Wang and Wang *Advances in Difference Equations* (2015) 2015:93
DOI 10.1186/s13662-015-0441-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Anti-periodic boundary value problem for first order impulsive delay difference equations

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Abstract

In this paper, we investigate the anti-periodic boundary value problem for first order impulsive delay difference equations. To begin with, we establish two comparison theorems. Then, by using these theorems, we prove the existence and uniqueness of solutions for the linear problem. Finally, by using the method of upper and lower solutions coupled with the monotone iterative technique, we obtain the new existence results of extremal solutions. Meanwhile, an example is given to illustrate the results obtained.

MSC: 39A10; 34B37

Keywords: anti-periodic boundary value problem; impulsive delay difference equations; monotone iterative technique; extremal solutions

1 Introduction

Impulsive differential equations are recognized as important models which describe many evolution processes that abruptly change their state at a certain moment. Such equations have extensive application in economics, physics, chemical technology, medicine, dynamic systems, optimal control, population dynamics and many other fields. The theory of impulsive differential equations has drawn much attention in recent years and is much richer than the corresponding theory of differential equations. For more information about the theory of important differential equations, see [1–4] and the references therein.

Anti-periodic boundary value problem is an important branch of boundary value problem, and it has recently become an interesting area of investigation. The existence and uniqueness of solutions for such a problem have received a great deal of attention, we refer the readers to [5–12] and the references therein. For the case of differential equations, Chen *et al.* [5] investigated the anti-periodic solutions for first order differential equations, Aftabizadeh *et al.* [7] discussed the anti-periodic boundary value problem for second order differential equations, Wang and Zhang [8] considered the anti-periodic problem for impulsive differential equations. Ahmad and Nieto [9] studied anti-periodic problem for impulsive functional differential equations. Moreover, for difference equations, a lot of results have been investigated in the literature [13–17]. For example, Liu [14] studied higher order functional difference equations with p-Laplacian. Immediately after this, he [15] studied higher order nonlinear periodic difference equations. However, we noticed that all these known results are related to anti-periodic problem for differential equations and to difference equations. Motivated by some recent work on anti-periodic problems and difference equations with impulse (see [18–23]), in this paper, we attempt to propose some results concerning the impulsive delay difference equations with anti-periodic boundary conditions

$$\begin{cases} \Delta u(n) = f(n, u(n), u(\theta(n))), & n \neq n_k, n \in J, \\ \Delta u(n_k) = I_k(u(n_k)), & k = 1, 2, \dots, p, \\ u(0) = -u(T), \end{cases} \tag{1}$$

where $\Delta u(n) = u(n+1) - u(n)$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\theta \in C[J, \mathbb{Z}]$, $0 \leq \theta(n) \leq n$, $J = [0, T] = \{0, 1, \dots, T\}$, $I_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, 2, \dots, p)$, $0 < n_1 < n_2 < \cdots < n_p < T$, T is a positive integer.

In [19], He and Zhang investigated first order impulsive difference equations with periodic boundary conditions. Wang and Wang [18] analyzed first order impulsive difference equations with linear boundary conditions. Zhang *et al.* [12] investigated impulsive anti-periodic boundary value problems for nonlinear q_k-difference equations. To the best of our knowledge, there are few results on the anti-periodic boundary value problem for impulsive delay difference equations. Hence, we are concerned with the existence of solutions for anti-periodic boundary value problem (1). In Section 2, we introduce the concept of upper and lower solutions and establish two comparison principles. In Section 3, we discuss the existence of solutions and uniqueness for the linear anti-periodic boundary value problem. Moreover, by using the monotone iterative technique and the method of upper and lower solutions, we obtain the existence theorem of extremal solutions for problem (1). Finally, an example is worked out to demonstrate the obtained results.

2 Comparison results

In this section, we introduce relative notation and some lemmas. Throughout this paper, let $\mathbb{N}$ denote the set of all natural numbers and let Ω denote the set of real-valued functions defined on J with the norm $\|u\| = \max_{n \in J} |u(n)|$ for $u \in \Omega$. For $\alpha, \beta \in \Omega$, we write $\alpha \leq \beta$ if $\alpha(n) \leq \beta(n)$ for all $n \in J$.

A function $u \in \Omega$ is said to be a solution of problem (1) if it satisfies (1).

Definition 2.1 Functions $\alpha, \beta \in \Omega$ are called lower and upper solutions of problem (1) if

$$\begin{cases} \Delta \alpha(n) \leq f(n, \alpha(n), \alpha(\theta(n))), & n \neq n_k, n \in J, \\ \Delta \alpha(n_k) \leq I_k(\alpha(n_k)), & k = 1, 2, \dots, p, \\ \alpha(0) \leq -\alpha(T), \end{cases}$$

and

$$\begin{cases} \Delta \beta(n) \geq f(n, \beta(n), \beta(\theta(n))), & n \neq n_k, n \in J, \\ \Delta \beta(n_k) \geq I_k(\beta(n_k)), & k = 1, 2, \dots, p, \\ \beta(0) \geq -\beta(T). \end{cases}$$

Lemma 2.1 ([19]) *Assume that*

(i) *the sequence n_k satisfies* $0 \leq n_0 < n_1 < \cdots < n_k < \cdots$ *with* $\lim_{k \to \infty} n_k = \infty$;

(ii) *for* $k \in \mathbb{N}$, $n \geq n_0$,

$$\begin{cases} \Delta m(n) \leq l_n m(n) + q_n, & n \neq n_k, \\ m(n_k + 1) \leq b_k m(n_k) + e_k, \end{cases}$$

where $\{l_n\}$ and $\{q_n\}$ are two real-valued consequences and $l_n > -1$, b_k and e_k are constants, and $b_k \geq 0$. Then

$$m(n) \leq m(n_0) \prod_{n_0<n_k<n} b_k \prod_{n_0<i<n, i\neq n_k, k\in\mathbb{N}} (1+l_i) + \sum_{i=n_0, i\neq n_k}^{n-1} \prod_{i<n_k<n} b_k \prod_{i<s<n, s\neq n_k} (1+l_s) q_i$$
$$+ \sum_{n_0<n_k<n} e_k \prod_{n_k<n_j<n} b_j \prod_{n_k<i<n, i\neq n_j, j\in\mathbb{N}} (1+l_i).$$

Remark 2.2 When k is finite, Lemma 2.1 also holds. In this paper, we only consider this case.

Next, we will establish two new comparison results which play an important role in the monotone iterative technique.

Lemma 2.2 *Let $m \in \Omega$ be such that*

$$\begin{cases} \Delta m(n) + Mm(n) + Nm(\theta(n)) \leq 0, & n \neq n_k, n \in J, \\ \Delta m(n_k) \leq -L_k m(n_k), & k = 1, 2, \ldots, p, \\ m(0) \leq 0, \end{cases}$$

where $N \geq 0$, $0 \leq L_k < M < 1$ for $k = 1, 2, \ldots, p$, and

$$N \sum_{i=0, i\neq n_k}^{T} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-1} - \prod_{k=1}^{p} (1-L_k) < M - 1. \tag{2}$$

Then $m(n) \leq 0$ on J.

Proof Let $v(n) = (1-M)^{-n} m(n)$, $n \in [0, T]$,

$$\Delta v(n) \leq -N(1-M)^{\theta(n)-n-1} v\big(\theta(n)\big), \tag{3}$$

$$v(n_k+1) \leq (1-M)^{-1}(1-L_k) v(n_k). \tag{4}$$

Obviously, $v(n) \leq 0$ implies $m(n) \leq 0$. So it suffices to show $v(n) \leq 0$ on J. Suppose on the contrary that there exists $n^* \in J$ such that $v(n^*) > 0$. Since $v(0) = m(0) \leq 0$, then $n^* \in (0, T]$. Let $\bar{n} \in [0, n^*)$ such that $v(\bar{n}) = \inf_{n\in[0,n^*)} v(n) = -\lambda \leq 0$. We suppose that $\bar{n} \neq n_k$ (if $\bar{n} = n_k$, the proof is similar). From (3) we have

$$\Delta v(n) \leq \lambda N(1-M)^{\theta(n)-n-1}, \quad n \in \big[0, n^*\big]. \tag{5}$$

By (4) and (5), using Lemma 2.1, we have for $n \in [\bar{n}, n^*]$

$$v(n) \leq v(\bar{n}) \prod_{\bar{n}<n_k<n} (1-M)^{-1}(1-L_k) + \lambda N \sum_{i=\bar{n}, i\neq n_k}^{n-1} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-2}$$
$$\leq -\lambda \prod_{k=1}^{p} (1-M)^{-1}(1-L_k) + \lambda N \sum_{i=0, i\neq n_k}^{T} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-2}.$$

Let $n = n^*$, we can get

$$\begin{aligned} v(n^*) &\le \lambda\left\{N \sum_{i=0, i\neq n_k}^{T} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-2} - \prod_{k=1}^{p}(1-M)^{-1}(1-L_k)\right\} \\ &= \lambda(1-M)^{-1}\left\{N \sum_{i=0, i\neq n_k}^{T} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-1} - \prod_{k=1}^{p}(1-L_k)\right\} \\ &< 0, \end{aligned}$$

which is a contradiction. Hence $v(n) \le 0$ on J, this completes the proof. □

Lemma 2.3 *Let* (2) *hold and* $m \in \Omega$,

$$\begin{cases} \Delta m(n) + Mm(n) + Nm(\theta(n)) \le 0, & n \neq n_k, n \in J, \\ \Delta m(n_k) \le -L_k m(n_k), & k = 1, 2, \dots, p, \\ m(0) \le -m(T), \end{cases} \tag{6}$$

where $N \ge 0$, $0 \le L_k < M < 1$ *for* $k = 1, 2, \dots, p$.

Then $m(n) \le 0$ *on* J.

Proof Suppose on the contrary that $m(n) > 0$ for some $n \in J$. Then there are two cases as follows.

Case 1: There exists $n^* \in J$ such that $m(n^*) > 0$ and $m(n) \ge 0$ for all $n \in J$. In this case, (6) implies $\Delta m(n) \le 0$ and $m(n_k + 1) \le (1 - L_k)m(n_k) \le m(n_k)$, $k = 1, 2, \dots, p$. So $m(n)$ is a nonincreasing function. Then we have $m(0) \ge m(T)$. Since $m(0) \le -m(T)$ and $m(n) \ge 0$ for all $n \in J$, we get $m(0) = m(T) = 0$, then $m(n) \equiv 0$, which is a contradiction with $m(n^*) > 0$.

Case 2: There exist n^* and n_* such that $m(n^*) > 0$ and $m(n_*) < 0$. The proof demonstrates that $m(0) \le 0$, so that we can apply Lemma 2.2 and affirm that $m(n) \le 0$ on J. If $m(0) > 0$, then $m(T) < 0$. Let $v(n) = (1 - M)^{-n}m(n)$ on J, we get $v(0) > 0$, $v(T) < 0$, $v(n^*) > 0$, $v(n_*) < 0$. Set $\min_{n\in J} v(n) = -\lambda$, then $\lambda > 0$. Without loss of generality, we shall suppose $v(n_*) = -\lambda$ (if for some k such that $v(n_k) = -\lambda$, the proof is similar), we have

$$\begin{cases} \Delta v(n) \le -N(1-M)^{\theta(n)-n-1} v(\theta(n)), & n \neq n_k, n \in J, \\ v(n_k + 1) \le (1-M)^{-1}(1-L_k)v(n_k), & k = 1, 2, \dots, p, \\ v(0) \le -v(T)(1-M)^T. \end{cases} \tag{7}$$

If $n_* < n^*$, using (7) and Lemma 2.1, for $n \in [n_*, n^*]$, we obtain

$$\begin{aligned} v(n) &\le v(n_*) \prod_{n_*<n_k<n} (1-M)^{-1}(1-L_k) + \lambda N \sum_{i=n_*, i\neq n_k}^{n-1} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-2} \\ &\le -\lambda \prod_{k=1}^{p}(1-M)^{-1}(1-L_k) + \lambda N \sum_{i=0, i\neq n_k}^{T} \prod_{i<n_k<n} (1-L_k)(1-M)^{\theta(i)-i-2}. \end{aligned}$$

Let $n = n^*$, we get

$$\begin{aligned} v(n^*) &\le \lambda\left\{N\sum_{i=0,i\neq n_k}^{T}\prod_{i<n_k<n}(1-L_k)(1-M)^{\theta(i)-i-2}-\prod_{k=1}^{p}(1-M)^{-1}(1-L_k)\right\} \\ &= \lambda(1-M)^{-1}\left\{N\sum_{i=0,i\neq n_k}^{T}\prod_{i<n_k<n}(1-L_k)(1-M)^{\theta(i)-i-1}-\prod_{k=1}^{p}(1-L_k)\right\} \\ &< 0, \end{aligned}$$

which is a contradiction.

If $n_* > n^*$, using (7) and Lemma 2.1, for $n \in [n_*, T]$, we get

$$v(n) \le v(n_*)\prod_{0<n_k<n}(1-M)^{-1}(1-L_k)+\lambda N\sum_{i=0,i\neq n_k}^{n-1}\prod_{i<n_k<n}(1-L_k)(1-M)^{\theta(i)-i-2}.$$

Let $n = T$, it then follows from (2) that

$$\begin{aligned} v(T) &\le v(n_*)\prod_{0<n_k<n}(1-M)^{-1}(1-L_k)+\lambda N\sum_{i=0,i\neq n_k}^{n-1}\prod_{i<n_k<n}(1-L_k)(1-M)^{\theta(i)-i-2} \\ &\le \lambda\left\{N\sum_{i=0,i\neq n_k}^{T}\prod_{i<n_k<n}(1-L_k)(1-M)^{\theta(i)-i-2}-\prod_{k=1}^{p}(1-M)^{-1}(1-L_k)\right\} \\ &= \lambda(1-M)^{-1}\left\{N\sum_{i=0,i\neq n_k}^{T}\prod_{i<n_k<n}(1-L_k)(1-M)^{\theta(i)-i-1}-\prod_{k=1}^{p}(1-L_k)\right\} \\ &< -\lambda = v(n_*), \end{aligned}$$

which is a contradiction with the definition of $v(n_*)$. So $m(0) \le 0$. By Lemma 2.2, we get $m(n) \le 0$ on J, the proof is complete. □

3 Main results

Let us consider the linear problem of (1) as follows:

$$\begin{cases} \Delta u(n) + Mu(n) + Nu(\theta(n)) = \sigma(n), & n \neq n_k, n \in J, \\ \Delta u(n_k) = -L_k u(n_k) + \gamma_k, & k = 1,2,\dots,p, \\ u(0) = -u(T), \end{cases} \tag{8}$$

where $N \ge 0$, $0 \le L_k < M < 1$, $\gamma_k \in \mathbb{R}$, $\sigma \in C(J,\mathbb{R})$.

Lemma 3.1 *A function $u \in \Omega$ is a solution of* (8) *if and only if u is a solution of the following impulsive summation equation*:

$$u(n) = \sum_{j=0,j\neq n_k}^{T-1} G(n,j)\big(\sigma(j) - Nu\big(\theta(j)\big)\big) + \sum_{0<n_k\le T-1} G(n,n_k)\big[(M-L_k)u(n_k)+\gamma_k\big], \tag{9}$$

where

$$G(n,j) = \frac{1}{1+(1-M)^T}\begin{cases} \frac{(1-M)^n}{(1-M)^{j+1}}, & 0 \le j \le n-1, \\ \frac{-(1-M)^{T+n}}{(1-M)^{j+1}}, & n \le j \le T-1. \end{cases}$$

Proof Assume that $u \in \Omega$ is a solution of (8). Set $v(n) = \frac{u(n)}{(1-M)^n}$, $n \in J$. From (8), we see that $v(n)$ satisfies

$$\begin{cases} v(n+1) = v(n) + \frac{\sigma(n) - Nv(\theta(n))(1-M)^{\theta(n)}}{(1-M)^{n+1}}, & n \neq n_k, n \in J, \\ \Delta v(n_k) = \frac{M-L_k}{1-M} v(n_k) + \frac{\gamma_k}{(1-M)^{n_k+1}}, & k = 1, 2, \ldots, p, \\ v(0) = -v(T)(1-M)^T. \end{cases} \tag{10}$$

By using (10), we have

$$\begin{aligned} v(n) = v(0) &+ \sum_{j=0, j\neq n_k}^{n-1} \frac{\sigma(j) - Nv(\theta(j))(1-M)^{\theta(j)}}{(1-M)^{j+1}} \\ &+ \sum_{0<n_k\leq n-1} \left(\frac{M-L_k}{1-M} v(n_k) + \frac{\gamma_k}{(1-M)^{n_k+1}} \right). \end{aligned} \tag{11}$$

If we set $n = T$ in (11), then we get

$$\begin{aligned} v(T) = v(0) &+ \sum_{j=0, j\neq n_k}^{T-1} \frac{\sigma(j) - Nv(\theta(j))(1-M)^{\theta(j)}}{(1-M)^{j+1}} \\ &+ \sum_{0<n_k\leq T-1} \left(\frac{M-L_k}{1-M} v(n_k) + \frac{\gamma_k}{(1-M)^{n_k+1}} \right). \end{aligned} \tag{12}$$

From the boundary condition $v(T) = \frac{-v(0)}{(1-M)^T}$, we obtain

$$\begin{aligned} v(0) = -\frac{(1-M)^T}{1+(1-M)^T} &\left[\sum_{j=0, j\neq n_k}^{T-1} \frac{\sigma(j) - Nv(\theta(j))(1-M)^{\theta(j)}}{(1-M)^{j+1}} \right. \\ &\left. + \sum_{0<n_k\leq T-1} \left(\frac{M-L_k}{1-M} v(n_k) + \frac{\gamma_k}{(1-M)^{n_k+1}} \right) \right]. \end{aligned} \tag{13}$$

Substituting (13) into (11) and using $v(n) = \frac{u(n)}{(1-M)^n}$, $n \in J$, we have

$$\begin{aligned} \frac{u(n)}{(1-M)^n} = {}& \frac{1}{1+(1-M)^T} \left(\sum_{j=0, j\neq n_k}^{n-1} \frac{\sigma(j) - Nu(\theta(j))}{(1-M)^{j+1}} + \sum_{0<n_k\leq n-1} \frac{(M-L_k)u(n_k) + \gamma_k}{(1-M)^{n_k+1}} \right) \\ & - \frac{(1-M)^T}{1+(1-M)^T} \left(\sum_{j=n, j\neq n_k}^{T-1} \frac{\sigma(j) - Nu(\theta(j))}{(1-M)^{j+1}} + \sum_{n<n_k\leq T-1} \frac{(M-L_k)u(n_k) + \gamma_k}{(1-M)^{n_k+1}} \right). \end{aligned}$$

Let

$$G(n,j) = \frac{1}{1+(1-M)^T} \begin{cases} \frac{(1-M)^n}{(1-M)^{j+1}}, & 0 \leq j \leq n-1, \\ \frac{-(1-M)^{T+n}}{(1-M)^{j+1}}, & n \leq j \leq T-1, \end{cases}$$

we see that u is a solution of (9). The proof is complete. □

Lemma 3.2 *Assume that constants* $0 < M < 1$, $N \geq 0$, $0 \leq L_k < 1$, $\gamma_k \in \mathbb{R}$, $\sigma \in C(J, \mathbb{R})$, *and*

$$\frac{1}{1 + (1 - M)^T} \left(TN + \sum_{k=1}^{p} |M - L_k| \right) < 1. \tag{14}$$

Then (8) *has a unique solution*.

Proof Define an operator $F : \Omega \to \Omega$ by

$$(Fu)(n) = \sum_{j=0, j \neq n_k}^{T-1} G(n,j)\big(\sigma(j) - Nu\big(\theta(j)\big)\big) + \sum_{0<n_k \leq T-1} G(n,n_k)\big[(M - L_k)u(n_k) + \gamma_k\big].$$

For any $u_1, u_2 \in \Omega$, we have

$$\begin{aligned} \big|Fu_1(n) - Fu_2(n)\big| &\leq \sum_{j=0, j \neq n_k}^{T-1} \big|G(n,j)\big| \big|N\big(u_1\big(\theta(j)\big) - u_2\big(\theta(j)\big)\big)\big| \\ &\quad + \sum_{0<n_k \leq T-1} \big|G(n,n_k)\big| \big|(M - L_k)\big(u_1(n_k) - u_2(n_k)\big)\big| \\ &\leq \frac{1}{1 + (1 - M)^T} \left(TN + \sum_{k=1}^{p} |M - L_k| \right) \|u_1 - u_2\|. \end{aligned}$$

Hence, $\|Fu_1(n) - Fu_2(n)\| = \max_{n \in J} |Fu_1(n) - Fu_2(n)| = \tau \|u_1 - u_2\|$, where

$$\tau = \frac{1}{1 + (1 - M)^T} \left(TN + \sum_{k=1}^{p} |M - L_k| \right) < 1.$$

By the Banach contraction principle, F has a unique fixed point. The proof is complete. □

Theorem 3.1 *Let the following conditions hold*:

(A_0) *Functions* $\alpha, \beta \in \Omega$ *are lower and upper solutions for* (1) *with* $\alpha \leq \beta$.

(A_1) *There exist* $N \geq 0$, $0 \leq L_k < M < 1$ *for* $k = 1, 2, \ldots, p$ *such that the function* $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ *satisfies*

$$f(n,x,y) - f(n,u,v) \geq -M(x - u) - N(y - v),$$

where $\alpha(n) \leq u \leq x \leq \beta(n)$, $\alpha(\theta(n)) \leq v(\theta(n)) \leq y(\theta(n)) \leq \beta(\theta(n))$, $n \in J$.

(A_2) *The functions* $I_k \in C(\mathbb{R}, \mathbb{R})$ *satisfy*

$$I_k(x) - I_k(y) \geq -L_k(x - y),$$

where $\alpha(n_k) \leq y \leq x \leq \beta(n_k)$, $0 \leq L_k < 1$, $k = 1, 2, \ldots, p$.

(A_3)

$$N \sum_{i=0, i \neq n_k}^{T} \prod_{i<n_k<T} (1 - L_k)(1 - M)^{\theta(i)-i-1} - \prod_{k=1}^{p} (1 - L_k) < M - 1.$$

(A_4)

$$\frac{1}{1+(1-M)^T}\left(TN+\sum_{k=1}^{p}|M-L_k|\right)<1.$$

Then there exists a solution u of problem (1) *such that $\alpha(n)\leq u(n)\leq\beta(n)$ on J.*

Proof We consider the following modified problem:

$$\begin{cases}\Delta u(n)+Mu(n)+Nu(\theta(n))=\sigma_q(n), & n\neq n_k,\ n\in J,\\ \Delta u(n_k)=-L_k u(n_k)+d_k, & k=1,2,\ldots,p,\\ u(0)=-u(T),\end{cases}\tag{15}$$

where

$$\begin{aligned}&\sigma_q(n)=f\bigl(n,q\bigl(n,u(n)\bigr),q\bigl(\theta(n),u\bigl(\theta(n)\bigr)\bigr)\bigr)+Mq\bigl(n,u(n)\bigr)+Nq\bigl(\theta(n),u\bigl(\theta(n)\bigr)\bigr),\\&d_k=I_k\bigl(q\bigl(n_k,u(n_k)\bigr)\bigr)+L_kq\bigl(n_k,u(n_k)\bigr),\\&q\bigl(n,u(n)\bigr)=\max\bigl\{\alpha(n),\min\bigl\{u,\beta(n)\bigr\}\bigr\}\quad\text{for } n\in J.\end{aligned}$$

We can easily see that if $\alpha(n)\leq u(n)\leq\beta(n)$ on J, then u is a solution of (1) if and only if u is a solution of (15). Indeed, suppose that $u\in\Omega$ is a solution of (15), we now prove that $\alpha(n)\leq u(n)$ on J. Set $m(n)=\alpha(n)-u(n)$, $n\in J$. Owing to (A_0)-(A_2), we acquire

$$\begin{aligned}\Delta m(n)&=\Delta\alpha(n)-\Delta u(n)\\&\leq f\bigl(n,\alpha(n),\alpha\bigl(\theta(n)\bigr)\bigr)-\bigl[-Mu(n)-Nu\bigl(\theta(n)\bigr)\\&\quad+f\bigl(n,q\bigl(n,u(n)\bigr),q\bigl(\theta(n),u\bigl(\theta(n)\bigr)\bigr)\bigr)+Mq\bigl(n,u(n)\bigr)+Nq\bigl(\theta(n),u\bigl(\theta(n)\bigr)\bigr)\bigr]\\&\leq -Mm(n)-Nm\bigl(\theta(n)\bigr),\\\Delta m(n_k)&=\Delta\alpha(n_k)-\Delta u(n_k)\\&\leq I_k\bigl(\alpha(n_k)\bigr)-I_k\bigl(q\bigl(n_k,u(n_k)\bigr)\bigr)-L_kq\bigl(n_k,u(n_k)\bigr)+L_ku(n_k)\\&\leq -L_km(n_k),\end{aligned}$$

and

$$m(0)=\alpha(0)-u(0)\leq-\alpha(T)+u(T)=-m(T).$$

By Lemma 2.3, we have $m(n)\leq 0$ on J, *i.e.*, $\alpha(n)\leq u(n)$ on J. Similarly, we can show that $u(n)\leq\beta(n)$ on J.

Next, we need to prove that problem (15) has a solution. To do this, we write problem (15) in the following way by Lemma 3.1:

$$u(n)=\sum_{j=0,j\neq n_k}^{T-1}G(n,j)\bigl(\sigma(j)-Nu\bigl(\theta(j)\bigr)\bigr)+\sum_{0<n_k\leq T-1}G(n,n_k)\bigl[(M-L_k)u(n_k)+d_k\bigr].$$

We define the continuous and compact operator $\mathcal{A}:\Omega\to\Omega$ by

$$\begin{aligned}[\mathcal{A}u](n) = {} & \sum_{j=0,j\neq n_k}^{T-1} G(n,j)\bigl(\sigma(j)-Nu\bigl(\theta(j)\bigr)\bigr) \\ & + \sum_{0<n_k\leq T-1} G(n,n_k)\bigl[(M-L_k)u(n_k)+d_k\bigr], \quad n\in J.\end{aligned}$$

The continuity of f and the definition of q imply that σ_q and d_k are bounded. We can choose constants $h>0$ and $w>0$ such that $|\sigma_q|\leq h$, $|d_k|\leq w$. For $\lambda\in(0,1)$, we find that any solution of $u=\lambda\mathcal{A}u$ satisfies

$$\begin{aligned}\|u\| &= \lambda\|\mathcal{A}u\| \\ &\leq \max_{n\in J}\sum_{j=0,j\neq n_k}^{T-1}\bigl|G(n,j)\bigr|\bigl[\bigl|\sigma_q(j)\bigr|+N\bigl|u\bigl(\theta(j)\bigr)\bigr|\bigr] \\ &\quad + \max_{n\in J}\sum_{0<n_k\leq T-1}\bigl|G(n,n_k)\bigr|\bigl[|M-L_k|\bigl|u(n_k)\bigr|+|d_k|\bigr] \\ &\leq \frac{hT}{1+(1-M)^T}+\frac{TN}{1+(1-M)^T}\|u\|+\frac{\sum_{k=1}^{p}|M-L_k|}{1+(1-M)^T}\|u\|+\frac{pw}{1+(1-M)^T}.\end{aligned}$$

Then by (A_4) we have

$$\|u\|\leq\left(1-\frac{1}{1+(1-M)^T}\left(TN+\sum_{k=1}^{p}|M-L_k|\right)\right)^{-1}\left(\frac{hT+pw}{1+(1-M)^T}\right).$$

From the Schaefer fixed-point theorem, $\mathcal{A}$ has at least a fixed point. It is clear that this fixed point is the solution of (15). Such a solution lies between α and β and in consequence is a solution of (1). The proof is complete. □

Theorem 3.2 *Let all assumptions of Theorem* 3.1 *hold. Then there exist monotone sequences* $\{\alpha_j(n)\}$, $\{\beta_j(n)\}$ *with* $\alpha_0=\alpha$, $\beta_0=\beta$ *such that* $\lim_{j\to\infty}\alpha_j(n)=\rho(n)$, $\lim_{j\to\infty}\beta_j(n)=r(n)$ *uniformly on* J, *and* $\rho(n)$, $r(n)$ *are the extremal solutions of problem* (1).

Proof Let

$$\begin{cases}\Delta\alpha_j(n)+M\alpha_j(n)+N\alpha_j(\theta(n)) \\ \quad = f(n,\alpha_{j-1}(n),\alpha_{j-1}(\theta(n)))+M\alpha_{j-1}(n)+N\alpha_{j-1}(\theta(n)), \quad n\neq n_k, n\in J, \\ \Delta\alpha_j(n_k)+L_k\alpha_j(n_k)=I_k(\alpha_{j-1}(n_k))+L_k\alpha_{j-1}(n_k), \quad k=1,2,\ldots,p, \\ \alpha_j(0)=-\alpha_j(T),\end{cases} \tag{16}$$

and

$$\begin{cases}\Delta\beta_j(n)+M\beta_j(n)+N\beta_j(\theta(n)) \\ \quad = f(n,\beta_{j-1}(n),\beta_{j-1}(\theta(n)))+M\beta_{j-1}(n)+N\beta_{j-1}(\theta(n)), \quad n\neq n_k, n\in J, \\ \Delta\beta_j(n_k)+L_k\beta_j(n_k)=I_k(\beta_{j-1}(n_k))+L_k\beta_{j-1}(n_k), \quad k=1,2,\ldots,p, \\ \beta_j(0)=-\beta_j(T),\end{cases} \tag{17}$$

for $j = 1, 2, \ldots$, where $\alpha_0 = \alpha$, $\beta_0 = \beta$. It follows from Lemma 3.2 that problems (16) and (17) have a unique solution, respectively.

First, we show that $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$.

Let $m = \alpha_0 - \alpha_1$, then owing to (A_0) and $\alpha_1(0) = -\alpha_1(T)$, we obtain

$$\begin{aligned}
\Delta m(n) &= \Delta\alpha_0(n) - \Delta\alpha_1(n) \\
&\le f\bigl(n, \alpha_0(n), \alpha_0\bigl(\theta(n)\bigr)\bigr) - \bigl[-M\alpha_1(n) - N\alpha_1\bigl(\theta(n)\bigr) \\
&\quad + f\bigl(n, \alpha_0(n), \alpha_0\bigl(\theta(n)\bigr)\bigr) + M\alpha_0(n) + N\alpha_0\bigl(\theta(n)\bigr)\bigr] \\
&\le -Mm(n) - Nm\bigl(\theta(n)\bigr), \\
\Delta m(n_k) &= \Delta\alpha_0(n_k) - \Delta\alpha_1(n_k) \\
&\le I_k\bigl(\alpha_0(n_k)\bigr) - \bigl[I_k\bigl(\alpha_0(n_k)\bigr) + L_k\alpha_0(n_k) - L_k\alpha_1(n_k)\bigr] \\
&\le -L_k m(n_k),
\end{aligned}$$

and

$$m(0) = \alpha_0(0) - \alpha_1(0) \le -\alpha_0(T) + \alpha_1(T) = -m(T).$$

It then follows from Lemma 2.3 that we have $m(n) \le 0$ on J, which implies $\alpha_0(n) \le \alpha_1(n)$ for $n \in J$. Similarly, we get $\beta_1(n) \le \beta_0(n)$ on J.

Next, take $m = \alpha_1 - \beta_1$, by using (A_0), (A_1), we have

$$\begin{aligned}
&\Delta m(n) + Mm(n) + Nm\bigl(\theta(n)\bigr) \\
&\quad = \Delta\alpha_1(n) + M\alpha_1(n) + N\alpha_1\bigl(\theta(n)\bigr) - \Delta\beta_1(n) - M\beta_1(n) - N\beta_1\bigl(\theta(n)\bigr) \\
&\quad \le f\bigl(n, \alpha_0(n), \alpha_0\bigl(\theta(n)\bigr)\bigr) + M\alpha_0(n) + N\alpha_0\bigl(\theta(n)\bigr) - f\bigl(n, \beta_0(n), \beta_0\bigl(\theta(n)\bigr)\bigr) \\
&\qquad - M\beta_1(n) - N\beta_1\bigl(\theta(n)\bigr) \le 0.
\end{aligned}$$

Noticing $\alpha_0 \le \beta_0$ and (A_2), we get

$$\begin{aligned}
\Delta m(n_k) &= -L_k\alpha_1(n_k) + I_k\bigl(\alpha_0(n_k)\bigr) + L_k\alpha_0(n_k) - I_k\bigl(\beta_0(n_k)\bigr) - L_k\beta_0(n_k) \\
&\le -L_k m(n_k), \\
m(0) &= \alpha_1(0) - \beta_1(0) = -\alpha_1(T) + \beta_1(T) = -m(T).
\end{aligned}$$

Again by Lemma 2.3 we get $m(n) \le 0$ on J, that is, $\alpha_1(n) \le \beta_1(n)$ for all $n \in J$. Thus we get $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$. Continuing this process, by induction, we can get the sequences $\{\alpha_j(n)\}$ and $\{\beta_j(n)\}$ such that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_j \le \cdots \le \beta_j \le \cdots \le \beta_1 \le \beta_0 \quad \text{on } J.$$

Clearly, the sequences $\{\alpha_j(n)\}$, $\{\beta_j(n)\}$ are uniformly bounded and equi-continuous. Since they are monotone sequences, by the Ascoli-Arzela theorem, we can get that the entire sequences $\{\alpha_j(n)\}$ and $\{\beta_j(n)\}$ converge uniformly and monotonically on J with $\lim_{j\to\infty} \alpha_j(n) = \rho(n)$ and $\lim_{j\to\infty} \beta_j(n) = r(n)$.

Obviously ρ, r are the solutions of (1). Next we prove that ρ, r are extremal solutions of (1), let $u \in \Omega$ be any solutions of problem (1) such that $\alpha_0 \le u \le \beta_0$. Suppose that there exists a positive integer j such that $\alpha_j \le u \le \beta_j$ on J. Setting $m = \alpha_{j+1} - u$, we have

$$\begin{aligned}
\Delta m(n) &= \Delta\alpha_{j+1}(n) - \Delta u(n) \\
&\le f\bigl(n, \alpha_j(n), \alpha_j\bigl(\theta(n)\bigr)\bigr) - M\alpha_{j+1}(n) - N\alpha_{j+1}\bigl(\theta(n)\bigr) \\
&\quad - f\bigl(n, u(n), u\bigl(\theta(n)\bigr)\bigr) + M\alpha_j(n) + N\alpha_j\bigl(\theta(n)\bigr) \\
&\le -M\bigl(\alpha_{j+1}(n) - u(n)\bigr) - N\bigl(\alpha_{j+1}\bigl(\theta(n)\bigr) - u\bigl(\theta(n)\bigr)\bigr) \\
&= -Mm(n) - Nm\bigl(\theta(n)\bigr), \quad n \neq n_k, n \in J, \\
\Delta m(n_k) &= \Delta\alpha_{j+1}(n_k) - \Delta u(n_k) \\
&= I_k\bigl(\alpha_j(n_k)\bigr) - I_k\bigl(u(n_k)\bigr) - L_k\alpha_{j+1}(n_k) + L_k\alpha_j(n_k) \\
&\le -L_k m(n_k), \quad k = 1, 2, \ldots, p, \\
m(0) &= \alpha_{j+1}(0) - u(0) = -\alpha_{j+1}(T) + u(T) = -m(T).
\end{aligned}$$

By Lemma 2.3, $m(n) \le 0$ on J, *i.e.*, $\alpha_{j+1}(n) \le u(n)$ $n \in J$. Similarly, one derives $\alpha_{j+1}(n) \le u(n) \le \beta_{j+1}(n)$ on J. Since $\alpha_0(n) \le u(n) \le \beta_0(n)$ on J, by induction we see that $\alpha_j(n) \le u(n) \le \beta_j(n)$ on J for every j. Taking the limit as $j \to \infty$, we conclude $\rho(n) \le u(n) \le r(n)$ on J. The proof is then finished. □

Example 3.1 Consider the equations

$$\begin{cases} \Delta u(n) = f(n, u(n), u(\theta(n))) = -u^2(n) - \frac{1}{200}u(\frac{1}{2}n) + \frac{2^{-n}}{25}, & n \in Z[0,3], n \neq n_1, \\ \Delta u(n_k) = -\frac{1}{2}u(n_1), \quad n_1 = 2, \\ u(0) = -u(3). \end{cases} \tag{18}$$

It is easy to verify that $\alpha = -\frac{1}{201}$ is a lower solution and $\beta(n) = \frac{1}{12}(4 - 2^{-n})$ is an upper solution for (18). Indeed,

$$\begin{aligned}
&\Delta\beta(n) = \frac{1}{12}\bigl(2^{-n} - 2^{-n-1}\bigr) \ge -\left[\frac{1}{12}\bigl(4 - 2^{-n}\bigr)\right]^2 - \frac{1}{200} \cdot \frac{1}{12}\bigl(4 - 2^{-\frac{1}{2}n}\bigr) + \frac{1}{25} \cdot 2^{-n}, \\
&\Delta\beta(2) = \frac{1}{12}\bigl(2^{-2} - 2^{-3}\bigr) > -\frac{1}{2} \cdot \frac{1}{12}\bigl(4 - 2^{-2}\bigr), \\
&\beta(0) = \frac{1}{12}(4 - 1) > -\frac{1}{12}\bigl(4 - 2^{-3}\bigr) = -\beta(3), \\
&f(n, x, y) - f(n, u, v) = -\bigl(x^2 - u^2\bigr) - \frac{1}{200}(y - v) \ge -\frac{2}{3}(x - u) - \frac{1}{200}(y - v),
\end{aligned}$$

for $\alpha \le u \le x \le \beta$. Taking $M = \frac{2}{3}$, $N = \frac{1}{200}$, $L_k = \frac{1}{2}$, we get

$$\prod_{k=1}^{p}(1 - L_k) + M - 1 = \frac{1}{2} + \frac{2}{3} - 1 = \frac{1}{6},$$

$$\begin{aligned}&\frac{1}{200}\sum_{i=0,i\neq n_k}^{3}\prod_{i<n_k<3}(1-L_k)(1-M)^{-\frac{1}{2}i-1}=\frac{1}{200}\sum_{i=0,i\neq n_k}^{3}\prod_{i<n_k<3}\frac{1}{2}\cdot\left(\frac{1}{3}\right)^{-\frac{1}{2}i-1}\\&\quad=\frac{1}{200}\sum_{i=0}^{1}\left(\frac{1}{3}\right)^{-\frac{1}{2}i-1}+\frac{1}{200}\sum_{i=2}^{3}\frac{1}{2}\cdot\left(\frac{1}{3}\right)^{-\frac{1}{2}i-1}<\frac{1}{6},\end{aligned}$$

which shows that condition (A_3) is satisfied, and

$$\frac{1}{1+(1-\frac{2}{3})^3}\left(3\cdot\frac{1}{200}+\sum_{k=1}^{p}\left(\frac{2}{3}-\frac{1}{2}\right)\right)<1,$$

which shows that condition (A_4) is satisfied. Hence, by Theorem 3.2, we obtain the existence of monotone sequences that converge to the extremal solutions of (18) in a function interval contained in $[\alpha, \beta]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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Acknowledgements
The authors sincerely thank the reviewers and the editors for their valuable suggestions and comments. This paper is supported by the National Natural Science Foundation of China (11271106), the Natural Science Foundation of Hebei Province, China (A2013201232) and the Science and Technology Research Projects of Higher Education Institutions of Hebei Province, China (Z2013038).

Received: 10 November 2014 Accepted: 13 March 2015 Published online: 20 March 2015

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