Aggregate Efficiency in Random Assignment Problems*

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Abstract

We introduce aggregate efficiency (AE) for random assignments (RA) by requiring higher expected numbers of agents be assigned to their more preferred choices. It is shown that the realizations of any aggregate efficient random assignment (AERA) must be an AE permutation matrix. While AE implies ordinally efficiency, the reverse does not hold. And there is no mechanism treating equals equally while satisfying weak strategyproofness and AE. But, a new mechanism, the reservation-1 (R1), is identified and shown to provide an improvement on grounds of AE over the probabilistic serial mechanism of Bogomolnaia and Moulin (2001). We prove that R1 is weakly strategyproof, ordinally efficient, and weak envy–free. Moreover, the characterization of R1 displays that it is the probabilistic serial mechanism updated by a principle decreed by the Turkish parliament concerning the random assignment of new doctors: Modifying the axioms of Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming) characterizing the probabilistic serial mechanism to satisfy this principle, fully characterizes R1.

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1 Introduction

Random assignment problems are allocation problems allotting some number of distinct indivisible alternatives among a population of agents with the use of a randomization device, e.g. the flip of a coin or the use of a dice, but without the use of monetary transfers. They constitute a non-negligible and often important aspect in our everyday life. Indeed, in recent years the surge of the use of random assignment methods by market designers and social planners has been significant. Relevant examples include student placement in public schools at various levels of education, organ transplantation, and the assignment of dormitory rooms. While many of these applications are implemented all over the world, Turkey, the country of our residence, features another important example: In the fields of medicine and education and justice, recent graduates are assigned to their places of duty via a random allotment arrangement.[1]

Among random assignment mechanisms, rules associating any (reported) preference profile with a stochastic distribution of alternatives to the agents, the *random priority* mechanism (henceforth, to be referred to as RP) is one of the most widely used and it has been analyzed extensively in Abdulkadiroglu and Sonmez (1998). It is also called the *random serial dictatorship* mechanism and defined as follows: A priority ranking of agents is selected uniformly, and following that rank every agent sequentially receives his favorite alternative among the ones that were not chosen by higher ranked agents. That study shows that even though the particular form of this mechanism is surprisingly simple, it is *strategyproof* (i.e. reporting the true preferences is a dominant strategy) and *ex–post efficient* (i.e. it can be represented by a probability distribution over efficient deterministic assignments). Another efficiency notion may be used when the problem at hand features von Neumann–Morgenstern utilities: A random assignment is *ex–ante efficient* if it is Pareto optimal with respect to

[1] We refer the reader to Roth and Sotomayor (1992) for a classic source on the subject. On the other hand, for more details on random assignment problems, we cite to Hylland and Zeckhauser (1979), Abdulkadiroglu and Sonmez (1998), Abdulkadiroglu and Sonmez (1999), Bogomolnaia and Moulin (2001), Bogomolnaia and Moulin (2002), Chen, Sonmez, and Unver (2002), Abdulkadiroglu and Sonmez (2003), Bogomolnaia and Moulin (2004), Roth, Sonmez, and Unver (2004), Ergin and Sonmez (2006), Katta and Sethuraman (2006), Kesten (2009), Kojima (2009), Yilmaz (2009), Yilmaz (2010), Kesten and Unver (2011), Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming).

the profile of von Neumann–Morgenstern utilities. Bogomolnaia and Moulin (2001) (henceforth, BM) shows that by using only the ordinal preference rankings some of the random assignments that are not ex–ante efficient may be identified even if agents' utility functions are not given. To that regard that study proposes *ordinal efficiency* which necessitates the consideration of (first order) *stochastic dominance*. A random assignment stochastically dominates another one whenever for all agents the probability of being allocated one of the top k ranked alternatives under the former is weakly higher than the one under the latter for all $k = 1, \ldots, K$ where K denotes the total number of available alternatives. A random assignment is ordinally efficient for a given profile of preferences if there is no random assignment stochastically dominating it for that given profile of preferences. BM shows that ex–ante efficiency implies ordinal efficiency and ordinal efficiency implies ex–post efficiency. The reverse directions of these two relations do not hold. Due to McLennan (2002), it is also known that if a random assignment is ordinally efficient then there is a profile of von Neumann–Morgenstern utilities such that that particular random assignment is ex–ante efficient. Motivated by its key finding that RP is not an ordinally efficient mechanism BM introduces and analyzes the *probabilistic serial* (henceforth, PS) mechanism. The outcome of the PS mechanism is identified using BM's *simultaneous eating algorithm* (SEA): Each object is considered as a continuum of probability shares. Agents "eat away" from their favorite objects simultaneously and at the same speed, and once the favorite object of an agent is gone he turns to his next favorite object, and so on. The amount of an object eaten away by an agent in this process is interpreted as the probability with which he is assigned this object under the PS mechanism. BM shows that PS satisfies ordinal efficiency but is not strategyproof. It satisfies the following weaker version: A random assignment mechanism is *weak strategyproof* whenever the random allocation sustained by an agent misrepresenting his preferences stochastically dominates the one he obtains under truthful revelation implies that the two allocations are the same. This shortcoming concerning incentives is made up by some gains in terms of envy–freeness, another relevant notion to judge the value–added of a random assignment mechanism. A random assignment mechanism is *envy–free* if it associates every profile of preferences with a random assignment in which the prescribed random allocation for any agent stochastically dominates that for another agent evaluated with the former's preferences. Meanwhile, relaxing this notion delivers *weak envy–freeness* by requiring that the prescribed random allocation for any agent satisfying the following: The random allocation of another agent stochastically dominating that of the agent at hand implies that the two random allocations are the same. The same study proves that while the PS mechanism involves envy–freeness, the RP rule is weakly envy–free (but not envy–free).

Insisting on ordinal efficiency may create unappealing features. When assigning 100 objects among a population of 100 both of the following assignments may be efficient: The first allocating 1 person to his best and 99 to their second best, and the second allotting 99 to their best and 1 to their second best.[2] Indeed, there are many instances where social planners and market designers evaluate a mechanism by considering how many agents are located into their first best, how many into their second choice, and so on. Often some statistics about how many agents are allocated their higher ranking choices is announced as a positive indicator of the performance of the system.[3]

The current paper introduces a new notion of efficiency, *aggregate efficiency*, tailored for situations in which social planners and market designers value the expected number of agents assigned to their higher ranked choices: We say that a random assignment *aggregate stochastic dominates* another whenever the expected number of agents placed into one of their top k choices under the former is weakly higher than that of the latter for $k = 1, \dots, K$. Moreover, a random assignment is aggregate efficient whenever another random allocation aggregate stochastic dominating the one under consideration implies that both of them assign

[2]Consider a situation where there are 100 agents and 100 objects denoted by $\{a_j\}_{j=1}^{100}$, on which the strict preference relations are as follows: Agent 1 strictly prefers a_1 to a_{100}, and a_{100} to any other alternative, and all other alternatives are ranked strictly lower and arbitrarily. Every other agent $i \neq 1$ strictly prefers a_{i-1} to a_i, and a_i to any other alternative, and all other alternatives are ranked strictly lower and arbitrarily. In this setting assigning each agent i to alternative a_i is (ordinally) efficient, and creates a situation in which one player (agent 1) gets his first best while all the other 99 players obtain their second ranked choice. On the other hand, assigning agent 1 to his second best alternative a_{100} and any other agent i to alternative a_{i-1} is also (ordinally) efficient and causes one agent to obtain his second best while 99 of them are allotted their first ranked choice.

[3]OSYM, the Turkish government agency responsible of administering the nation-wide university admission examination and allocating students to programs, includes the percentage of students allocated to one of their top three choices in their press conferences. Moreover, Featherstone (2011), an independent study that was brought to our attention when the final draft of this paper was being prepared, observes that reports by NYC Department of Education 2009 and San Fransisco Unified School District 2011 also include such aspects.

the same expected number of agents into any one of their top k choices for $k = 1, \ldots, K$.

We establish that the notion of aggregate efficiency implies ordinal efficiency. Yet the reverse does not hold and there are no logical relations between ex–ante efficiency and aggregate efficiency. After proving the existence of aggregate efficiency, we show that Gale's conjecture, the incompatibility of strategyproof and efficient mechanisms treating equals equally, takes a new form: The search for an aggregate efficient and weak strategyproof mechanism treating equals equally is futile.

On the other hand, we prove that there is a weak strategyproof, weak envy–free, and ordinally efficient mechanism, the reservation–1 mechanism (henceforth, R1), that displays a better performance on grounds of aggregate efficiency when compared to the PS mechanism. The outcome of the R1 mechanism is also identified using the SEA with an important modification that provides agents reservation rights for their most favorite alternatives. That is, the algorithm starts with agents "eating away" from their favorite objects simultaneously all at the same speed while no agent (who is finished with his favorite alternative) is allowed to start eating an alternative that is a favorite for some other agent. Once these favorite objects are gone, the algorithm proceeds exactly as the unmodified SEA does. Naturally, the amount of an object eaten away by an agent in this process is interpreted as the probability with which he is assigned this object under the R1 mechanism.

A characterization of the R1 mechanism is provided along the lines of a recent important study, Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming). This establishes that the R1 mechanism is nothing but the PS mechanism modified to satisfy a principle decreed by the Turkish parliament on the issue of the random assignment of new doctors to their places of duty.

The organization of the paper is as follows: The next section provides intuition and motivation for the efficiency notion proposed and contains an elucidative discussion of our results. Then section 3 presents the model. In section 4 we analyze aggregate efficiency and obtain some impossibility results. Section 5 introduces and contains the detailed analysis and full characterization of the R1 mechanism.

$I.$	a	b	c
1	1	0	0
2	0	1	0
3	0	0	1

$II.$	a	b	c
1	0	1	0
2	1	0	0
3	0	0	1

$III.$	a	b	c
1	1	0	0
2	0	0	1
3	0	1	0

$IV.$	a	b	c
1	0	0	1
2	1	0	0
3	0	1	0

Table 1: The deterministic efficient assignments.

2 Aggregate Efficiency and the R1 Mechanism

In order to facilitate an easier reading and more motivation we wish to introduce the notion of aggregate efficiency and present our results in the context of the following simple example with 3 agents and 3 alternatives. The set of players is $N = \{1, 2, 3\}$ and the set of alternatives $A = \{a, b, c\}$. The preferences of agents are given by $a \succ_1 b \succ_1 c$, $a \succ_2 b \succ_2 c$, and $b \succ_3 a \succ_3 c$, where $x \succ_i y$ denotes agent i strictly preferring x to y.

The deterministic efficient assignments are given in table 1. In fact, in matrices I and II player 3 is assigned to c, his least preferred alternative, and one of players 1 and 2 get his favorite alternative a while the other consumes his second best, alternative b. Therefore, one player is given his most favorite one his second best and one his worst. On the other hand, in permutation matrices III and IV two players are achieving their first best while one player has to bear his least preferred alternative.

When the society values the number of agents allocated to their higher ranked alternatives, the dismissal of the efficient matrices I and II can be justified on grounds of an "aggregate" efficiency notion. Consequently, both III and IV can be labeled as aggregate efficient deterministic assignments because there are no other permutation matrices that beat them on grounds of this efficiency notion. Moreover, when one extends this analysis to random assignment settings, this notion implies that no strictly positive weights should be given to permutation matrices I and II. Indeed, in this example any convex combination of III and IV would be aggregate efficient.[4]

On the other hand, when one employs the RP rule and/or the PS mechanism the resulting random assignments coincide and are given by the table on the left hand side of table 2. It

[4]In general, convex combinations of aggregate efficient deterministic assignments are not necessarily aggregate efficient.

$N\backslash A$	a	b	c
1	1/2	1/6	1/3
2	1/2	1/6	1/3
3	0	2/3	1/3

$N\backslash A$	a	b	c
1	1/2	0	1/2
2	1/2	0	1/2
3	0	1	0

Table 2: Two random assignments.

should be pointed out that under the RP and PS mechanisms the permutation matrices I and II are realized with a probability of 1/6 each and III and IV with a probability of 1/3 each. Hence, the expected number of agents ranked into their top choices is 5/3 and the top two 2 and, naturally, the top three 3. Noticing that the same figures are given by 2,2, and 3 for the aggregate efficient random assignment, this example establishes that both the RP and PS are not aggregate efficient as they are aggregate stochastic dominated. This follows from $(2, 2, 3) \geq (5/3, 2, 3)$ and $(2, 2, 3) \neq (5/3, 2, 3)$. The same example also shows that there are ordinally efficient random assignments, the one given by RP and PS, which are not aggregate efficient.

After proving the existence of an aggregate efficient random assignment, we show that the set of aggregate efficient random assignments is a subset of the set of ordinally efficient random allotments and that aggregate efficient random assignments are decomposed only to aggregate efficient permutation matrices. These establish that in any realized state of the world the outcome of an aggregate efficient random assignment must be not only be efficient but also aggregate efficient. Furthermore, ex–ante efficiency and aggregate efficiency are not logically related, i.e. these two notions of efficiency do not have any containment relations between each other. In general, there are von Neumann–Morgenstern utility profiles for which the first of two ordinally efficient random allotments is aggregate efficient and not ex–ante efficient and the second ex–ante efficient but not aggregate efficient.[5] On the other hand, it needs to be mentioned that using McLennan (2002) and our result that aggregate efficiency implies ordinal efficiency it can be concluded that for every aggregate efficient random assignment there exists a profile of von Neumann–Morgenstern utilities with which

[5]We refer the reader to example 1 in the proof of Theorem 4 which is obtained from the above example by a particular choice of von Neumann–Morgenstern utilities.

that particular random assignment is ex–ante efficient.

These findings, naturally, makes one wonder about aggregate efficient and strategyproof mechanisms. Yet one should not forget Gale's conjecture about the incompatibility of efficiency and strategyproofness. It is useful to remind the reader that considering deterministic environments Zhou (1990) proves that efficiency and strategyproofness cannot be simultaneously satisfied by a mechanism treating equals equally. BM extends this result to random assignment problems and prove that there is no mechanism treating equals equally and satisfying ordinal efficiency and strategyproofness. Thus, the mechanism they propose, the PS mechanism, being weak strategyproof is of significance.

In the current study we show that Gale's conjecture takes a new form: We prove that there is no mechanism treating equals equally and satisfying aggregate efficiency and weak strategyproofness. Moreover, another impossibility result involves a weaker notion of envy–freeness and a stronger efficiency concept: There is no mechanism satisfying aggregate efficiency and weak envy–freeness.

While these results ensure that the search for an aggregate efficient and weak strategyproof mechanism satisfying the equal treatment property is futile, they do not rule out the possibility of an improvement upon the PS mechanism in terms of the notion of aggregate efficiency. Indeed, it turns out that a relevant and interesting observation can be found in Turkey in the context of the random assignment mechanism used in the allotment of new doctors to their specific places of duty. The Turkish lawmaker decrees that the following principle has to be obeyed: (1) whenever a new doctor is the only one ranking a place of duty as the highest, then he is allocated that particular place of duty; and (2) if there are more than one new doctors ranking a particular place of duty as their highest, then one of them is selected with a random draw.[6] This requirement, which we name *condition T*, results in the bistochastic matrix on the right hand side of table 2.

The above example establishes that the RP and PS do not satisfy condition T and are both not aggregate efficient. Meanwhile, it also shows that there are ordinally efficient random assignments that are not aggregate efficient.[7] While condition T produced an aggregate

[6] We refer the reader to the Official Journal of Republic of Turkey 16 November 1996 issue number 22819.

[7] Considering the example given in BM (Bogomolnaia and Moulin 2001, p.298), one can easily show that the resulting random assignments of the PS and the aggregate efficiency coincide while both are different

efficient allocation in this example, in general we also show that there are situations in which there exists an ordinally efficient random allocation satisfying condition T but not aggregate efficiency, and there is an aggregate efficient random allotment that do not satisfy condition T.[8]

On the other hand, imposing condition T on the PS mechanism produces a weak strategyproof rule that is weak envy-free and outperforms the PS mechanism in terms of aggregate efficiency: the R1 mechanism. We prove that this mechanism aggregate stochastic dominates the PS mechanism and preserves all of the important properties of the PS mechanism with the exception of envy–freeness: The R1 mechanism is weak strategyproof and ordinally efficient and weak envy–free (but not envy–free).

Imposing condition T in the characterization of the R1 mechanism involves the modification of two axioms of a recent and important study, Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming) (HHKKU, hereafter). These two axioms, ordinal fairness and non-wastefulness, fully characterize the PS mechanism. As elegantly put by some of these authors in the working paper version of this study (Kesten, Kurino, and Unver 2011), *ordinal fairness* follows "whenever an agent is assigned some object with positive probability, his surplus at this object is no greater than that of any other agent at the same object"; and *non-wastefulness* whenever "the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object".[9]

The current study provides a full characterization of the R1 mechanism by employing versions of these axioms modified to make them satisfy condition T. Indeed, the imposition of condition T on the PS is obtained as follows: HHKKU's axiom concerning efficiency (non-wastefulness) is modified to satisfy the first part of condition T and their fairness axiom (ordinal fairness) is updated by the second part. Consequently, our axioms are T-ordinal fairness and T-non-wastefulness are obtained. A random assignment is *T-ordinally fair* if each favorite object has to be assigned with equal probabilities to agents preferring it as the first choice, and whenever an agent is assigned with positive probability some object

from the outcome of the RP.

[8]See the first example in the proof of Theorem 5.

[9]The surplus of an agent in a random allotment for a given object is the cumulative probability that he is assigned an alternative at least as good as the current one.

that is not a favorite by any one of the agents then his surplus at this object is no greater than that of any other agent at the same object. On the other hand, a random assignment is *T-non-wasteful* if each one of the favorite alternatives are fully assigned to those agents preferring it as their first choice, and the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object.

Why not R2? Naturally this is a relevant follow-up question. That is why not allow agents to have two reservations, not just one. We prove that doing so eliminates weak strategyproofness a key property that we do not wish to sacrifice.

A recent and independent study, Featherstone (2011), was brought to our attention when the final draft of this paper was being prepared.[10] It deserves special emphasis. We should point out that that study is also concerned with aggregate efficiency (which it refers to as the *rank efficiency*) and some important parts of our results involving the analysis of aggregate efficiency are common. On the other hand, the two papers differ extensively after developing this efficiency notion. We restrict attention to the identification and characterization of a tangible weakly strategyproof and ordinally efficient mechanism with better aggregate efficiency performances than the PS mechanism (while not completely giving up envy–freeness). On the other hand, Featherstone (2011) analyzes and characterizes aggregate efficient mechanisms (at the expense of weak strategyproofness) and concentrates on special cases given by low information environments, like the ones given in Roth and Rothblum (1999). Moreover, an empirical analysis about costs of strategyproofness is considered and it is established that "it would be a mistake to not at least consider using a rank efficient mechanism" even at the expense of strategyproofness.

3 The Model

Let A be a finite set of indivisible objects and $N = \{1, 2, ..., n\}$ be a finite set of agents, with the requirement that $|A| \geq |N|$. A *random assignment* (alternatively, an allocation) $P = [p_{ia}]_{i \in N, a \in A}$ is a matrix where $p_{ia} \in [0, 1]$ denotes the probability of agent i being allocated an object a. $\sum_{a \in A} p_{ia} = 1$ and $\sum_{i \in N} p_{ia} \leq 1$. Let the set of all random assignments

[10] We thank Umut Mert Dur in that regard.

be denoted by $\mathcal{P}$. On the other hand, a *preference profile* is denoted with $\succ \equiv (\succ_i)_{i \in N}$, where $\succ_i$ is the strict preference relation of agent i on A. Let $\succeq_i$ denote the weak preference relation induced by $\succ_i$. We assume that preferences are linear orders, i.e., for all $a, b \in A$, $a \succeq_i b \Leftrightarrow a = b$ or $a \succ_i b$. We denote the set of all such preference relations of agent i by Π_i, and the set of all such preference profiles by Π. For any $\succ$ in Π, we define *favorite alternatives* as $F(\succ) = \{a \in A | \exists i \in N : a \succeq_i b, \forall b \in A\}$. For all agents $i \in N$, define *most preferred alternative of agent i* as $F_i(\succ) = \{a \in A | a \succeq_i b, \forall b \in A\}$ and define *the set of agents preferring alternative a as their first choices*, $F_a(\succ) = \{i \in N | a \succeq_i b, \forall b \in A\}$. Given a preference profile $\succ$ in Π, define the *weak upper contour set* of agent $i \in N$ at object $a \in A$ by $U(a, \succ_i) = \{b \in A : b \succeq_i a\}$ and given $P \in \mathcal{P}$ let $U(a, P, \succ_i) = \sum_{b \succeq_i a} P_{ib}$ denote the *surplus of agent i at a under P*, i.e. the probability that i is assigned an object at least as good as a under P_i.

Next, we define *ex-ante efficiency* and *ex-post efficiency*: Let $(u_i)_{i \in N}$ be a profile of von Neumann-Morgenstern utility functions, where each individual one is a real valued function on A and the corresponding preferences over $\mathcal{P}$ is obtained by the comparison of expected utilities where $u_i(P_i) = \sum_{a \in A} p_{ia} u_i(a)$. Given a profile of preferences $\succ$ in Π and an associated profile of von Neumann-Morgenstern utilities $u = (u_i)_{i \in N}$, we say that a random assignment $P \in \mathcal{P}$ is (1) ex-ante efficient at u if and only if P is Pareto optimal in $\mathcal{P}$ at u; and (2) ex-post efficient at $\succ$ whenever its decomposition involves only efficient deterministic assignments.

Given two allocations P and Q, we say that P *stochastically dominates Q for agent i*, and denote it by $P_i \succ_i^{sd} Q_i$, if and only if $U(a, P_i, \succ_i) \geq U(a, Q_i, \succ_i)$ for all $a \in A$. Moreover, P *stochastically dominates* Q if and only if $P_i \succ_i^{sd} Q_i$ for all $i \in N$. Furthermore, given preference profile $\succ$ in Π, a random assignment $P \in \mathcal{P}$ is said to be *ordinally efficient* if and only if for any given $P' \in \mathcal{P}$, $P' \succ^{sd} P$ implies $P' = P$.

We say that an allocation $P \in \mathcal{P}$ is *envy-free* for a given preference profile $\succ$ if and only if we have that for all $i, j \in N$, $P_i \succ_i^{sd} P_j$. Moreover, it is *weakly envy-free* if and only if $P_j \succ_i^{sd} P_i$ implies $P_i = P_j$.

A *mechanism* is a function mapping preference profiles to random assignments. Given a mechanism $\varphi : \Pi \to \mathcal{P}$, we say φ is *strategy-proof* if for all $\succ$ in Π and for all $i \in N$ we have

$\varphi_i(\succ) \succ_i^{sd} \varphi_i(\succ_i', \succ_{-i})$ for all $\succ_i'$ in Π_i. Furthermore, φ is *weakly strategy-proof* if for all $\succ$ in Π and for all $i \in N$, $\varphi_i(\succ_i', \succ_{-i}) \succ_i^{sd} \varphi_i(\succ)$ implies $\varphi_i(\succ_i', \succ_{-i}) = \varphi_i(\succ)$ for all $\succ_i'$ in Π_i.

Next we introduce *aggregate efficiency*: For a given preference profile $\succ$ in Π, define r_{ik} as the most preferred k objects in A by agent $i \in N$. Moreover, for a given random assignment $P \in \mathcal{P}$ let an aggregate efficiency vector be defined by $w^P = (w_1^P, ..., w_{|A|}^P)$ in $\mathbb{R}^{|A|}$ where $w_k^P = \sum_{i \in N} \sum_{a \in r_{ik}} P_{ia}$. Consequently, given a preference profile $\succ$ in Π, we say that a random assignment P *aggregate stochastically dominates* Q, if $w^P \geq w^Q$, and we denote this by $P \succ^{asd} Q$. Finally, given a preference profile $\succ$ in Π, a random assignment $P \in \mathcal{P}$ is *aggregate efficient* whenever $P' \succ^{asd} P$ for some $P' \in \mathcal{P}$ implies $w^P = w^{P'}$.

Moreover, aggregate stochastic domination between mechanisms is defined as follows: A *mechanism* $\varphi : \Pi \to \mathcal{P}$ *aggregate stochastically dominates another mechanism* $\varphi' : \Pi \to \mathcal{P}$, if for all $\succ$ in Π we have $\varphi'(\succ) \succ^{asd} \varphi(\succ)$ implies $w^{\varphi'(\succ)} = w^{\varphi(\succ)}$, and there exist $\succ^*$ such that $\varphi(\succ^*) \succ^{asd} \varphi'(\succ^*)$ and $w^{\varphi'(\succ^*)} \neq w^{\varphi(\succ^*)}$.

4 Aggregate Efficiency and Impossibility Results

Due to BM it is well known that ex-ante efficiency implies ordinal efficiency which in turn implies ex-post efficiency, demanding that every possible realization of the random assignment has to be an efficient deterministic assignment. On the other hand, McLennan (2002) establishes that if a random assignment is ordinally efficient then there exists a profile of von Neumann-Morgenstern utilities for which this random assignment is ex-ante efficient.

After handling the existence question in Theorem 1, we prove that every aggregate efficient random assignment has to be ordinally efficient, (hence, ex-post efficient), and there are ordinally efficient random assignments that are not aggregate efficient (Theorem 3). Moreover, we also show that an aggregate efficient random assignment resolves only into aggregate efficient deterministic assignments (Theorem 2). Therefore, combining these two results formally establishes that every possible realization of an aggregate efficient random assignment can involve only an aggregate efficient deterministic assignment. While aggregate efficiency has these useful properties, an interesting finding emerges when considering the relation of ex-ante efficiency with aggregate efficiency: Ex-ante efficiency does not imply aggregate efficiency (Theorem 4). On the other hand, due to Theorem 3 and McLennan (2002), we know that if a random assignment is aggregate efficient then there exists a profile of von Neumann-Morgenstern utilities for which this random assignment is ex-ante efficient.

Next, we consider the relation between strategyproofness and aggregate efficiency. Zhou (1990) proves Gale's conjecture about the incompatibility of Pareto efficiency and strategyproofness in one–sided deterministic matching problems: "When there are n objects to be assigned to n agents, for $n \geq 3$, there exits no mechanism that satisfies symmetry (equal treatment of equals), Pareto optimality, and strategyproofness." Moreover, BM shows that this incompatibility arises in random allocation problems as a tradeoff between ordinal efficiency and strategyproofness: They show that there is no mechanism treating equals equally which satisfies ordinal efficiency and strategyproofness. We, therefore, ask whether or not similar conclusions hold with the stronger efficiency concept introduced in the current study. Indeed, in Theorem 5 we show that the inevitable trade–off between efficiency and strategyproofness concepts (when attention is restricted to mechanisms treating equals equally) prevails: Aggregate efficiency and weak–strategy proofness are incompatible with the equal treatment property. In other words, when one strengthens the efficiency notion and weakens the strategyproofness concept, there are no changes regarding this impossibility result. This, in turn, points to an updated version of the classic trade–off between efficiency and strategyproofness, this time between aggregate efficiency and weak strategyproofness. Moreover, Theorem 6 points to another impossibility: Aggregate efficiency and the property of weak envy-free are not compatible.

The existence of aggregate efficient random assignments follows from the Theorem 1 which is presented without a proof. This is because the result is an immediate consequence of the acyclicity of the order on random assignments defined in $\mathbb{R}^{|A|}$ and compactness of $\mathcal{P}$, the set of all random assignments on A.

Theorem 1 *Given any preference profile $\succ$ in Π, there exist an aggregate efficient random assignment P in $\mathcal{P}$.*

Theorem 1 also establishes the existence of an aggregate efficient mechanism: For every given preference structure, using Theorem 1 one can simply construct an aggregate efficient mechanism by picking an aggregate efficient allocation for each possible preference structure.

The following 3 Theorems present the results discussed above.

Theorem 2 *An aggregate efficient random assignment can only be decomposed into aggregate efficient permutation matrices.*

Proof. The Von-Neumann Birkhoff Theorem tells us that a matrix is identifying a random assignment if and only if it can be written as a convex combination of permutation matrices. Therefore, any aggregate efficient random assignment $P \in \mathcal{P}$, can be written as a convex combination of permutation matrices $(P_1, P_2, ..., P_k)$, as $P = \sum_{\ell=1}^{k} \lambda_\ell P_\ell$. We claim that all these permutation matrices have to be aggregate efficient. Suppose not, then there exists a permutation matrix P_c for some $c \in \{1, 2, ..., k\}$ that is not aggregate efficient; so, P'_c aggregated stochastically dominates P_c. Hence, P' defined through $(P_1, P_2, ..., P'_c, ..., P_k)$ as the following convex combination $\lambda_c P'_c + \sum_{\ell \neq c} \lambda_\ell P_\ell$ aggregate stochastically dominates P. Consequently, P is not aggregate efficient. ■

Theorem 3 *The set of aggregate efficient random assignments is a subset of the set of ordinally efficient random assignments. Moreover, this containment relation may be strict.*

Proof. For any $P \in \mathcal{P}$ that is not ordinally efficient, it must be that there exists $P' \neq P$ and $i \in N$ such that $P_i \neq P'_i$ and $P'_i \succ_i^{sd} P_i$, which is if and only if $\sum_{b \succeq_i a} P'_{ib} \geq \sum_{b \succeq_i a} P_{ib}$ for all $a \in A$ and there exists $a' \in A$ such that this inequality holds strictly. Hence, $\sum_{a \in r_{jk}} P'_{ja} \geq \sum_{a \in r_{jk}} P_{ja}$, for all $k \leq |A|$ and for all $j \in N$, yet there exists $m \leq |A|$ such that this inequality holds strictly for agent $i \in N$, because otherwise $P' = P$. Therefore, $\sum_{j \in N} \sum_{a \in r_{jk}} P'_{ja} \geq \sum_{j \in N} \sum_{a \in r_{jk}} P_{ja}$ for all k and this inequality holds strictly for $k = m$. Hence, P' aggregated stochastically dominates P, implying that P is not aggregate efficient.

In order to see the second part, consider the example supplied both in the introduction and as 2 in the proof of Theorem 9. ■

Theorem 4 *Given any preference profile $\succ$ in Π and associated von Neumann-Morgenstern utility profile $u = (u_i)_{i \in N}$, an ex-ante efficient random assignment does not need to be aggregate efficient. Moreover, for every aggregate efficient random assignment P in $\mathcal{P}$, there exists a von Neumann-Morgenstern utility profile $\tilde{u} = (\tilde{u}_i)_{i \in N}$ such that P is ex-ante efficient at $\tilde{u}$.*

Proof. The first part is due to the following example which is a cardinal version of the example supplied both in section 2 and as Example 2 in the proof of Theorem 9:

$W(\succ)$	a	b	c	d
1	0.5	0	0.5	0
2	0.5	0	0.5	0
3	0	1	0	0
4	0	0	0	1

$W(\succ')$	a	b	c	d
1	0.5	0	0.5	0
2	0.5	0	0.5	0
3	0	0.5	0	0.5
4	0	0.5	0	0.5

Table 3: Aggregate efficient allocations for $\succ$ and $\succ'$.

Example 1 *Let* $N = \{1, 2, 3\}$, $A = \{a, b, c\}$ *and* $a \succ_1 b \succ_1 c$, $a \succ_2 b \succ_2 c$, $b \succ_3 a \succ_3 c$. *Let* $u_i = (u_{ia}, u_{ib}, u_{ic})$ *be the vector which denotes the utilities of agent* $i \in \{1, 2, 3\}$ *from getting objects* $\{a, b, c\}$ *respectively. Suppose* $u_1 = u_2 = (10, 8, 1)$, *and* $u_3 = (8, 10, 6)$.

Then any aggregate efficient allocation P *can be denoted by a number* $x \in [0, 1]$ *such that* $P_1(\succ) = (x, 0, 1 - x)$, $P_2(\succ) = (1 - x, 0, x)$ *and* $P_3(\succ) = (0, 1, 0)$. *So, the sum of expected utilities of agents under any aggregate efficient allocation is equal to* 21. *However, let* R *be an allocation such that* $R_1 = (1, 0, 0)$, $R_2 = (0, 1, 0)$ *and* $R_3 = (0, 0, 1)$. *Then, sum of expected utilities of agents under* R *is equal to* 24.

The second part of the Theorem follows from Theorem 3 and McLennan (2002). ■

The following Theorem establishes the incompatibility of aggregate efficiency with weak strategyproofness and equal treatment of equals:

Theorem 5 *Suppose that* $N \geq 4$. *Then, there is no mechanism treating equals equally and satisfies aggregate efficiency and weak strategyproofness.*

Proof. Consider the following example: $N = \{1, 2, 3, 4\}$ and $A = \{a, b, c, d\}$ where $a \succ_i c \succ_i b \succ_i d$, for $i = 1, 2$, and $b \succ_3 c \succ_3 d \succ_3 a$ and $b \succ_4 d \succ_4 c \succ_4 a$. The unique aggregate efficient allocation satisfying the equal treatment property is given in first table in table 3.

Now, consider $\succ'$ where $\succ'_i = \succ_i$ for $i \in \{1, 2, 3\}$ and $\succ'_4 = \succ_3$. That is to say, $\succ'$ is preference structure constructed with the deviation of fourth agent to third agents' preferences. Then a mechanism satisfying aggregate efficiency and equal treatment of equals should assign the allocation on the right. This deviation is profitable for player 4 because it results in an ordinally better allocation for the fourth agent when compared with stating his true preference.

This example can be embedded into other problems with more than four agents as follows: Suppose there are n agents and n objects. Let, first four agents' are called old agents and their first four preferences are exactly like the above example and the order of preferences of other objects are not important for old agents. Also, any new object is declared as a first choice by exactly one new agent. Then a mechanism satisfying welfare efficiency and equal treatment of equals should assign old agents exactly the allocation given above and assign each new agent his most preferred object. Hence, the same deviation is still profitable for fourth agent as in example above. ■

In the following Theorem, we establish that aggregate efficiency and the property of weak envy-free are not compatible without the need to employ the equal treatment property:

Theorem 6 *Suppose that $N \geq 4$. Then, there is no mechanism satisfying aggregate efficiency and weak envy-freeness.*

Proof. Consider the example in the proof of Theorem 5 and notice that any aggregate efficient mechanism must assign object b to the third agent and object d to the fourth. Then clearly forth agent strictly envies the third. As discussed in the proof of Theorem 5, this example can be generalized to other problems with more agents and more alternatives. ■

5 The R1 Mechanism

Theorem 5 tells us that there is no mechanism treating equals equally and satisfying weak strategyproofness and aggregate efficiency. On the other hand, BM shows that the PS mechanism satisfies ordinal efficiency and weak strategyproofness and equal treatment of equals (while it is shown not to be strategyproof). Therefore, the PS mechanism cannot satisfy aggregate efficiency whenever $N \geq 4$.

Following the same of thinking as in BM, it is a plausible question to ask whether or not the PS mechanism can be beaten on grounds of aggregate efficiency when one restricts attention to weakly strategyproof mechanisms treating equals equally. It turns out that the answer to this question is positive.

Theorems 8 and 9 establish that there exists a mechanism, the R1, treating equals equally and satisfying weak strategyproofness while displaying a better performance with respect to aggregate efficiency. Moreover, this mechanism is weakly envy-free.

The *R1 mechanism* is defined via the following algorithm: Given a problem $(\succ, A)$, each alternative $a \in A$ is interpreted as infinitely divisible with total supply of 1 unit. Agents can eat one object at a time and the eating speed values of agents are all equal. On the other hand, a distinct feature is that each agent has a reservation right for his most preferred object. That is, each agent $i \in N$ starts to eat away from $F_i(\succ)$ until it is depleted. When agent i's best alternative is exhausted, then he starts to eat from their most preferred object in $A \setminus F(\succ)$, until that one is also depleted. Then, he continues with the next best among the nondepleted ones in $A \setminus F(\succ)$. That is, once $F(\succ)$ is depleted, the R1 mechanism behaves exactly as the PS mechanism. The algorithm terminates when each agent has eaten exactly 1 total unit of objects. The allocation of an agent i by R1 is then given by the amount of each object he has eaten until the algorithm terminates. Let $R1(\succ) \in \mathcal{P}$ denote the random assignment obtained as a result of R1 for a preference profile given by $\succ$ in Π.

A natural follow-up questions is about why we are not allowing agents to have two reservations. In fact, why not R2? The interesting finding is that, doing so eliminates weak strategyproofness. Hence, clearly this provides sufficiently strong reasons for the dismissal of the R2 mechanisms. The formal execution is in Appendix B.

A further interesting and motivating observation emerges when one considers the characterization of the R1 mechanism: It is nothing but the PS mechanism modified to satisfy a principle set forth by the Turkish parliament. This principle, which we call *condition T* is outlined in the Official Journal of Republic of Turkey 16 November 1996 issue number 22819, and it decrees that: (1) whenever a new doctor (an agent) is the only one ranking a place of duty (an alternative) as the highest, then he is allocated that particular place of duty; and (2) if there are more than one new doctors ranking a particular place of duty as their highest, then one of them is selected with a random draw.

In a recent and important study, Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming) provides a full characterization for PS mechanism by using only two axioms, one related to fairness and the other one to efficiency. These are namely ordinal fairness and non-wastefulness. Ordinal fairness follows "whenever an agent is assigned some object with positive probability, his surplus at this object is no greater than that of any other agent at the same object"; and non-wastefulness whenever "the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object".

Theorem 7 of the current study provides a full characterization of the R1 mechanism which employs versions of these axioms modified to make them satisfy condition T. These axioms are T-ordinal fairness and T-non-wastefulness:

Definition 1 *Given $\succ$ in Π, a random assignment $P \in \mathcal{P}$ is T-non-wasteful at $\succ$ if $\sum_{i \in F_a(\succ)} P_{i,a} = 1$ for all $a \in F(\succ)$; and, for all $i \in N$ and for all $a \in A$ such that $P_{i,a} > 0$ we have $\sum_{j \in N} P_{j,b} = 1$ for all $b \in A$ with $b \succ_i a$.*

Definition 2 *Given $\succ$ in Π, a random assignment $P \in \mathcal{P}$ is T-ordinally fair at $\succ$ if $a = F_i(\succ)$ with $i \in N$ and $a \in A$ implies $U(a, P, \succ_i) \leq U(a, P, \succ_j)$ for all $j \in F_a(\succ)$, and for all $i, j \in N$ and for all alternatives $a \neq F_i(\succ)$ with $P_{i,a} > 0$ it must be that $U(a, P, \succ_i)) \leq U(a, P, \succ_j)$.*

The next Theorem the proof of which is deferred to the Appendix, renders a full characterization of the R1 mechanism:

Theorem 7 *A mechanism is T-ordinally fair and T-non-wasteful if and only if it is R1.*

In what follows, we provide some important properties of the R1 mechanism. In fact, it is useful to point out that under the R1 mechanism any alternative $a \in F(\succ)$ will be allocated only to agents in $F_a(\succ)$, and with equal probabilities. That is, the R1 mechanism obeys condition T while satisfying the important properties of weak strategyproofness, ordinal efficiency, and weak envy-freeness. These are stated in the following Theorem the proof of which is in the Appendix.

Theorem 8 *R1 mechanism satisfies condition T, weak strategyproofness, ordinally efficency and weak envy-freeness.*

It is important to emphasize that due to Theorem 5 we know that there exists no mechanism treating equals equally and satisfying aggregate efficiency and weak strategyproofness. Moreover, due to Theorem 8 we know that the R1 mechanism is weak strategyproof and ordinally efficient, while being weakly envy-free (which clearly implies the equal treatment property). Therefore, it is not aggregate efficient. On the other hand, recalling that the PS mechanism is weakly strategyproof and ordinally efficient and envy-free, one may wonder whether or not the "slack" created by relaxing envy-freeness to weak envy-freeness is useful for some other property. The answer is affirmative, and the property that gets strengthened concerns aggregate efficiency. To be precise, the R1 mechanism satisfies condition T, and moreover, we show that the R1 mechanism aggregate stochastically dominates the PS mechanism.

Theorem 9 *The R1 mechanism aggregate stochastic dominates the probabilistic serial mechanism.*

Proof. We prove this Theorem by showing that there is no preference profile where $PS(\succ) \succ^{asd} R1(\succ)$ and $w^{PS(\succ)} \neq w^{R1(\succ)}$, where $PS : \Pi \rightarrow \mathcal{P}$ denotes the PS mechanism. Moreover, using the example given in the introduction we establish that there exists $\tilde{\succ} \in \Pi$ with $R1(\tilde{\succ}) \succ^{asd} PS(\tilde{\succ})$ and $w^{PS(\tilde{\succ})} \neq w^{R1(\tilde{\succ})}$.

In order to show that there does not exist a preference profile $\succ$ in Π such that $PS(\succ) \succ^{asd} R1(\succ)$ and $w^{PS(\succ)} \neq w^{R1(\succ)}$, suppose (for a contradiction) that there exists a $\succ^*$ for which $PS(\succ^*) \succ^{asd} R1(\succ^*)$. By the definition of aggregate stochastic domination, there should be a strict difference between the allocations, therefore $PS(\succ^*) \neq R1(\succ^*)$. Note that if $PS(\succ^*)_{ia} = R1(\succ^*)_{ia}$ for all $a \in F(\succ^*)$, then $PS(\succ^*) = R1(\succ^*)$, since the R1 algorithm proceeds exactly the same as PS after the favorite alternatives are allocated (i.e. R1 behaves the same as PS for all $a \notin F(\succ^*)$). Therefore, there exist $i \in N$ and $a = F_i(\succ^*)$ such that $PS(\succ^*)_{ia} \neq R1(\succ^*)_{ia}$. Hence, by the defining property of R1, $PS(\succ^*)_{ia} \neq R1(\succ^*)_{ia}$ implies $R1(\succ^*)_{iF_i(\succ^*)} > PS(\succ^*)_{iF_i(\succ^*)}$ (since the reservation right can lead only to an increase in the allocation of a good to an agent who prefers it as his favorite object), so $\sum_{i=1}^{N} PS(\succ^*)_{i,F_i(\succ^*)} < \sum_{i=1}^{N} R1(\succ^*)_{i,F_i(\succ^*)}$ and $PS(\succ^*) \not\succ^{asd} R1(\succ^*)$, delivering the desired contradiction.

Example 2 *Let* $N = \{1, 2, 3\}$, $A = \{a, b, c\}$ *and* $a \succ_1 b \succ_1 c$, $a \succ_2 b \succ_2 c$, $b \succ_3 a \succ_3 c$. *Then* $R1_1(\succ) = R1_2(\succ) = [1/2, 0, 1/2]$ *and* $R1_3(\succ) = [1, 0, 0]$. *Where as* $PS_1(\succ) = PS_2(\succ) = [1/2, 1/6, 1/3]$ *and* $PS_3(\succ) = [2/3, 0, 1/3]$. *When we compare these two allocation,* $R1$ *allocates* 2 *alternatives to agents viewing them as the first choice preferences and* 1 *alternative to an agent ranking it the least while* PS *distributes one to each. That is to say,* $w^{R1(\succ)} = (2, 2, 3)$ *and* $w^{PS(\succ)} = (1, 2, 3)$.

This finishes the proof of Theorem 9. ■

A Proofs

A.1 Proof of Theorem 7

First, we propose an eating algorithm that can be used for any allocation at any preference structure $\succ$ in Π, which will be key to the proof.

Fix a preference structure $\succ$. Then any allocation $P(\succ)$ can be simulated by an eating function defined as follows:

Think each object as an infinitely divisible good with a quota 1. Each agent eats away from his most preferred object among the objects that are assigned him with a positive probability. When an agent eats his assigned probability from some object, he starts to eat away from his next preferred object that is assigned him with a positive probability. Each agent eats with a same speed until the algorithm ends at time 1 when each agent has eaten exactly 1 total unit of objects. Therefore we define the eating function $f : [0, 1) \times N \to A$ such that for all $f^P_\succ(t, i) = \{a \in U(t) | a \succeq_i b, \forall b \in U(t)\}$ where $U(t) = \{a \in A : U(a, P, \succ_i) > t\}$. This function identifies the object that agent i eats at time t.

Proof. First we will show that there exist an allocation which satisfies T-ordinal fairness and aggregate non-wastefulness for any preference $\succ$. In particular, we will show that it is $R1(\succ)$.

For all $a \in F(\succ)$, $R1(\succ)_{ia} = 1/|F_a(\succ)|$ for all $i \in F_a(\succ)$. Therefore it must be that $\sum_{i \in F_a(\succ)} R1(\succ)_{ia} = 1$ for all $a \in F(\succ)$. Then, let i be any player in N and a, b be any objects in A such that $P_{i,a} > 0$ and $b \succ_i a$. Then, from the ordinal efficiency of $R1(\succ)$ (due to Theorem 8), $\sum_{j \in N} P_{j,b} = 1$. Hence, $R1(\succ)$ is aggregate non-wasteful.

For all $i \in N$ and $a \in A$ such that $a = F_i(\succ)$, we have $U(a, P, \succ_i) \leq U(a, P, \succ_j)$ for all $j \in F_a(\succ)$ since $R1(\succ)_{ja} = R1(\succ)_{ia} = 1/|F_a(\succ)|$ (by aggregate non-wastefulness of $R1(\succ)$). Now, we have to show that for each object $a \notin F(\succ)$ and all $i, j \in N$ with $P_{ia} > 0$, we have $U(a, P, \succ_i) \leq U(a, P, \succ_j)$. Suppose not, then there exists $t^* \in [0, U(a, P, \succ_i))$ such that $a \succ_j f^{R1}_\succ(t^*, j)$. However, this means that agent j has eaten from an object less preferred to a while the unfavorite object a is not exhausted, contradicting with $R1$ mechanism. Therefore $R1(\succ)$ is T-ordinally fair.

We have shown that there exists an aggregate non-wasteful and T-ordinally fair allocation for every preference structure. Now we will show that there is no other allocation than $R1$ satisfying these two properties by showing that if there is an allocation satisfying these properties, it should be characterized by the same eating function with $R1$.

Fix a preference profile $\succ$, and let $P \in \mathcal{P}$ be any T-ordinally fair and aggregate non-wasteful allocation at $\succ$. We will show that $f^P_\succ(t,i) = f^{R1}_\succ(t,i)$ for all $t \in [0,1)$ and $i \in N$. Suppose not, then there exists a time, say t^*, where the eating functions proceed same until t^*, but starts to differ at t^*. Formally, $t^* = \max\{t \in [0,1) : f^P_\succ(t',i) = f^{R1}_\succ(t',i)$ for all $t' < t$ for all $i \in N\}$. Then there exist an agent i such that $f^P_\succ(t^*,i) \neq f^{R1}_\succ(t^*,i)$. Let $f^{R1}_\succ(t^*,i) = a$ and $f^P_\succ(t^*,i) = b$.

Aggregate non-wastefulness restricts that all favorite goods has to be assigned completely, $\sum_{i \in F_a(\succ)} P_{ia} = 1$, and must be assigned equally among the agents who prefers that object as his favorite good, $P_{iF_i(\succ)} = 1/|F_{F_i(\succ)}(\succ)|$ follows from the definition of T-ordinal fairness. Therefore, the eating functions should be exactly same for favorite goods, hence a and b cannot be in $F(\succ)$. Note that, the eating functions are same until t^*, therefore with the fact that b is not exhausted at t^* and agent i eats from a in $R1$ at t^* where $b \notin F(\succ)$, $a \succ_i b$. Then, there are two cases to consider: (i) There exist $j \in N$ and $t > t^*$ such that $f^P_\succ(t,j) = a$. Then $U(a,P,\succ_j) \geq t > t^* \geq U(a,P,\succ_i)$ contradicting with T-ordinal fairness of P, since $a \notin F(\succ)$ and $P_{ja} > 0$. (ii) Otherwise, $\sum_{k \in N} P_{ka} < \sum_{k \in N} R1_{ka} = 1$ and $P_{ib} > 0$ contradicts with aggregate non-wastefulness of P.

Therefore there exist not such $t^* \in [0,1)$ and hence $f^P_\succ(t,i) = f^{R1}_\succ(t,i)$ for all $t \in [0,1)$ implying $P = R1(\succ)$. ■

A.2 Proof of Theorem 8

In the eating algorithm given in proof of Theorem 7, define $t^\succ(a)$ to be time at which object a is exhausted at preference $\succ \in \Pi$ (under $R1$ mechanism), i.e.

$$t^\succ(a) = \begin{cases} \sup\{t \in [0,1) : f^{R1}_\succ(t,i) = a\} & \text{if } \sum_i R1(\succ)_{ia} = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Claim 1 *R1 mechanism satisfies ordinal efficiency.*

Proof. Fix a preference profile $\succ$. Suppose $R1(\succ)$ is not ordinally efficient. Then there exists an allocation $Q \neq R1(\succ)$ such that $Q \succ^{sd} R1(\succ)$. First of all, we claim that there do not exist an agent $i \in N$ and objects $a, b \in A$ such that $R1(\succ)_{ib} > 0$ and $a \succ_i b$, $\sum_{j \in N} R1(\succ)_{ja} < 1$. If otherwise, a cannot be in $F(\succ)$ since R1 mechanism assigns favourable objects completely. On the other hand, object a must be in $F(\succ)$ since agent i starts to eat away from object b when object a is available in the simultaneous eating algorithm for R1 mechanism. End of claim.

$Q \succ^{sd} R1(\succ)$ and $Q \neq R1(\succ)$ implies that there exist an agent $i_1 \in N$ such that $Q_{i_1} \succ_{i_1}^{sd} R1(\succ)_{i_1}$ and $Q_{i_1} \neq R1(\succ)_{i_1}$. Then, there are objects $a_1, a_2 \in A$ such that $a_2 \succ_{i_1} a_1$ and $Q_{i_1 a_2} > R1(\succ)_{i_1 a_2}$ and $Q_{i_1 a_1} < R1(\succ)_{i_1 a_1}$. Since, $\sum_{j \in N} R1(\succ)_{ja_2} = 1$, there exist an agent $i_2 \neq i_1$ such that $Q_{i_2 a_2} < R1(\succ)_{i_2 a_2}$. Since $Q_{i_2} \succ^{sd} R1(\succ)_{i_2}$ and $Q_{i_2 a_2} < R1(\succ)_{i_2 a_2}$, then there exist an object $a_3 \in A$ such that $Q_{i_2 a_3} > R1(\succ)_{i_2 a_3}$. Hence, we can successively define sets $\{i_1, i_2, .., i_n\}$ and $\{a_1, a_2, .., a_n, a_{n+1}\}$ where $a_{n+1} = a_m$ for some $m < n$.

Now consider the agents $(i_m, i_{m+1}, ..., i_n)$ and objects $(a_m, a_{m+1}, ..., a_{n+1})$ such that $a_{k+1} \succ_{i_k} a_k$ for every $m \leq k \leq n$ and $a_{n+1} = a_m$. Any object a_k is not in $F(\succ)$, since $a_{k+1} \succ_{i_k} a_k$ and $R1(\succ)_{i_k a_k} > 0$. Then $t^{\succ}(a_k) > t^{\succ}(a_{k+1})$ for all $m \leq k \leq n$, since object a_{k+1} must be unavailable when agent i_k eats away from object a_k. Therefore $t^{\succ}(a_m) > t^{\succ}(a_{m+1}) > ... > t^{\succ}(a_{n+1}) = t^{\succ}(a_m)$. Contradiction. ■

Claim 2 *R1 mechanism satisfies weak strategy proofness.*

Proof. Fix a preference structure $\succ$ and take an agent $i \in N$ and let $\succ_i$: $a_1 \succ a_2 \succ ... \succ a_n$. Suppose there exists a deviation $\succ_i'$ such that $R1_i(\succ') \succ_i^{sd} R1_i(\succ)$, where $\succ' = (\succ_i', \succ_{-i})$. We will show that $R1_i(\succ) = R1_i(\succ')$ with induction, first by showing that for any given deviation, $R1(\succ)$ and $R1(\succ')$ coincides on the interval $[0, t^{\succ}(a_1))$ in the basis step. Then we will assume that for any a_m such that $1 \leq m < n$ and $R1(\succ)_{ia_m} > 0$, eating algorithms coincide on the interval $[0, t^{\succ}(a_m))$ such that $t^{\succ}(a_m) < 1$. Set $f_{\succ}^{R1}(t^{\succ}(a_m), i) = a_l$. We will prove that eating algorithms coincide on the interval $[0, \min(t^{\succ}(a_l), 1))$, which completes this proof in an inductive manner.

Basis: Note that if $R1(\succ)_{ia_1} = 1$, then it is clear that $R1(\succ) = R1_i(\succ')$. If $R1(\succ)_{ia_1} < 1$, then there exist $j \in N \setminus \{i\}$ such that $j \in F_{a_1}(\succ)$. So if $i \notin F_{a_1}(\succ')$, then $R1(\succ')_{ia_1} =$

$0 < R1(\succ)_{ia_1}$. Contradiction. Hence, i cannot misreport his favorite object, and so $F_j(\succ') = F_j(\succ)$ for all $j \in N$. So, the eating algorithms coincide on the interval $[0, t^{\succ}(a_1))$.

Inductive Step: Suppose for some a_m such that $1 \leq m < n$ and $R1(\succ)_{ia_m} > 0$, eating algorithms coincide on the interval $[0, t^{\succ}(a_m))$ such that $t^{\succ}(a_m) < 1$. Set $f_{\succ}^{R1}(t^{\succ}(a_m), i) = a_l$, and note that $a_l \succeq b$ for all b such that $b \notin F(\succ)$ and $t^{\succ}(b) > t^{\succ}(a_m)$, that is to say a_l is the most preferred object for agent i among the available ones at $t^{\succ}(a_m)$. We will prove that eating algorithms coincide on the interval $[0, \min\{t^{\succ}(a_l), 1\})$.

Note that, we assumed that $R1_i(\succ') \succ_i^{sd} R1_i(\succ)$, so it must be true that $R1(\succ)_{ia_l} \leq R1(\succ')_{ia_l}$ and therefore $\min\{t^{\succ}(a_l), 1\} \leq \min\{t^{\succ'}(a_l), 1\}$. If there is no agent $j \in N \setminus \{i\}$ and time $t', t^{\succ}(a_m) \leq t' < t^{\succ}(a_l)$ such that $f_{\succ}^{R1}(t', j) = a_l$, then the surplus of agent i at object a_l is equal to 1, hence $R1(\succ)_i = R1(\succ')_i$. So, there exist an agent $j \in N \setminus \{i\}$, and a time $t' \in [t^{\succ}(a_m), t^{\succ}(a_l))$ such that $f_{\succ}^{R1}(t', j) = a_l$.

If there is no agent $j, j \neq i$ and a time $t' \in [t^{\succ}(a_m), t^{\succ}(a_l))$ such that $f_{\succ}^{R1}(t', j) = a_l$ and $f_{\succ'}^{R1}(t', j) = b$, where $b \neq a_l$, then there are two cases,

(i) if $t^{\succ}(a_l) < 1$, we claim $t^{\succ'}(a_l) = t^{\succ}(a_l)$. First note that $t^{\succ'}(a_l) < t^{\succ}(a_l)$, contradicts with the assumption $R1_i(\succ') \succ_i^{sd} R1_i(\succ)$. Also, if $t^{\succ'}(a_l) > t^{\succ}(a_l)$, then for all $j \in N \setminus \{i\}$ such that $R1(\succ)_{ja_l} > 0$, $R1(\succ')_{ja_l} > R1(\succ)_{ja_l}$, implying $R1(\succ')_{ia_l} < R1(\succ)_{ia_l}$ and hence $R1_i(\succ') \not\succ^{sd} R1_i(\succ)$. Contradiction. So, $t^{\succ'}(a_l) = t^{\succ}(a_l)$ and hence $f_{\succ}^{R1}(t, i) = f_{\succ'}^{R1}(t', i) = a_l$ for all t' such that $t^{\succ}(a_m) \leq t' < t^{\succ}(a_l)$ implying that eating algorithms coincide on the interval $[0, \min\{t^{\succ}(a_l), 1\})$.

(ii) if $t^{\succ}(a_l) \geq 1$, then $\sum_{b \succeq a_l} R1(\succ)_{ib} = 1$. Hence $R1(\succ') \succ^{sd} R1(\succ)$ implies $\sum_{b \succeq a_l} R1(\succ')_{ib} = 1$. Therefore, $f_{\succ}^{R1}(t, i) = f_{\succ'}^{R1}(t', i) = a_l$ for all t' such that $t^{\succ}(a_m) \leq t' < 1$ implying that eating algorithms coincide on the interval $[0, \min\{t^{\succ}(a_l), 1\})$.

We have shown that there exist an agent $j \in N \setminus \{i\}$ and a time $t' \in [t^{\succ}(a_m), t^{\succ}(a_l))$ such that $f_{\succ}^{R1}(t', j) = a_l$ and $f_{\succ'}^{R1}(t', j) = b$, where $b \neq a_l$. Then, $b \succ_j a_l$ and note that b and a_l cannot be in either $F(\succ)$ or $F(\succ')$. Hence b is not available at t' under $\succ$ but available under $\succ'$, so $t^{\succ}(b) < t^{\succ'}(b)$. Let G be the set of objects such that $g \neq a_l$ such that $t^{\succ}(g) < t^{\succ'}(g)$. Note that G is nonempty (b is in G) and let y be the object in G with minimal $t^{\succ}(y)$. Note that $t^{\succ}(a_m) \leq t^{\succ}(y) < t^{\succ}(a_l)$, since the algorithm proceeds same until $t^{\succ}(a_m)$

and $t^{\succ}(y) \leq t^{\succ}(b)$ from the definition of y and $t^{\succ}(b) < t^{\succ}(a_l)$ from $b \succ_j a_l$ and $R1(\succ)_{ja_l} > 0$.

Suppose there exist an agent k and a time $t', t' < t^{\succ}(y)$ such that $f^{R1}_{\succ}(t', k) = y$ and $f^{R1}_{\succ'}(t', k) = c$, where $c \neq y$. (Otherwise, verify that $t^{\succ}(y) = t^{\succ'}(y)$. Contradiction.) Note that $k \neq i$ since agent i eats object a_l for all t', $t^{\succ}(a_m) \leq t' < t^{\succ}(a_l)$ at $\succ$ since $t^{\succ}(y) < t^{\succ}(a_l)$.Then, $c \succ_k y$ and $t^{\succ}(c) < t' < t^{\succ}(y)$. Also, $t^{\succ}(c) < t^{\succ'}(c)$ since $c \succ_k y$ and $f^{R1}_{\succ}(t', k) = y$ and $f^{R1}_{\succ'}(t', k) = c$. Then c must be the minimal in G. A contradiction.

Therefore there cannot be any agent starting to eat from any other object between $t^{\succ}(a_m)$ and $t^{\succ}(a_l)$, implying that each algorithm proceed same until $\min\{t^{\succ}(a_l), 1\}$.

■

Claim 3 *R1 mechanism satisfies weak envy-freeness.*

Proof. Fix a preference structure $\succ$. Take $i, j \in N$ such that $R1(\succ)_j \succ^{sd}_i R1(\succ)_i$, we will show that $R1(\succ)_j = R1(\succ)_i$. If $F_i(\succ) \neq F_j(\succ)$, then $R1(\succ)_{jF_i(\succ)} = 0 < R1(\succ)_{iF_i(\succ)}$. A contradiction. So, $f^{R1}_{\succ}(t', i) = f^{R1}_{\succ}(t', j) = F_i(\succ)$ for all t' such that $0 \leq t' < t^{\succ}(F_i(\succ))$.

Suppose $f^{R1}_{\succ}(t', i) = f^{R1}_{\succ}(t', j)$ for all t',$0 \leq t' < t^{\succ}(a) < 1$ for some object $a \in A$ with $R1(\succ)_{ia} > 0$. Then let $f^{R1}_{\succ}(t^{\succ}(a), i) = b$ and $f^{R1}_{\succ}(t^{\succ}(a), j) = c$, where $b \neq c$. Note that $b, c \notin F(\succ)$. Then $b \succ_i c$ and $f^{R1}_{\succ}(t', i) = b$ for all t' such that $t^{\succ}(a) \leq t' < \min(1, t^{\succ}(b))$. Hence, clearly $R1(\succ)_{ib} > R1(\succ)_{jb}$ and so $\sum_{m \succeq_i b} R1(\succ)_{im} > \sum_{m \succeq_i b} R1(\succ)_{jm}$. A contradiction. Hence we can conclude that b and c should be the same good, inductively implying that $R1(\succ)_i = R1(\succ)_j$ ■

B The R2 Mechanism

We can define R2 mechanism as follows:

A plausible extension of R1 mechanism is to give reservation right to agents not only for their top choice but also to their second most preferred object. Similar to the reservation right for top choice in R1 mechanism, an agent can put reservation to an object if he ranks that object as his second choice in R2 mechanism. This reservation right can be introduced in various ways but we will prefer to define it as general as possible as follows: An agent can put reservation on his second choice as much as his remaining quota if he is the first person who eat from that object through the eating algorithm. We leave the following questions about the reservation rights in R2 mechanism open, since our counter example covers all possible versions of reservation rights for the second most preferred object: "Can an agent put reservation when he is not the first one that eats from an object even he ranks that object as his second choice?" or "what happens if there are more than one agent who can put reservation?" or "Can a group of agents put reservation, when another group is eating from that object?".

Note that, we can extend R1 algorithm by giving two reservation rights to agents where they will put these reservations to not only their top two choices, but the first two goods they eat through the algorithm. Again, the following example will show that this mechanism is also not weak strategy proof.

Example 3 *Let* $N = \{1, 2, 3, 4, 5\}$ *and* $A = \{a, b, c, d, e\}$, *and the preferences are given by* $a \succ_1 b \succ_1 c \succ_1 d \succ_1 e$, $a \succ_2 c \succ_2 b \succ_2 d \succ_2 e$, $a \succ_3 c \succ_3 d \succ_3 e \succ_3 b$ *and* $\succ_3 = \succ_4 = \succ_5$.

All possible version of R2 allocation is given by:

N/A	a	b	c	d	e
1	1/5	4/5	0	0	0
2	1/5	1/5	1/4	1/10	1/4
3	1/5	0	1/4	3/10	1/4
4	1/5	0	1/4	3/10	1/4
5	1/5	0	1/4	3/10	1/4

When the second agent reports his ranking as $a \succ'_2 c \succ'_2 d \succ'_2 b \succ'_2 e$, *the R2 allocation, for all possible versions, for reported preferences is given by:*

N/A	a	b	c	d	e
1	1/5	4/5	0	0	0
2	1/5	1/5	1/4	1/4	1/10
3	1/5	0	1/4	1/4	3/10
4	1/5	0	1/4	1/4	3/10
5	1/5	0	1/4	1/4	3/10

Therefore, there is a profitable deviation where the second agent gets an allocation that stochastically dominates his previous allocation in truthful submission.

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