Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2014, Article ID 597630, 6 pages
http://dx.doi.org/10.1155/2014/597630

*Research Article*

# Point-Symmetric Multivariate Density Function and Its Decomposition

**Kiyotaka Iki and Sadao Tomizawa**

*Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science, Noda City, Chiba 278-8510, Japan*

Correspondence should be addressed to Kiyotaka Iki; iki@is.noda.tus.ac.jp

Received 13 December 2013; Revised 21 April 2014; Accepted 23 April 2014; Published 13 May 2014

Academic Editor: Chin-Shang Li



For a $T$-variate density function, the present paper defines the point-symmetry, quasi-point-symmetry of order $k$ ($< T$), and the marginal point-symmetry of order $k$ and gives the theorem that the density function is $T$-variate point-symmetric if and only if it is quasi-point-symmetric and marginal point-symmetric of order $k$. The theorem is illustrated for the multivariate normal density function.

## 1. Introduction

For square contingency tables, it is known that the symmetry model holds if and only if both the quasi-symmetry and the marginal homogeneity models hold (e.g., see Caussinus [1]; Tomizawa and Tahata [2]). For multiway contingency tables, Bhapkar and Darroch [3] defined the complete symmetry, quasi-symmetry, and marginal symmetry models and showed that the complete symmetry model holds if and only if both the quasi-symmetry and the marginal symmetry models hold.

Tomizawa et al. [4] gave a similar decomposition for the bivariate density function (instead of cell probabilities). Iki et al. [5] showed a similar decomposition for the multivariate density function.

On the other hand, for contingency tables, Wall and Lienert [6] defined the point-symmetry model for the cell probabilities, and Tomizawa [7] gave the theorem that the point-symmetry model holds for the cell probabilities if and only if both the quasi-point-symmetry and the marginal point-symmetry models hold (see also Tahata and Tomizawa [8]).

Tomizawa and Konuma [9] gave a similar decomposition for the bivariate point-symmetric density function. Now, we are interested in extending the decomposition of the point-symmetric density function to multivariate case.

In the present paper, we define the point-symmetry, quasi-point-symmetry, and marginal point-symmetry for the multivariate density function and decompose the point-symmetry into quasi-point-symmetry and marginal point-symmetry. Section 2 provides the decomposition for the trivariate density function. Section 3 extends the decomposition to multivariate density function. Section 4 illustrates our decomposition for the multivariate normal distribution.

## 2. Decomposition of Trivariate Density Function

Let $X_1, X_2$, and $X_3$ be three continuous random variables with a density function $f(x_1, x_2, x_3)$, where

$$\begin{aligned} f(x_1, x_2, x_3) > 0 \quad \text{for} \quad (x_1, x_2, x_3) \in D^3, \\ f(x_1, x_2, x_3) = 0 \quad \text{for} \quad (x_1, x_2, x_3) \notin D^3, \end{aligned} \tag{1}$$

with

$$D^3 = \{(x_1, x_2, x_3) \mid a_i < x_i < b_i;\ i = 1, 2, 3\}, \tag{2}$$

and where $a_i = -\infty$ and $b_i = +\infty$, or $a_i$ and $b_i$ are finite. Let $(c_1, c_2, c_3)$ denote a given point in domain $D^3$, where $c_i = (a_i + b_i)/2$ if $a_i$ and $b_i$ are finite. Let $x_i^* = 2c_i - x_i$ when $X_i = x_i$

for $i = 1, 2, 3$. For example, when $X_2 = 10$ with $c_2 = 3$, then $10^* = 2 \times 3 - 10 = -4$. Note that, for $i = 1, 2, 3$, (i) $x_i^*$ is the symmetrical value of $x_i$ with respect to $c_i$, (ii) $(x_i^*)^* = x_i$, and (iii) $c_i^* = c_i$.

We will define the point-symmetry (denoted by $\text{PS}^3$) of density function with respect to the point $(c_1, c_2, c_3)$ by

$$f(x_1^*, x_2^*, x_3^*) = f(x_1, x_2, x_3) \quad \text{for every } (x_1, x_2, x_3) \in R^3. \tag{3}$$

Let $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$ be the marginal density functions of $X_1, X_2$, and $X_3$, respectively. For the density function $f(x_1, x_2, x_3)$, we will define the marginal point-symmetry of order 1 (denoted by $\text{MP}_1^3$) by

$$f_{X_i}(x_i^*) = f_{X_i}(x_i) \quad \text{for } i = 1, 2, 3;\ (x_1, x_2, x_3) \in R^3. \tag{4}$$

Let $f_{X_i X_j}(x_i, x_j)$ be the marginal density function of $(X_i, X_j)$ for $1 \le i < j \le 3$. We define the marginal point-symmetry of order 2 (denoted by $\text{MP}_2^3$) by

$$\begin{gathered} f_{X_i X_j}(x_i^*, x_j^*) = f_{X_i X_j}(x_i, x_j) \\ \text{for } 1 \le i < j \le 3; \quad (x_1, x_2, x_3) \in R^3. \end{gathered} \tag{5}$$

Note that $\text{MP}_2^3$ implies $\text{MP}_1^3$.

We can express the density function as

$$\begin{aligned} f(x_1, x_2, x_3) = {} & \mu \alpha_1(x_1) \alpha_2(x_2) \alpha_3(x_3) \\ & \times \beta_{12}(x_1, x_2) \beta_{13}(x_1, x_3) \beta_{23}(x_2, x_3) \\ & \times \gamma(x_1, x_2, x_3), \end{aligned} \tag{6}$$

where $(x_1, x_2, x_3) \in D^3$, and

$$\begin{gathered} \alpha_1(c_1) = 1, \qquad \beta_{12}(c_1, x_2) = \beta_{12}(x_1, c_2) = 1, \\ \gamma(c_1, x_2, x_3) = \gamma(x_1, c_2, x_3) = \gamma(x_1, x_2, c_3) = 1, \end{gathered} \tag{7}$$

with similar properties of $\alpha_2, \alpha_3, \beta_{13}$, and $\beta_{23}$. The terms $\alpha_i$ $(i = 1, 2, 3)$ correspond to main effects of the variable $X_i$, $\beta_{ij}$ $(i \ne j)$ to interaction effects of $X_i$ and $X_j$, and $\gamma$ to interaction effect of $X_1$, $X_2$, and $X_3$. We see

$$\begin{gathered} \mu = f(c_1, c_2, c_3), \qquad \alpha_1(x_1) = \frac{f(x_1, c_2, c_3)}{f(c_1, c_2, c_3)}, \\ \beta_{12}(x_1, x_2) = \frac{f(x_1, x_2, c_3) f(c_1, c_2, c_3)}{f(x_1, c_2, c_3) f(c_1, x_2, c_3)}, \\ \gamma(x_1, x_2, x_3) \\ = \frac{f(x_1, x_2, x_3) f(x_1, c_2, c_3) f(c_1, x_2, c_3) f(c_1, c_2, x_3)}{f(c_1, c_2, c_3) f(x_1, x_2, c_3) f(x_1, c_2, x_3) f(c_1, x_2, x_3)}, \end{gathered} \tag{8}$$

with similar properties of $\alpha_2, \alpha_3, \beta_{13}$, and $\beta_{23}$. The term $\alpha_1(x_1)$ indicates the odds of density function with respect to $X_1$-values with $(X_2, X_3) = (c_2, c_3)$. Note that

$$\begin{aligned} & \beta_{12}(x_1, x_2) \\ & \quad = \frac{(f(x_1, x_2, c_3) / f(c_1, x_2, c_3))}{(f(x_1, c_2, c_3) / f(c_1, c_2, c_3))} \\ & \quad = \frac{(f(x_1, x_2, c_3) / f(x_1, c_2, c_3))}{(f(c_1, x_2, c_3) / f(c_1, c_2, c_3))}, \\ & \gamma(x_1, x_2, x_3) \\ & \quad = \frac{(f(x_1, x_2, x_3) f(c_1, c_2, x_3) / f(x_1, c_2, x_3) f(c_1, x_2, x_3))}{(f(x_1, x_2, c_3) f(c_1, c_2, c_3) / f(x_1, c_2, c_3) f(c_1, x_2, c_3))} \\ & \quad = \frac{(f(x_1, x_2, x_3) f(c_1, x_2, c_3) / f(x_1, x_2, c_3) f(c_1, x_2, x_3))}{(f(x_1, c_2, x_3) f(c_1, c_2, c_3) / f(x_1, c_2, c_3) f(c_1, c_2, x_3))} \\ & \quad = \frac{(f(x_1, x_2, x_3) f(x_1, c_2, c_3) / f(x_1, x_2, c_3) f(x_1, c_2, x_3))}{(f(c_1, x_2, x_3) f(c_1, c_2, c_3) / f(c_1, x_2, c_3) f(c_1, c_2, x_3))}. \end{aligned} \tag{9}$$

Thus, $\beta_{12}(x_1, x_2)$ indicates the odds ratio of density function with respect to $(X_1, X_2)$-values with $X_3 = c_3$. Also $\gamma(x_1, x_2, x_3)$ indicates the ratio of odds ratios of density function, that is, the ratio of odds ratio with respect to $(X_1, X_2)$-values with $X_3 = x_3$ to that with $X_3 = c_3$ (or the ratio of odds ratio with respect to $(X_i, X_j)$-values with $X_k = x_k$ to that with $X_k = c_k$, where $(i, j, k) = (1, 3, 2)$ and $(2, 3, 1)$).

The density function is $\text{PS}^3$ if and only if it is expressed as form (6) with

$$\begin{gathered} \alpha_i(x_i^*) = \alpha_i(x_i) \quad \text{for } i = 1, 2, 3, \\ \beta_{ij}(x_i^*, x_j^*) = \beta_{ij}(x_i, x_j) \quad \text{for } 1 \le i < j \le 3, \\ \gamma(x_1^*, x_2^*, x_3^*) = \gamma(x_1, x_2, x_3). \end{gathered} \tag{10}$$

We will define the quasi-point-symmetry of order 1 (denoted by $\text{QP}_1^3$) by (6) with

$$\begin{gathered} \beta_{ij}(x_i^*, x_j^*) = \beta_{ij}(x_i, x_j) \quad \text{for } 1 \le i < j \le 3, \\ \gamma(x_1^*, x_2^*, x_3^*) = \gamma(x_1, x_2, x_3). \end{gathered} \tag{11}$$

The $\text{QP}_1^3$ is equivalent to

$$\begin{gathered} \theta(s_1, s_2; t_1, t_2; u) = \theta(s_1^*, s_2^*; t_1^*, t_2^*; u^*), \\ \theta(s_1, s_2; u; t_1, t_2) = \theta(s_1^*, s_2^*; u^*; t_1^*, t_2^*), \\ \theta(u; s_1, s_2; t_1, t_2) = \theta(u^*; s_1^*, s_2^*; t_1^*, t_2^*), \end{gathered} \tag{12}$$

where

$$\theta(s_1,s_2;t_1,t_2;u)=\frac{f(s_1,t_1,u)f(s_2,t_2,u)}{f(s_1,t_2,u)f(s_2,t_1,u)},$$
$$\theta(s_1,s_2;u;t_1,t_2)=\frac{f(s_1,u,t_1)f(s_2,u,t_2)}{f(s_1,u,t_2)f(s_2,u,t_1)}, \tag{13}$$
$$\theta(u;s_1,s_2;t_1,t_2)=\frac{f(u,s_1,t_1)f(u,s_2,t_2)}{f(u,s_1,t_2)f(u,s_2,t_1)},$$

with $(s_i,t_j,u)\in D^3$ and so on. Therefore, $QP_1^3$ indicates that the density function is point-symmetric with respect to the odds ratio.

Also, we will define the quasi-point-symmetry of order 2 (denoted by $QP_2^3$) by (6) with

$$\gamma(x_1^*,x_2^*,x_3^*)=\gamma(x_1,x_2,x_3). \tag{14}$$

The $QP_2^3$ is equivalent to

$$\frac{\theta(s_1,s_2;t_1,t_2;u_1)}{\theta(s_1,s_2;t_1,t_2;u_2)}=\frac{\theta(s_1^*,s_2^*;t_1^*,t_2^*;u_1^*)}{\theta(s_1^*,s_2^*;t_1^*,t_2^*;u_2^*)},$$
$$\frac{\theta(s_1,s_2;u_1;t_1,t_2)}{\theta(s_1,s_2;u_2;t_1,t_2)}=\frac{\theta(s_1^*,s_2^*;u_1^*;t_1^*,t_2^*)}{\theta(s_1^*,s_2^*;u_2^*;t_1^*,t_2^*)}, \tag{15}$$
$$\frac{\theta(u_1;s_1,s_2;t_1,t_2)}{\theta(u_2;s_1,s_2;t_1,t_2)}=\frac{\theta(u_1^*;s_1^*,s_2^*;t_1^*,t_2^*)}{\theta(u_2^*;s_1^*,s_2^*;t_1^*,t_2^*)}.$$

Therefore, $QP_2^3$ indicates that the density function is point-symmetric with respect to the ratio of odds ratios. We note that $QP_1^3$ implies $QP_2^3$. We obtain the following theorem.

**Theorem 1.** *For $k$ fixed ($k = 1,2$), the trivariate density function $f(x_1,x_2,x_3)$ is $PS^3$ if and only if it is both $QP_k^3$ and $MP_k^3$.*

*Proof.* Consider the case of $k=1$. If a density function is $PS^3$, then it satisfies $QP_1^3$ and $MP_1^3$. Assume that it is both $QP_1^3$ and $MP_1^3$, and then we will show that it satisfies $PS^3$.

Let $X_1$, $X_2$, and $X_3$ be three continuous random variables with a density function $h(x_1,x_2,x_3)$ which satisfies both $QP_1^3$ and $MP_1^3$. Therefore, we see

$$\begin{aligned}\log h(x_1,x_2,x_3) &= \log\mu+\log\alpha_1(x_1)+\log\alpha_2(x_2)\\ &\quad+\log\alpha_3(x_3)+\log\beta_{12}(x_1,x_2)+\log\beta_{13}(x_1,x_3)\\ &\quad+\log\beta_{23}(x_2,x_3)+\log\gamma(x_1,x_2,x_3),\end{aligned} \tag{16}$$

where $(x_1,x_2,x_3)\in D^3$, $\beta_{ij}(x_i^*,x_j^*)=\beta_{ij}(x_i,x_j)$ $(1\le i<j\le 3)$, and $\gamma(x_1^*,x_2^*,x_3^*)=\gamma(x_1,x_2,x_3)$.

Let

$$g(x_1,x_2,x_3)=\frac{1}{\Delta}w(x_1,x_2,x_3), \tag{17}$$

where

$$\begin{aligned}\log w(x_1,x_2,x_3) &= \log\beta_{12}(x_1,x_2)+\log\beta_{13}(x_1,x_3)\\ &\quad+\log\beta_{23}(x_2,x_3)+\log\gamma(x_1,x_2,x_3),\\ g(x_1^*,x_2^*,x_3^*) &= g(x_1,x_2,x_3),\\ \Delta &= \iiint w(x_1,x_2,x_3)\,dx_1dx_2dx_3.\end{aligned} \tag{18}$$

Note that $\iiint g(x_1,x_2,x_3)dx_1dx_2dx_3=1$. Then we have

$$\log\left(\frac{h(x_1,x_2,x_3)}{g(x_1,x_2,x_3)}\right)=\nu+\log\alpha_1(x_1)+\log\alpha_2(x_2)+\log\alpha_3(x_3), \tag{19}$$

where $\nu=\log\Delta+\log\mu$.

Since $h(x_1,x_2,x_3)$ satisfies $MP_1^3$, we see

$$h_{X_i}(x_i^*)=h_{X_i}(x_i)\quad(i=1,2,3), \tag{20}$$

where $(x_1,x_2,x_3)\in D^3$ and $h_{X_i}(x_i)$ is the marginal density function of $X_i$ for $i=1,2,3$. Denote (20) as $h_{X_i}(x_i^*)=h_{X_i}(x_i)$ $(=h_{X_i}^{(0)}(x_i))$ for $i=1,2,3$.

Consider the arbitrary density function $f(x_1,x_2,x_3)$ satisfying $MP_1^3$ with

$$f_{X_i}(x_i^*)=f_{X_i}(x_i)=h_{X_i}^{(0)}(x_i)\quad(i=1,2,3). \tag{21}$$

From (19), (20), and (21), we have

$$\iiint\{f(x_1,x_2,x_3)-h(x_1,x_2,x_3)\}\times\log\left(\frac{h(x_1,x_2,x_3)}{g(x_1,x_2,x_3)}\right)dx_1dx_2dx_3=0. \tag{22}$$

Using (22), we obtain

$$I(f,g)=I(h,g)+I(f,h), \tag{23}$$

where $I(\cdot,\cdot)$ is the Kullback-Leibler information; that is,

$$I(f,g)=\iiint f(x_1,x_2,x_3)\times\log\left(\frac{f(x_1,x_2,x_3)}{g(x_1,x_2,x_3)}\right)dx_1dx_2dx_3. \tag{24}$$

For $g$ fixed, we see

$$\min_f I(f,g)=I(h,g), \tag{25}$$

and then $h$ uniquely minimizes $I(f,g)$.

Let $\tilde{h}(x_1,x_2,x_3)=h(x_1^*,x_2^*,x_3^*)$ for $(x_1,x_2,x_3)\in D^3$. Since $\tilde{h}(x_1,x_2,x_3)$ satisfies $QP_1^3$, we see

$$\begin{aligned}\log\tilde{h}(x_1,x_2,x_3) &= \log\mu+\log\alpha_1(x_1^*)+\log\alpha_2(x_2^*)+\log\alpha_3(x_3^*)\\ &\quad+\log\beta_{12}(x_1,x_2)+\log\beta_{13}(x_1,x_3)\\ &\quad+\log\beta_{23}(x_2,x_3)+\log\gamma(x_1,x_2,x_3).\end{aligned} \tag{26}$$

Since $\widetilde{h}(x_1, x_2, x_3)$ satisfies $MP_1^3$, we see

$$\widetilde{h}_{X_i}(x_i^*) = \widetilde{h}_{X_i}(x_i) = h_{X_i}^{(0)}(x_i) \quad (i = 1, 2, 3), \tag{27}$$

where $(x_1, x_2, x_3) \in D^3$.

Consider the arbitrary density function $f(x_1, x_2, x_3)$ satisfying $MP_1^3$ with

$$f_{X_i}(x_i^*) = f_{X_i}(x_i) = h_{X_i}^{(0)}(x_i) \quad (i = 1, 2, 3), \tag{28}$$

where $(x_1, x_2, x_3) \in D^3$. In a similar way, we see

$$\iiint \left\{ f(x_1, x_2, x_3) - \widetilde{h}(x_1, x_2, x_3) \right\} \times \log\left( \frac{\widetilde{h}(x_1, x_2, x_3)}{g(x_1, x_2, x_3)} \right) dx_1 dx_2 dx_3 = 0. \tag{29}$$

Thus, we obtain

$$I(f, g) = I(\widetilde{h}, g) + I(f, \widetilde{h}). \tag{30}$$

For $g$ fixed, we see

$$\min_f I(f, g) = I(\widetilde{h}, g), \tag{31}$$

and then $\widetilde{h}$ uniquely minimizes $I(f, g)$. Therefore, we see $h(x_1, x_2, x_3) = \widetilde{h}(x_1, x_2, x_3)$. Thus, $h(x_1, x_2, x_3) = h(x_1^*, x_2^*, x_3^*)$. Namely, $h(x_1, x_2, x_3)$ satisfies $PS^3$. The case of $k = 2$ can be proved in a similar way as the case of $k = 1$. So the proof is completed. □

## 3. Decomposition of Multivariate Density Function

Let $X_1, \ldots, X_T$ be $T$ continuous random variables with a density function $f(x_1, \ldots, x_T)$, where $f(x_1, \ldots, x_T) > 0$ for $(x_1, \ldots, x_T) \in D^T$ and $D^T$ is defined in a similar way to $D^3$. Let $(c_1, \ldots, c_T)$ denote a given point in $D^T$, where $c_i = (a_i + b_i)/2$ if $a_i$ and $b_i$ are finite. Let $x_i^* = 2c_i - x_i$ when $X_i = x_i$ for $i = 1, \ldots, T$. For the density function $f(x_1, \ldots, x_T)$, we will define the point-symmetry (denoted by $PS^T$) with respect to the point $(c_1, \ldots, c_T)$ by

$$f(x_1^*, \ldots, x_T^*) = f(x_1, \ldots, x_T) \quad \text{for every } (x_1, \ldots, x_T) \in R^T. \tag{32}$$

Also, for $k = 1, \ldots, T-1$, we will define the marginal point-symmetry of order $k$ (denoted by $MP_k^T$) by

$$f_{X_{i_1} \cdots X_{i_k}}(x_{i_1}^*, \ldots, x_{i_k}^*) = f_{X_{i_1} \cdots X_{i_k}}(x_{i_1}, \ldots, x_{i_k}) \quad (1 \le i_1 < \cdots < i_k \le T), \tag{33}$$

where $f_{X_{i_1} \cdots X_{i_k}}$ is the marginal density function of $(X_{i_1}, \ldots, X_{i_k})$. We note that $MP_{k+1}^T$ implies $MP_k^T$ $(k = 1, \ldots, T-2)$.

We can express the density function as

$$\begin{aligned} f(x_1, \ldots, x_T) &= \mu \left[ \prod_{i_1=1}^{T} \alpha_{i_1}(x_{i_1}) \right] \left[ \prod\prod_{1 \le i_1 < i_2 \le T} \alpha_{i_1 i_2}(x_{i_1}, x_{i_2}) \right] \times \cdots \\ &\quad \times \left[ \prod \cdots \prod_{1 \le i_1 < \cdots < i_{T-1} \le T} \alpha_{i_1 \cdots i_{T-1}}(x_{i_1}, \ldots, x_{i_{T-1}}) \right] \\ &\quad \times \alpha_{1 \cdots T}(x_1, \ldots, x_T), \end{aligned} \tag{34}$$

where $(x_1, \ldots, x_T) \in D^T$, and

$$\left\{ \alpha_i(c_i) = \alpha_{i_1 i_2}(c_{i_1}, x_{i_2}) = \cdots = \alpha_{1 \cdots T}(x_1, \ldots, x_{T-1}, c_T) = 1 \right\}. \tag{35}$$

Then, the density function $f(x_1, \ldots, x_T)$ being $PS^T$ is also expressed as (34) with

$$\alpha_{i_1 \cdots i_m}(x_{i_1}^*, \ldots, x_{i_m}^*) = \alpha_{i_1 \cdots i_m}(x_{i_1}, \ldots, x_{i_m}) \quad (m = 1, \ldots, T;\ 1 \le i_1 < \cdots < i_m \le T). \tag{36}$$

For $k = 1, \ldots, T-1$, we will define the quasi-point-symmetry of order $k$ (denoted by $QP_k^T$) by (34) with

$$\alpha_{i_1 \cdots i_m}(x_{i_1}^*, \ldots, x_{i_m}^*) = \alpha_{i_1 \cdots i_m}(x_{i_1}, \ldots, x_{i_m}) \quad (m = k+1, \ldots, T;\ 1 \le i_1 < \cdots < i_m \le T). \tag{37}$$

We note that $QP_k^T$ implies $QP_{k+1}^T$ $(k = 1, \ldots, T-2)$. Then we obtain the following theorem.

**Theorem 2.** *For $k$ fixed $(k = 1, \ldots, T-1)$, the multivariate density function $f(x_1, \ldots, x_T)$ is $PS^T$ if and only if it is both $QP_k^T$ and $MP_k^T$.*

The proof of Theorem 2 is omitted because it is obtained in a similar way to the proof of Theorem 1.

## 4. Point-Symmetry of Multivariate Normal Density Function

Consider a $T$-dimensional random vector $X = (X_1, \ldots, X_T)'$ having a normal distribution with mean vector $\mu = (\mu_1, \ldots, \mu_T)'$ and covariance matrix $\Sigma$. The density function is

$$f(x_1, \ldots, x_T) = \frac{1}{(2\pi)^{T/2} |\Sigma|^{1/2}} \times \exp\left\{ -\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu) \right\}. \tag{38}$$

Denote $\Sigma^{-1}$ by $A = (a_{ij})$ with $a_{ij} = a_{ji}$. Then the density function can be expressed as

$$f(x_1, \ldots, x_T) = C \exp\left\{ -\frac{1}{2} H \right\}, \tag{39}$$

where $C$ is positive constant and

$$H = \sum_{s=1}^{T} a_{ss} x_s^2 + \sum_{s \neq t} a_{st} x_s x_t - 2 \sum_{s=1}^{T} \sum_{t=1}^{T} a_{st} \mu_s x_t. \quad (40)$$

For an arbitrary given point $(c_1, \ldots, c_T)$, we set $\tilde{x}_i = x_i - c_i$ and $\tilde{\mu}_i = \mu_i - c_i$ $(i = 1, \ldots, T)$. Then noting that $x_i - \mu_i = \tilde{x}_i - \tilde{\mu}_i$ $(i = 1, \ldots, T)$, we see

$$f(x_1, \ldots, x_T) = \widetilde{C} \exp\left\{-\frac{1}{2} \widetilde{H}\right\}, \quad (41)$$

where $\widetilde{C}$ is positive constant and

$$\widetilde{H} = \sum_{s=1}^{T} a_{ss} \tilde{x}_s^2 + \sum_{s \neq t} a_{st} \tilde{x}_s \tilde{x}_t - 2 \sum_{s=1}^{T} \sum_{t=1}^{T} a_{st} \tilde{\mu}_s \tilde{x}_t. \quad (42)$$

Thus,

$$\begin{aligned} \alpha_i(x_i) &= \frac{f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_T)}{f(c_1, \ldots, c_T)} \\ &= \exp\left\{-\frac{1}{2}\left(a_{ii} \tilde{x}_i^2 - 2 \sum_{s=1}^{T} a_{si} \tilde{\mu}_s \tilde{x}_i\right)\right\} \quad (i = 1, \ldots, T), \\ \alpha_{ij}(x_i, x_j) &= \left(f\left(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_{j-1}, x_j, c_{j+1}, \ldots, c_T\right) \times f(c_1, \ldots, c_T)\right) \\ &\quad \times \left(f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_T) \times f\left(c_1, \ldots, c_{j-1}, x_j, c_{j+1}, \ldots, c_T\right)\right)^{-1} \\ &= \exp\left(-\frac{1}{2} a_{ij} \tilde{x}_i \tilde{x}_j\right) \quad (i < j), \end{aligned} \quad (43)$$

and for $m = 3, \ldots, T$,

$$\alpha_{i_1 \cdots i_m}\left(x_{i_1}, \ldots, x_{i_m}\right) = 1 \quad \left(1 \leq i_1 < \cdots < i_m \leq T\right). \quad (44)$$

Since $x_i^* = 2c_i - x_i$ $(i = 1, \ldots, T)$, we see

$$\begin{aligned} \alpha_{ij}\left(x_i^*, x_j^*\right) &= \exp\left\{-\frac{1}{2} a_{ij} \left(x_i^* - c_i\right)\left(x_j^* - c_j\right)\right\} \\ &= \exp\left\{-\frac{1}{2} a_{ij} \left(x_i - c_i\right)\left(x_j - c_j\right)\right\} \\ &= \alpha_{ij}\left(x_i, x_j\right) \quad (i < j). \end{aligned} \quad (45)$$

Therefore, the normal density function $f(x_1, \ldots, x_T)$ is $\mathrm{QP}_k^T$ for $k = 1, \ldots, T-1$, without depending on the value of $(c_1, \ldots, c_T)$ and on the values of parameters $\mu$ and $\Sigma$. Thus, we see from Theorem 2 that, for $k$ fixed $(k = 1, \ldots, T-1)$, the normal density function $f(x_1, \ldots, x_T)$ is $\mathrm{PS}^T$ if and only if $f(x_1, \ldots, x_T)$ is $\mathrm{MP}_k^T$. Therefore, we see that the normal density function $f(x_1, \ldots, x_T)$ is not $\mathrm{PS}^T$ with respect to the point $(c_1, \ldots, c_T)$ where $(c_1, \ldots, c_T) \neq (\mu_1, \ldots, \mu_T)$, and it is $\mathrm{PS}^T$ only with respect to $(c_1, \ldots, c_T) = (\mu_1, \ldots, \mu_T)$ without depending on the value of $\Sigma$. We see from Theorem 2 that when the normal density function $f(x_1, \ldots, x_T)$ is not $\mathrm{PS}^T$, it is caused by the lack of the structure of $\mathrm{MP}_k^T$.

## 5. Discussion

When a density function $f(x_1, \ldots, x_T)$ is not point-symmetric, Theorem 2 may be useful for knowing the reason, that is, for $k$ fixed, which structure of quasi-point-symmetry of order $k$ and marginal point-symmetry of order $k$ is lacking.

For symmetry of a multivariate distribution, there are various kinds of symmetry such as spherical symmetry, elliptical symmetry, and central symmetry (see, e.g., Kotz et al. [10, pages 5338–5341], Fang et al. [11, Chapter 2], Muirhead [12, pages 32–34], and Tong [13, Chapter 4]). The $\mathrm{PS}^T$ described in the present paper is equivalent to the central symmetry. Also, for the $T$-variate spherical (elliptical) distribution, the probability density function is $\mathrm{PS}^T$ with respect to the mean vector, although when the density function is $\mathrm{PS}^T$, the distribution is not always spherical (elliptical). Thus, for the $T$-variate spherical (elliptical) distribution, the density function is $\mathrm{QP}_k^T$ and $\mathrm{MP}_k^T$ $(k = 1, \ldots, T-1)$ with respect to the mean vector. We point out that, as described in Section 4, for $T$-variate normal distribution, the density function is $\mathrm{QP}_k^T$ $(k = 1, \ldots, T-1)$ with respect to the arbitrary point $(c_1, \ldots, c_T)$ (not only mean vector $(\mu_1, \ldots, \mu_T)$).

Testing spherical symmetry and elliptical symmetry is described in, for example, Fang and Zhang [14, Chapter 5], Muirhead [12, page 333], and Kotz et al. [10, pages 5341-5342]. Heathcote et al. [15] gave a procedure for testing a general multivariate distribution for symmetry about a point which indicates that the imaginary part of the characteristic function of centered variable vanishes identically. Although the readers may be interested in seeing the comparison of both approaches and the decomposition of $\mathrm{PS}^T$ into $\mathrm{QP}_k^T$ and $\mathrm{MP}_k^T$, it seems difficult.

As (6), we have considered the multiplicative form of probability density function by the terms of the odds, the odds ratios, the ratios of odds ratios, and so on; as an analog to the log-linear model for the analysis of categorical data (Agresti [16, Chapter 9]). Although the readers also may be interested in the additive form of density function for point-symmetry, it seems difficult to consider it.

On discrete probability, the concept of odds ratio is important. Also it is important to use the odds ratio on probability density function (corresponding to a continuous random variable). For example, for bivariate probability density function $f(x, y)$, the odds ratio

$$\beta_{12}(x_1, x_2) = \frac{f(x_1, x_2) f(c_1, c_2)}{f(x_1, c_2) f(c_1, x_2)} \quad (46)$$

equals 1 for any $x_1$, $x_2$ and fixed $c_1$ and $c_2$, if and only if two variables are independent. So we are interested in how the structures of odds ratios, the ratios of odds ratios, and so on, of probability density function, are, for example, the point-symmetry. Note that Holland and Wang [17], Kotz et al. [18, page 74], and Tong [13, Chapter 4] discuss the properties of bivariate probability density function using the odds ratios, for example, as the local dependence function and the totally positive of density function, although the details are omitted.

In Section 4, we have shown that, for the $T$-variate normal distribution, the density function is always $\mathrm{QP}_k^T$ but not $\mathrm{MP}_k^T$ (thus not $\mathrm{PS}^T$) with respect to the arbitrary point $(c_1, \ldots, c_T)$ where $(c_1, \ldots, c_T)$ is not equal to mean vector $(\mu_1, \ldots, \mu_T)$. The readers may be interested in the probability density function such that it is not $\mathrm{QP}_k^T$ but it is $\mathrm{MP}_k^T$. Consider the following density function:

$$f(x_1, \ldots, x_T) = \frac{1}{C(2\pi)^T}\left[\sum_{k=1}^{T}\left(\prod_{\substack{i=1\\ i\neq k}}^{T} x_i\right)\cos x_k + C\right], \quad (47)$$

for $-\pi \le x_i \le \pi$ $(i = 1, \ldots, T)$ with $C$ satisfying $f(x_1, \ldots, x_T) > 0$. When $T$ is odd, the density function is $\mathrm{PS}^T$ with respect to the point $(0, \ldots, 0)$ because $f(x_1^*, \ldots, x_T^*)$ equals $f(x_1, \ldots, x_T)$ for $x_i^* = -x_i$ $(i = 1, \ldots, T)$. Thus, from Theorem 2, when $T$ is odd, this density function is $\mathrm{QP}_k^T$ and $\mathrm{MP}_k^T$ $(k = 1, \ldots, T-1)$. However, when $T$ is even, the density function (47) is not $\mathrm{PS}^T$. Also, for $k = 1, \ldots, T-1$, the marginal density function of $(X_{i_1}, \ldots, X_{i_k})$ is

$$f(x_{i_1}, \ldots, x_{i_k}) = \frac{1}{(2\pi)^k} \quad (-\pi \le x_{i_l} \le \pi;\ l = 1, \ldots, k). \quad (48)$$

Namely, this is the uniform distribution. Therefore, the density function (47) is always $\mathrm{MP}_k^T$ $(k = 1, \ldots, T-1)$ with respect to the point $(0, \ldots, 0)$ without depending on whether $T$ is odd or even. In addition, when $T$ is even, the density function (47) is not $\mathrm{QP}_k^T$ $(k = 1, \ldots, T-1)$, because then $\alpha_{i_1 \cdots i_{T-1}}(x_{i_1}^*, \ldots, x_{i_{T-1}}^*) \neq \alpha_{i_1 \cdots i_{T-1}}(x_{i_1}, \ldots, x_{i_{T-1}})$ and $\alpha_{1\cdots T}(x_1^*, \ldots, x_T^*) \neq \alpha_{1\cdots T}(x_1, \ldots, x_T)$ for $x_m^* = -x_m$ $(m = 1, \ldots, T)$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

The authors would like to thank two referees for their many helpful comments.

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