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# AN EXTREME FUNCTION FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, the subclass $\mathcal{B}$ of $\mathcal{A}$ concerned with the class of uniformly convex functions or the class $\mathcal{S}_p$ is defined. By virtue of some properties of uniformly convex functions and the class $\mathcal{S}_p$, an extreme function of the class $\mathcal{B}$ and its power series are considered.

## 1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at $\zeta$ in $\mathbb{U}$ onto the convex arc (or the starlike arc) with respect to $f(\zeta)$. These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Rønning [4] and [5], that is, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}).$$

It is independently studied by Ma and Minda [3]. Further, a function $f(z) \in \mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

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This class $\mathcal{S}_p$ was introduced by Rønning [4]. We easily know that the relation $f(z) \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{S}_p$. In view of these classes, we introduce the subclass $\mathcal{B}$ of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{z}{f(z)}\right) > \left|\frac{z}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

We try to derive some properties of functions $f(z)$ belonging to the class $\mathcal{B}$.

**Remark 1.1.** For $f(z) \in \mathcal{B}$, we write $w(z) = \dfrac{f(z)}{z} = u + iv$, then $w$ lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by a kind of teardrop-shape domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.$$

## 2. An extreme function for the class $\mathcal{B}$

In this section, we would like to exhibit an extreme function of the class $\mathcal{B}$ and its power series. For our results, we need to recall here some properties of the class $\mathcal{S}_p$.

**Lemma 2.1. (Rønning [4])**. *The extremal function $f(z)$ for the class $\mathcal{S}_p$ is given by*

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

By using the expansion of logarithmic part of $\dfrac{zf'(z)}{f(z)}$ in Lemma 2.1, we get

**Lemma 2.2. (Rønning [4])**. *The power series of $\dfrac{zf'(z)}{f(z)}$ is following*

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \\ &= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\,\frac{1}{2n+1-2k}\right)z^n.\end{aligned}$$

From Remark 1.1 and Lemma 2.1, we have the first result for the class $\mathcal{B}$.

**Theorem 2.1.** *The extreme function $f(z)$ for the class $\mathcal{B}$ is given by*

$$f(z)=\frac{z}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

*Proof.* Let us consider the function $\frac{f(z)}{z}$ as given by

$$\frac{f(z)}{z}=\frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

It sufficies to show that $\frac{f(z)}{z}$ maps $\mathbb{U}$ onto the interior of the domain such that

$$u^4-2u^3+2u^2v^2-2uv^2+v^4+v^2<0,$$

implying that $\frac{f(z)}{z}$ maps the unit circle onto the boundary of the domain. Taking $z=e^{i\theta}$, we obtain that

$$\begin{aligned}
\frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}&=\frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right)\right)^2}\\
&=\frac{1}{1+\frac{2}{\pi^2}\left(\log i-\log\left(\tan\frac{\theta}{4}\right)\right)^2}\\
&=\frac{1}{\frac{1}{2}+\frac{2}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2-i\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}\\
&=\frac{\frac{1}{2}+\frac{2}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2}{\frac{1}{4}+\frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2+\frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}\\
&\quad+i\,\frac{\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}{\frac{1}{4}+\frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2+\frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}.
\end{aligned}$$

Writing $\frac{f(z)}{z}=u+iv$, we see that

$$\log\left(\tan\frac{\theta}{4}\right)=\frac{\pi(u\pm\sqrt{u^2-v^2})}{2v}.$$

Thus we have

$$v=\frac{\dfrac{2}{\pi}\log\left(\tan\dfrac{\theta}{4}\right)}{\dfrac{1}{4}+\dfrac{6}{\pi^2}\left(\log\left(\tan\dfrac{\theta}{4}\right)\right)^2+\dfrac{4}{\pi^4}\left(\log\left(\tan\dfrac{\theta}{4}\right)\right)^4}$$

$$=\frac{\dfrac{2}{\pi}\dfrac{\pi(u\pm\sqrt{u^2-v^2})}{2v}}{\dfrac{1}{4}+\dfrac{6}{\pi^2}\left(\dfrac{\pi(u\pm\sqrt{u^2-v^2})}{2v}\right)^2+\dfrac{4}{\pi^4}\left(\dfrac{\pi(u\pm\sqrt{u^2-v^2})}{2v}\right)^4}.$$

Therefore, we arrive that

$$u^4-2u^3+2u^2v^2-2uv^2+v^4+v^2=0.$$

This completes the proof of the theorem. □

Considering the power series of the function $f(z)$ in Theorem 2.1, we derive

**Theorem 2.2.** *The power series of the extreme function for the class $\mathcal{B}$ is given by*

$$f(z)=\frac{z}{1+\dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$=z+\sum_{n=2}^{\infty}\sum_{p=1}^{n-1}(-1)^p\left(\frac{8}{\pi^2}\right)^p\sum_{m_1=1}^{n-p}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\,\frac{1}{2m_1+1-2k}\right)$$

$$\times\sum_{m_2=1}^{n+1-p-m_1}\left(\sum_{k=1}^{m_2}\frac{1}{2k-1}\,\frac{1}{2m_2+1-2k}\right)\times\cdots$$

$$\times\sum_{m_{p-1}=1}^{n-2-A_{p-2}}\left(\sum_{k=1}^{m_{p-1}}\frac{1}{2k-1}\,\frac{1}{2m_{p-1}+1-2k}\right)\left(\sum_{k=1}^{n-1-A_{p-1}}\frac{1}{2k-1}\,\frac{1}{2(n-A_{p-1})-1-2k}\right)z^n,$$

*where* $A_p=\sum_{l=1}^{p}m_l$.

*Proof.* Let us suppose that

$$\frac{f(z)}{z}=\frac{1}{1+\dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

as the proof of Theorem 2.1. Then from Lemma 2.2, we have

$$\frac{f(z)}{z}=\frac{1}{1+\dfrac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\dfrac{1}{2k-1}\,\dfrac{1}{2n+1-2k}\right)z^n}$$

$$
\begin{aligned}
&= 1 - \frac{8}{\pi^2}\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\ \frac{1}{2n+1-2k}\right)z^n\right) \\
&\quad + \left(\frac{8}{\pi^2}\right)^2\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\ \frac{1}{2n+1-2k}\right)z^n\right)^2 \\
&\quad - \left(\frac{8}{\pi^2}\right)^3\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\ \frac{1}{2n+1-2k}\right)z^n\right)^3 + \cdots \\
&\quad + (-1)^n\left(\frac{8}{\pi^2}\right)^n\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\ \frac{1}{2n+1-2k}\right)z^n\right)^n + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \frac{8}{\pi^2}\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}z \\
&+ \left\{-\frac{8}{\pi^2}\sum_{k=1}^{2}\frac{1}{2k-1}\ \frac{1}{5-2k} + \left(\frac{8}{\pi^2}\right)^2\left(\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}\right)\left(\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}\right)\right\}z^2 \\
&+ \left[-\frac{8}{\pi^2}\sum_{k=1}^{3}\frac{1}{2k-1}\ \frac{1}{7-2k} + \left\{\left(\frac{8}{\pi^2}\right)^2\left(\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}\right)\left(\sum_{k=1}^{2}\frac{1}{2k-1}\ \frac{1}{5-2k}\right)\right.\right. \\
&\quad \left.\left. + \left(\sum_{k=1}^{2}\frac{1}{2k-1}\ \frac{1}{5-2k}\right)\left(\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}\right)\right\} - \left(\frac{8}{\pi^2}\right)^3\left(\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}\right)^3\right]z^3 \\
&+ \cdots \\
&+ \left\{-\frac{8}{\pi^2}\sum_{k=1}^{n}\frac{1}{2k-1}\ \frac{1}{2n+1-2k}\right. \\
&\quad + \left(\frac{8}{\pi^2}\right)^2\sum_{m_1=1}^{n-1}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\ \frac{1}{2m_1+1-2k}\right)\left(\sum_{k=1}^{n-m_1}\frac{1}{2k-1}\ \frac{1}{2n-2m_1+1-2k}\right) \\
&\quad - \left(\frac{8}{\pi^2}\right)^3\sum_{m_1=1}^{n-2}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\,\frac{1}{2m_1+1-2k}\right)\sum_{m_2=1}^{n-1-m_1}\left(\sum_{k=1}^{m_2}\frac{1}{2k-1}\ \frac{1}{2m_2+1-2k}\right) \\
&\qquad\qquad \times\left(\sum_{k=1}^{n-m_1-m_2}\frac{1}{2k-1}\ \frac{1}{2n-2m_1-2m_2+1-2k}\right) \\
&\quad + \cdots \\
&+ (-1)^p\left(\frac{8}{\pi^2}\right)^p\sum_{m_1=1}^{n+1-p}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\ \frac{1}{2m_1+1-2k}\right)\sum_{m_2=1}^{n+2-p-m_1}\left(\sum_{k=1}^{m_2}\frac{1}{2k-1}\ \frac{1}{2m_2+1-2k}\right) \\
&\times\cdots\times\sum_{m_{p-1}=1}^{n-1-A_{p-2}}\left(\sum_{k=1}^{m_{p-1}}\frac{1}{2k-1}\ \frac{1}{2m_{p-1}+1-2k}\right)\left(\sum_{k=1}^{n-A_{p-1}}\frac{1}{2k-1}\ \frac{1}{2(n-A_{p-1})+1-2k}\right) \\
&\qquad\qquad\qquad\qquad \left. + \cdots + \left(\sum_{k=1}^{1}\frac{1}{2k-1}\ \frac{1}{3-2k}\right)^n\right\}z^n \\
&+ \cdots \qquad\qquad \left(A_p = \sum_{l=1}^{p} m_l\right)
\end{aligned}
$$

$$= 1 + \sum_{n=2}^{\infty} \sum_{p=1}^{n} (-1)^p \left(\frac{8}{\pi^2}\right)^p \sum_{m_1=1}^{n+1-p} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \, \frac{1}{2m_1+1-2k} \right)$$
$$\times \sum_{m_2=1}^{n+2-p-m_1} \left( \sum_{k=1}^{m_2} \frac{1}{2k-1} \, \frac{1}{2m_2+1-2k} \right) \times \cdots$$
$$\times \sum_{m_{p-1}=1}^{n-1-A_{p-2}} \left( \sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \, \frac{1}{2m_{p-1}+1-2k} \right) \left( \sum_{k=1}^{n-A_{p-1}} \frac{1}{2k-1} \, \frac{1}{2(n-A_{p-1})+1-2k} \right) z^n.$$

This completes the proof of the theorem. □

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