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# SOME RESULTS ON COLOMBEAU PRODUCT OF DISTRIBUTIONS

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ABSTRACT. In this paper some results on singular products of distributions are derived. The results are obtained in Colombeau algebra of generalized functions, which is most relevant algebraic construction for tackling nonlinear problems of Schwartz distributions.

## 1. INTRODUCTION

Because of the large employment of distributions in the natural sciences and other mathematical fields where products of distributions with coinciding singularities often appear, the problem of multiplication of Schwartz distributions has been for a long time interest of many researchers.

One of the most useful aspects of Schwartz's theory of distributions in applications is that discontinuous functions can be handled as easily as continuous or differentiable functions which provides a powerful tool in formulating and solving many problems of various fields of science and engineering [12]. In applications the results sometimes show that multiplications of two generalized functions are not always allowed, so there have been made many attempts to define products of distributions, or rather to enlarge the range of existing products [15]. Several attempts have been made to include the distributions into differential algebras (as an example one can see [21]). Colombeau in [2] describes the problem of multiplication of distributions and shows how to overcome it. His theory was primarly been used for dealing with nonlinear partial differential equations with singularities and was developed during the years and it has now a big appliance in a different fields (physics, geometry, etc. see [19, 13, 14, 20, 23, 18, 17, 22]).

The origin of the construction of Colombeau algebra (introduced in [3, 1]) lies in Schwartz's impossibility result [24], i.e in searching for an associative and commutative algebra $(\mathcal{A}(\Omega), +, \circ)$, where $\Omega$ is an open set in $\mathbf{R}^n$, satisfying following properties:

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1) The space $\mathcal{D}'(\Omega)$ of distributions over $\Omega$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unit in $\mathcal{A}(\Omega)$;

2) There exist derivation operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$, $i = 1, 2, ..., n$, that are linear and satisfy the Leibnitz rule;

3) $\partial_i|_{\mathcal{D}'(\Omega)}$, $i = 1, 2, ..., n$, is the usual partial derivative;

4) $\circ|_{C(\Omega)\times C(\Omega)}$ is the usual product of functions.

The second condition means that $\mathcal{A}(\Omega)$ is differential algebra. It is shown in [26] that there isn't any algebra satisfying these conditions. The optimal solution of this problem was constructing differential algebra $\mathcal{A}(\Omega)$ satisfying 1) - 3) and 4') where $C(\Omega)$ is replaced with $C^\infty(\Omega)$, i.e.

4') $\circ|_{C^\infty(\Omega)\times C^\infty(\Omega)}$ is the usual product of functions

and it was done by J.F. Colombeau ([3, 1]).

The differential Colombeau algebra $\mathcal{G}(\mathbf{R})$ as a powerful tool for treating linear and nonlinear problems including singularities has almost the optimal properties while the problem of multiplication of Schwartz distributions is concerned: it is an associative differential algebra of generalized functions, contains the algebra of smooth functions as a subalgebra (elements of this algebra are some equivalence classes of nets of smooth functions), the distribution space $\mathcal{D}'$ is linearly embedded in it as a subspace and the multiplication is compatible with the operations of differentiation and products with $C^\infty$ - differentiable functions. The notion 'association' in $\mathcal{G}$ is a faithful generalization of the equality of distributions and enables obtaining results in terms of distributions again. About embedding of the space of distributions into the space of Colombeau generalized functions one can read papers [16, 21, 11]. We can see some products of distributions in a Colombeau algebra in [10, 5, 6, 8, 9, 4, 7] and other papers written by this author.

Following this approach, we evaluate in this paper some products of distributions with coinciding singularities, as embedded in Colombeau algebra, in terms of associated distributions. The results obtained in this way can be reformulated as regularized products in the classical distribution theory.

## 2. Colombeau algebra

In this section we will introduce basic notations and definitions from Colombeau theory.

Let $\mathbf{N_0}$ denoted the set of non-negative integers, i.e. $\mathbf{N_0} = \mathbf{N} \cup \{0\}$. Let $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$; $i, j \in \mathbf{N_0}$. Then, for $q \in \mathbf{N_0}$ we denote

$$A_q(\mathbf{R}) = \left\{ \varphi(x) \in \mathcal{D}(\mathbf{R}) \,\middle|\, \int_{\mathbf{R}} x^j \varphi(x)\, dx = \delta_{0j},\ j = 0, 1, ..., q \right\}$$

where $\mathcal{D}(\mathbf{R})$ is the space of all $C^\infty$ functions $\varphi : \mathbf{R} \to \mathbf{C}$ with compact support. The elements of the set $A_q(\mathbf{R})$ are called *test functions*.

It is obvious that $A_1 \supset A_2 \supset A_3...$ . Also, $A_k \neq \emptyset$ for all $k \in \mathbf{N}$ (this is proved in [3]).

For $\varphi \in A_q(\mathbf{R})$ and $\varepsilon > 0$ it is denoted $\varphi_\varepsilon = \dfrac{1}{\varepsilon}\varphi\left(\dfrac{x}{\varepsilon}\right)$ and $\overset{\vee}{\varphi}(x) = \varphi(-x)$.

**Definition 2.1.** *Let $\mathcal{E}(\mathbf{R})$ be the algebra of functions $f(\varphi, x) : A_0(\mathbf{R}) \times \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable for fixed 'parameter' $\varphi$. The generalized functions of Colombeau are elements of the quotient algebra*

$$\mathcal{G} \equiv \mathcal{G}(\mathbf{R}) = \frac{\mathcal{E}_M[\mathbf{R}]}{\mathcal{I}[\mathbf{R}]}$$

.

Here $\mathcal{E}_M[\mathbf{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset $K$ of $\mathbf{R}$ and any $p \in \mathbf{N_0}$ there is a $q \in \mathbf{N}$ such that, for each $\varphi \in A_q(\mathbf{R})$ there are $c > 0$, $\eta > 0$ and it holds:

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \le c\varepsilon^{-q}$$

for $0 < \varepsilon < \eta$.

$\mathcal{I}[\mathbf{R}]$ is an ideal of $\mathcal{E}_M[\mathbf{R}]$ consisting of all functions $f(\varphi, x)$ such that for each compact subset $K$ of $\mathbf{R}$ and any $p \in \mathbf{N_0}$ there is a $q \in \mathbf{N}$ such that for every $r \ge q$ and each $\varphi \in A_r(\mathbf{R})$ there are $c > 0$, $\eta > 0$ and it holds:

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \le c\varepsilon^{r-q}$$

for $0 < \varepsilon < \eta$.

The Colombeau algebra $\mathcal{G}(\mathbf{R})$ contains the distributions on $\mathbf{R}$ canonically embedded as a $\mathbf{C}$ - vector subspace by the map:

$$i : \mathcal{D}'(\mathbf{R}) \to \mathcal{G}(\mathbf{R}) : u \to \widetilde{u} = \left\{ \widetilde{u}(\varphi, x) = \left(u * \overset{\vee}{\varphi}\right)(x) : \varphi \in A_q(\mathbf{R}) \right\}$$

where $*$ denotes the convolution product of two distributions and is given by: $(f * g)(x) = \int_{\mathbf{R}} f(y)\, g(x-y) dy$.

According to the above, we can also write: $\widetilde{u}(\varphi, x) = \langle u(y), \varphi(y-x) \rangle$ where $\langle u, \varphi \rangle$ denotes the integral $\int_{\mathbf{R}} u(x)\varphi(x)\, dx$.

**Definition 2.2.** *(a) Generalized functions $f, g \in \mathcal{G}(\mathbf{R})$ are said to be associated, denoted $f \approx g$, if for some representatives $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbf{R})$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q(\mathbf{R})$*

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} |f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)| \psi(x)\, dx = 0$$

*(b) A generalized function $f \in \mathcal{G}$ is said to admit some $u \in \mathcal{D}'(\mathbf{R})$ as 'associated distribution', denoted $f \approx u$, if for some representative $f(\varphi_\varepsilon, x)$ of $f$ and any $\psi(x) \in \mathcal{D}(\mathbf{R})$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q(\mathbf{R})$*

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} f(\varphi_\varepsilon, x)\psi(x)\, dx = \langle u, \psi \rangle$$

These definitions are independent of the representatives chosen and the distribution associated, if it exists, is unique. The association is a faithful generalization of the equality of distributions.

By *Colombeau product of distributions* is meant the product of their embedding in $\mathcal{G}$ whenever the result admits an associated distribution.

If the regularized model product of two distributions exists, then their Colombeau product also exists and it is same with the first one.

The relation $f \approx u$ is asymmetric, the distribution $u$ stands on the r.h.s.; the relation $f \approx \widetilde{u}$ is an equivalent relation in $\mathcal{G}$ so it is symmetric in $\mathcal{G}$ and it can also be written as $f - \widetilde{u} \approx 0$.

We denote with $\widetilde{\ln|x|}$ the embedding into $\mathcal{G}(\mathbf{R})$ of the distribution $\ln|x|$ and with $\widetilde{\delta^{(s-1)}}(x)$ the embedding of the $\delta^{(s-1)}(x)$, i.e the embedding of the $(s-1)$ - st derivative of the Dirac delta function.

## 3. Results on some products of distributions

**Theorem 3.1.** *The product of the generalized functions* $\ln|x|$ *and* $\delta^{(s-1)}(x)$ *for* $s = 0, 1, 2, \ldots$ *in* $\mathcal{G}(\mathbf{R})$ *admits associated distributions and it holds:*

$$\widetilde{\ln|x|} \cdot \widetilde{\delta^{(s-1)}(x)} \approx \frac{(-1)^s}{s} \delta^{(s-1)}(x) \tag{3.1}$$

*Proof.* For given $\varphi \in A_0(\mathbf{R})$ we suppose that $\operatorname{supp}\varphi(x) \subseteq [-l, l]$, without lost of generality. Then using the embedding rule and the substitution $v = (y-x)/\varepsilon$ we have the representatives of the distribution $\ln|x|$ in Colombeau algebra:

$$\widetilde{\ln|x|}(\varphi_\varepsilon, x) = \varepsilon^{-1} \int_{x-l\varepsilon}^{x+l\varepsilon} \ln|y| \varphi\left(\frac{y-x}{\varepsilon}\right) dy = \int_{-l}^{l} \ln|x + \varepsilon v| \varphi(v) dv \,,$$

Similar,

$$\widetilde{\delta^{(s-1)}}(\varphi_\varepsilon, x) = \frac{(-1)^{s-1}}{\varepsilon^s} \varphi^{(s-1)}\left(-\frac{x}{\varepsilon}\right) .$$

Then, for any $\psi(x) \in \mathcal{D}(\mathbf{R})$ we have:

$$\begin{aligned} \langle \widetilde{\ln|x|}(\varphi_\varepsilon, x) \cdot \widetilde{\delta^{(s-1)}}(\varphi_\varepsilon, x), \psi(x) \rangle &= \int_{-\infty}^{\infty} \widetilde{\ln|x|}(\varphi_\varepsilon, x)\, \widetilde{\delta^{(s-1)}}(\varphi_\varepsilon, x)\, \psi(x) dx \\ &= \frac{(-1)^{s-1}}{\varepsilon^s} \int_{-l\varepsilon}^{l\varepsilon} \left( \int_{-l}^{l} \ln|x + \varepsilon v| \varphi(v) dv \right) \varphi^{(s-1)}(-x/\varepsilon) \psi(x) dx \\ &= \frac{(-1)^s}{\varepsilon^{s-1}} \int_{-l}^{l} \varphi^{(s-1)}(u) \psi(-\varepsilon u) \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| \varphi(v) dv du \,. \end{aligned} \tag{3.2}$$

using the substitution $u = -x/\varepsilon$. By the Taylor theorem we have that

$$\psi(-\varepsilon\omega) = \sum_{k=0}^{s-1} \frac{\psi^{(k)}(0)}{k!} (-\varepsilon\omega)^k + \frac{\psi^{(s)}(\eta\omega)}{(s)!} (-\varepsilon\eta)^s \tag{3.3}$$

for $\eta \in (0,1)$. Using this for (3.2) we have:

$$\langle \widetilde{\ln|x|}\,(\varphi_\varepsilon, x)\cdot \widetilde{\delta^{(s-1)}}\,(\varphi_\varepsilon, x)\,, \psi(x)\rangle = \sum_{i=0}^{s-1} \frac{(-1)^{s+i}\psi^{(i)}(0)}{i!\varepsilon^{s-i-1}} J_i + O(\varepsilon)\,.$$

where $J_i = \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| u^i \varphi^{(s-1)}(u)du$ and $i = 0, 1, \ldots, s-1$. Next using integration by part we have:

$$\begin{aligned} J_i &= \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| u^i \varphi^{(s-1)}(u)du \\ &= \frac{1}{i+1}\int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| \varphi^{(s-1)}(u)d\left(u^{i+1} - v^{i+1}\right) \\ &= -\frac{1}{i+1}\int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \Big[\left(u^{i+1} - v^{i+1}\right)\ln|\varepsilon v - \varepsilon u| \varphi^{(s)}(u)du \\ &\quad -\frac{1}{i+1}\int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \frac{u^{i+1} - v^{i+1}}{u - v}\varphi^{(s-1)}(u)du\Big]\,. \end{aligned}$$

The first term above is zero, and we have

$$(i+1)J_i = \sum_{k=0}^{i} \int_{-l}^{l} v^{i-k}\varphi(v)dv \int_{-l}^{l} u^k \varphi^{(s-1)}(u)du = \int_{-l}^{l} u^i \varphi^{(s-1)}(u)du\,.$$

So, the only non-zero term we have it for $i = s-1$ and that is $J_{s-1} = \dfrac{(-1)^{s-1}(s-1)!}{s}$.

$$\begin{aligned} \langle \widetilde{\ln|x|}\,(\varphi_\varepsilon, x)\cdot \widetilde{\delta^{(s-1)}}\,(\varphi_\varepsilon, x)\,, \psi(x)\rangle &= \frac{(-1)^s\psi^{(s-1)}(0)}{s} + O(\varepsilon) \\ &= \frac{(-1)^s}{s}\langle \delta^{(s-1)}(x), \psi(x)\rangle + O(\varepsilon)\,. \end{aligned}$$

Therefor passing to the limit, as $\varepsilon \to 0$, we obtain equation (3.1) proving the theorem. □

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