ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **1** (2012), no.1, 15–21
ISSN 1857-8365, UDC: 517.544.8:517.546

STARLIKE FUNCTION WITH RESPECT TO A BOUNDARY POINT DEFINED BY SUBORDINATION

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Presented at the 8^{th} International Symposium GEOMETRIC FUNCTION THEORY AND APPLICATIONS, 27-31 August 2012, Ohrid, Republic of Macedonia.

Abstract. For a starlike univalent function φ, the class of functions f that are starlike with respect to a boundary point satisfying the subordination

$$2\frac{zf'(z)}{f(z)}+\frac{1+z}{1-z}\prec\varphi(z)$$

is investigated. The integral representation, growth and distortion theorem are proved by relating this functions with Ma and Minda starlike functions. Some earlier results are shown to be special case of the results obtained.

1. Introduction and Motivation

Let $\mathbb{D}=\{z:|z|<1\}$ be the open unit disk of the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the class of analytic function f normalized by $f(0)=0$ and $f'(0)=1$. Let w_0 be an interior or a boundary point of a set $\mathcal{D}$ in $\mathbb{C}$. The set $\mathcal{D}$ is starlike with respect to w_0 if the line segment joining w_0 to every other point in $\mathcal{D}$ lies in the interior of $\mathcal{D}$. If a function $f\in\mathcal{A}$ maps $\mathbb{D}$ onto a starlike domain with respect to origin, then f is a starlike function. The class of starlike functions with respect to origin is denoted by $\mathcal{S}^*$. Analytically,

$$\mathcal{S}^*:=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf'(z)}{f(z)}>0\right\}.$$

Robertson [12] took a leap forward with the characterization of the class $\mathcal{S}^*$ and defined the class $\mathcal{S}_b^*$ of starlike functions with respect to a boundary point. Geometrically, it is the characterization of function $f\in\mathcal{S}_b=\{f(z)=1+d_1z+d_2z^2+\cdots|f\text{univalent}\}$ such that $f(\mathbb{D})$ is starlike with respect to the boundary point $f(1):=\lim_{r\to1^-}f(r)=0$ and lies in a half plane. The analytic description given by Robertson was

$$\mathcal{S}_b^*:=\left\{f\in\mathcal{S}_b:\operatorname{Re}\left(2\frac{zf'(z)}{f(z)}+\frac{1+z}{1-z}\right)>0\right\}.$$

2010 *Mathematics Subject Classification.* 30C45, 30C50, 30C80.
Key words and phrases. Subordination, starlike with respect to boundary point.

This was partially proved in[12]. It was only in 1984 that the characterization was validated by Lyzzaik [9]. Todorov [15] associated this class with functional $f(z)/(1-z)$ and obtained a structured formula and coefficient estimates in the year 1986. Later, Silverman and Silvia, [14] gave a full description of the class of univalent functions on $\mathbb{D}$, the image of which is star-shaped with respect to boundary point. Since then, this class of starlike functions with respect to a boundary point gained notable interest among geometric function theorist and also other researchers. See [1, 3, 4, 5, 7, 8].

On the other hand, Ma and Minda [10] gave a unified presentation of the class starlike using the method of subordination. For two functions h and g in $\mathcal{A}$, the function h is subordinate to g, written $h(z) \prec g(z)$, if there exists a function $w \in \mathcal{A}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $h(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $h(z) \prec g(z)$ is equivalent to $h(0) = g(0)$ and $h(\mathbb{D}) \subset g(\mathbb{D})$. A function $h \in \mathcal{A}$ is starlike if $zh'(z)/h(z)$ is subordinated to $(1+z)/(1-z)$. Ma and Minda [10] introduced the class

$$\mathcal{S}^*(\varphi) = \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec \varphi(z) \right\},$$

where φ is an analytic function with positive real part in $\mathbb{D}$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. A function $f \in \mathcal{S}^*(\varphi)$ is called Ma and Minda starlike (with respect to φ). The class $\mathcal{S}^*(\beta)$ consisting of starlike functions of order β, $0 \le \beta < 1$ and the class $\mathcal{S}^*(A, B)$ of Janowski starlike functions are special cases of $\mathcal{S}^*(\varphi)$ when $\varphi(z) := (1+(1-2\beta)z)/(1-z)$ and $\varphi(z) := (1+Az)/(1+Bz)$ for $-1 \le B < A \le 1$ respectively.

In the same direction and motivated mainly by [9, 10] and [12], we consider the following class.

Definition 1.1. *Let $f \in \mathcal{S}_b$. Also let φ be an analytic function with positive real part $\mathbb{D}$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. The function $f \in \mathcal{S}_b^*(\varphi)$ if the subordination*

$$2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \prec \varphi(z), \qquad z \in \mathbb{D} \tag{1.1}$$

holds.

For $\varphi(z) = (1+Az)/(1+Bz)$, $(-1 \le B < A \le 1)$ denote the class $\mathcal{S}_b^*(\varphi)$ by $\mathcal{S}_b^*(A, B)$. For $0 \le \beta < 1$, $A = 1 - 2\beta$ and $B = -1$, denote $\mathcal{S}_b^*(A, B)$ by $\mathcal{S}_b^*(\mu, \beta)$.

The class $\mathcal{S}_b^*(\varphi)$ defined by subordination is investigated to obtain representation, estimates for f and f' and subordination conditions. We obtained some interesting result in a wider context and our approach is mainly based on [10].

2. Representation for the class $\mathcal{S}_b^*(\varphi)$

Theorem 2.1. *The function $f \in \mathcal{S}_b^*(\varphi)$ if and only if there exist p satisfying $p \prec \varphi$ such that*

$$f(z) = (1-z)\exp\left(\frac{1}{2}\int_0^z \frac{p(\zeta)-1}{\zeta}d\zeta\right).$$

Proof. Let $f \in \mathcal{S}_b^*(\varphi)$. Then define $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = 2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}.$$

Then $f \in \mathcal{S}_b^*(\varphi)$ implies that $p \prec \varphi$. Rewriting the above equation and integrating from 0 to z, it follows that

$$\log\left(\frac{f(z)}{1-z}\right)^2 = \int_0^z \frac{p(\zeta)-1}{\zeta}d\zeta.$$

The desired result follows from this. The converse follows easily. $\square$

3. Estimates for f and f' in the class $\mathcal{S}_b^*(\varphi)$

Theorem 3.1. *Let h_φ be the analytic function with $h_\varphi(0)=0$, $h'_\varphi(0)=1$ satisfying the equation $zh'_\varphi(z)/h_\varphi(z)=\varphi(z)$. If $f \in \mathcal{S}_b^*(\varphi)$ then*

$$\left(\frac{-h_\varphi(-r)}{r}\right)^{\frac{1}{2}}|1-z| \le |f(z)| \le \left(\frac{h_\varphi(r)}{r}\right)^{\frac{1}{2}}|1-z|, \qquad |z|=r. \tag{3.1}$$

Proof. Define the function $h \in \mathcal{A}$ by

$$h(z) = \frac{z}{(1-z)^2}f(z)^2, \qquad z \in \mathbb{D}. \tag{3.2}$$

Since f is univalent and $f(1) := \lim_{r\to 1^-} f(r) = 0$, it is clear that $f(z) \neq 0$ in $\mathbb{D}$. Therefore, the function h is well-defined and analytic in $\mathbb{D}$. A computation shows that

$$\frac{zh'(z)}{h(z)} = 2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}. \tag{3.3}$$

Hence we have the relation $f \in \mathcal{S}_b^*(\varphi)$ if and only if $h \in \mathcal{S}^*(\varphi)$. Ma and Minda [10, Corollary 1'] have shown that for $h \in \mathcal{S}^*(\varphi)$, $-h_\varphi(-r) \le |h(z)| \le h_\varphi(r)$, for $|z|=r$. Using this inequality for h in (3.2), gives

$$-h_\varphi(-r) \le \left|\frac{z}{(1-z)^2}f(z)^2\right| \le h_\varphi(r), \qquad |z|=r.$$

and hence the desired result follows. $\square$

If $f \in \mathcal{S}_b^*(A,B)$ and hence

$$h_\varphi(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ z\exp(Az), & B = 0, \end{cases}$$

then

$$|1-z|(1-Br)^{\frac{A-B}{2B}} \le |f(z)| \le |1-z|^2(1+Br)^{\frac{A-B}{2B}} \quad \text{for} \quad B \neq 0$$
$$|1-z|\exp\left(\frac{-Ar}{2}\right) \le |f(z)| \le |1-z|\exp\left(\frac{Ar}{2}\right) \quad \text{for} \quad B = 0.$$

If $f \in \mathcal{S}_b^*(\beta)$ and

$$h_\varphi(z) = \frac{z}{(1-z)^{2-2\beta}}$$

then

$$\frac{|1-z|}{(1+r)^{1-\beta}} \le |f(z)| \le \frac{|1-z|}{(1-r)^{1-\beta}}.$$

Theorem 3.2. *Let $\varphi(z) = zh'_\varphi(z)/h_\varphi(z)$ and $f \in \mathcal{S}_b^*(\varphi)$. Then, for $|z|=r$*

$$\left|\arg\frac{f(z)}{1-z}\right| \le \frac{1}{2}\max_{|z|=r} \arg\frac{h_\varphi(z)}{z}.$$

Similar results for $f \in \mathcal{S}_b^*(A,B)$ and $f \in \mathcal{S}_b^*(\beta)$ could be easily obtained.

Theorem 3.3. *Let* $\varphi(z) = zh'_{\varphi}(z)/h_{\varphi}(z)$ *and*

$$\min_{|z|=r} |\varphi(z)| = \varphi(-r) \quad \text{and} \quad \max_{|z|=r} |\varphi(z)| = \varphi(r). \tag{3.4}$$

Also let

$$H_{\varphi_1} = \frac{|1-z|}{2r}\left(\frac{h_{\varphi}(-r)}{-r}\right)^{\frac{1}{2}}\left(-\left|\frac{1+z}{1-z}\right| + \varphi(-r)\right)$$

and

$$H_{\varphi_2} = \frac{|1-z|}{2r}\left(\frac{h_{\varphi}(r)}{r}\right)^{\frac{1}{2}}\left(\left|\frac{1+z}{1-z}\right| + \varphi(r)\right).$$

If $f \in \mathcal{S}_b^*(\varphi)$ then

$$H_{\varphi_1} \leq |f'(z)| \leq H_{\varphi_2}.$$

Proof. By Definition 1.1, for $f \in \mathcal{S}_b^*(\varphi)$, we have

$$2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \prec \varphi(z), \qquad z \in \mathbb{D}.$$

When (3.4) holds, the above subordination indicates that

$$\varphi(-r) \leq \left|2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right| \leq \varphi(r) \qquad |z| = r.$$

Rewriting the above inequality and combining with Theorem 3.1, the desired results follows. $\square$

We could accordingly derive similar results for $f \in \mathcal{S}_b^*(A,B)$ and $f \in \mathcal{S}_b^*(\beta)$.

4. Necessary and Sufficient Condition

Theorem 4.1. *Let* φ *be a convex univalent function defined on* $\mathbb{D}$. *The function* $f \in \mathcal{S}_b^*(\varphi)$ *if and only if for all* $|s| \leq 1$, $|t| \leq 1$,

$$\frac{s}{t}\left(\frac{1-tz}{1-sz}\right)^2\left(\frac{f(sz)}{f(tz)}\right)^2 \prec \frac{F(sz)}{F(tz)}$$

where $F(z) = z\exp\left(\int_0^z ((\varphi(\zeta)-1)/\zeta)d\zeta\right)$.

Proof. Ruscheweyh [13, Theorem 1] have shown that for φ a convex univalent function, F as in the hypothesis and $h \in \mathcal{A}$, $zh'(z)/h(z) \prec \varphi(z)$ if and only if for all $|s| \leq 1$, $|t| \leq 1$

$$\frac{h(sz)}{h(tz)} \prec \frac{F(sz)}{F(tz)}. \tag{4.1}$$

From the relation (3.3), we know that $f \in \mathcal{S}_b^*(\varphi)$ if and only if $h \in \mathcal{S}^*(\varphi)$. Substituting (3.2) in (4.1), we have

$$\frac{\frac{sz}{(1-sz)^2}f(sz)^2}{\frac{tz}{(1-tz)^2}f(tz)^2} \prec \frac{F(sz)}{F(tz)}$$

and hence the desired result follows. $\square$

Theorem 4.2 is the special cases of Theorem 4.1 when $s = 1$ and $t = 0$. However, we prove the below without the convexity assumption on φ.

Theorem 4.2. *If $f \in \mathcal{S}_b^*(\varphi)$ then*

$$\left(\frac{f(z)}{1-z}\right)^2 \prec \frac{h_\varphi(z)}{z},$$

where $h_\varphi(z) = z\exp\left(\int_0^z ((\varphi(\zeta)-1)/\zeta)d\zeta\right)$.

Proof. Clearly $zh_\varphi'(z)/h_\varphi(z) = \varphi(z)$. If $h \in \mathcal{S}^*(\varphi)$, then $zh'(z)/h(z) \prec zh_\varphi'(z)/h_\varphi(z)$. Therefore by [10, Theorem 1'] $h(z)/z \prec h_\varphi(z)/z$. Let $h(z)$ be defined as in (3.2) and hence we arrive to the desired conclusion. □

Similar results in this section for $f \in \mathcal{S}_b^*(A,B)$ and $f \in \mathcal{S}_b^*(\beta)$ could be easily obtained.

5. Coefficient Estimate for $f \in \mathcal{S}_b^*(\varphi)$

Theorem 5.1. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$. If $f \in \mathcal{S}_b^*(\varphi)$, then the coefficients d_1, d_2, d_3 satisfy the following inequalities:*

$$|d_1| \le \frac{B_1}{2} + 1$$

$$|d_2| \le \frac{B_1}{4}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{2}\right|\right\} + \frac{B_1}{2}$$

$$|d_3| \le \frac{B_1}{6}H\left(\frac{6B_1^2+16B_2}{8B_1}, \frac{B_1^3+6B_1B_2+8B_3}{8B_1}\right) + \frac{B_1}{4}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{2}\right|\right\}$$

where $H(q_1,q_2)$[1] *is as defined in* [11] *(see also* [2, Lemma 3]*) and*

$$|d_2 - \nu d_1^2| \le \begin{cases} \frac{B_1}{4}\left(\frac{B_2}{B_1} - (2\nu-1)\frac{B_1}{2}\right) + (2\nu+1)\frac{B_1}{2} + 2\nu, & \nu \le \sigma_1; \\ \frac{B_1}{4} + (2\nu+1)\frac{B_1}{2} + 2\nu, & \sigma_1 \le \nu \le \sigma_2; \\ \frac{B_1}{4}\left((2\nu-1)\frac{B_1}{2} - \frac{B_2}{B_1}\right) + (2\nu+1)\frac{B_1}{2} + 2\nu, & \nu \ge \sigma_2. \end{cases}$$

where

$$\sigma_1 = \frac{1}{B_1}\left(\frac{B_2}{B_1} - 1\right) + \frac{1}{2} \qquad \sigma_2 = \frac{1}{B_1}\left(\frac{B_2}{B_1} + 1\right) + \frac{1}{2}.$$

Proof. Define the function $g(z) = 1 + g_1z + g_2z^2 + \cdots$ by $g(z) = f(z)/(1-z)$. Then, a computation shows that

$$2\frac{zg'(z)}{g(z)} + 1 = 2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}.$$

Since $f \in \mathcal{S}_b^*(\varphi)$, there is an analytic function $w(z) = w_1z + w_2z^2 + \cdots$ such that

$$2\frac{zg'(z)}{g(z)} + 1 = \varphi(w(z)).$$

Comparing the coefficients of z, z^2 and z^3, we see that

$$g_1 = \frac{B_1w_1}{2}, \qquad g_2 = \frac{B_1}{4}\left(w_2 + \left(\frac{B_2}{B_1} + \frac{B_1}{2}\right)w_1^2\right) \quad \text{and}$$

$$g_3 = \frac{B_1}{6}\left(w_3 + \left(\frac{6B_1^2+16B_2}{8B_1}\right)w_1w_2 + \left(\frac{B_1^3+6B_1B_2+8B_3}{8B_1}\right)w_1^3\right).$$

[1]The expression for H is too lengthy to be reproduced here. See [11] or [2] for the full expression.

In view of the well known inequality $|w_1| \le 1$, we have $|g_1| \le B_1/2$. Applying [6, inequality 7, p.10] and [2, Lemma 3] (see also [11]), we get

$$|g_2| \le \frac{B_1}{4} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{2}\right|\right\}$$

and

$$|g_3| \le \frac{B_1}{6} H\left(\frac{6B_1^2 + 16B_2}{8B_1}, \frac{B_1^3 + 6B_1B_2 + 8B_3}{8B_1}\right)$$

respectively. Also, we see that by applying [2, Lemma 1] (see also [10]) to inequality

$$g_2 - \nu g_1^2 = \frac{B_1}{4}\left(w_2 - \left((2\nu - 1)\frac{B_1}{2} - \frac{B_2}{B_1}\right) w_1^2\right)$$

yields

$$|g_2 - \nu g_1^2| \le \begin{cases} \frac{B_1}{4}\left(\frac{B_2}{B_1} - (2\nu-1)\frac{B_1}{2}\right), & \nu \le \sigma_1; \\ \frac{B_1}{4}, & \sigma_1 \le \nu \le \sigma_2; \\ \frac{B_1}{4}\left((2\nu-1)\frac{B_1}{2} - \frac{B_2}{B_1}\right), & \nu \ge \sigma_2 \end{cases}$$

for σ_1 and σ_2 as in the hypothesis. Todorov, in [15] shows that for $g(z) = 1 + \sum_1^\infty g_n z^n$ the coefficient $g_n = 1 + d_1 + d_2 + \cdots + d_n$ and hence from the above relation the desired results are obtained. □

Remark 5.1. *When $\varphi(z) = (1+z)/(1-z)$, our results coincides with* [15, Corollary 2.3].

Acknowledgment

The work here is supported by UKM-ST-06-FRGS00244-2010.

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