsilon_3 \cosh(h_{RQ} - l_{RQ}) < l_{RQ},$$

$$h_{RQ} < l_{RQ} + \operatorname{arcosh} \frac{l_{RQ}}{\varepsilon_3},$$

which leads us to a contradiction with the unboundedness of h_{RQ}.

The length of $\widehat{T_RT_Q}$ is less than the length ε_3 of $\widehat{T'_RT'_Q}$ when the inequality

$$h'_T > h_T\Big(= \frac{h_{RQ}}{2}\Big) \tag{4.6}$$

is satisfied, which is equivalent to the validity of

$$\cosh h'_T > \cosh \frac{h_{RQ}}{2},$$

and, by (4.3), is also equivalent to

$$\frac{\varepsilon_3}{l_O} > \cosh \frac{h_{RQ}}{2}. \tag{4.7}$$

Due to the following property of the hyperbolic cosine: $\cosh 2x = \cosh^2 x + \sinh^2 x$, we see that

$$\cosh^2\Big(\frac{h_{RQ}}{2}\Big) \leq \cosh h_{RQ}.$$

Hence, the validity of the formula

$$\cosh h_{RQ} < \frac{\varepsilon_3^2}{l_O^2} \tag{4.8}$$

implies the validity of (4.7).

Let us exclude l_O from (4.8) with the help of (4.5).

At first, we perform a series of modifications of (4.5). By the formula for the hyperbolic cosine of the sum of two angles, we get

$$\cosh h_{RQ} \cosh l_{RQ} - \sinh h_{RQ} \sinh l_{RQ} < \frac{l_{RQ}}{l_O}.$$

Then, as $\sinh x > 0$ for each $x > 0$, and because $\cosh x > \sinh x$ and $\cosh x > 0$ for all $x \in \mathbb{R}$, we obtain

$$\cosh h_{RQ}(\cosh l_{RQ} - \sinh l_{RQ}) < \frac{l_{RQ}}{l_O},$$

and the definitions of the hyperbolic sine and cosine,

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}, \tag{4.9}$$

imply

$$\cosh h_{RQ} < \frac{e^{l_{RQ}} l_{RQ}}{l_O}.$$

It means that the validity of the formula

$$\frac{e^{l_{RQ}} l_{RQ}}{l_O} < \frac{\varepsilon_3^2}{l_O^2} \tag{4.10}$$

implies the validity of (4.8). We rewrite the condition (4.10) in a more convenient form:

$$l_O < \frac{\varepsilon_3^2}{e^{l_{RQ}} l_{RQ}}. \tag{4.11}$$

By (4.5), we know that

$$l_O < \frac{l_{RQ}}{\cosh(h_{RQ} - l_{RQ})}.$$

Hence, the validity of

$$\frac{l_{RQ}}{\cosh(h_{RQ} - l_{RQ})} < \frac{\varepsilon_3^2}{e^{l_{RQ}} l_{RQ}} \tag{4.12}$$

implies the validity of (4.10).

We can now conclude that the condition

$$h_{RQ} > h_{int}^{ort}$$

obtained from (4.12) implies (4.6). □

Again, supposing $h_{RQ}^+ \geq h_{RQ}^-$ we see that the condition

$$h = h_{RQ}^+ + h_{RQ}^- > 2\left(l_{RQ}^+ + \operatorname{arcosh} \frac{e^{l_{RQ}^+}(l_{RQ}^+)^2}{\varepsilon_3^2}\right)$$

implies

$$h_{RQ}^+ > l_{RQ}^+ + \operatorname{arcosh} \frac{e^{l_{RQ}^+}(l_{RQ}^+)^2}{\varepsilon_3^2}$$

and, by Lemma 4.10 applied to the quadrilateral $O_R O_Q R^+ Q^+$, the length of $\widehat{T_R^+ T_Q^+}$ is less than ε_3. Similarly, if $h_{RQ}^+ < h_{RQ}^-$ and

$$h > 2\left(l_{RQ}^- + \operatorname{arcosh} \frac{e^{l_{RQ}^-}(l_{RQ}^-)^2}{\varepsilon_3^2}\right),$$

then the length of $\widehat{T_R^- T_Q^-}$ is less than ε_3.

Applying now the reasoning made just before the formulation of Lemma 4.10, we obtain Lemma 4.9.

4.6 Consideration of Situation 2 in case the *orthogonality* condition holds true

Lemma 4.11. *Let a cylinder of the type Cyl do not contain a closed geodesic and possess a fundamental domain $R^+R^-Q^+Q^- \subset \mathbb{H}^2$ satisfying the orthogonality condition. Define by l_{RQ}^+ and l_{RQ}^- the lengths of the sides R^+Q^+ and R^-Q^-, and by h the length of the sides R^+R^- and Q^+Q^-. Then*

$$h < \max\left\{\left(l_{RQ}^+ + \ln \frac{2l_{RQ}^+}{l_{RQ}^-}\right), \left(l_{RQ}^- + \ln \frac{2l_{RQ}^-}{l_{RQ}^+}\right)\right\}.$$

Proof. We will use notation developed in Section 4.5. In these terms, the fact that a cylinder of the type Cyl does not contain a closed geodesic means that the segment $O_R O_Q$ lies outside the fundamental domain $R^+R^-Q^+Q^- \subset \mathbb{H}^2$ of the cylinder.

First, we assume that $h_{RQ}^+ \geq h_{RQ}^-$, then

$$h = h_{RQ}^+ - h_{RQ}^-, \tag{4.13}$$

which distinguishes Situation 2 from Situation 1 when the orthogonality condition is satisfied (compare (4.13) with (4.1)).

Given the quadrilateral $O_R O_Q R^- Q^-$, Remarks 4.6 and 4.7 imply

$$l_O \cosh h_{RQ}^- > l_{RQ}^-,$$

then, by the definition of the hyperbolic cosine (4.9), we have

$$\frac{e^{h_{RQ}^-} + e^{-h_{RQ}^-}}{2} > \frac{l_{RQ}^-}{l_O},$$

and, as $e^{h_{RQ}^-} \geq e^{-h_{RQ}^-}$ for $h_{RQ}^- \geq 0$, we obtain

$$e^{h_{RQ}^-} > \frac{l_{RQ}^-}{l_O}. \tag{4.14}$$

If $h_{RQ}^+ \leq l_{RQ}^+$ then, by (4.13),

$$h \leq l_{RQ}^+ \tag{4.15}$$

as well.

Assume that $h^+_{RQ} > l^+_{RQ}$. By Remarks 4.6 and 4.8 applied to the quadrilateral $O_R O_Q R^+ Q^+$, we get

$$l_O \cosh(h^+_{RQ} - l^+_{RQ}) < l^+_{RQ},$$

and, by (4.13),

$$l_O \cosh(h^-_{RQ} + h - l^+_{RQ}) < l^+_{RQ}$$

then, the definition of the hyperbolic cosine (4.9) gives us

$$e^{h^-_{RQ}} e^h e^{-l^+_{RQ}} + e^{-h^-_{RQ}} e^{-h} e^{l^+_{RQ}} < \frac{2l^+_{RQ}}{l_O}.$$

Let us weaken the obtained inequality:

$$e^{h^-_{RQ}} e^h e^{-l^+_{RQ}} < \frac{2l^+_{RQ}}{l_O},$$

and, together with (4.14), we get

$$\frac{l^-_{RQ}}{l_O} e^h e^{-l^+_{RQ}} < \frac{2l^+_{RQ}}{l_O},$$

$$e^h < \frac{2l^+_{RQ}}{l^-_{RQ}} e^{l^+_{RQ}},$$

$$h < l^+_{RQ} + \ln \frac{2l^+_{RQ}}{l^-_{RQ}}. \tag{4.16}$$

Note that the inequality (4.15) is stronger than (4.16).

Assuming $h^+_{RQ} < h^-_{RQ}$, we just need to interchange the upper indices $+$ and $-$ in the formula (4.16):

$$h < l^-_{RQ} + \ln \frac{2l^-_{RQ}}{l^+_{RQ}}.$$

□

4.7 Proof of Theorem 4.2 in the general case

Let a quadrilateral $R_0^+ R_0^- R_1^+ R_1^- \subset \mathbb{H}^2$ with $h \overset{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^+, R_0^-) = \mathrm{d}_{\mathbb{H}^2}(R_1^+, R_1^-)$, $l^+ \overset{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^+, R_1^+)$, and $l^- \overset{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^-, R_1^-)$ be a fundamental domain in $\mathbb{H}^2$ of a cylinder Cyl_0 of the type Cyl. Denote by χ_{R_0} and χ_{R_1} the hyperbolic straight lines in $\mathbb{H}^2$ containing the segments $R_0^+ R_0^-$ and $R_1^+ R_1^-$ correspondingly. Then, by Lemma 4.3 applied to the points $R_0^+ \in \chi_{R_0}$ and $R_1^+ \in \chi_{R_1}$ there is a a unique hyperbolic straight line $\chi_O \subset \mathbb{H}^2$ intersecting χ_{R_0} at a point O_0, χ_{R_1} at a point O_1, such that R_0^+ and R_1^+ lie in the same half-plane with respect to χ_O, $h^+ \overset{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^+, O_0) = \mathrm{d}_{\mathbb{H}^2}(R_1^+, O_1)$, and the angles of intersection $\angle(\chi_O, \chi_{R_0})$ and $\angle(\chi_O, \chi_{R_1})$ are equal to some $\alpha \in (0, \pi/2)$. Denote also $h^- \overset{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^-, O_0) = \mathrm{d}_{\mathbb{H}^2}(R_1^-, O_1)$ and $l_O \overset{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(O_0, O_1)$.

Let the hyperbolic isometry $\bar{\chi}_O$ of $\mathbb{H}^2$ send O_0 to O_1 leaving the geodesic χ_O invariant. Note that $\bar{\chi}_O$ sends also R_0^+ to R_1^+ and R_0^- to R_1^-. We define points $R_i^+ \overset{\text{def}}{=} \bar{\chi}_O^i . R_0^+$, $R_i^- \overset{\text{def}}{=} \bar{\chi}_O^i . R_0^-$, and $O_i \stackrel{\text{def}}{=} \bar{\chi}_O^i.O_0$ for $i \in \mathbb{Z}$, where the symbol $\bar{\chi}_O^i$ stands for the isometry $\bar{\chi}_O$ applied i times when i is a positive integer, and for the inverse isometry $\bar{\chi}_O^{-1}$ applied $-i$ times when $i < 0$. Denote by χ_{R_i} the hyperbolic straight line containing the segment $R_i^+R_i^-$, $i \in \mathbb{Z}$. Construct the curves $\nu_+ \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{Z}} R_i^+R_{i+1}^+$ and $\nu_- \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{Z}} R_i^-R_{i+1}^-$ of the geodesic segments $R_i^+R_{i+1}^+$ and $R_i^-R_{i+1}^-$, $i \in \mathbb{Z}$. Remark that for each $i \in \mathbb{Z}$ the quadrilateral $R_i^+R_i^-R_{i+1}^+R_{i+1}^- \subset \mathbb{H}^2$ serves as a fundamental domain of the cylinder Cyl_0 in $\mathbb{H}^2$, and the connected domain between the curves ν_+ and ν_- of the hyperbolic plane is a universal covering of Cyl_0 in $\mathbb{H}^2$. By construction, $\mathrm{d}_{\mathbb{H}^2}(R_i^+, R_i^-) = h$, $\mathrm{d}_{\mathbb{H}^2}(R_i^+, O_i) = h^+$, $\mathrm{d}_{\mathbb{H}^2}(R_i^-, O_i) = h^-$, $\mathrm{d}_{\mathbb{H}^2}(R_i^+, R_{i+1}^+) = l^+$, $\mathrm{d}_{\mathbb{H}^2}(R_i^-, R_{i+1}^-) = l^-$, $\angle(\chi_O, \chi_{R_i}) = \alpha$, $i \in \mathbb{Z}$.

Let us construct a family of hyperbolic straight lines χ_i^+ passing through R_i^+ and orthogonal to χ_O, $i \in \mathbb{Z}$. Define the points of intersection $O_i^+ \stackrel{\text{def}}{=} \chi_i^+ \cap \chi_O$, $T_i^- \stackrel{\text{def}}{=} \chi_i^+ \cap \nu_-$, $i \in \mathbb{Z}$. Note that, by construction, the connected sets Ξ_i^+ bounded by χ_{i+1}^+, ν_+, χ_i^+, and ν_- are fundamental domains of the cylinder Cyl_0 in $\mathbb{H}^2$, $i \in \mathbb{Z}$.

Remark 4.12. *The geodesic segment $R_{i+1}^+R_{i+1}^-$ lies inside the fundamental domain $\Xi_i^+ \subset \mathbb{H}^2$ of a cylinder Cyl_0 of the type Cyl; on the other hand, the geodesic segment $R_i^+T_i^-$ lies inside the fundamental domain $R_i^+R_i^-R_{i+1}^+R_{i+1}^- \subset \mathbb{H}^2$ of the same cylinder Cyl_0, $i \in \mathbb{Z}$.*

Proof. Since for every integer i the hyperbolic straight lines χ_i^+ are orthogonal to the geodesic χ_O corresponding to the closed geodesic χ° of the unbounded cylinder $Cyl_0^\circ = \mathbb{H}^2/\langle\bar{\chi}_O\rangle$ which contains Cyl_0 (see also Section 4.2), the projection on Cyl_0 of a path $\xi \subset \Xi_i^+$ connecting any point P^u of the upper boundary $\partial\Xi_i^+ \cap \nu_+ (= R_i^+R_{i+1}^+)$ of Ξ_i^+ with any point P^l of its lower boundary $\partial\Xi_i^+ \cap \nu_-$ does not make a full turn around Cyl_0.

Let us fix $i \in \mathbb{Z}$. As $\Xi_i^+ \subset \mathbb{H}^2$ is a fundamental domain of Cyl_0, the lower boundary $\partial\Xi_i^+ \cap \nu_-$ of Ξ_i^+ must contain at least one and at most two points of the family $\{R_j^- \in \mathbb{H}^2 | j \in \mathbb{Z}\}$ corresponding to one point on Cyl_0. Consider the point R_{i+1}^- of this family. By Remark 4.4, the length of the segment $R_{i+1}^+R_{i+1}^-$ is the smallest one among the lengths of all the segments $R_{i+1}^+R_j^-$, $j \in \mathbb{Z}$. Hence, the projection on Cyl_0 of $R_{i+1}^+R_{i+1}^-$ does not make a full turn around Cyl_0 (otherwise, there would be a path shorter than $R_{i+1}^+R_{i+1}^-$ among the segments $R_{i+1}^+R_j^-$, $j \in \mathbb{Z}$). Since $\alpha \in (0, \pi/2)$, we conclude that $R_{i+1}^+R_{i+1}^- \subset \Xi_i^+$. Similarly, $R_i^+R_i^- \subset \Xi_{i-1}^+$. Hence, $R_i^+T_i^- \subset R_i^+R_i^-R_{i+1}^+R_{i+1}^-$. □

Similarly, we construct a family of hyperbolic straight lines χ_i^- passing through R_i^- and orthogonal to χ_O, $i \in \mathbb{Z}$, and define the points of intersection $O_i^- \stackrel{\text{def}}{=} \chi_i^- \cap \chi_O$, $T_i^+ \stackrel{\text{def}}{=} \chi_i^- \cap \nu_+$, $i \in \mathbb{Z}$. By construction, the connected sets Ξ_i^- bounded by χ_{i+1}^-, ν_+, χ_i^-, and ν_- are fundamental domains of the cylinder Cyl_0 in $\mathbb{H}^2$ and, by analogy with Remark 4.12, the following statement holds true.

Remark 4.13. *The geodesic segment $R_i^+R_i^-$ lies inside the fundamental domain $\Xi_i^- \subset \mathbb{H}^2$ of a cylinder Cyl_0 of the type Cyl; on the other hand, the geodesic segment $R_{i+1}^-T_{i+1}^+$ lies inside the fundamental domain $R_i^+R_i^-R_{i+1}^+R_{i+1}^- \subset \mathbb{H}^2$ of the same cylinder Cyl_0, $i \in \mathbb{Z}$.*

Also, define $h_O^+ \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_i^+, O_i^+)$, $h_O^- \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_i^-, O_i^-)$, and note that $\mathrm{d}_{\mathbb{H}^2}(O_i, O_{i+1}) = \mathrm{d}_{\mathbb{H}^2}(O_i^+, O_{i+1}^+) = \mathrm{d}_{\mathbb{H}^2}(O_i^-, O_{i+1}^-) = l_O$, $i \in \mathbb{Z}$.

4.7.1 Consideration of Situation 1 in the general case

In this section, we demonstrate

Lemma 4.14. *Let a cylinder of the type Cyl contain a closed geodesic and possess a fundamental domain* $R_0^+R_1^+R_0^-R_0^- \subset \mathbb{H}^2$*. Define by* l^+ *and* l^- *the lengths of the sides* $R_0^+R_1^+$ *and* $R_0^-R_1^-$*, and by h the length of* $R_0^+R_0^-$ *and* $R_1^+R_1^-$*. Then the condition*

$$h \geq 2\max\left\{\operatorname{arcosh}\left[\cosh l^+ \cosh\left(l^+ + \operatorname{arcosh}\frac{e^{l^+}(l^+)^2}{\varepsilon_3^2}\right)\right],\right.$$

$$\left.\operatorname{arcosh}\left[\cosh l^- \cosh\left(l^- + \operatorname{arcosh}\frac{e^{l^-}(l^-)^2}{\varepsilon_3^2}\right)\right]\right\}. \tag{4.17}$$

guarantees that there is a path in $R_0^+R_1^+R_0^-R_0^-$ *connecting the midpoints of* $R_0^+R_0^-$ *and* $R_1^+R_1^-$*, and such that its length is smaller than the Margulis constant* ε_3*.*

As we consider Situation 1, we suppose that $O_i \in R_i^-R_i^+$ for $i \in \mathbb{Z}$ and, consequently,

$$h = h^- + h^+. \tag{4.18}$$

For all $i \in \mathbb{Z}$, let us denote the midpoint of the segment $R_i^+R_i^-$ by R_i^{mid}, the midpoints of $R_i^+O_i$ and $R_i^-O_i$ by R_i^{mid+} and R_i^{mid-}, the midpoints of $R_i^+O_i^+$ and $R_i^-O_i^-$ by O_i^{mid+} and O_i^{mid-}. Denote the distances from the points R_i^{mid} to the straight hyperbolic line χ_O by d, from R_i^{mid+} to χ_O by d^+, from R_i^{mid-} to χ_O by d^- and note that, by construction, the distances from the points O_i^{mid+} to χ_O are equal to $h_O^+/2$ and from the points O_i^{mid-} to χ_O are equal to $h_O^-/2$, $i \in \mathbb{Z}$.

Denote by $\hat{\chi}$ a curve in $\mathbb{H}^2$ at the distance d from χ_O and passing through the points R_i^{mid} for all i integers; by $\hat{\chi}_R^+$ a curve in $\mathbb{H}^2$ at the distance d^+ from χ_O and passing through the points R_i^{mid+}; by $\hat{\chi}_R^-$ a curve in $\mathbb{H}^2$ at the distance d^- from χ_O and passing through the points R_i^{mid-}; by $\hat{\chi}_O^+$ a curve in $\mathbb{H}^2$ at the distance $h_O^+/2$ from χ_O and passing through the points O_i^{mid+}; by $\hat{\chi}_O^-$ a curve in $\mathbb{H}^2$ at the distance $h_O^-/2$ from χ_O and passing through the points O_i^{mid-}, $i \in \mathbb{Z}$.

Remark 4.15. *In the notation defined above, the inequalities*

$$d^+ \leq \frac{h_O^+}{2} \quad \textit{and} \quad d^- \leq \frac{h_O^-}{2} \tag{4.19}$$

hold true.

Proof. Define by $\hat{R}_0^{mid+}$ the orthogonal projection of the point R_0^{mid+} on $\chi_O \subset \mathbb{H}^2$ and consider the hyperbolic triangles $\triangle O_0O_0^+R_0^+$ and $\triangle O_0\hat{R}_0^{mid+}R_0^{mid+}$. Recall that $\mathrm{d}_{\mathbb{H}^2}(R_0^+, O_0^+) = h_O^+$, $\mathrm{d}_{\mathbb{H}^2}(R_0^{mid+}, \hat{R}_0^{mid+}) = d^+$, $\mathrm{d}_{\mathbb{H}^2}(R_0^+, O_0) = h^+$, $\mathrm{d}_{\mathbb{H}^2}(R_0^{mid+}, O_0) = h^+/2$, $\angle R_0^+O_0O_0^+ = \angle R_0^{mid+}O_0\hat{R}_0^{mid+} = \alpha$, and $\angle O_0O_0^+R_0^+ = \angle O_0\hat{R}_0^{mid+}R_0^{mid+} = \pi/2$.

Applying Hyperbolic Law of Sines to $\triangle O_0O_0^+R_0^+$ and $\triangle O_0\hat{R}_0^{mid+}R_0^{mid+}$, we obtain the formulas

$$\frac{\sin\alpha}{\sinh h_O^+} = \frac{\sin\frac{\pi}{2}}{\sinh h^+}$$

and

$$\frac{\sin\alpha}{\sinh d^+} = \frac{\sin\frac{\pi}{2}}{\sinh\frac{h^+}{2}},$$

or, after simplification,

$$\sinh h_O^+ = \sin\alpha \sinh h^+ \tag{4.20}$$

and

$$\sinh d^+ = \sin\alpha \sinh \frac{h^+}{2}. \tag{4.21}$$

Note that when the formula

$$\sinh d^+ \leq \sinh \frac{h_O^+}{2} \tag{4.22}$$

holds true, the first relation in (4.19) is satisfied.

By (4.21), (4.22) is equivalent to

$$\sin\alpha \sinh \frac{h^+}{2} \leq \sinh \frac{h_O^+}{2}. \tag{4.23}$$

Due to the following property of the hyperbolic sine: $\sinh 2x = 2\sinh x \cosh x$, from (4.20) we get

$$2\sinh \frac{h_O^+}{2} \cosh \frac{h_O^+}{2} = 2\sin\alpha \sinh \frac{h^+}{2} \cosh \frac{h^+}{2} \tag{4.24}$$

As $h_O^+ \leq h^+$ by construction and the function $\cosh x$ is monotonically increasing for $x \geq 0$, then it is true that $\cosh(h_O^+/2) \leq \cosh(h^+/2)$ and, by (4.20), we obtain

$$\sinh \frac{h_O^+}{2} \cosh \frac{h^+}{2} \geq \sin\alpha \sinh \frac{h^+}{2} \cosh \frac{h^+}{2}. \tag{4.25}$$

Simplifying (4.25), we see that the condition (4.23) is satisfied. Hence, the first inequality in (4.19) holds true.

The validity of the second relation in (4.19) we prove by the same method. □

Together with constructions made above, Remark 4.15 means geometrically that the curve $\hat{\chi}$ lies inside the connected domain of the hyperbolic plane bounded by the curves $\hat{\chi}_R^+$ and $\hat{\chi}_R^-$ which is embedded into the connected domain bounded by $\hat{\chi}_O^+$ and $\hat{\chi}_O^-$ which is embedded, in its turn, into the connected domain bounded by ν_+ and ν_-.

By Remark 4.6, the length of the path $\widehat{R_i^{mid} R_{i+1}^{mid}}$ connecting the points R_i^{mid} and R_{i+1}^{mid} on the curve $\hat{\chi}$ is $\hat{l} = l_O \cosh d$, the length of the path $\widehat{R_i^{mid+} R_{i+1}^{mid+}} \subset \hat{\chi}_R^+$ connecting the points R_i^{mid+} and R_{i+1}^{mid+} is $\hat{l}_R^+ = l_O \cosh d^+$, the length of the path $\widehat{R_i^{mid-} R_{i+1}^{mid-}} \subset \hat{\chi}_R^-$ connecting the points R_i^{mid-} and R_{i+1}^{mid-} is $\hat{l}_R^- = l_O \cosh d^-$, the length of the path $\widehat{O_i^{mid+} O_{i+1}^{mid+}} \subset \hat{\chi}_O^+$ connecting the points O_i^{mid+} and O_{i+1}^{mid+} is $\hat{l}_O^+ = l_O \cosh(h_O^+/2)$, and the length of the path $\widehat{O_i^{mid-} O_{i+1}^{mid-}} \subset \hat{\chi}_O^-$ connecting the points O_i^{mid-} and O_{i+1}^{mid-} is $\hat{l}_O^- = l_O \cosh(h_O^-/2)$, $i \in \mathbb{Z}$.

Assume that $R_i^{mid} \in R_i^+ O_i$, $i \in \mathbb{Z}$. According to Remark 4.15, we have

$$l_O \leq \hat{l} \leq \hat{l}_R^+ \leq \hat{l}_O^+ \leq l^+. \tag{4.26}$$

Otherwise $R_i^{mid} \in R_i^- O_i$, $i \in \mathbb{Z}$ and

$$l_O \leq \hat{l} \leq \hat{l}_R^- \leq \hat{l}_O^- \leq l^- \tag{4.27}$$

(remind that we consider Situation 1). Hence, if we prove that for h big enough $\hat{l}_O^+ < \varepsilon_3$ and $\hat{l}_O^- < \varepsilon_3$, then $\hat{l} < \varepsilon_3$ and the projection of the path $\widehat{R_i^{mid} R_{i+1}^{mid}} \subset \mathbb{H}^2$ on the cylinder Cyl_0 is a closed curve which is shorter than the Margulis constant ε_3 and which passes through the midpoint R^{mid} of the segment $R^+R^- \subset Cyl_0$ corresponding to $R_i^+R_i^- \subset \mathbb{H}^2$, $i \in \mathbb{Z}$ (compare it with the reasoning made in the proof of Lemma 4.9).

First, fixing l^+ let us find a condition on h^+ which will guarantee $\hat{l}_O^+$ to be less than ε_3.

By Remark 4.12, the geodesic segment $R_0^+T_0^-$ lies inside the fundamental domain $R_0^+R_0^-R_1^+R_1^- \subset \mathbb{H}^2$. Hence, the point O_0^+ of intersection of $R_0^+T_0^-$ with χ_O belongs to the geodesic segment O_0O_1.

Denote $l_{O_0^+O_0} \overset{\text{def}}{=} d_{\mathbb{H}^2}(O_0^+, O_0)$ and consider the right-angled triangle $\triangle O_0O_0^+R_0^+$. Hyperbolic Pythagorean Theorem implies:

$$\cosh h^+ = \cosh h_O^+ \cosh l_{O_0^+O_0}. \tag{4.28}$$

Since $O_0O_0^+ \subset O_0O_1$, the inequality $l_{O_0^+O_0} \leq l_O$ holds true and, together with (4.28) gives us

$$\cosh h^+ \leq \cosh h_O^+ \cosh l_O,$$

and, by (4.26),

$$\cosh h^+ \leq \cosh h_O^+ \cosh l^+,$$

or, in other form,

$$\cosh h_O^+ \geq \frac{\cosh h^+}{\cosh l^+}. \tag{4.29}$$

It means that, once we take h^+ to satisfy the condition

$$\cosh h^+ \geq \cosh l^+ \cosh\left(l^+ + \operatorname{arcosh}\frac{e^{l^+}(l^+)^2}{\varepsilon_3^2}\right), \tag{4.30}$$

then, according to (4.29),

$$h_O^+ \geq l^+ + \operatorname{arcosh}\frac{e^{l^+}(l^+)^2}{\varepsilon_3^2},$$

and, by Lemma 4.10 applied to the quadrilateral $O_0^+O_1^+R_0^+R_1^+$, we conclude that

$$\hat{l}_O^+ \leq \varepsilon_3. \tag{4.31}$$

Similarly, if we take h^- to verify the inequality

$$\cosh h^- \geq \cosh l^- \cosh\left(l^- + \operatorname{arcosh}\frac{e^{l^-}(l^-)^2}{\varepsilon_3^2}\right), \tag{4.32}$$

then

$$\hat{l}_O^- \leq \varepsilon_3. \tag{4.33}$$

Finally, let the condition (4.17) be satisfied. Supposing $h^+ \geq h^-$, we have $\widehat{R_0^{mid}R_1^{mid}} \subset O_0^+O_1^+R_0^+R_1^+$ and, by (4.18), the inequality (4.30) holds true, which implies (4.31) and, due to (4.26), leads as to the validity of the condition

$$\hat{l} \leq \varepsilon_3 \tag{4.34}$$

(compare this reasoning with the proof of Lemma 4.9). On the other hand, if $h^+ < h^-$ then $\widehat{R_0^{mid}R_1^{mid}} \subset O_0^-O_1^-R_0^-R_1^-$ and, by (4.18), the inequality (4.32) holds true, which implies (4.33) and, due to (4.27), leads as to the validity of (4.34).

Lemma 4.14 is proved.

4.7.2 Consideration of Situation 2 in the general case

Lemma 4.16. *Let a cylinder of the type Cyl do not contain a closed geodesic and possess a fundamental domain $R_0^+R_1^+R_0^-R_0^- \subset \mathbb{H}^2$. Define by l^+ and l^- the lengths of the sides $R_0^+R_1^+$ and $R_0^-R_1^-$, and by h the length of $R_0^+R_0^-$ and $R_1^+R_1^-$. Then*

$$h < \max\left\{\left(l^+ + l^- + \ln\frac{2l^+}{l^-}\right), \left(l^+ + l^- + \ln\frac{2l^-}{l^+}\right)\right\}.$$

Proof. We will use notation developed in Section 4.7. In these terms, the fact that a cylinder of the type Cyl does not contain a closed geodesic means that the segment O_0O_1 lies outside the fundamental domain $R_0^+R_1^+R_0^-R_0^- \subset \mathbb{H}^2$ of the cylinder.

First, we suppose that $h^+ \geq h^-$, then

$$h = h^+ - h^-, \tag{4.35}$$

which distinguishes Situation 2 from Situation 1 (compare (4.35) with (4.18)).

Denote

$$h_O \stackrel{\text{def}}{=} h_O^+ - h_O^-, \tag{4.36}$$

construct a curve $\hat{\chi}^- \subset \mathbb{H}^2$ at the distance h_O^- from χ_O and passing through the points R_i^-, and define the points of intersection $K_i^- \stackrel{\text{def}}{=} \chi_i^+ \cap \hat{\chi}^-$, $i \in \mathbb{Z}$. By construction, the lengths $l_{R_i^+K_i^-}$ and $l_{O_i^+K_i^-}$ of the segments $R_i^+K_i^- \subset R_i^+O_i^+$ and $O_i^+K_i^- \subset R_i^+O_i^+$ are equal to

$$l_{R_i^+K_i^-} = h_O \quad \text{and} \quad l_{O_i^+K_i^-} = h_O^-, \tag{4.37}$$

$i \in \mathbb{Z}$. Define also the path $\widehat{R_i^-K_i^-}$ connecting the points R_i^- and K_i^- on the curve $\hat{\chi}^-$, $i \in \mathbb{Z}$.

By Remark 4.12, the geodesic segment $R_0^+K_0^- \subset R_0^+T_0^-$ lies inside the fundamental domain $R_0^+R_0^-R_1^+R_1^- \subset \mathbb{H}^2$. Hence, the path $\widehat{R_i^-K_i^-}$ is contained in the hyperbolic ball $B_{R_0^-}(l^-)$ (also, we see that the segment $R_0^-R_1^-$ is a radius of $B_{R_0^-}(l^-)$), and the length $l_{R_0^-K_0^-}$ of the segment $R_0^-K_0^- \subset R_0^-R_1^-$ satisfies the following inequality:

$$l_{R_0^-K_0^-} \leq l^-. \tag{4.38}$$

Applying the triangle inequality to $\triangle R_0^+R_0^-K_0^-$, we get:

$$h \leq l_{R_0^-K_0^-} + l_{R_0^+K_0^-},$$

and, by (4.37) and (4.38),

$$h \leq l^- + h_O. \tag{4.39}$$

Let us now estimate the parameter h_O from above.

Given the quadrilateral $O_0^-O_1^-R_0^-R_1^-$, Remarks 4.6 and 4.7 imply

$$l_O \cosh h_O^- > l^-,$$

then, by the definition of the hyperbolic cosine (4.9), we have

$$\frac{e^{h_O^-} + e^{-h_O^-}}{2} > \frac{l^-}{l_O},$$

and, as $e^{h_O^-} \geq e^{-h_O^-}$ for $h_O^- \geq 0$, we obtain

$$e^{h_O^-} > \frac{l^-}{l_O}. \tag{4.40}$$

If $h_O^+ \leq l^+$ then, by (4.36),

$$h_O \leq l^+ \tag{4.41}$$

as well.

Assume that $h_O^+ > l^+$. By Remarks 4.6 and 4.8 applied to the quadrilateral $O_0^+ O_1^+ R_0^+ R_1^+$, we get

$$l_O \cosh(h_O^+ - l^+) < l^+,$$

and, by (4.36),

$$l_O \cosh(h_O^- + h_O - l^+) < l^+,$$

then the definition of the hyperbolic cosine (4.9) gives us

$$e^{h_O^-} e^{h_O} e^{-l^+} + e^{-h_O^-} e^{-h_O} e^{l^+} < \frac{2l^+}{l_O}.$$

Let us weaken the obtained inequality:

$$e^{h_O^-} e^{h_O} e^{-l^+} < \frac{2l^+}{l_O},$$

and, together with (4.40), we get

$$\frac{l^-}{l_O} e^{h_O} e^{-l^+} < \frac{2l^+}{l_O},$$

$$e^{h_O} < \frac{2l^+}{l^-} e^{l^+},$$

$$h_O < l^+ + \ln \frac{2l^+}{l^-}. \tag{4.42}$$

Note that the inequality (4.41) is stronger than (4.42). Mixing and (4.42) we get:

$$h < l^- + l^+ + \ln \frac{2l^+}{l^-}. \tag{4.43}$$

Supposing $h^+ < h^-$, we just need to interchange the upper indices $+$ and $-$ in the formula (4.43):

$$h < l^- + l^+ + \ln \frac{2l^-}{l^+}.$$

□

4.7.3 Finalizing the proof of Theorem 4.2

Consider some points $P^+ \in c_1^+ \cap c_2^+$ and $P^- \in c_1^- \cap c_2^-$. As in Section 4.1, construct the cylinders Cyl_1 and Cyl_2 of the type Cyl homotopically equivalent to the pairs of curves (c_1^+, c_1^-) and (c_2^+, c_2^-), with the upper boundaries of the lengths l_1^+ and l_2^+, with the lower boundaries of the lengths l_1^- and l_2^-, and such that the hyperbolic geodesic segment $P^+P^- \subset \mathcal{M}^\circ$ lies in the intersection $Cyl_1 \cap Cyl_2$.

If Situation 2 is realized for at least one of the cylinders Cyl_1 and Cyl_2, than Lemma 4.16 implies that

$$d(\mathcal{S}^+,\mathcal{S}^-) < \max\left\{\left(l_1^+ + l_1^- + \ln\frac{2l_1^+}{l_1^-}\right), \left(l_1^+ + l_1^- + \ln\frac{2l_1^-}{l_1^+}\right), \left(l_2^+ + l_2^- + \ln\frac{2l_2^+}{l_2^-}\right), \left(l_2^+ + l_2^- + \ln\frac{2l_2^-}{l_2^+}\right)\right\}.$$

Otherwise, Situation 1 is realized for both cylinders Cyl_1 and Cyl_2 and, once we suppose

$$d(\mathcal{S}^+,\mathcal{S}^-) < 2\max\left\{\operatorname{arcosh}\left[\cosh l_1^+ \cosh\left(l_1^+ + \operatorname{arcosh}\frac{e^{l_1^+}(l_1^+)^2}{\varepsilon_3^2}\right)\right],\right.$$

$$\operatorname{arcosh}\left[\cosh l_1^- \cosh\left(l_1^- + \operatorname{arcosh}\frac{e^{l_1^-}(l_1^-)^2}{\varepsilon_3^2}\right)\right], \operatorname{arcosh}\left[\cosh l_2^+ \cosh\left(l_2^+ + \operatorname{arcosh}\frac{e^{l_2^+}(l_2^+)^2}{\varepsilon_3^2}\right)\right],$$

$$\left.\operatorname{arcosh}\left[\cosh l_2^- \cosh\left(l_2^- + \operatorname{arcosh}\frac{e^{l_2^-}(l_2^-)^2}{\varepsilon_3^2}\right)\right]\right\},$$

by Lemma 4.14, there are curves $cur_1 \subset Cyl_1$ and $cur_2 \subset Cyl_2$ with the lengths less than the Margulis constant ε_3, both passing through the midpoint of the segment P^+P^-. Thus, we come to a contradiction with Margulis Lemma. Theorem 4.2 is proved. □

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