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ON TRIADS OF COMPOSITIONS IN AN EVEN–DIMENSIONAL SPACE WITH A SYMMETRIC AFFINE CONNECTION

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Abstract. Let $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$ be a net, defined by the independent vector fields $\underset{\alpha}{v}^{\beta}(\alpha = 1,2,\dots,n)$, in a space with a symmetric affine connection A_n. An apparatus for studying of more than one composition for which $\underset{\alpha}{v}^{\beta}$ are eigen-vectors of the matrices of their affinors is developed in [5], [8]. This apparatus is applied in A_{3n} for studying of a triad of compositions without common basic manifolds [7].
In the present paper two triads of compositions $X_n \times \overline{X}_n$, $X_n \times Y_n$, $X_n \times Z_n$ and $X_n \times \overline{X}_n$, $Y_n \times \overline{X}_n$, $Z_n \times \overline{X}_n$, such that each of them has one common basic manifold, are introduced in A_{2n} with the help of the independent vector fields $\underset{\alpha}{v}^{\beta}(\alpha = 1,2,\dots,2n)$. Characteristics of each of the triads of compositions are obtained in the cases when they are cartesian or chebyshevian. The spaces admitting such triads of compositions are characterized.

1. Preliminaries

Let A_N be a space with a symmetric affine connection. The connection coefficients are denoted by $\Gamma^{\sigma}_{\alpha\beta}$. In A_N we consider a composition $X_n \times X_m$ $(n+m=N)$ of two basic differentiable manifolds. Two positions $P(X_n)$ and $P(X_m)$ of the basic manifolds pass thought any point of the space A_N [2, 3, 4].

It is known that a composition is completely defined by the field of the affinor a^{β}_{α}, satisfying the condition [2, 3]

$$a^{\beta}_{\sigma}a^{\sigma}_{\alpha} = \delta^{\beta}_{\alpha}. \tag{1.1}$$

The affinor a^{β}_{α} is called the composition affinor [3, 4]. The projecting affinors $\overset{1}{a}{}^{\beta}_{\alpha}$, $\overset{2}{a}{}^{\beta}_{\alpha}$ [4] are defined by, $\overset{1}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^{\beta}_{\alpha} + a^{\beta}_{\alpha})$, $\overset{2}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^{\beta}_{\alpha} - a^{\beta}_{\alpha})$ and they satisfy the conditions $\overset{1}{a}{}^{\beta}_{\alpha} + \overset{2}{a}{}^{\beta}_{\alpha} = \delta^{\beta}_{\alpha}$ and $\overset{1}{a}{}^{\beta}_{\alpha} - \overset{2}{a}{}^{\beta}_{\alpha} = a^{\beta}_{\alpha}$.

The following characteristics for some special types of compositions are given in [3]:

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Composition of the type $(c,-)$ or composition of the type $(-,c)$, for which the positions $P(X_n)$ or $P(X_m)$ are parallelly translated along any line in the space A_N, is characterized by the condition

$$\overset{1}{a}{}_\nu^\sigma \nabla_\alpha \overset{1}{a}{}_\sigma^\beta = 0 \quad \text{or} \quad \overset{2}{a}{}_\nu^\sigma \nabla_\alpha \overset{2}{a}{}_\sigma^\beta = 0. \tag{1.2}$$

Composition of the type (c,c), for which the positions $P(X_n)$ and $P(X_m)$ are both parallelly translated along any line in the space A_N, is characterized by both conditions in (1.2) or by

$$\nabla_\alpha a_\beta^\sigma = 0. \tag{1.3}$$

Composition of the type $(ch,-)$ or composition of the type $(-,ch)$, for which the positions $P(X_n)$ or $P(X_m)$ are parallelly translated along any line of the manifold X_m or X_n, is characterized by the condition

$$\overset{2}{a}{}_\alpha^\sigma \overset{1}{a}{}_\delta^\nu \nabla_\sigma \overset{1}{a}{}_\nu^\beta = 0 \quad \text{or} \quad \overset{1}{a}{}_\alpha^\sigma \overset{2}{a}{}_\delta^\nu \nabla_\sigma \overset{2}{a}{}_\nu^\beta = 0. \tag{1.4}$$

Composition of the type (ch,ch), for which the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line of the manifolds X_m and X_n, respectively, is characterized by both conditions in (1.4) or by

$$\nabla_{[\alpha} a_{\beta]}^\sigma = 0. \tag{1.5}$$

Let us consider an even-dimensional space A_{2n} equipped with a symmetric affine connection. Let $\underset{\sigma}{v}^\alpha (\sigma = 1,2,...,2n)$ be independent vector fields. The reciprocal covectors $\overset{\sigma}{v}_\alpha$ are defined by

$$\underset{\sigma}{v}^\alpha \overset{\sigma}{v}_\beta = \delta_\beta^\alpha \quad \Leftrightarrow \quad \underset{\sigma}{v}^\alpha \overset{\nu}{v}_\alpha = \delta_\sigma^\nu. \tag{1.6}$$

We introduce the following notations

$$\begin{aligned} &\alpha,\beta,\gamma,\sigma,\nu,\delta,... = 1,2,...,2n; \\ &i,j,s,k,... = 1,2,...,n; \quad \bar{i},\bar{j},\bar{s},\bar{k},... = n+1,n+2,....,2n. \end{aligned} \tag{1.7}$$

Following [5], [8], we consider the affinor

$$a_\alpha^\beta = \underset{i}{v}^\beta \overset{i}{v}_\alpha - \underset{\bar{i}}{v}^\beta \overset{\bar{i}}{v}_\alpha. \tag{1.8}$$

According to (1.6), it follows $a_\sigma^\alpha a_\beta^\sigma = \delta_\beta^\alpha$, i.e. the affinor (1.8) defines a composition $X_n \times \overline{X}_n$ in A_{2n}. Denote by $P(X_n)$ and $P(\overline{X}_n)$ the positions of this composition. The projecting affinors of the composition $X_n \times \overline{X}_n$ are of the form [5]:

$$\overset{1}{a}{}_\alpha^\beta = \underset{i}{v}^\beta \overset{i}{v}_\alpha, \qquad \overset{2}{a}{}_\alpha^\beta = \underset{\bar{i}}{v}^\beta \overset{\bar{i}}{v}_\alpha. \tag{1.9}$$

According to [6], the following derivative equations are valid

$$\nabla_\sigma \underset{\alpha}{v}^\beta = \underset{\alpha}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}^\beta, \qquad \nabla_\sigma \overset{\alpha}{v}_\beta = -\underset{\nu}{\overset{\alpha}{T}}{}_\sigma \overset{\nu}{v}_\beta. \tag{1.10}$$

The lines defined by the vector field $\underset{\alpha}{v}^\sigma$ are denoted by $\left(\underset{\alpha}{v}\right)$, and the net defined by the lines $\left(\underset{\alpha}{v}\right)$ $(\alpha = 1,2,...,2n)$ is denoted by $\left(\underset{1}{v},\underset{2}{v},...,\underset{2n}{v}\right)$. If we choose $\left(\underset{1}{v},\underset{2}{v},...,\underset{2n}{v}\right)$ to be the coordinate net, then

$$\begin{aligned}(1.11)\qquad & \underset{1}{v}^{\sigma}(1,0,0,\dots,0,0),\ \underset{2}{v}^{\sigma}(0,1,0,\dots,0,0),\dots,\underset{2n}{v}^{\sigma}(0,0,0,\dots,0,1),\\ & \overset{1}{v}_{\sigma}(1,0,0,\dots,0,0),\ \overset{2}{v}_{\sigma}(0,1,0,\dots,0,0),\dots,\overset{2n}{v}_{\sigma}(0,0,0,\dots,0,1).\end{aligned}$$

In this case the net $\left(\underset{1}{v},\underset{2}{v},\dots,\underset{2n}{v}\right)$ defines a coordinate system which is adapted to the composition $X_n \times \overline{X}_n$ in the meaning of [2].

In the parameters of the coordinate net the following equality holds [5], [8]

$$(1.12)\qquad \Gamma^{\sigma}_{\alpha\beta} = \underset{\beta}{\overset{\sigma}{T}}{}_{\alpha}.$$

2. Triads of compositions with one common basic manifold

Let the composition $X_n \times \overline{X}_n$ be defined by the affinor (1.8). Let us consider the following affinors

$$\begin{aligned}(2.1)\qquad & f^{\beta}_{\alpha} = \underset{i}{w}^{\beta}\,\overset{i}{w}_{\alpha} - \underset{\bar{i}}{w}^{\beta}\,\overset{\bar{i}}{w}_{\alpha}, \qquad h^{\beta}_{\alpha} = \underset{i}{z}^{\beta}\,\overset{i}{z}_{\alpha} - \underset{\bar{i}}{z}^{\beta}\,\overset{\bar{i}}{z}_{\alpha},\\ & F^{\beta}_{\alpha} = \underset{i}{x}^{\beta}\,\overset{i}{x}_{\alpha} - \underset{\bar{i}}{x}^{\beta}\,\overset{\bar{i}}{x}_{\alpha}, \qquad H^{\beta}_{\alpha} = \underset{i}{y}^{\beta}\,\overset{i}{y}_{\alpha} - \underset{\bar{i}}{y}^{\beta}\,\overset{\bar{i}}{y}_{\alpha}.\end{aligned}$$

where

(2.2)

$$\underset{i}{w}^{\alpha} = \underset{i}{v}^{\alpha}, \qquad \underset{\bar{i}}{w}^{\alpha} = \tfrac{1}{\sqrt{2}}\left(\underset{\bar{i}-n}{v}^{\alpha} + \underset{\bar{i}}{v}^{\alpha}\right), \quad \overset{i}{w}_{\alpha} = \overset{i}{v}_{\alpha} - \overset{n+i}{v}_{\alpha}, \quad \overset{\bar{i}}{w}_{\alpha} = \sqrt{2}\,\overset{\bar{i}}{v}_{\alpha};$$

$$\underset{i}{z}^{\alpha} = \underset{i}{v}^{\alpha}, \qquad \underset{\bar{i}}{z}^{\alpha} = \tfrac{1}{\sqrt{2}}\left(\underset{\bar{i}-n}{v}^{\alpha} - \underset{\bar{i}}{v}^{\alpha}\right), \quad \overset{i}{z}_{\alpha} = \overset{i}{v}_{\alpha} + \overset{n+i}{v}_{\alpha}, \quad \overset{\bar{i}}{z}_{\alpha} = -\sqrt{2}\,\overset{\bar{i}}{v}_{\alpha};$$

$$\underset{i}{x}^{\alpha} = \tfrac{1}{\sqrt{2}}\left(\underset{i}{v}^{\alpha} + \underset{n+i}{v}^{\alpha}\right), \quad \underset{\bar{i}}{x}^{\alpha} = \underset{\bar{i}}{v}^{\alpha}, \qquad \overset{i}{x}_{\alpha} = \sqrt{2}\,\overset{i}{v}_{\alpha}, \qquad \overset{\bar{i}}{x}_{\alpha} = \overset{\bar{i}}{v}_{\alpha} - \overset{\bar{i}-n}{v}_{\alpha};$$

$$\underset{i}{y}^{\alpha} = \tfrac{1}{\sqrt{2}}\left(\underset{i}{v}^{\alpha} - \underset{n+i}{v}^{\alpha}\right), \quad \underset{\bar{i}}{y}^{\alpha} = \underset{\bar{i}}{v}^{\alpha}, \qquad \overset{i}{y}_{\alpha} = \sqrt{2}\,\overset{i}{v}_{\alpha}, \qquad \overset{\bar{i}}{y}_{\alpha} = \overset{\bar{i}}{v}_{\alpha} + \overset{\bar{i}-n}{v}_{\alpha}.$$

and

$$(2.3)\qquad \overset{\alpha}{w}_{\sigma}\,\underset{\beta}{w}^{\sigma} = \delta^{\alpha}_{\beta}, \qquad \overset{\alpha}{z}_{\sigma}\,\underset{\beta}{z}^{\sigma} = \delta^{\alpha}_{\beta}, \qquad \overset{\alpha}{x}_{\sigma}\,\underset{\beta}{x}^{\sigma} = \delta^{\alpha}_{\beta}, \qquad \overset{\alpha}{y}_{\sigma}\,\underset{\beta}{y}^{\sigma} = \delta^{\alpha}_{\beta}.$$

From (2.1), (2.2) and (2.3) it follows that the affinors $f^{\beta}_{\alpha}, h^{\beta}_{\alpha}, F^{\beta}_{\alpha}, H^{\beta}_{\alpha}$ define compositions. Let us denote with $X_n \times Y_n$, $X_n \times Z_n$, $Y_n \times \bar{X}_n$, $Z_n \times \bar{X}_n$, the compositions define by the affinors $f^{\beta}_{\alpha}, h^{\beta}_{\alpha}, F^{\beta}_{\alpha}, H^{\beta}_{\alpha}$, respectively. The triad of compositions $X_n \times \bar{X}_n$, $X_n \times Y_n$, $X_n \times Z_n$ have one common basic manifold X_n, and the triad of compositions $X_n \times \bar{X}_n$, $Y_n \times \bar{X}_n$, $Z_n \times \bar{X}_n$ have one common basic manifold $\bar{X}_n$.

By (1.8), (2.1), (2.2), (2.3) we obtain

$$(2.4)\qquad f^{\beta}_{\alpha} = a^{\beta}_{\alpha} - 2d^{\beta}_{\alpha}, \quad h^{\beta}_{\alpha} = a^{\beta}_{\alpha} + 2d^{\beta}_{\alpha}, \quad F^{\beta}_{\alpha} = a^{\beta}_{\alpha} + 2\underset{1}{d}{}^{\beta}_{\alpha}, \quad H^{\beta}_{\alpha} = a^{\beta}_{\alpha} - 2\underset{1}{d}{}^{\beta}_{\alpha},$$

where

$$(2.5)\qquad d^{\beta}_{\alpha} = \underset{i}{v}^{\beta}\,\overset{n+i}{v}_{\alpha}, \qquad \underset{1}{d}{}^{\beta}_{\alpha} = -\underset{n+i}{v}^{\beta}\,\overset{i}{v}_{\alpha}.$$

We denote by $\overset{1}{f}{}^{\beta}_{\alpha}$, $\overset{2}{f}{}^{\beta}_{\alpha}$ the projecting affinors of the composition $X_n \times Y_n$, by $\overset{1}{h}{}^{\beta}_{\alpha}$, $\overset{2}{h}{}^{\beta}_{\alpha}$ the projecting affinors of the composition $X_n \times Z_n$, by $\overset{1}{F}{}^{\beta}_{\alpha}$, $\overset{2}{F}{}^{\beta}_{\alpha}$ the projecting affinors of the composition $Y_n \times \overline{X}_n$ and by $\overset{1}{H}{}^{\beta}_{\alpha}$, $\overset{2}{H}{}^{\beta}_{\alpha}$ the projecting affinors of the composition $Z_n \times \overline{X}_n$.

From (2.1), (2.2), (2.3), (2.4), (2.5) we obtain

$$\begin{aligned} &\overset{1}{f}{}^{\beta}_{\alpha} = \overset{1}{a}{}^{\beta}_{\alpha} - d^{\beta}_{\alpha}, \quad \overset{2}{f}{}^{\beta}_{\alpha} = \overset{2}{a}{}^{\beta}_{\alpha} + d^{\beta}_{\alpha}, \qquad \overset{1}{h}{}^{\beta}_{\alpha} = \overset{1}{a}{}^{\beta}_{\alpha} + d^{\beta}_{\alpha}, \quad \overset{2}{h}{}^{\beta}_{\alpha} = \overset{2}{a}{}^{\beta}_{\alpha} - d^{\beta}_{\alpha} \\ &\overset{1}{F}{}^{\beta}_{\alpha} = \overset{1}{a}{}^{\beta}_{\alpha} + \underset{1}{d}{}^{\beta}_{\alpha}, \quad \overset{2}{F}{}^{\beta}_{\alpha} = \overset{2}{a}{}^{\beta}_{\alpha} - \underset{1}{d}{}^{\beta}_{\alpha}, \qquad \overset{1}{H}{}^{\beta}_{\alpha} = \overset{1}{a}{}^{\beta}_{\alpha} - \underset{1}{d}{}^{\beta}_{\alpha}, \quad \overset{2}{H}{}^{\beta}_{\alpha} = \overset{2}{a}{}^{\beta}_{\alpha} + \underset{1}{d}{}^{\beta}_{\alpha}, \end{aligned} \tag{2.6}$$

where $\overset{1}{a}{}^{\beta}_{\alpha}$, $\overset{2}{a}{}^{\beta}_{\alpha}$ are the projecting affinors of the composition $X_n \times \overline{X}_n$.

3. Special types of triads of compositions

Lemma 3.1. *If the composition $X_n \times \overline{X}_n$ is of the type (c,c), the conditions $\nabla_\sigma\ d^{\beta}_{\alpha} = 0$ and $\nabla_\sigma\ \underset{1}{d}{}^{\beta}_{\alpha} = 0$ are equivalent.*

Proof. Let the equalities

$$\nabla_\sigma\ a^{\beta}_{\alpha} = 0, \quad \nabla_\sigma\ d^{\beta}_{\alpha} = 0. \tag{3.1}$$

hold. According to (1.3), the first one of the equalities (3.1) is a necessary and sufficient condition for the composition $X_n \times \overline{X}_n$ to be of the type (c,c). By (2.4), (2.6) and (3.1) we obtain

$$\nabla_\sigma\ f^{\beta}_{\alpha} = 0, \quad \nabla_\sigma\ h^{\beta}_{\alpha} = 0, \tag{3.2}$$

i.e. the compositions $X_n \times Y_n$ and $X_n \times Z_n$ are also of the type (c,c). Hence, the positions $P(X_n)$, $P(\overline{X}_n)$, $P(Y_n)$ and $P(Z_n)$ are parallelly translated along any line in the space A_{2n}. From here it follows that the compositions $Y_n \times \overline{X}_n$ and $Z_n \times \overline{X}_n$ are of the type (c,c), too. Then, according to (1.3), we have

$$\nabla_\sigma\ F^{\beta}_{\alpha} = 0, \quad \nabla_\sigma\ H^{\beta}_{\alpha} = 0. \tag{3.3}$$

By (2.4), (3.1) and (3.3) we obtain $\nabla_\sigma\ \underset{1}{d}{}^{\beta}_{\alpha} = 0$.

The statement that from $\nabla_\sigma\ a^{\beta}_{\alpha} = 0$ and $\nabla_\sigma\ \underset{1}{d}{}^{\beta}_{\alpha} = 0$ it follows $\nabla_\sigma\ d^{\beta}_{\alpha} = 0$ is proved analogously. ☐

Theorem 3.1. *If the compositions $X_n \times \overline{X}_n$, $X_n \times Y_n$, $X_n \times Z_n$, $Y_n \times \overline{X}_n$ and $Z_n \times \overline{X}_n$ are of the type (c,c), then the space A_{2n} is affine.*

Proof. According to Lemma 3.1, the compositions $X_n \times \overline{X}_n$, $X_n \times Y_n$, $X_n \times Z_n$, $Y_n \times \overline{X}_n$ and $Z_n \times \overline{X}_n$ are of the type (c,c) iff the conditions (3.1) hold. Having in mind (1.8), (2.5) and (3.2), the equalities (3.1) can be written in the form

$$\nabla_\sigma \left(\underset{i}{v}{}^{\beta} \overset{i}{v}_{\alpha} - \underset{\bar{i}}{v}{}^{\beta} \overset{\bar{i}}{v}_{\alpha} \right) = 0, \qquad \nabla_\sigma \left(\underset{i}{v}{}^{\beta}\ \overset{n+i}{v}{}_{\alpha} \right) = 0. \tag{3.4}$$

Taking into account (1.10), the equalities (3.4) take the form $\underset{j}{\overset{\bar{k}}{T}}{}_{\sigma} = 0$, $\underset{\bar{j}}{\overset{k}{T}}{}_{\sigma} = 0$, $\underset{j}{\overset{k}{T}}{}_{\sigma} - \underset{n+j}{\overset{n+k}{T}}{}_{\sigma} = 0$.

We choose $\left(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{2n}{v}\right)$ for the coordinate net. According to (1.12), the last equalities are equivalent to $\Gamma^{\bar{k}}_{\sigma j} = 0$, $\Gamma^{k}_{\sigma \bar{j}} = 0$, $\Gamma^{k}_{\sigma j} - \Gamma^{n+k}_{\sigma\ n+j} = 0$, from where we obtain $\Gamma^{\sigma}_{\alpha\beta} = 0$. Hence, A_{2n} is an affine space [1]. □

The following statement is obvious

Theorem 3.2. *If the compositions $X_n \times \overline{X}_n$, $X_n \times Y_n$ and $X_n \times Z_n$ are of the type (ch, ch), then they are also of the type (c, ch). If the compositions $X_n \times \overline{X}_n$, $Y_n \times \overline{X}_n$ and $Z_n \times \overline{X}_n$ are of the type (ch, ch), then they are also of the type (ch, c).*

Theorem 3.3. *The following hold:*

(i) *If the compositions $X_n \times \overline{X}_n$, $X_n \times Y_n$ and $X_n \times Z_n$ are of the type (c, ch), the curvature tensor satisfies $R_{\alpha\beta i.}{}^{\sigma} = 0$.*

(ii) *If the compositions $X_n \times \overline{X}_n$, $Y_n \times \overline{X}_n$ and $Z_n \times \overline{X}_n$ are of the type (ch, c), the curvature tensor satisfies $R_{\alpha\beta\bar{i}.}{}^{\sigma} = 0$.*

Proof. (i) Let the compositions $X_n \times \overline{X}_n$, $X_n \times Y_n$ and $X_n \times Z_n$ be of the type (c, ch). Then, the position $P(X_n)$ is parallelly translated along any line in A_{2n}, and the positions $P(\overline{X}_n)$, $P(Y_n)$ and $P(Z_n)$ are parallelly translated along any line of the manifold X_n. According to (1.2) and (1.4), the three compositions are of the type (c, ch) iff the following conditions

$$\overset{1}{a}{}^{\sigma}_{\nu} \nabla_{\alpha} \overset{1}{a}{}^{\beta}_{\sigma} = 0, \quad \overset{1}{a}{}^{\sigma}_{\alpha} \overset{2}{a}{}^{\nu}_{\delta} \nabla_{\sigma} \overset{2}{a}{}^{\beta}_{\nu} = 0, \quad \overset{1}{f}{}^{\sigma}_{\alpha} \overset{2}{f}{}^{\nu}_{\delta} \nabla_{\sigma} \overset{2}{f}{}^{\beta}_{\nu} = 0, \quad \overset{1}{h}{}^{\sigma}_{\alpha} \overset{2}{h}{}^{\nu}_{\delta} \nabla_{\sigma} \overset{2}{h}{}^{\beta}_{\nu} = 0 \tag{3.5}$$

hold.

By (1.8), (1.10) and (2.6) the equalities (3.5) take the form

$$\underset{j}{\overset{\bar{i}}{T}}{}_{\sigma} = 0, \qquad \underset{\bar{j}}{\overset{i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} = 0, \qquad \underset{\bar{j}}{\overset{i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} + \underset{\bar{j}-n}{\overset{i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} - \underset{\bar{j}}{\overset{n+i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} - \underset{\bar{j}-n}{\overset{n+i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} = 0,$$

$$\underset{\bar{j}}{\overset{i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} - \underset{\bar{j}-n}{\overset{i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} + \underset{\bar{j}}{\overset{n+i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} - \underset{\bar{j}-n}{\overset{n+i}{T}}{}_{\sigma}\, \underset{s}{v}{}^{\sigma} = 0.$$

Let the net $\left(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{2n}{v}\right)$ be chosen for the coordinate net. Then, according to (1.11), the last equalities can be written in the form

$$\underset{j}{\overset{\bar{i}}{T}}{}_{\sigma} = 0, \quad \underset{\bar{j}}{\overset{i}{T}}{}_{s} = 0, \quad \underset{\bar{j}}{\overset{i}{T}}{}_{s} + \underset{\bar{j}-n}{\overset{i}{T}}{}_{s} - \underset{\bar{j}}{\overset{n+i}{T}}{}_{s} - \underset{\bar{j}-n}{\overset{n+i}{T}}{}_{s} = 0, \quad \underset{\bar{j}}{\overset{i}{T}}{}_{s} - \underset{\bar{j}-n}{\overset{i}{T}}{}_{s} + \underset{\bar{j}}{\overset{n+i}{T}}{}_{s} - \underset{\bar{j}-n}{\overset{n+i}{T}}{}_{s} = 0. \tag{3.6}$$

By (1.12) and (3.6) we obtain $\Gamma^{\bar{i}}_{\alpha j} = 0$, $\Gamma^{i}_{s\bar{j}} = 0$, $\Gamma^{i}_{s\bar{j}} - \Gamma^{n+i}_{s\ \bar{j}-n} = 0$, $\Gamma^{i}_{s\ \bar{j}-n} - \Gamma^{n+i}_{s\bar{j}} = 0$, from where it follows that $\Gamma^{\sigma}_{\alpha i} = 0$. Then, for the curvature tensor we get $R_{\alpha\beta i.}{}^{\sigma} = 0$.

Condition (ii) is proved analogously. □

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