



## Biadjoint wires

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## ABSTRACT

Biadjoint scalar field theory has been the subject of much recent study, due to a number of applications in field and string theory. The catalogue of exact non-linear solutions of this theory is relatively unexplored, despite having a role to play in extending known relationships between gauge and gravity theories, such as the double copy. In this paper, we present new solutions of biadjoint scalar theory, corresponding to singular line configurations in four spacetime dimensions, with a power-law dependence on the cylindrical radius. For a certain choice of common gauge group (SU(2)), a family of infinitely degenerate solutions is found, whose existence can be traced to the global symmetry of the theory. We also present extended solutions, in which the pure power-law divergence is partially screened by a form factor.

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## 1. Introduction

To the best of our current experimental knowledge, the forces of nature are described by (quantum) field theories, making the latter the subject of intense ongoing scrutiny. Recently, a number of intriguing relationships have been discovered between field theories whose physics is very different. One such correspondence is the *double copy* [1–3], that relates (non-abelian) gauge theories and gravity. Although originally formulated for scattering amplitudes [2,4–41], it has subsequently been extended to classical solutions [42–65], curved space [66–68] and double field theory [65]. At tree-level, the double copy has a string theoretic justification [69]. More generally, double-copy-like ideas have a natural representation in terms of the so-called CHY equations [70,71], which themselves emerge from ambitwistor string theory [72], and have been investigated at loop level [73].

In all of the above contexts, an additional theory makes a crucial appearance, containing a single scalar field  $\Phi^{aa'}$  carrying two adjoint indices associated with a pair of (in principle distinct) Lie groups. This *biadjoint scalar field theory* can be described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \Phi^{aa'} \partial_\mu \Phi^{aa'} + \frac{y}{3} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}, \quad (1)$$

where  $f^{abc}$  and  $\tilde{f}^{a'b'c'}$  are the structure constants associated with the Lie groups, and we adopt the summation convention for repeated indices. The above cubic Lagrangian leads to the quadratic field equation

$$\partial_\mu \partial^\mu \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0. \quad (2)$$

Although the biadjoint theory is not a physical theory by itself, mounting evidence suggests that its dynamical information is inherited, at least in part, by gauge and gravity theories. For example, amplitudes in biadjoint scalar theory are related to those in (non-abelian) gauge theory by a process known as the *zeroth copy*. A similar procedure holds for classical solutions, in those cases in which the single copy between gravity and gauge theory is also known (see the above references for further details). An ever-increasing web of theories related by similar correspondences is currently being established, where a recent summary can be found in figure 1 of ref. [64].

The above correspondences involve perturbative solutions of the respective theories, and / or those with only positive powers of the coupling constant. It is natural to ponder whether or not the relationships can be extended to the fully nonperturbative regime. There are a number of ways of approaching this issue. Firstly, one may consider how symmetries match up on both sides of the double copy (see e.g. the recent ref. [74] for a discussion). Secondly, one may catalogue fully non-linear solutions of various field theories, before trying to match them up in some way. How to do the latter is unclear, as all previous examples of how to perform the double copy involve solutions of the linearised biadjoint equation. However, the elucidation of new nonlinear solutions is

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certainly achievable, and a necessary step in probing nonperturbative aspects of the double copy any further. A wide literature already exists on nonlinear solutions in gauge and gravity theories (see e.g. refs. [75–77] for reviews). Much less is known about exact solutions of the biadjoint scalar theory of eq. (2).

Some first non-linear solutions of biadjoint theory were presented in ref. [78]. They included a spherically symmetric monopole-like object in the case in which both Lie groups are the same, where the field  $\Phi^{aa'}$  has a power-law behaviour, diverging at the origin. An additional (and more general) solution was found when the common gauge group is SU(2), although again possessing spherical symmetry. Extended solutions were found in ref. [79], where the power-law behaviour of the field was dressed by a non-trivial form factor, which partially screens (but does not remove<sup>1</sup>) the singular behaviour at the origin.

The aim of this paper is to go beyond spherically symmetric solutions of the biadjoint field equation. More specifically, we will consider cylindrically symmetric solutions, depending only on the cylindrical polar radius  $\rho$ . A number of non-trivial results will be presented. First, we will find a power-law solution for the case in which both Lie groups are the same. This mirrors the monopole-like solution found in ref. [78], and can be interpreted as a wiry object localised on the  $z$ -axis. We will look for a more general solution when the common Lie group is SU(2), finding a one-parameter family of solutions, again as in ref. [78] for the spherically symmetric case. Unlike the latter, however, we will see that the family of SU(2) solutions is degenerate in energy, which can be traced to the symmetry of the biadjoint theory in this case. Next, we will consider dressed solutions, for which the power-law behaviour in  $\rho$  is modified by a form factor. We will find, as in ref. [79], that such form factors can indeed be obtained, and have the effect of screening the divergent behaviour of the wire. Our results will be important for future studies of the nonperturbative double copy, as well as being of interest in their own right, given the multiple contexts in which the biadjoint theory arises.

The structure of the paper is as follows. In section 2, we derive power-law line solutions for a general common gauge group  $G$ , and for the case in which this group is SU(2). In section 3, we consider dressed solutions. We discuss our results and conclude in section 4.

## 2. Power-law wire solutions

### 2.1. Solution for generic common gauge group

Let us first consider a common gauge group  $G$ , such that  $f^{abc} = \bar{f}^{abc}$  in eqs. (1), (2). Adopting cylindrical polar coordinates  $(\rho, z, \phi)$ , we may look for a static cylindrically symmetric solution by making the ansatz

$$\Phi^{aa'} = \delta^{aa'} f(\rho), \tag{3}$$

where the structure constants are normalised according to

$$f^{abc} f^{a'bc} = T_A \delta^{aa'}. \tag{4}$$

One then finds

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + y T_A f^2(\rho) = 0, \tag{5}$$

which has the non-trivial power-law solution

$$\Phi^{aa'} = -\frac{4\delta^{aa'}}{y T_A \rho^2}. \tag{6}$$

As for the spherically symmetric monopole-like object found in ref. [78], this has an inverse power of the coupling constant  $y$ , and thus is nonperturbative (i.e. it vanishes at weak coupling). It goes like the inverse square of the cylindrical radius  $\rho$ , where this dependence can also be surmised by dimensional analysis. The solution is singular as  $\rho \rightarrow 0$ , corresponding to a line defect in the field, localised on the  $z$ -axis. To further illustrate this, we may calculate the energy of the field, for which eq. (1) implies the Hamiltonian density (see ref. [78] for further details)

$$\mathcal{H} = \frac{160}{3} \frac{\mathcal{N}}{y^2 T_A^2} \frac{1}{\rho^6}, \tag{7}$$

where  $\mathcal{N}$  is the dimension of the common gauge group  $G$ . This diverges as  $\rho \rightarrow 0$  due to the singularity in the field, but is well-behaved as  $\rho \rightarrow \infty$ . We can thus place a cutoff  $\rho_0$  around the wire, and calculate an energy per unit length

$$\frac{E}{L} = \frac{80\pi\mathcal{N}}{3y^2 T_A^2 \rho_0^4}. \tag{8}$$

As in the spherically symmetric case of ref. [78], we see that this is well-defined. The different behaviour with respect to the cutoff compared with that reference stems from the fact that in the wire case, we must calculate the energy per unit length, rather than the energy itself.

### 2.2. Solutions for SU(2)×SU(2)

For the special case in which the common gauge group  $G$  is SU(2), one may write a more general form for the field. The structure constants in that case are equal to the Levi-Cevita tensor  $f^{abc} = \epsilon^{abc}$ , allowing the possibility of mixing spatial and gauge indices. This was used in refs. [78,79] to find novel spherically symmetric solutions, and a similar (but not identical) ansatz can be used here. A notable feature in the present case – which is shared by similar solutions in Yang–Mills theory [81] – is that the requirement of cylindrical symmetry, together with the mixing of spatial and gauge indices, means that choosing a special direction in space (the  $z$ -axis) picks out a special direction (the 3-direction) in the gauge space, suggesting the following ansatz:

$$\begin{aligned} \Phi^{33} &= f_1(\rho), & \Phi^{ij} &= f_2(\rho)\delta^{ij} + f_3(\rho)x^i x^j + f_4(\rho)\epsilon^{3ij}, \\ \Phi^{3i} &= \Phi^{i3} = 0. \end{aligned} \tag{9}$$

Here and in the following, we use indices  $i, j, k \dots \in (1, 2)$ , as distinct from indices  $a, b, c \in (1, 2, 3)$ . Substituting eq. (9) into eq. (2), one obtains the four coupled non-linear ordinary differential equations

$$\begin{aligned} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \bar{f}_2}{\partial \rho} \right) + 2\bar{f}_3 + 2\bar{f}_1 (\bar{f}_2 + \rho^2 \bar{f}_3) &= 0; \\ \frac{\partial^2 \bar{f}_3}{\partial \rho^2} + \frac{5}{\rho} \frac{\partial \bar{f}_3}{\partial \rho} - 2\bar{f}_1 \bar{f}_3 &= 0; \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \bar{f}_4}{\partial \rho} \right) + 2\bar{f}_1 \bar{f}_4 &= 0; \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \bar{f}_1}{\partial \rho} \right) + 2(\bar{f}_2^2 + \bar{f}_4^2 + \rho^2 \bar{f}_2 \bar{f}_3) &= 0, \end{aligned} \tag{10}$$

<sup>1</sup> Finite energy solutions are forbidden in scalar field theories, by Derrick's theorem [80].

where, following ref. [78], we have defined the convenient combinations

$$f_i(\rho) = \frac{\bar{f}_i(\rho)}{y}. \quad (11)$$

We may find a power-law solution to eqs. (10) by writing

$$\bar{f}_i = k_i \rho^{\alpha_i}, \quad (12)$$

substitution of which straightforwardly yields

$$\alpha_1 = \alpha_2 = \alpha_4 = -2, \quad \alpha_3 = -4, \quad (13)$$

as well as the nonlinear simultaneous equations

$$\begin{aligned} 2k_2 + k_3 + k_1(k_2 + k_3) &= k_4(2 + k_1) \\ &= 2k_1 + k_2^2 + k_4^2 + k_2 k_3 = k_1 k_3 = 0. \end{aligned} \quad (14)$$

Setting  $k_2 = -2k$ , the general solution of these equations implies

$$\Phi^{33} = -\frac{2}{y\rho^2}, \quad \Phi^{ij} = -\frac{2}{y\rho^2} \left[ k\delta^{ij} \mp \sqrt{1 - k^2} \epsilon^{3ij} \right]. \quad (15)$$

There is thus a continuously infinite family of solutions. Note that the previous solution of eq. (6) emerges as the special case  $k = 1$ . As for that case, one may calculate the energy associated with eq. (15), subject to a cutoff being applied for low cylindrical radius. An explicit calculation reveals an energy per unit length

$$\frac{E}{L} = \frac{20\pi}{y^2 \rho_0^4}, \quad (16)$$

regardless of the value of  $k$ . Hence, the family of solutions obtained in eq. (15) is degenerate. Furthermore, eq. (16) agrees with eq. (8) for the specific SU(2) values  $\mathcal{N} = 3$ ,  $T_A = 2$ , as it should given the degeneracy, and the fact that the previous solution emerges from the present one as a special case.

One may understand the degeneracy of the family of solutions of eq. (15) as follows. First, we may write  $k = \cos\theta$ , and introduce a matrix  $\Phi$ , whose components are  $\Phi^{aa'}$ :

$$\Phi = -\frac{2}{y\rho^2} \begin{pmatrix} \cos\theta & \mp \sin\theta & 0 \\ \pm \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (17)$$

This is a rotation about the 3-axis in gauge space by angle  $\theta$ . Next, we may write the Lagrangian of eq. (1), for the special case of SU(2), as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial^\mu \Phi^{aa'} \partial_\mu \Phi^{aa'} + \frac{y}{3} \epsilon^{abc} \epsilon^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'} \\ &= \frac{1}{2} \text{Tr} [(\partial^\mu \Phi)^T (\partial_\mu \Phi)] + 2y \det[\Phi]. \end{aligned} \quad (18)$$

This is manifestly invariant under the transformation  $\Phi \rightarrow \mathbf{R}_1^T \Phi \mathbf{R}_2$ , for arbitrary rotation matrices  $\mathbf{R}_i$ . Thus, one may start with the solution of eq. (6) for SU(2), which in the present notation reads

$$\Phi = -\frac{2}{y\rho^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (19)$$

before acting on it with rotations about the 3-axis to generate the family of solutions of eq. (15). The rotation will not change the energy, given the above-mentioned symmetry, which is none other than the global symmetry associated with the SU(2) biadjoint theory.<sup>2</sup> Here we have considered rotating eq. (19) about the 3-axis.

<sup>2</sup> A similar interpretation can be applied to the spherically symmetric SU(2)×SU(2) solution in ref. [78], which consists of an infinitesimal rotation of the analogue of eq. (6).

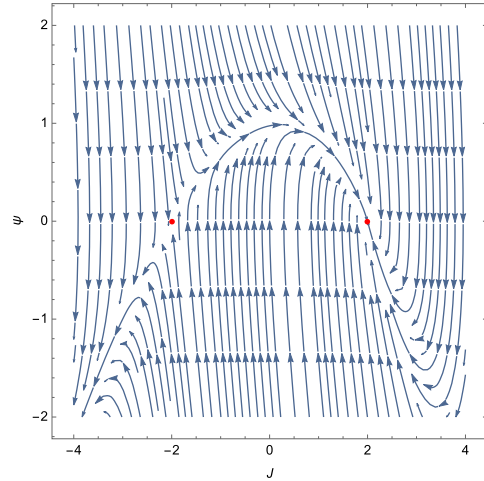


Fig. 1. Integral curves of the vector field of eq. (24). Bounded curves (or fixed points) correspond to solutions for  $J(\xi)$  that are finite for all  $\xi$ .

However, eq. (18) is invariant under arbitrary rotations, and thus we could also rotate around the 1- or 2-axes, leading to a different correspondence between the spatial and gauge directions.

### 3. Dressed wires

In the preceding section, we have found pure power-law wire solutions of the biadjoint theory. This is not the whole story, however. In gauge theories, there is a cornucopia of solutions in which a pure power-law divergence can be dressed with a non-trivial form factor, where the latter can be interpreted as providing some internal structure (see e.g. refs. [75,76]). Furthermore, it can have the effect of screening the divergent behaviour near singular regions of the field. Reference [79] found that such solutions also exist in the biadjoint theory, namely that the spherically symmetric monopole solution can be dressed with a screening function. Given that our aim is to catalogue – both qualitatively and quantitatively – the solutions that are possible in biadjoint theory, it is both interesting and useful to check whether or not such dressed behaviour is possible also for the wire solution.

To this end, let us reconsider the ansatz of eqs. (3), (11), and define<sup>3</sup>

$$J(\rho) = 2 + \rho^2 \bar{f}(\rho). \quad (20)$$

Dimensional analysis fixes  $\bar{f}(\rho) \sim \rho^{-2}$ , so that  $J(\rho)$  must be finite for all  $\rho$ . Substituting eq. (20) into eq. (5) yields<sup>4</sup>

$$\rho^2 \frac{d^2 J(\rho)}{d\rho^2} - 3\rho \frac{dJ(\rho)}{d\rho} + J(\rho)^2 - 4 = 0. \quad (21)$$

Upon transforming to  $\xi$  via

$$\rho = e^{-\xi}, \quad (22)$$

eq. (21) becomes an ODE with constant coefficients:

$$\frac{d^2 J(\xi)}{d\xi^2} + 4 \frac{dJ(\xi)}{d\xi} + J^2(\xi) - 4 = 0. \quad (23)$$

<sup>3</sup> Strictly speaking, the left-hand side of eq. (20) should be a function of a dimensionless ratio  $\rho/\bar{\rho}$  for some length scale  $\bar{\rho}$ , but we may take the latter to be unity without loss of generality in what follows.

<sup>4</sup> Varying the constant term on the right-hand side of eq. (20) would lead to an additional term linear in  $J$ , which it is convenient to remove by the above choice.

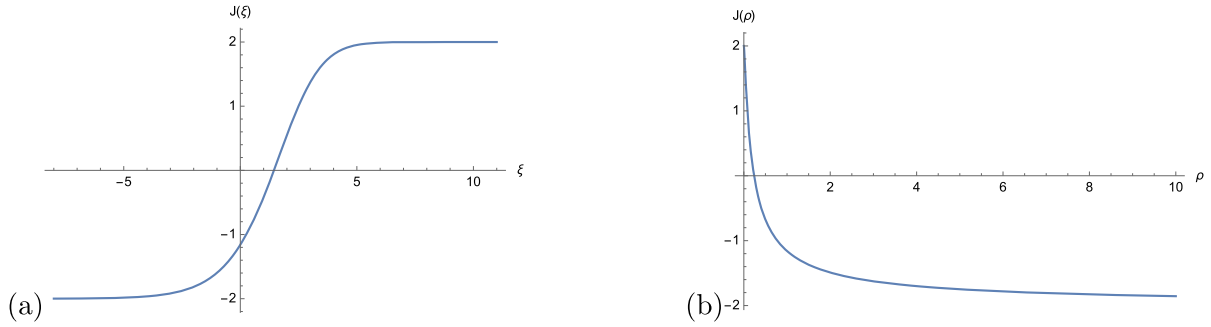


Fig. 2. (a) Numerical solution of  $J(\xi)$  from eq. (20), with boundary conditions as described in the text; (b) Behaviour of  $J$  as a function of the cylindrical radius  $\rho$ .

This can be related to an Abel equation of the second kind, albeit one with no analytic solution (see also ref. [79]). We may instead look for a numerical solution, as follows. One may write eq. (23) as a set of coupled first order equations by defining

$$\left(\frac{dJ}{d\xi}, \frac{d\psi}{d\xi}\right) = (\psi, 4 - 4\psi - J^2). \tag{24}$$

Next, one may plot the integral curves of this vector field in the  $(J, \psi)$  plane, where any bounded curves correspond to solutions for  $J(\xi)$  that remain finite for all  $\xi$  (and thus  $\rho$ ). We show this vector field in Fig. 1, whose examination yields the following solutions:

1.  $J(\xi) = -2$ : this corresponds to the solution of eq. (6).
2.  $J(\xi) = 2$ : this yields the trivial solution  $\Phi^{aa'} = 0$ .
3.  $J(\xi) \rightarrow \pm 2$  as  $\xi \rightarrow \pm\infty$  respectively. This is a non-trivial form factor, corresponding to the curve that flows from  $(-2, 0)$  to  $(2, 0)$  in Fig. 1.

It is worth remarking that the structure of Fig. 1 is qualitatively similar to that of the spherically symmetric case of ref. [79], although the non-trivial form factor (in case 3)  $J(\xi)$  is different. We can solve for  $J(\xi)$  in the asymptotic limits  $\xi \rightarrow \pm\infty$  by writing

$$J(\xi) = \pm 2 + \chi_{\pm}(\xi), \quad \xi \rightarrow \pm\infty. \tag{25}$$

In each limit, we may neglect terms in  $\chi_{\pm}^2$ , obtaining approximate solutions (for finite  $J(\xi)$ )

$$J(\xi) \simeq \begin{cases} -2 + c_1 e^{(2\sqrt{2}-2)\xi}, & \xi \rightarrow -\infty \\ +2 + c_2 e^{-2\xi} + c_3 \xi e^{-2\xi}, & \xi \rightarrow +\infty \end{cases} \tag{26}$$

or in terms of  $\rho$ :

$$J(\rho) \simeq \begin{cases} -2 + c_1 \rho^{2-2\sqrt{2}}, & \rho \rightarrow \infty \\ +2 + (c_2 - c_3 \log \rho) \rho^2, & \rho \rightarrow 0. \end{cases} \tag{27}$$

We can solve for the complete form of  $J(\xi)$  numerically, upon choosing  $c_1 = 1$ .<sup>5</sup> We plot this numerical solution in Fig. 2. The boundary condition  $J(\rho) \rightarrow 2$  as  $\rho \rightarrow 0$  means that the divergence of the wire solution near the  $z$ -axis is partially screened. Indeed, we find an energy per unit length of

$$\frac{E}{L} = \frac{\pi c_3^2 \mathcal{N}}{T_A^2 y^2} \frac{1}{\rho_0} + \mathcal{O}(\rho^0), \tag{28}$$

which is less singular than the undressed result of eq. (8), showing that physical behaviour (i.e. the energy) is also screened. One

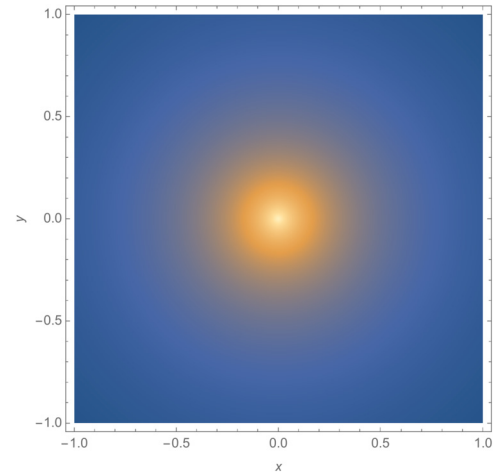


Fig. 3. The profile function  $\rho^2 \bar{f}(\rho)$  in the  $(x, y)$  plane for a dressed wire solution situated on the  $z$ -axis.

may wonder whether or not it is possible to completely screen the divergence on the  $z$ -axis. However, as in the spherically symmetric case of ref. [78], this is forbidden by Derrick’s theorem.

In order to visualise the dressed wire, we show in Fig. 3 the combination  $\rho^2 \bar{f}(\rho)$  in the  $(x, y)$  plane i.e. the profile function of eq. (3), with the singular factor  $\rho^2$  removed. This corresponds to looking down on the wire, which is situated at the origin. Finally, we note that it would also be possible to generate dressed solutions in the  $SU(2) \times SU(2)$  theory of section 2.2, by rotating the solution obtained here, as in eq. (17).

#### 4. Conclusion

In this paper, we have obtained new solutions of the biadjoint scalar field theory, which occurs in a number of contexts, including the double copy relationship between gauge theories and gravity. Although much is already known about the perturbative sector of biadjoint theory, much less is known about its nonperturbative properties, making any results of interest in their own right. Furthermore, solutions such as that found in this study open up the possibility to try to extend the double copy to a fully nonperturbative regime.

We have focused specifically on cylindrically symmetric solutions, finding a power-law solution corresponding to a wire-like object localised on the  $z$ -axis, for the case in which both Lie groups in the biadjoint theory are the same. For the case in which this common group is  $SU(2)$ , an infinite family of solutions is possible. Such solutions are degenerate, an effect which can be traced to the global symmetry of the biadjoint theory for this choice of gauge group.

<sup>5</sup> A different choice of  $c_1$  amounts to shifting  $\xi$  by a constant, or rescaling  $\rho$ , neither of which affect the qualitative shape of Fig. 2.

As well as pure power-law solutions, we also constructed dressed wires, in which a form factor partially screens the divergent behaviour of the field near the core. This is further evidence of a potentially rich spectrum of nonperturbative solutions in bi-adjoint theory, whose properties mimic those found in gauge theories. Whether or not the relationship with gauge theory can be made precise is the subject of ongoing research, and we hope that the results of this investigation will prove useful in this regard.

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