FINITELY PRESENTED CENTRE-BY-METABELIAN LIE ALGEBRAS

R.M. BRYANT AND J.R.J. GROVES

To Bernhard Neumann on his ninetieth birthday

It is shown that finitely presented centre-by-metabelian Lie algebras are Abelian-by-finite-dimensional.

1. INTRODUCTION

In [7], the second author proved that a finitely presented centre-by-metabelian group is Abelian-by-polycyclic. The proof of this result used the fact, proved by Bieri and Strebel in [2], that a finitely presented soluble group with an infinite cyclic quotient is an HNN extension with finitely generated base group. In [3], Bieri and Strebel deduced another proof of the result of [7] as a corollary of their work on finitely presented soluble groups, particularly the fact that a metabelian quotient of a finitely presented soluble group is again finitely presented.

The aim of this note is to prove a similar result for Lie algebras.

Theorem. A finitely presented centre-by-metabelian Lie algebra is Abelian-by-finite-dimensional.

The key tools quoted above do not seem to be available for Lie algebras. The closest result of which we are aware is one of Wasserman [8, Theorem 9.1] which is similar to the result quoted from [2]. But the consequences of this result do not appear to be sufficiently powerful to obtain results for Lie algebras analogous to those for groups. We have therefore needed to take a substantially different approach.

The authors have shown in [6] that a finitely presented soluble Lie algebra of characteristic 2 which satisfies the maximal condition for ideals must be of finite dimension. Because a finitely generated Abelian-by-finite-dimensional Lie algebra must satisfy the maximal condition for ideals [1, Corollary 11.1.8], the Theorem implies that all finitely presented centre-by-metabelian Lie algebras of characteristic 2 are of finite dimension.

The main step in the proof of the Theorem is to show that a metabelian quotient of a finitely presented centre-by-metabelian Lie algebra is again finitely presented. It would be interesting to know to what extent this can be generalised.

Received 29th April, 1999
Work supported by ARC research grant S69813080 and by a Royal Society travel grant.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/99 $A2.00+0.00.
QUESTION. Is it true that a metabelian quotient of a finitely presented soluble Lie algebra is again finitely presented?

An affirmative answer to the corresponding question for groups is given by [3, Theorem B].

2. QUOTIENTS OF FINITELY PRESENTED LIE ALGEBRAS

Throughout this paper $K$ denotes an arbitrary field, and all tensor and exterior products are taken over $K$. If $L$ is any Lie algebra over $K$ we write $K[L]$ for the enveloping algebra of $L$. Also, we write $L'$ for the subalgebra $[L, L]$ and $L''$ for $[L', L']$.

Let $L$ be a finitely presented Lie algebra over $K$, and suppose that $A$ and $B$ are ideals of $L$ such that $B \subseteq A$ and $A/B$ is Abelian. Set $R = K[L/A]$ and $M = A/B$. Then $M$ has a natural structure as a (right) $R$-module via

$$(a + B)(l + A) = [a, l] + B$$

for all $a \in A$ and $l \in L$.

The $R$-module structure on $M$ carries over to an $R \otimes R$-module structure on the tensor square $M \otimes M$. There is an algebra homomorphism $\delta : R \to R \otimes R$ given by $x\delta = x \otimes 1 + 1 \otimes x$ for all $x \in L/A$. In fact, as is well known, $\delta$ is an embedding (it has right inverse $\iota \otimes \epsilon$ where $\iota : R \to R$ is the identity map and $\epsilon : R \to K$ is the augmentation map). We call $\delta$ the diagonal embedding. Let $\tilde{R} = R\delta$. Thus $M \otimes M$ is an $\tilde{R}$-module, and therefore an $R$-module. The action of $R$ on $M \otimes M$ is called the diagonal action. It induces an action of $R$ on the exterior square $M \wedge M$ given by

$$(m \wedge n)x = (mx) \wedge n + m \wedge (nx)$$

for all $m, n \in M$ and $x \in L/A$. The action of $L$ on itself carries over to an action of $R$ on $B/[B, A]$ via

$$(b + [B, A])(l + A) = [b, l] + [B, A]$$

for all $b \in B$ and $l \in L$. There is a linear map $\gamma : M \wedge M \to B/[B, A]$ satisfying

$$(a_1 + B) \wedge (a_2 + B) \mapsto [a_1, a_2] + [B, A]$$

for all $a_1, a_2 \in A$, and it is easily verified (via the Jacobi identity) that $\gamma$ is a homomorphism of $R$-modules.

**Lemma.** With the notation above, suppose that $L$ is finitely presented and that $L/A$ is of finite dimension. Then the kernel of $\gamma$ is a finitely generated $R$-module.

**Proof:** It is possible to prove this by means of the spectral sequence associated to the extension $A \to L \to L/A$, but we provide an elementary proof. Let $F$ be a finitely
generated free Lie algebra such that there is an epimorphism \( \pi : F \to L \). Let \( U, V \) and \( W \) denote the complete inverse images under \( \pi \) of \( A, B \) and \( \{0\} \), respectively. Note that we can then identify \( L/A \) with \( F/U \) and hence \( R \) with \( K[F/U] \).

The correspondence given by

\[
(u_1 \pi + B) \wedge (u_2 \pi + B) \mapsto [u_1, u_2] + [V, U],
\]

for all \( u_1, u_2 \in U \), leads to an explicit \( R \)-module isomorphism between \( M \wedge M \) and \( U'/[V, U] \). For details, see [6, Section 2.2]. The epimorphism \( \pi \) also induces an isomorphism of \( B/[B, A] \) with \( V/([V, U] + W) \), and this is again an \( R \)-module isomorphism. We can thus identify \( \gamma \) with the map

\[
U'/[V, U] \to V/([V, U] + W)
\]

induced from the inclusion of \( U' \) into \( V \). Therefore

\[
\ker \gamma \cong \left( U' \cap \left( [V, U] + W \right) \right)/[V, U] = \left( [V, U] + (U' \cap W) \right)/[V, U]
\]

\[
\cong (U' \cap W)/\left( [V, U] \cap (U' \cap W) \right) = (U' \cap W)/( [V, U] \cap W).
\]

We must show that this section of \( F \) is finitely generated as an \( R \)-module.

Since \( L \) is finitely presented, \( W \) is finitely generated as an ideal of \( F \). Hence \( W/[W, U] \) is finitely generated as an \( R \)-module. But \( (U' \cap W)/([V, U] \cap W) \) is isomorphic to an \( R \)-section of \( W/[W, U] \). Since \( L/A \) is of finite dimension, \( R \) is Noetherian (see, for example [5, Proposition 6 of I.2.6]) and so this section is also finitely generated, as required.

Observe that this lemma, although technical in nature, has some important consequences in special cases. For example, if \( B/[B, A] \) is finite dimensional, then we may deduce that \( M \wedge M \) is finitely generated as an \( R \)-module and so, using [6, Theorem A], that \( L/B \) is also finitely presented. The following is another special case where we can deduce that \( L/B \) is finitely presented.

**Proposition.** Let \( L \) be a finitely presented centre-by-metabelian Lie algebra over the field \( K \). Let \( A \) and \( B \) be ideals of \( L \) with \( B \subseteq A \) such that \( L/A \) and \( A/B \) are Abelian and \( B \) is central; and write \( M = A/B \). Then \( M \wedge M \) is finitely generated as a \( K[L/A] \)-module. As a consequence, \( L/B \) is finitely presented (so, taking \( B = L'' \), we have that \( L/L'' \) is finitely presented).

**Proof:** Observe that the last sentence of the Proposition follows from [6, Theorem A]. Write \( R = K[L/A] \) and let \( I \) denote the augmentation ideal of \( R \) (that is, the ideal of \( R \) generated by the elements of \( L/A \)). By the Lemma, the kernel of \( \gamma : M \wedge M \to B \)
is finitely generated as an $R$-module. Since $B$ is central in $L$ it is trivial as an $R$-module (by which we mean that each element of $L/A$ has zero action on $B$). Thus the $R$-module $(M \wedge M)I$ is contained in the kernel of $\gamma$. Therefore, by the Lemma, it is finitely generated. We shall use this to show that $M \wedge M$ is finitely generated as an $R$-module.

By [6, Lemmas 2.1 and 2.2], we may assume that $K$ is algebraically closed. We shall use arguments similar to those of [6, Proposition 2.4]. By [4, Theorem 1 of IV.1.4], $M$ has a finite series of submodules

$$\{0\} = M_0 \leq M_1 \leq M_2 \leq \ldots \leq M_k = M,$$

where each quotient $M_i/M_{i-1}$ is isomorphic to an $R$-module of the form $R/P_i$ where $P_i$ is a prime ideal of $R$. Further, by [4, Theorem 2 of IV.1.4], each $P_i$ contains a prime ideal $Q_i$ of $R$ which is associated to $M$.

It will clearly suffice to show that $M \otimes M$ is finitely generated as an $R$-module under the diagonal action. But the series above for $M$ yields a finite series of $R$-submodules of $M \otimes M$ in which each quotient is of the form $R/P_i \otimes R/P_j$ (here, of course, $R$ acts via the diagonal embedding of $R$ into $R \otimes R$). Since $R/P_i \otimes R/P_j$ is a quotient of $R/Q_i \otimes R/Q_j$, it suffices to prove that each $R/Q_i \otimes R/Q_j$ is finitely generated as an $R$-module.

Suppose firstly that $R/Q_i \otimes R/Q_j$ is trivial as an $R$-module. Then, for each $x \in L/A$,

$$0 = ((1 + Q_i) \otimes (1 + Q_j))x = (x + Q_i) \otimes (1 + Q_j) + (1 + Q_i) \otimes (x + Q_j).$$

But this implies that $x + Q_i \in K + Q_i$ and $x + Q_j \in K + Q_j$ for each $x \in L/A$, so that $R/Q_i$ and $R/Q_j$ have dimension 1. It is then clear that $R/Q_i \otimes R/Q_j$ is finitely generated as an $R$-module.

Thus we can assume that $R/Q_i \otimes R/Q_j$ is not trivial as an $R$-module. Choose an element $x$ of $L/A$ which has non-zero action on $R/Q_i \otimes R/Q_j$. We observe for future reference that, because $K$ is assumed algebraically closed and because $R/Q_i$ and $R/Q_j$ are integral domains, $R/Q_i \otimes R/Q_j$ is also an integral domain (see [9, Corollary 1 to Theorem 40 of Chapter III]). Thus multiplication in $R/Q_i \otimes R/Q_j$ by the image of $x$ is a monomorphism of $R$-modules.

Suppose that $Q_i \neq Q_j$. Because $Q_i$ and $Q_j$ are associated prime ideals of $M$, there are elements $m_i$ and $m_j$ of $M$ such that the submodules $m_iR$ and $m_jR$ are isomorphic to $R/Q_i$ and $R/Q_j$, respectively. Further, because $Q_i$ and $Q_j$ are distinct, these submodules intersect trivially, and so $m_iR + m_jR \cong R/Q_i \oplus R/Q_j$. Since $R/Q_i \otimes R/Q_j$ is isomorphic to a submodule of $\Lambda^2 (R/Q_i \otimes R/Q_j)$, it follows that $R/Q_i \otimes R/Q_j$ is
isomorphic to a submodule of $M \land M$. Therefore $(R/Q_i \otimes R/Q_j)x$ is isomorphic to a submodule of $(M \land M)I$ and is finitely generated. But

$$R/Q_i \otimes R/Q_j \cong (R/Q_i \otimes R/Q_j)x.$$  

Thus $R/Q_i \otimes R/Q_j$ is finitely generated.

Suppose now that $Q_i = Q_j$. Because $Q_i$ is an associated prime ideal of $M$, there is an isomorphic copy of $R/Q_i$ in $M$. Thus $(R/Q_i \land R/Q_i)x$ is isomorphic to a submodule of $(M \land M)I$ and is finitely generated. It is standard, and easily verified, that the linear map induced by $a \land b \mapsto a \otimes b - b \otimes a$ (for all $a, b \in R/Q_i$) yields an $R$-monomorphism from $R/Q_i \land R/Q_i$ to $R/Q_i \otimes R/Q_i$. Thus multiplication in $R/Q_i \land R/Q_i$ by the image of $x$ is a monomorphism of $R$-modules. Therefore $R/Q_i \land R/Q_i$ is isomorphic to $(R/Q_i \land R/Q_i)x$ and is finitely generated. It follows, by [6, Theorem A], that $R/Q_i \otimes R/Q_i$ is finitely generated as an $R$-module, which completes the proof of the Proposition.

3. PROOF OF THE THEOREM

We use the notation preceding the statement of the Lemma with $A = L'$ and $B = L''$. Here $L''$ is central in $L$. It will sometimes be convenient to consider $M \land M$ and $M \otimes M$ as $\tilde{R}$-modules rather than $R$-modules (recall that $\tilde{R} = R\delta \subseteq R \otimes R$). By the Proposition, $M \land M$ is finitely generated as an $\tilde{R}$-module. Hence, by [6, Theorem A], $M \otimes M$ is also finitely generated as an $\tilde{R}$-module.

Let $\{w_1, \ldots, w_k\}$ be a finite generating set for $M \otimes M$ as an $R \otimes R$-module and, for $i = 1, \ldots, k$, let $J_i$ be the annihilator of $w_i$ in $R \otimes R$. Further, let $J$ be the annihilator of $M \otimes M$. Thus $J = J_1 \cap \cdots \cap J_k$ and

$$(R \otimes R)/J_i \cong w_i(R \otimes R) \leq M \otimes M.$$  

Since $M \otimes M$ is finitely generated as an $\tilde{R}$-module, so is $(R \otimes R)/J_i$. Thus $(R \otimes R)/J$ is also finitely generated as an $\tilde{R}$-module.

Let $\tilde{T}$ be the augmentation ideal of $\tilde{R}$ and let $\tilde{I}$ be the ideal of $R \otimes R$ generated by $\tilde{T}$. Then $(R \otimes R)/(\tilde{I} + J)$ is both finitely generated and trivial as an $\tilde{R}$-module and so is of finite dimension. Let $T = \{t \in R : t \otimes 1 \in \tilde{I} + J\}$. Then $T$ is an ideal of $R$ such that $R/T$ is of finite dimension.

Let $\sigma : M \otimes M \rightarrow L''$ be the homomorphism of $R$-modules satisfying

$$(a_1 + L'') \otimes (a_2 + L'') \mapsto [a_1, a_2]$$  

for all $a_1, a_2 \in L'$. Since $L''$ is a trivial $R$-module, $(M \otimes M)\tilde{I}$ is contained in the kernel of $\sigma$. But

$$MT \otimes M \subseteq (M \otimes M)(\tilde{I} + J) = (M \otimes M)\tilde{I}.$$
Thus \((MT \otimes M)\sigma = \{0\}\).

Let \(H\) be the subspace of \(L\) such that \(L'' \leq H \leq L'\) and \(H/L'' = MT\). Since \(T\) is an ideal of \(R\), \(H\) is an ideal of \(L\). From the definition of \(\sigma\) we find \((MT \otimes M)\sigma = [H, L']\). Thus \([H, L'] = \{0\}\) and, since \(H \leq L'\), it follows that \(H\) is Abelian. Since \(T\) is of finite co-dimension in \(R\) and \(M\) is a finitely generated \(R\)-module, \(MT\) is of finite co-dimension in \(M\). Thus \(H\) is of finite co-dimension in \(L'\) and so also in \(L\). Therefore \(L\) is Abelian-by-finite-dimensional, which completes the proof of the Theorem. \(\square\)

**References**


