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SOME RESULTS OVER THE FIRST DERIVATIVE OF ANALYTIC FUNCTIONS

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ABSTRACT. Let f be an analytic function in the open unit disk and normalized such that $f(z) = z + a_n z^n + \ldots$, $n \in \mathbb{N}$, $n \geq 2$. In this work we use differential subordinations to study the expression

$$z\,\frac{f'(z)-1}{f(z)-z}$$

and give estimates of $|f'(z)-1|$. Also, sufficient conditions for a function to be with bounded turning are obtained and some open problems are posed. This work is a continuation of the results published in [7].

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}(\mathbb{D})$ be the class of functions that are just analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_n$, $n \in \mathbb{N}$, $n \geq 2$, be its subclass consisting of functions f that are analytic in $\mathbb{D}$ and normalized such that $f(z) = z + a_n z^n + \ldots$, $a_n \neq 0$. Further, $f \in \mathcal{A} \equiv \mathcal{A}_1$ is such that $f(0) = f'(0) - 1 = 0$.

Now, $f \in \mathcal{A}$ is a *starlike function* if and only if $\mathrm{Re}[zf'(z)/f(z)] > 0$, $z \in \mathbb{D}$. All such functions are univalent and the corresponding class is denoted by S^*. Another subclasses of univalent functions are R_α $(0 \leq \alpha < 1)$ and $\mathcal{R}(\beta)$ $(0 < \alpha \leq 1)$ consisting of functions $f \in \mathcal{A}$ such that $\mathrm{Re}\, f'(z) > \alpha$ $(z \in \mathbb{D})$ and $|\arg f'(z)| < \alpha\pi/2$ $(z \in \mathbb{D})$, respectively. The special case $\mathbb{R} \equiv \mathbb{R}_0 = \mathcal{R}(1)$ is the well known *class of functions with bounded turning*. The interest for this class comes from the result of Krzyz [6] that S^* does not contain R and R does not contain S^*. In this paper, using a method from the theory of differential subordinations (valuable references on this topic are [1] and [2]), we will receive criteria over the expression

$$z\,\frac{f'(z)-1}{f(z)-z}$$

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that will embed a function $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $n \geq 2$, in the class R. This work is a continuation of the results published in [7]. From the theory of first-order differential subordinations we will use the following lemma:

Lemma 1.1. [4] *Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:*

(i) Q is starlike in the unit disk $\mathbb{D}$,

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re} \left[\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right] > 0, \ z \in \mathbb{D}.$

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{1.1}$$

then $p(z) \prec q(z)$, and q is the best dominant of (1.1)).

2. Main results and consequences

Using Lemma 1.1 we will receive conclusions that will later lead to criteria for a function f to be in the class R.

Theorem 2.1. *Let $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $n \geq 2$, such that $f(z) \neq z$ for all $z \in \mathbb{D} \setminus \{0\}$, and let $a_n = \frac{f^{(n)}(0)}{n!}$. If $0 < |\lambda| \leq |a_n|$ and*

$$z\,\frac{f'(z) - 1}{f(z) - z} - n \prec \frac{\lambda z}{a_n + \lambda z} =: h_1(z), \tag{2.1}$$

then

$$\frac{f(z) - z}{z^n} \prec a_n + \lambda z, \tag{2.2}$$

and the function $a_n + \lambda z$ is the best dominant of (2.1). Even more,

$$\left|\frac{f(z) - z}{z^n} - a_n\right| < \lambda, \ z \in \mathbb{D}, \tag{2.3}$$

and this conclusion is sharp, i.e. in the inequality (2.3) the parameter $|\lambda|$ can not be replaced by a smaller number so that the implication holds.

Proof. Let choose $\theta(w) = 0$, $\phi(w) = \frac{1}{w}$, $p(z) = \frac{f(z)-z}{z^n}$ and $q(z) = a_n + \lambda z$. Then $\theta,\ \phi \in \mathcal{H}(D)$, where $D = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $D \supset q(\mathbb{D})$ since $0 < |\lambda| \leq |a_n|$. Further,

$$Q(z) = h(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)} = \frac{\lambda z}{a_n + \lambda z}$$

and

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \frac{1}{1 + \lambda/a_n z} > 0, \quad z \in \mathbb{D}.$$

So, q is univalent in $\mathbb{D}$, $p \in \mathcal{H}(\mathbb{D})$, $p(0) = q(0) = a_n$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$, i.e. $p(\mathbb{D}) \subset D$, and all the conditions of Lemma 1.1 are satisfied. Concerning that the subordinations (1.1) and (2.1) are equivalent, we receive subordination (2.2) and its equivalent inequality (2.3).

For the sharpness of our result, let assume that subordination (2.1) and inequality $\left|\frac{f(z)-z}{z^n}-a_n\right|<|\lambda_1|$, $z\in\mathbb{D}$, hold, i.e. $\frac{f(z)-z}{z^n}\prec a_n+\lambda_1 z$. But, the function $a_n+\lambda z$ is the best dominant of (2.1), meaning that $a_n+\lambda z\prec a_n+\lambda_1 z$, i.e. $|\lambda|\le|\lambda_1|$. □

It is easy to verify that if $0<|\lambda|<|a_n|$, then $h_1(\mathbb{D})$ (where h_1 was defined in (2.1)) is an open disk with the center

$$c=\frac{1}{2}\cdot\left[h_1\left(e^{i\arg(a_n/\lambda)}\right)+h\left(-e^{i\arg(a_n/\lambda)}\right)\right]=\frac{|\lambda|^2}{|\lambda|^2-|a_n|^2}, \tag{2.4}$$

and radius

$$r=\left|h_1\left(e^{i\arg(a_n/\lambda)}\right)-c\right|=\frac{|\lambda|\cdot|a_n|}{|a_n|^2-|\lambda|^2}. \tag{2.5}$$

Therefore, Theorem 2.1 brings the following corollary.

Corollary 2.1. *Let $f\in\mathcal{A}_n$, $n\in\mathbb{N}$, $n\ge 2$, such that $f(z)\ne z$ for all $z\in\mathbb{D}\setminus\{0\}$, and let $a_n=\frac{f^{(n)}(0)}{n!}$. If $0<|\lambda|<|a_n|$ and*

$$\left|z\,\frac{f'(z)-1}{f(z)-z}-n-\frac{|\lambda|^2}{|\lambda|^2-|a_n|^2}\right|<\frac{|\lambda|\cdot|a_n|}{|a_n|^2-|\lambda|^2},\quad z\in\mathbb{D},$$

then

$$\left|\frac{f(z)-z}{z^n}-a_n\right|<|\lambda|,\quad z\in\mathbb{D}.$$

This implication is sharp (the radius of the open disk from the conclusion is the smallest possible so that the corresponding implication holds) due to the function $f(z)=z+a_nz^n+\lambda z^{n+1}$, $a_n\ne 0$.

In the case when $n=2$ we receive

Corollary 2.2. *Let $f\in\mathcal{A}_2$, and $\lambda\in\mathbb{C}$ with $\frac{4}{5}|a_2|\le|\lambda|<|a_2|$, where $a_2=\frac{f''(0)}{2}$. Also, let denote*

$$\mu:=\begin{cases}-2+\frac{|\lambda|}{|a_2|-|\lambda|}, & \text{if } \frac{4}{5}|a_2|\le|\lambda|\le\sqrt{\frac{2}{3}}\,|a_2|\\ 2+\frac{|\lambda|}{|a_2|+|\lambda|}, & \text{if } \sqrt{\frac{2}{3}}\,|a_2|\le|\lambda|<|a_2|\end{cases}.$$

If

$$|f'(z)-1|<\mu\left|\frac{f(z)}{z}-1\right|,\quad z\in\mathbb{D}\setminus\{0\}, \tag{2.6}$$

then

$$\left|\frac{f(z)-z}{z^2}-a_2\right|<\eta_1:=|\lambda|,\quad z\in\mathbb{D}, \tag{2.7}$$

and

$$|f'(z)-1|<\eta_2,\quad z\in\mathbb{D}, \tag{2.8}$$

where

$$\eta_2:=\begin{cases}\frac{(|a_2|+|\lambda|)\cdot(2|a_2|-3|\lambda|)}{|\lambda|-|a_2|}, & \text{if } \frac{4}{5}|a_2|\le|\lambda|\le\sqrt{\frac{2}{3}}\,|a_2|,\\ 2|a_2|+3|\lambda|, & \text{if } \sqrt{\frac{2}{3}}\,|a_2|\le|\lambda|<|a_2|.\end{cases}$$

Moreover, the implication (2.6) $\Rightarrow$ (2.7) *is sharp for* $\sqrt{\frac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$, *and the implication* (2.6) $\Rightarrow$ (2.8) *is sharp for* $\sqrt{\frac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$, *i.e. for these ranges of* $|\lambda|$, *the values* η_1 *and* η_2 *are the smallest ones so that the corresponding implications hold.*

Also, if $\eta_2 < 1$, *then* f *is univalent with bounded turning, i.e.* $f \in \mathrm{R}_{\alpha_1}$ *and* $f \in \mathcal{R}(\alpha_2)$, *where* $\alpha_1 = 1 - \eta_2$ *and* $\alpha_2 = \arcsin \eta_2$.

Proof. First we will prove inequality (2.7). The assumption (2.6) leads to

$$|f'(z) - 1| < \mu \cdot \left|\frac{f(z)}{z} - 1\right| = \mu \cdot \left|\frac{f(z) - z}{z}\right|, \quad z \in \mathbb{D} \setminus \{0\},$$

meaning that $\frac{f(z)-z}{z} \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, hence $f(z) \neq z$ for all $z \in \mathbb{D} \setminus \{0\}$. Also, the inequality (2.6) implies

$$\left|z\,\frac{f'(z) - 1}{f(z) - z}\right| < \mu, \quad z \in \mathbb{D} \setminus \{0\},$$

and letting $z \to 0$ in the above inequality we obtain that $\mu \ge 2$ is a necessary condition for the above inequality to hold in the case $z = 0$.

It is easy to check that

$$\mu = \frac{|\lambda| \cdot |a_2| - \left|3|\lambda|^2 - 2|a_2|^2\right|}{|a_2|^2 - |\lambda|^2} = r - |2 + c|,$$

where c and r are defined as in (2.4) and (2.5), respectively, and that $\mu \ge 2$ whenever $|\lambda| \ge \frac{4}{5}|a_2|$. Further, we can write

$$\left|z\,\frac{f'(z) - 1}{f(z) - z} - (2 + c) + (2 + c)\right| < \mu, \quad z \in \mathbb{D} \setminus \{0\},$$

and it follows that

$$\left|z\,\frac{f'(z) - 1}{f(z) - z} - (2 + c)\right| < \mu + |2 + c| = r, \quad z \in \mathbb{D} \setminus \{0\}.$$

The above inequality holds for $z = 0$, since $|c| < \mu + |2 + c| = r$ for $0 < |\lambda| < |a_2|$, and thus, from the first part of the Theorem 2.1'(i) for the special case $n = 2$ we get (2.7)).

From the assumption (2.6) we get

$$|f'(z) - 1| < \mu \cdot \left|\frac{f(z)}{z} - 1\right| < \mu \cdot \left|\frac{f(z) - z}{z^2}\right|, \quad z \in \mathbb{D} \setminus \{0\}, \tag{2.9}$$

and the inequality (2.8) follows from (2.7) and (2.9)), having in mind that $\eta_2 = \mu\,(|a_2| + \lambda)$.

The implication (2.6) $\Rightarrow$ (2.7) is sharp for $\sqrt{\frac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$, and the implication (2.6) $\Rightarrow$ (2.8) is sharp for $\sqrt{\frac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$, since for the function $f(z) = z + a_2 z^2 + \lambda z^3$ we have:

$$|f'(z) - 1| = |z| \cdot |2a_2 + 3\lambda z| < 2|a_2| + 3|\lambda|, \quad z \in \mathbb{D},$$
$$\left|\frac{f(z)}{z} - 1\right| = |z| \cdot |a_2 + \lambda z|$$

and

$$\left|\frac{f(z) - z}{z^2} - a_2\right| = |\lambda| \cdot |z| < |\lambda|, \; z \in \mathbb{D}.$$

The assertion (2.6) is equivalent to

$$\mu > \left|\frac{3\lambda z + 2a_2}{\lambda z + a_2}\right|, \quad z \in \mathbb{D} \setminus \{0\},$$

and a simple computation shows that

$$\sup\left\{\left|\frac{3\lambda z + 2a_2}{\lambda z + a_2}\right| : \ z \in \mathbb{D} \setminus \{0\}\right\} = 2 + \frac{|\lambda|}{|a_2| + |\lambda|},$$

whenever $|\lambda| < |a_2|$, hence

$$\mu \geq 2 + \frac{|\lambda|}{|a_2| + |\lambda|},$$

which holds for $\sqrt{\frac{2}{3}}\,|a_2| \leq |\lambda| < |a_2|$.

Since

$$-2 + \frac{|\lambda|}{|a_2| - |\lambda|} < 2 + \frac{|\lambda|}{|a_2| + |\lambda|}, \quad \text{if} \quad |\lambda| < \sqrt{\frac{2}{3}}\,|a_2|,$$

the function $f(z) = z + a_2 z^2 + \lambda z^3$ shows that the implication (2.6) $\Rightarrow$ (2.7) is not sharp for $\frac{4}{5}\,|a_2| < |\lambda| < \sqrt{\frac{2}{3}}\,|a_2|$.

Finally, from (2.7) and the definitions of the classes R_α and $R(\alpha)$ we receive $f \in R_{\alpha_1}$ and $f \in R(\alpha_2)$. $\square$

For $\eta_2 = 1$, Corollary 2.2 reduces to the next example:

Example 2.1. *Let $f \in \mathcal{A}_2$, with $\frac{1}{5} < |a_2| \leq \frac{5}{18}$, where $a_2 = \frac{f''(0)}{2}$. Also, let*

$$\mu_* := \begin{cases} \frac{3}{1+|a_2|}, & \textit{if} \quad 0.2 = \frac{1}{5} < |a_2| \leq \frac{1}{2+\sqrt{6}} = 0.22474\ldots, \\ \frac{1}{|\lambda|_* + |a_2|}, & \textit{if} \quad \frac{1}{2+\sqrt{6}} \leq |a_2| \leq \frac{5}{18} = 0.27\ldots, \end{cases}$$

where

$$|\lambda_*| := \frac{-(1+|a_2|) + \sqrt{25|a_2|^2 + 14|a_2| + 1}}{6}.$$

If

$$|f'(z) - 1| < \mu_* \cdot \left|\frac{f(z)}{z} - 1\right|, \quad z \in \mathbb{D} \setminus \{0\}, \tag{2.10}$$

then

$$|f'(z) - 1| < 1, \quad z \in \mathbb{D}. \tag{2.11}$$

This implication is sharp for $\frac{1}{5} < |a_2| \leq \frac{1}{2+\sqrt{6}} = 0.22474\ldots$. Also, the function f is univalent with bounded turning, i.e. $f \in \mathrm{R}$.

Proof. We need to prove that conditions of Corollary 2.2, in the case $\eta_2 = 1$, are equivalent to the assumptions of this example.

For the case when $\frac{4}{5}\,|a_2| \leq |\lambda| \leq \sqrt{\frac{2}{3}}\,|a_2|$, then $\eta_2 = \mu_*\left(|a_2| + |\lambda|\right) = 1$ if and only if $\mu_* = \frac{1}{|\lambda| + |a_2|}$, i.e.

$$-2\left(|\lambda| + |a_2|\right) + \frac{\lambda\left(|a_2| + |\lambda|\right)}{|a_2| - |\lambda|} = 1,$$

or in other words

$$|\lambda| = |\lambda_*| := \frac{-\left(1 + |a_2|\right) + \sqrt{25|a_2|^2 + 14|a_2| + 1}}{6}.$$

Here, we considered only the positive sign of the square root since the negative one leads to negative values of $|\lambda|$. Further, the inequalities

$$\frac{4}{5}|a_2| \le |\lambda| = |\lambda_*| \le \sqrt{\frac{2}{3}}|a_2|$$

are equivalent to

$$0.22474\ldots = \frac{1}{2+\sqrt{6}} \le |a_2| \le \frac{5}{18} = 0.27\ldots.$$

In a similar way, for the case $\sqrt{\frac{2}{3}}|a_2| \le |\lambda| < |a_2|$ we have $\eta_2 = 1$ if and only if $3|\lambda| + 2|a_2| = 1$, i.e. $|\lambda| = \frac{1-2|a_2|}{3}$. A simple calculus shows that

$$\sqrt{\frac{2}{3}}|a_2| \le |\lambda| < |a_2|$$

is equivalent to

$$0.2 = \frac{1}{5} < |a_2| \le \frac{1}{2+\sqrt{6}} = 0.22474\ldots,$$

which completes the proof. □

Remark 2.1. *Weather implications* (2.6) $\Rightarrow$ (2.8) *for* $\frac{4}{5}|a_2| \le |\lambda| < \sqrt{\frac{2}{3}}|a_2|$ *(Corollary 2.2 and* (2.10) $\Rightarrow$ (2.11) *for* $\frac{1}{5} < |a_2| \le \frac{1}{2+\sqrt{6}} = 0.22474\ldots$ *(Example 2.1 are sharp are still* ***open problems****.*

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