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MIKUSIŃSKI'S OPERATIONAL CALCULUS APPROACH TO THE DISTRIBUTIONAL STIELTJES TRANSFORM

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ABSTRACT. We consider a space $\mathcal{M}$ which was introduced by Yosida to provide a simplified version for Mikusiński operational calculus. The classical Stieltjes transform is extended to a subspace of $\mathcal{M}$ and then studied. Some Abelian type theorems are presented.

1. INTRODUCTION

The ring of continuous complex-valued functions on the real line which vanish on $(-\infty, 0)$, denoted by $C_+(\mathbb{R})$, with addition and convolution has no zero divisors by Titchmarch's theorem. The quotient field of $C_+(\mathbb{R})$ is called the field of Mikusiński operators [6].

Yosida [10] constructed a space $\mathcal{M}$ in order to provide a simplified version for Mikusiński's operational calculus without using Titchmarch's convolution theorem. Even though the space $\mathcal{M}$ does not give the full space of Mikusiński operators, it contains many of the important operators needed for applications.

In this note, we use the space $\mathcal{M}(r) \subset \mathcal{M}$ to extend the classical Stieltjes transform. It turns out that $\mathcal{M}(r)$ is isomorphic to the space of distributions $J'(r)$. Roughly speaking, a distribution T, which is supported on $[0, \infty)$, is in $J'(r)$ provided there exist $k \in \mathbb{N}$ and a locally integrable function f satisfying a growth condition at infinity such that T is the k^{th} distributional derivative of f.

The space $J'(r)$, and variations of $J'(r)$, have been investigated by several authors [2, 4, 5, 7, 8, 9] in regards to extending the Stieltjes transform.

While the construction of $J'(r)$ requires a space of testing functions, the concept of a dual space, and functional analysis, the construction of $\mathcal{M}(r)$ is algebraic, elementary, and only requires elementary calculus.

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2. Preliminaries

Let $C_+(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$ which vanish on the interval $(-\infty, 0)$.

For f, $g \in C_+(\mathbb{R})$, the convolution is given by

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx. \tag{2.1}$$

Let H denote the Heaviside function. That is, $H(t) = 1$ for $t \geq 0$ and zero otherwise.

For each $n \in \mathbb{N}$, we denote by H^n the function $H * \cdots * H$ where H is repeated n times.

The space $\mathcal{M}$ is defined as follows.

$$\mathcal{M} = \left\{ \frac{f}{H^k} : f \in C_+(\mathbb{R}), k \in \mathbb{N} \right\}.$$

Two elements of $\mathcal{M}$ are equal, denoted $\frac{f}{H^n} = \frac{g}{H^m}$, if and only if $H^m * f = H^n * g$.

Addition, multiplication, and scalar multiplication are defined in the natural way, and $\mathcal{M}$ with these operations is a commutative algebra with identity $\delta = \frac{H^2}{H^2}$.

$$\frac{f}{H^n} + \frac{g}{H^m} = \frac{H^m * f + H^n * g}{H^{n+m}} \tag{2.2}$$

$$\frac{f}{H^n} * \frac{g}{H^m} = \frac{f * g}{H^{n+m}} \tag{2.3}$$

$$\alpha \frac{f}{H^n} = \frac{\alpha f}{H^n}, \quad \alpha \in \mathbb{C}. \tag{2.4}$$

The generalized derivative is defined as follows.
Let $W = \frac{f}{H^k} \in \mathcal{M}$. Then, $DW = \frac{f}{H^{k+1}}$.

Remark 2.1. *For the construction of $\mathcal{M}$, the space of locally integrable functions which vanish on $(-\infty, 0)$ could have been used instead of $C_+(\mathbb{R})$. Also notice by identifying $f \in L^1_{loc}(\mathbb{R}^+)$ with $\frac{H*f}{H} \in \mathcal{M}$, $L^1_{loc}(\mathbb{R}^+)$ can be considered a subspace of $\mathcal{M}$.*

3. Stieltjes Transform

For $k = 0, 1, 2, \ldots$

$$\mathcal{M}_k(r) = \left\{ \frac{f}{H^k} \in \mathcal{M} : f(t)\, t^{-r-k+\alpha} \text{ is bounded as } t \to \infty \text{ for some } \alpha > 0 \right\} \tag{3.1}$$

$$\mathcal{M}(r) = \bigcup_{k=0}^{\infty} \mathcal{M}_k(r) \tag{3.2}$$

Let $W \in \mathcal{M}(r)$. That is, $W = \frac{f}{H^k} \in \mathcal{M}_k(r)$, for some $k \in \mathbb{N}$. For $r > -1$, define the Stieltjes transform of index r by

$$\Lambda_z^r W = (r+1)_k \int_0^\infty \frac{f(t)}{(t+z)^{r+k+1}}\, dt, \quad z \in \mathbb{C}\backslash(-\infty, 0], \tag{3.3}$$

where $(r+1)_k = \frac{\Gamma(r+k+1)}{\Gamma(r+1)} = (r+1)(r+2)\cdots(r+k)$ and Γ is the gamma function.

Remark 3.1.

(1) *The definition for the Stieltjes transform is well-defined. This follows by observing the following. First,* $\frac{f}{H^k} = \frac{g}{H^n}$ $(n \geq k)$ *if and only if* $g = H^{n-k} * f$. *Also, for* $m \in \mathbb{N}$,

$$\Lambda_z^r \left(\frac{f}{H^k}\right) = \Lambda_z^r \left(\frac{H^m * f}{H^{m+k}}\right), \; z \in \mathbb{C}\backslash(-\infty, 0].$$

(2) *Notice that the Stieltjes transform* Λ_z^r *is consistent with the classical Stieltjes transform* S_z^r. *That is, if* $f \in L^1_{loc}(\mathbb{R}^+)$ *such that* f *satisfies the growth condition in (3.1) with* $k = 0$, *then* $S_z^r f = \Lambda_z^r \left(\frac{H*f}{H}\right)$, *where* $S_z^r f = \int_0^\infty \frac{f(t)}{(t+z)^{r+1}}\, dt$.

The Stieltjes transform can be obtained by iteration of the Laplace transform.

Theorem 3.1. *Let* $W = \dfrac{f}{H^k} \in \mathcal{M}(r)$. *Then,* $\Lambda_z^r W = \dfrac{1}{\Gamma(r+1)} \displaystyle\int_0^\infty e^{-zt} t^r \widehat{W}(t)\, dt$, $\mathrm{Re}\,(z) > 0$, *where*

$$\widehat{W}(t) = t^k \widehat{f}(t) = t^k \int_0^\infty e^{-t\sigma} f(\sigma) d\sigma. \tag{3.4}$$

Proof.

$$\frac{1}{\Gamma(r+1)} \int_0^\infty e^{-zt} t^r \widehat{W}(t)\, dt = \frac{1}{\Gamma(r+1)} \int_0^\infty \int_0^\infty e^{-(z+\sigma)t}\, t^{r+k} f(\sigma) d\sigma\, dt \tag{3.5}$$

Because of the growth condition on f, the interchanging of the order of integration is justified.

Hence,

(3.6)

$$\begin{aligned} \frac{1}{\Gamma(r+1)} \int_0^\infty \int_0^\infty e^{-(z+\sigma)t}\, t^{r+k} f(\sigma) d\sigma\, dt &= \frac{1}{\Gamma(r+1)} \int_0^\infty f(\sigma) \left(\int_0^\infty e^{-(z+\sigma)t}\, t^{r+k}\, dt\right) d\sigma \\ &= \frac{\Gamma(r+k+1)}{\Gamma(r+1)} \int_0^\infty \frac{f(\sigma)}{(\sigma+z)^{r+k+1}}\, d\sigma \\ &= \Lambda_z^r W, \; \mathrm{Re}\, z > 0. \end{aligned}$$

Therefore, by (3.5) and (3.6),

$$\Lambda_z^r W = \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-zt} t^r \widehat{W}(t)\, dt, \; \mathrm{Re}\,(z) > 0\,.$$

□

Remark 3.2. *The Laplace transform operator (3.4) has similar properties as the classical Laplace transform (see* [1]*).*

The proofs of the following properties follow directly by using the previous theorem and the properties of the Laplace transform.

Properties. Let $W = \frac{f}{H^k} \in \mathcal{M}(r)$. Then for $r > -1$ and $z \in \mathbb{C}\backslash(-\infty, 0]$,

(1) $\Lambda_z^r \tau_c W = \Lambda_{z+c}^r W$, $c > 0$ and $\tau_c W = \frac{\tau_c f}{H^k}$, $\tau_c f(t) = f(t-c)$.
(2) $\Lambda_z^r D^m W = (r+1)_m \Lambda_z^{r+m} W$, $m = 1, 2, \ldots$
(3) $\frac{d^m}{dz^m} \Lambda_z^r W = (-1)^m (r+1)_m \Lambda_z^{r+m} W = (-1)^m \Lambda_z^r D^m W$, $m = 1, 2, \ldots$
(4) $\Lambda_z^{r+1}(tW) = \Lambda_z^r W - z\Lambda_z^{r+1} W$, where $tW = \frac{tf}{H^k} - \frac{kf}{H^{k-1}}$, $k \geq 2$.

Theorem 3.2. *Let $W \in \mathcal{M}(r)$. Then, there exist positive numbers α and β such that*

(i) $\Lambda_z^r W = o(z^{-\alpha})$ *as* $z \to 0$, $|arg\, z| \leq \psi < \frac{\pi}{2}$.
(ii) $\Lambda_z^r W = o(z^{-\beta})$ *as* $z \to \infty$, $|arg\, z| \leq \psi < \frac{\pi}{2}$.

Proof. Let $W = \frac{f}{H^k} \in \mathcal{M}(r)$, where for some positive constants M, α, and γ,

$$|f(t)| \leq M\, t^{r+k-\alpha}, \quad \text{for } t \geq \gamma.$$

(i) $\Lambda_z^r W = \frac{1}{\Gamma(r+1)} (t^{r+k} \widehat{f}(t))^\wedge(z)$, $\operatorname{Re} z > 0$.

Now,

$$\frac{t^{r+k} \widehat{f}(t)}{t^{r+k}} = \widehat{f}(t) \to 0 \text{ as } t \to \infty.$$

Therefore, by a classical Abelian theorem for the Laplace transform [3],

$$\frac{z^{r+k+1}(t^{r+k}\widehat{f}(t))^\wedge}{\Gamma(r+k+1)} \to 0 \text{ as } z \to 0,\ |argz| \leq \psi < \frac{\pi}{2}.$$

Thus,

$$\lim_{\substack{z \to 0 \\ |argz| \leq \psi < \frac{\pi}{2}}} z^{r+k+1} \Lambda_z^r W = 0.$$

This completes the proof of (i). Now, for the proof of (ii). There exist $A > 0$ and $B > 0$ such that

$$|t^{r+k} \widehat{f}(t)| \leq At^{r+k} + \frac{B}{t^{1-\alpha}},\ t > 0 \text{ (see [7], p. 211)}.$$

Thus, the function $t^{r+k} \widehat{f}(t)$ is locally integrable on $[0, \infty)$.

Now,

$$\begin{aligned} |\Lambda_z^r W| &\leq \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-t \operatorname{Re} z}\, t^{r+k} |\widehat{f}(t)|\, dt \\ &\leq \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-t \operatorname{Re} z} \left(At^{r+k} + \frac{B}{t^{1-\alpha}} \right) dt \\ &= \frac{C}{(\operatorname{Re} z)^{r+k+1}} + \frac{D}{(\operatorname{Re} z)^\alpha},\ \operatorname{Re} z > 0, \end{aligned}$$

for some positive constants C, D.

Thus,

$$\lim_{\substack{z \to \infty \\ |argz| \leq \psi < \frac{\pi}{2}}} z^\beta \Lambda_z^r W = 0, \text{ where } \beta = \frac{1}{2} \min\{\alpha, r+k+1\}.$$

This completes the proof of the theorem. □

4. Localization

Definition 4.1. *Let $W = \frac{f}{H^k} \in \mathcal{M}$. W is said to vanish on an open interval (a, b), denoted $W(t) = 0$ on (a, b), provided there exists a polynomial p with degree at most $k-1$ such that $p(t) = f(t)$ for $a < t < b$.*

The support of $W \in \mathcal{M}$, denoted supp W, is the complement of the largest open set on which W vanishes.

Remark 4.1.

(1) *The definition of W vanishing on an interval does not depend on the representation of W.*
(2) *The notion of an element of $\mathcal{M}$ vanishing on an interval is consistent with the notion of a function vanishing on an interval. That is, $f(t) = 0$ for $a < t < b$ if and only if $W_f(t) = 0$ on (a, b), where $f \in C_+(\mathbb{R})$ and $W_f = \frac{H*f}{H}$.*
(3) *It follows that if $W(t) = 0$ on (a, b), where $a < 0$, then $f(t) = 0$ for all $a < t < b$, where $W = \frac{f}{H^k}$.*

Example 4.1. *Recall $\delta = \frac{H^2}{H^2}$. Notice that $H^2(t) = t$ on the open interval $(0, \infty)$. Thus, $\delta(t) = 0$ on $(0, \infty)$. Also, $H(t) = 0$ on $(-\infty, 0)$. So, $\delta(t) = 0$ on $(-\infty, 0)$. Therefore, supp $\delta = \{0\}$.*

Example 4.2. *Let $W = \frac{f}{H^3}$, where* $f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ t+2 & t \ge 2 \\ 0 & t < 0. \end{cases}$
Then W has compact support. Notice that W vanishes on $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$, and hence, supp $W = \{0\} \cup \{2\}$.

Theorem 4.1. *Let $W \in \mathcal{M}$. If $DW(t) = 0$ on (a, b), then W is constant on (a, b).*

Proof. Let $W = \frac{f}{H^k}$ such that $DW = \frac{f}{H^{k+1}} = 0$ on (a, b). Therefore, there exists a polynomial $p(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_k t^k$ such that $f(t) = p(t)$, for all $a < t < b$. That is,

$$f - k!\alpha_k H^{k+1} = \alpha_0 H + \alpha_1 H^2 + \cdots + (k-1)!\alpha_{k-1} H^k \text{ on } (a, b).$$

Thus,

$$\frac{f}{H^k} - \frac{k!\alpha_k H^{k+1}}{H^k} = 0 \text{ on } (a, b).$$

That is, $W = \frac{f}{H^k} = k!\alpha_k H$ on (a, b). □

5. Abelian Theorems

As an application, we establish some Abelian type theorems.

Let $W, V \in \mathcal{M}$. Then, $W(t) = V(t)$ on (a, b) provided $(W - V)(t) = 0$ on (a, b).

Definition 5.1. *Let $W \in \mathcal{M}$ and $\xi, \lambda \in \mathbb{C}$ where $\operatorname{Re}\lambda > -1$. W is said to be equivalent at the origin (infinity) to ξt^λ, denoted $W(t) \sim \xi t^\lambda$ as $t \to 0^+$ $(t \to \infty)$, provided there exist an interval (a, b) with $a < 0$ and $b > 0$ ($a > 0$ and $b = \infty$) and $g \in L^1_{loc}(\mathbb{R}^+)$ such that $W(t) = W_g(t)$ on (a, b), where $W_g = \frac{H*g}{H} \in \mathcal{M}$, and $\frac{g(t)}{t^\lambda} \to \xi$ as $t \to 0^+$ $(t \to \infty)$.*

Lemma 5.1. *Let $k \in \mathbb{N}$, $\alpha > 0$, and $r > -1$. If $f \in L^1_{loc}(\mathbb{R}^+)$ such that $f(t)t^{-r-k+\alpha}$ is bounded on $[b,\infty)$ (for some $b > 0$), then*
$\int_b^\infty \frac{f(t)}{(t+z)^{r+k+1}}\, dt$ *is bounded in the half-plane* $\operatorname{Re} z > 0$.

The following is an initial value theorem.

Theorem 5.1. *Let $W \in \mathcal{M}(r)$ and $\nu > -1$. If $W(t) \sim \xi t^\nu$ as $t \to 0^+$, then for $r > \nu$,*

$$\lim_{\substack{z\to 0\\ |argz|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

Proof. Since $W(t) \sim \xi t^\nu$ as $t \to 0^+$, $W(t) = W_g(t)$ on $(-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$, where $g \in L^1_{loc}(\mathbb{R}^+)$ and $\frac{g(t)}{t^\nu} \to \xi$ as $t \to 0^+$. We may assume that $g(t) = 0$ on $[\varepsilon,\infty)$.

Now, $W = W_g + V$, where for some $k \in \mathbb{N}$, $V \in \mathcal{M}_k(r)$ and supp $V \subseteq [\varepsilon,\infty)$. Thus, by a classical Abelian theorem for the Stieltjes transform and the previous lemma, we obtain

$$\lim_{\substack{z\to 0\\ |argz|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r g}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi,$$

and,

$$\lim_{\substack{z\to 0\\ |argz|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r V}{\Gamma(r-\nu)\Gamma(\nu+1)} = 0.$$

Therefore,

$$\lim_{\substack{z\to 0\\ |argz|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

□

Lemma 5.2. *Let $a > 0$ and $k \in \mathbb{N}$. Then, for $n = 0,1,2,\dots,k-1$ and $r > \nu > -1$,*

$$\lim_{\substack{z\to\infty\\ \operatorname{Re} z>0}} z^{r-\nu}\int_a^\infty \frac{t^n}{(t+z)^{r+k+1}}\, dt = 0.$$

Proof. Follows by induction on k. □

Now, the final value theorem.

Theorem 5.2. *Let $W \in \mathcal{M}$ and $\nu > -1$. If $W(t) \sim \xi t^\nu$ as $t \to \infty$, then for $r > \nu$,*

$$\lim_{\substack{z\to\infty\\ |argz|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

Proof. Since $W(t) \sim \xi t^\nu$ as $t \to \infty$, there exist $k \in \mathbb{N}$, $c > 0$, a polynomial p, and $g \in L^1_{loc}(\mathbb{R}^+)$ such that supp $g \subseteq [c,\infty)$, deg $p \le k-1$, and $f(t) = (H^k * g)(t) + p(t)$ on (c,∞) with $\frac{g(t)}{t^\nu} \to \xi$ as $t \to \infty$.

It follows that $W \in \mathcal{M}_k(r)$ and that $W = W_g + V$, where $V = \frac{f-H^k*g}{H^k} \in \mathcal{M}_k(r)$ and supp $V \subseteq [0,c]$. By using a classical Abelian theorem and noting that $\Lambda_z^r W_g$ is the same as the classical Stieltjes transform of g, we obtain

$$\lim_{\substack{z\to\infty\\ |argz|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W_g}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

Now, letting $T = f - H^k * g$, we obtain

$$z^{r-\nu}\Lambda_z^r V = (r+1)_k\, z^{r-\nu} \int_0^\infty \frac{T(t)}{(t+z)^{r+k+1}}\, dt$$
$$= (r+1)_k\, z^{r-\nu} \int_0^c \frac{T(t)}{(t+z)^{r+k+1}}\, dt + (r+1)_k\, z^{r-\nu} \int_c^\infty \frac{p(t)}{(t+z)^{r+k+1}}\, dt.$$

By the previous lemma, for Re $z > 0$, it follows that the limit of the second term converges to zero as $z \to \infty$. Now, for Re $z > 0$,

$$\left| z^{r-\nu} \int_0^c \frac{T(t)}{(t+z)^{r+k+1}}\, dt \right| \leq |z|^{-k-\nu-1} \int_0^c |T(t)|\, dt \to 0 \text{ as } z \to \infty.$$

The proof of the theorem is completed by observing that

$$z^{r-\nu}\Lambda_z^r W = z^{r-\nu}\Lambda_z^r W_g + z^{r-\nu}\Lambda_z^r V.$$

□

As a final remark, the map $\frac{f}{H^k} \to D^k f$ is a well-defined linear bijection from $\mathcal{M}(r)$ onto $J'(r)$, where D denotes the distributional differential operator [11] and

$$J'(r) = \{D^k f : k \in \mathbb{N}, f \in L^1_{loc}(\mathbb{R}^+), f(t)t^{-r-k+\alpha} \text{ bdd as } t \to \infty \text{ for some } \alpha > 0\}.$$

Moreover, the Stieltjes transform for $\frac{f}{H^k} \in \mathcal{M}(r)$ and the Stieltjes transform for $D^k f \in J'(r)$ are equivalent.

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