Quasi-Spline Sheaves and their Contact Ideals

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Abstract

Quasi-Spline Sheaves and their Contact Ideals

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We research quasi-spline sheaves, which are an algebraic geometric generalization of spline spaces. Spline spaces are vector spaces of splines that are defined over some polyhedral complex in real space, and the dimension and basis for them are of interest. Billera found that certain spline spaces are determined by ideals that are affine forms that vanish on the intersections of the maximal faces of the complex. These ideals correspond to contact ideals of a quasi-spline sheaf, and we ask if quasi-spline sheaves are determined by contact ideals in the same way. A quasi-spline sheaf or spline space can be defined by identifying ideals, called ideal difference-conditions. We find that the contact ideals are a canonical example of ideal difference-conditions for a quasi-spline sheaf. Next, we ask how to find the contact ideals of a quasi-spline sheaf when only ideal difference-conditions are given. Last, we find a complex for any quasi-spline sheaf and try to figure out when this gives a resolution for the quasi-spline sheaf. If it is a resolution, it gives an alternative way to compute the dimension of a quasi-spline sheaf, or any spline space.

1. Introduction

My main research project involves studying *quasi-spline sheaves*, which are sheaves of splines, using algebraic geometry and commutative algebra. Quasi-spline sheaves are generalizations of spline spaces to algebraic geometry. Spline spaces are vector spaces of splines over a polyhedral complex embedded in real space. Billera found that some splines are completely determined by ideals in the base ring. We look at the these ideals in an algebraic-geometric setting of spline sheaves, and find just how relevant these are.

A spline function is a piece-wise defined polynomial that satisfies smoothness conditions where the polynomial pieces connect. Splines have the benefits of polynomials when it comes to approximations of functions on a closed set, and being easy to store and manipulate using a computer. Unlike polynomials, they do not exhibit the same kinds of wild ocillations. Splines have played a central role in approximation theory and numerical analysis for many years. More recently, they have been used in computer graphics and computer modeling. [Sch07] A particular basis of splines, B-splines, are very useful in the finite element method in the field of partial differential equations as they are used to approximate solutions over local regions. [Höl03]

A spline space is a vector space of splines that are continuously differentiable up to a particular order defined over a region $\Omega \subseteq \mathbb{R}^d$. Let $S^r_m(\Delta)$ be the set of all piecewise polynomial functions on Δ of degree at most m and smooth of order r. In another words, all functions $f : \Delta \to \mathbb{R}$ such that

- $f\big|_\sigma$ is a polynomial of degree $\leq m$, for each $\sigma \in \Delta$ and
- f is continuously differentiable up to order r

[Bil88]

The most fundamental questions for spline spaces includes its dimension and basis. In Chapter 2 we discuss the past efforts for solving these problems.

The univarite case is where the splines of the spline space are univariate, and the bivariate case is when the splines are bivariate, and so on. The univarite case is solved for any polyhedral complex $\Delta \subset \mathbb{R}^n$, and the bivariate case has been worked on a while and there are many results. When it comes to the trivariate case, things become much more difficult, and there aren't many results. When working with the bivariate or trivariate case, we find the the dimension of a spline space depends on the degree d, the smoothness conditions denoted by r, and even the 'geometry' of the space, or how it is embedded. In some cases, the dimension can depend on the location of the vertices counterintuitively [Alf00]. It was first noted in [MS75] that the dimension depends on the geometry of the triangulation.

Then we look at Billera's work, the first algebraic treatment of spline spaces. The ideas here will permeate through my own research. We find that when Δ is a strongly connected d-complex such that all links of simplices are strongly connected complexes, $F \in S^r_m(\Delta)$ if and only if $\ell^{r+1}|(p_1 - p_2)$ as above for each pair σ_1, σ_2 of adjcacent d-simplices in Δ, where ℓ is any nontrivial affine form which vanishes on τ and $p_i = F\big|_{\sigma_i}$, $i = 1, 2$. In this way, the affine spans of $\sigma_i \cap \sigma_j$ for all i, j, determine $S^r_d(\Delta)$. [Bil88, Theorem 2.4]. So in this sense, we can say two splines spaces are isomorphic if they are determined by the same ideals ℓ_τ^{r+1} (see [KS14]).

Then we look at how the dimensions of spline spaces have been researched by mathematicians like Billera, Alfeld, and others. After that, I give a summary of the homology and cohomology theories for splines spaces.

Last, we look into a more modern approach done by Clarke and Foucart [CS13], using commutative algebra to study spline spaces and find bases for them. In this paper, a spline space is identified as an R-algebra, where $R = \mathbb{R}[x_1, \ldots, x_n]$. The spline space is shown to be an ideal in the ring $R[y_1, \ldots, y_s]/\langle y_1, \ldots, y_s \rangle$. We find the the generators of the splines spaces are actually coefficients of the y_i-linear terms of the generating set for this ideal. The generating set can be computed using a Gröbner basis. Using this, Clarke and Foucart were able to write code that gives a basis for spline spaces, and other things like the Hilbert series for the spline space.

Chapter 3 will go over the basics of algebra, sheaf theory, and scheme theory, that we use in research. The idea here is to introduce whatever is necessary to discuss my research after quasi-spline sheaves are defined. First we look into ring theory and the results we will need here, then sheaf theory. Once in sheaf theory, we look into cohomology and introduce Čech cohomology, which we emulate later on with quasi-spline sheaves. Last, we look into scheme theory.

In Chapter 4 we introduce quasi-spline sheaves and what contact ideals are. We look into what has been done with quasi-spline sheaves so far, including work done by Clarke proving the existence of the moduli space of quasi-spline sheaves amongst other things.

Let Y be a scheme and let $s \in \mathbb{N}$. We define a **quasi-spline sheaf** $\mathcal{S}$ as a quasi-coherent $\mathcal{O}_Y$-subalgebra of $\mathcal{O}_Y^s$. We can take the global spectrum of $\mathcal{S}$, Spec $\mathcal{S}$, creating what is called a **quasi-spline scheme**.

Let's go over one example of a quasi-spline sheaf. Let $Y = \operatorname{Spec} \mathbb{R}[x]$. Then we can take the sheaf associated with the $\mathbb{R}[x]$-module

$$\mathcal{S} = \{(g_1, g_2) \ : \ g_1 - g_2 \in (x^2)\}$$

which is contained in $(\mathbb{R}[x])^2$. Geometrically, this can be thought of the splines with continuous derivatives over the subdivision of $\mathbb{R}$, $(-\infty, 0] \cup [0, \infty)$. [Cla15]

To any quasi-spline scheme X, there are subschemes $K_{ij} \subseteq Y$ locally defined by the quasi-coherent ideals

$$\mathcal{J}_{ij} = \langle g_i - g_j \mid (g_1, \ldots, g_s) \in S \rangle.$$

We refer to them as the **contact subschemes** of X. In the example above, the contact ideals are all equal to the ideal (x^2).

The example above illustrated another way we can define quasi-spline sheaves, which is done by giving ideal sheaves. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf over a scheme Y. We say that $\mathcal{S}$

is determined by *ideal-difference conditions* $(I_{ij})_{ij}$, where I_{ij} are ideal sheaves over Y for each i, j such that $1 \leq i < j \leq s$, when $\mathcal{S}$ is locally defined as:

$$\mathcal{S} = \{(g_1, \ldots, g_s) \in \mathcal{O}_Y^s \ : \ g_i - g_j \in I_{ij}\}$$

The concept of ideal difference-conditions is not new, as it can be found in [Sch79]. We find that ideal difference-conditions are the same thing as smoothness conditions in Clarke and Foucart's paper [CS13].

Typically when studying spline spaces, we are given a geometric object where the splines are defined with smoothness conditions. Then we ask what the dimension of the entire vector space of splines is, or what the basis is. We can reverse the question when working in great generality, i.e scheme theory. Instead of working with splines on a given region, a region cut into pieces via smooth conditions for the splines, we can ask what is the most optimal subdivision of the given region, or for the best smoothness conditions. [Cla15]

This question can be rephrased as how to minimize a functional on $C^r(\Omega)$ where Ω is some domain embedded in some base scheme Y. Each point of the moduli space cooresponds to a subdivision of a region of Ω with a spline sheaf over that. The moduli space of spline schemes can be thought as the set of all subdivisions of a region Ω. As these spaces become more understood, techniques for optimal subdivisions of Ω could be found. Existence of these moduli spaces has been proved in certain cases [Cla15].

In Chapter 5, I introduce my own research. Billera found that certain spline spaces are determined by ideals, these ideals correspond to contact ideals, so we ask if contact ideals determine quasi-spline sheaves in the same way. We find that this isn't necessarily true, and find an example where it occurs.

For the rest of this thesis and every quasi-spline sheaf we work with, it will be **assumed that the quasi-spline sheaf is determined by its contact ideals** unless specified. That is, a quasi-spline sheaf $\mathcal{S}$, with contact ideals $\mathcal{J}_{ij}$'s, will be locally defined as such:

$$\mathcal{S} = \{(g_1, \ldots, g_s) \in O_Y^s \ : \ g_i - g_j \in \mathcal{J}_{ij} \text{ for all } i, j\}$$

Next, we look into ideal difference-conditions and discuss just how ideal difference-conditions relate to the contact ideals of a quasi-spline sheaf. The contact ideals are the canonical ideals, the smallest ideals, that generate a particular quasi-spline sheaf. After that, we look into when ideal difference-conditions are precisly the contact ideals. We find that ideal difference conditions are contact ideals when the conditions

$$\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj} \text{ for all } i, j, k$$

are satisfied for all ideals $\mathcal{J}_{ij}$ that give ideal difference-conditions to the quasi-spline sheaf. This is fairly significant, because it gives us a certain way to determine when the ideals we have are actually the contact ideals.

Next, we try to find a way to obtain contact ideals when all we have are ideal difference-conditions. This is useful because its so much easier to give ideal difference-conditions. To do this, we introduce a pinching operation, which seeks to find how contact ideals are changed when one ideal difference-conditions is placed on the original quasi-spline sheaf. Let $\mathcal{J}_{ij}$'s be the contact ideals for a s-sheeted quasi-spline sheaf $\mathcal{S}$. Let I_p be an ideal sheaf, then we can pinch the spline sheaf and create a new one that is locally defined as such:

$$\mathcal{S}' = \{(g_1, \ldots, g_s) \in \mathcal{S} \ : \ g_a - g_b \in I_p\}$$

when $a, b \in \{1, \ldots, s\}$. We conjecture that the contact ideals for $\mathcal{S}'$ are given by the formulas: for all i, j

$$\mathcal{J}'_{ab} = \mathcal{J}_{ab} \cap \mathcal{J}_P$$

$$\mathcal{J}'_{ai} = \mathcal{J}_{ai} \cap (\mathcal{J}_{bi} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{aj} = \mathcal{J}_{aj} \cap (\mathcal{J}_{bj} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{bi} = \mathcal{J}_{bi} \cap (\mathcal{J}_{ai} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{bj} = \mathcal{J}_{bj} \cap (\mathcal{J}_{aj} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{ij} = \mathcal{J}_{ij} \cap (\mathcal{J}'_{ai} + \mathcal{J}'_{aj}) \cap (\mathcal{J}'_{bi} + \mathcal{J}'_{bj}).$$

We find that these formulas are not true all the time. But by using code in Sage, these formulas do seem to be true the a lot of the times. If the pinching formulas do work using all ideals that give ideal difference-conditions for a quasi-spline sheaf, these formulas can be used to give recursive procedure for finding what the contact ideals are.

After this, we look into other ways to solve the problem of finding the contact ideals when ideal difference-conditions are given. Its easier to find closed formulas for the contact ideals when the number of sheets is 3. Indeed, if the ideal difference-conditions for $\mathcal{S}$ are given by I_{ij}'s, the contact ideals $\mathcal{J}_{ij}$'s are exactly:

$$\mathcal{J}_{ij} = I_{ij} \cap (I_{ik} + I_{kj})$$

where $k \neq i, j$. But this proof can't be generalized to larger values of s.

After this, we use what we have learned so far to conjecture some other formulas. As before, let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with ideal difference-conditions given by I_{ij}'s. Then define

$$I^1_{ij} = I_{ij} \cap \left(\bigcap_k (I_{ik} + I_{kj}) \right)$$

and for any $n \geq 2$, let I^n_{ij} be defined as:

$$I^n_{ij} = I^{n-1}_{ij} \cap \left(\bigcap_k (I^{n-1}_{ik} + I^{n-1}_{kj}) \right)$$

Notice if this sequence of ideals stabilizes, that is, if $I_{ij}^m = I_{ij}^{m+1}$, it is necessarily the case that the ideals I_{ij}^m's satisfy the conditions above identifying that these ideals are actually the contact ideals for $\mathcal{S}$. I conjecture that this sequence will always stabilize before or at $s-2$, so that the contact ideals are

$$\mathcal{J}_{ij} = I_{ij}^{s-2}$$

I've used Sage and coded this up, and I haven't found a counter example to this. Notice how it works for $s = 3$.

Last, we define a complex for a quasi-spline sheaf based on the contact subschemes. We suggest a method of proving that this complex is actually a resolution, but this only goes so far. This complex is defined similarly to a Čech complex: for each p,

$$C^p(\mathcal{S}) = \bigoplus_{i_0<\cdots<i_p} i_*\mathcal{O}_{X_{i_0,\ldots,i_p}}$$

where $X_{i_0\ldots i_p} = K_{i_0i_1} \cap K_{i_1i_2} \cap \cdots \cap K_{i_{p-1}i_p}$, an intersection of contact subschemes, and $\mathcal{O}_{X_{i_0\ldots i_p}}$ is the structure sheaf of these closed subschemes.

We aim to prove that the complex

$$0 \to \mathcal{S} \to C^0(\mathcal{S}) \xrightarrow{d_0} C^1(\mathcal{S}) \xrightarrow{d_1} C^2(\mathcal{S}) \to \ldots$$

is a resolution to $\mathcal{S}$, or that the sequence is exact. Notice that $C^0(\mathcal{S}) = \mathcal{O}_Y^s$ and $C^1(\mathcal{S}) = \bigoplus_{ij} \mathcal{O}_{K_{ij}}$, so that exactness at the first couple step implies that $\mathcal{S}$ is determined by its contact ideals. We've shown this doesn't always happen, so its not necessarily the case this complex is a resolution. But we want to know if it is a resolution when $\mathcal{S}$ *is* determined by its contact ideals.

It is easy to see that $\text{im}\,(d_i) \subseteq \ker(d_{i+1})$ for any $i \geq 0$, but it is difficult to show the other containment. We suggested a Chinese Remainder Theorem-esque argument, but this only goes so far. We know for sure that the complex is a resolution for quasi-spline sheaves determined by their contact ideals when $s = 3$ and almost for $s = 4$.

For the future, I want to give more complete answers for the variety of conjectures I have about the relationship that contact ideals have with quasi-spline sheaves. It would be nice to have a complete answer for the pinching formulas, to know *when* the formulas are correct. Further, I'd like to get a definite answer as to when my conjectured formulas for the contact ideal in terms of ideal difference-conditions are actually the contact ideals. Last, I want to know if the complex of a quasi-spline sheaf is actually a resolution.

2. Spline Spaces

Splines are piece-wise defined polynomials that satisfies certain smoothness conditions where the polynomial pieces connect.

A spline function is piece-wise defined polynomial that satisfies certain smoothness conditions where the polynomial pieces connect. Let

$$f(x) = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

then this function is a C^0 function on the real line. Indeed, f is continuous at the origin, the place where the two polynomial pieces intersect. Notice that the two pieces, $g_1(x) = x$ and $g_2(x) = -x$ are such that x divides $g_1(x) - g_2(x)$. If we wanted the function to be a C^1 function on the real line, we would require that x^2 divides $g_1(x) - g_2(x)$, like if the spline function was

$$f(x) = \begin{cases} x^2 & : x \geq 0 \\ 0 & : x < 0 \end{cases}$$

Algebraically speaking, if we want a C^0 function, we want $g_1 - g_2 \in (x)$, where (x) is the ideal of $\mathbb{R}[x]$ generated by x. If we want a C^1 function, we want $g_1 - g_2 \in (x^2)$.

Next, we can ask what vector spaces of these spline functions would look like. Take for example, the vector space of splines functions that are C^1 over the real line, where each spline function is split into three pieces, where the first and second piece intersect at the origin, and the second and third pieces intersect at the point 3 on the real line. This is equivalent to finding a three-tuple (g_1, g_2, g_3) where $g_i \in \mathbb{R}[x]$ for all i, $g_1 - g_2 \in (x^2)$ and $g_2 - g_3 \in ((x-3)^2)$.

First, there are traditional analytic techniques to work with splines, as can be seen in Schumaker's book [Sch07] . But here, we are more focused on the algebraic treatments of spline spaces.

This subsection is for studying spline spaces, which are vector spaces of spline spaces that are usually over a polyhedral complex in real space. First we will discuss some of the prerequisites to understanding splines spaces. We give a definition for spline space defined over a polyhedral complex.

The most important questions when studying spline spaces is finding a basis for it and the dimension for the spline space. There has been much work done in both of these when the splines are bivariate and the problem is solved for when the splines are univariate. Research done on splines spaces in three or more variables is relatively new and there isn't too much that has been done here.

Next, we discuss the research that Billera had done over spline spaces. Billera was the first mathematician to employ algebra and algebraic geometry, and other homological algebraic techniques to studying spline spaces. After this, we discuss the homology and cohomology theories for spline spaces. Last, we look at a more recent treatement of splines spaces using commutative algebra, done by Clarke and Foucart.

2.1 Preliminaries for Spline Spaces

Spline spaces can be defined over many kinds of domains in real space, but typically they are defined on a polyhedral complex in real space.

First, we will introduce a definition of spline space that is defined over Δ, a simplicial complex in $\mathbb{R}^d$ found in Billera's paper [Bil88]. Let Δ be a finite, pure d-dimensional, strongly connected, simplicial complex embedded in $\mathbb{R}^d$. After that, we show a definition of a spline space defined overa polyhedral complex, a generalized version of the splines space defined on a simplicial complex.

We define what a simplex is, then a simplicial complex.

A **simplex** is a generalization of a triangle or tetrahedron to arbitrary dimensions. A k-simplex is a k-dimensional polytope which is the convex hull of its $k + 1$ vertices. More formally, suppose the $k+1$ points $x_0, \ldots x_k \in \mathbb{R}^n$ are affinely independent, which means $u_1 - u_0, \ldots, u_k - u_0$ are linearly independent. Then the simplex determined by them is the set of points

$$\Delta = \{t_0 x_0 + \cdots + t_k x_k \mid \theta_0 \geq 0,\ 0 \leq i \leq k,\ \sum_{i=0}^{k} t_i = 1\}$$

[Hat02] So a 0-simplex is a point, a 1-simplex is a line segment, 2-simplex is a triangle, and 3-simplex is a tetrahedron.

Let Δ be a finite d-dimensional **simplicial complex** embedded in $\mathbb{R}^d$. The domain Δ can be thought of a triangulation of a compact region in $\mathbb{R}^d$, where each simplex of Δ is a convex hull of its vertices. The domain Δ is embedded in $\mathbb{R}^d$, so we say $\Delta \subseteq \mathbb{R}^d$. You can think of a triangulation of a topological space to be a simplicial complex that is topologically homeomorphic to the original topological space. [Bil88] Further, we assume Δ is *pure*, which is to say, the maximal simplex has dimension d. Also, we assume that Δ is connected.

Then we have the definition of a spline space over that simplicial complex as it is written in [Bil88]. Let $S^r_m(\Delta)$ be the set of all piecewise polynomial functions on Δ of degree at most m and smooth of order r. In another words, all functions $f : \Delta \to \mathbb{R}$ such that

- $f\big|_\sigma$ is a polynomial of degree $\leq m$, for each $\sigma \in \Delta$ and
- f is continuously differentiable up to order r.

More generally, we can also build spline spaces on polyhedral complexes.

A **polyhedral complex** $\Delta \subseteq \mathbb{R}^d$ is a finite set of complex polytopes in $\mathbb{R}^d$ where any polytope contains every face of it and any intersection of any two polytopes is a face of each. [Yuz92] [BR92]

A spline space can be defined on a polyhedral complex like this: let Δ be a finite polyhedral complex embedded in real space $\mathbb{R}^d$, then we define $C^r(\Delta)$ to be the set of all piecewise polynomial functions over Δ which are smooth of order r. Then the spline space, $S^r_m(\Delta)$ denotes the subset of $C^r(\Delta)$ consisting of functions whose polynomials are of degree at most m. [BR91]

We assume a polydral complex Δ will be purely embedded in $\mathbb{R}^d$, meaning that all of its maximal polytopes have dimension d. [Yuz92]

2.2 Billera on Spline Spaces

Now we will look into how Billera worked with spline spaces. Again, he was the first to use algebraic techniques to study these things.

Billera defines a spline space on a simplicial complex. Let Δ be a finite, pure d-dimensional simplicial complex embedded in $\mathbb{R}^d$ and is connected. [Bil88]

Let $S^r_m(\Delta)$ be the set of all piecewise polynomial functions on Δ of degree at most m and smooth of order r. In another words, all functions $f : \Delta \to \mathbb{R}$ such that

- $f\big|_\sigma$ is a polynomial of degree $\leq m$, for each $\sigma \in \Delta$ and
- f is continuously differentiable up to order r.

These functions f are called *splines*. The set $S^r_m(\Delta)$ is a vector space over R. [Bil88]

When Δ is a polyhedral complex, in [BR92] Billera/Rose show that $C^r(\Delta)$ is a $\mathbb{R}$-algebra with some conditions for when $C^r(\Delta)$ is free. This makes it easier to find the dimensions of subalgebras $S^r_m(\Delta)$. If Δ is a manifold with boundary then $C^r(\Delta)$ will be free.

Next, Billera finds a way to determine $S^r_m(\Delta)$ by looking at the intersection of the maximal faces of Δ. First, we discover that every spline of $S^r_m(\Delta)$ satisfies some algebraic conditions.

Lemma 1 (Lemma 2.2 of [Bil88]). Suppose $F \in S^r_m(\Delta)$ for some d-complex $\Delta \subset \mathbb{R}^d$ and $r \geq 0$. Let $\sigma_1, \sigma_2 \in \Delta$ are two d-simplices such that $\tau = \sigma_1 \cap \sigma_2$ has dimension $d-1$. Then if ℓ is a nontrivial affine form which vanishes on τ, we have

$$\ell^{r+1} | (p_1 - p_2)$$

where $p_i = F\big|_{\sigma_i}$, $i = 1, 2$.

Now we try to find sufficient conditions for a function to be in $S^r_m(\Delta)$. But to do this, we need to define a link of a simplex in Δ.

Let a d-complex Δ be *strongly connected* if for any d-simplices $\sigma, \sigma' \in \Delta$, there is a sequence of d-simplices

$$\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k = \sigma'$$

such that for all $i < k$, $\sigma_i \cap \sigma_{i+1}$ has dimension $d - 1$. Here, σ_i and σ_{i+1} are said to be adjacent to each other. [Bil88]

For every $\tau \in \Delta$, we define the link of τ, $\text{lk}(\tau) = \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$. Now we can give sufficient conditions for a piecewise polynomial to be in a spline space.

Theorem 2 (Theorem 2.4 [Bil88]). Suppose Δ is strongly connected d-complex such that all links of simplices are also strongly connected complexes. Let F be a piecewise polynomial such that $F\big|_\sigma$ is a polynomial of degree $\leq m$ for each $\sigma \in \Delta$. Then $F \in S^r_m(\Delta)$ if and only if $\ell^{r+1} | (p_1 - p_2)$ as above for each pair σ_1, σ_2 of adjcacent d-simplices in Δ.

In this way, the affine spans of $\sigma_i \cap \sigma_j$ for all i, j, determine $S^r_d(\Delta)$. [Bil88, Theorem 2.4]. So in this sense, we can say two splines spaces are isomorphic if they are determined by the same ideals ℓ_τ^{r+1} (see [KS14]). We will find later that two different sets of ideals might determine the same spline space, even though their ideals are different.

There is a similar result when Δ is a polyhedral complex, see [BR91], but it's required that Δ be *hereditary*. The forward direction, as in the Theorem below, is easy to show, but we need Δ to be hereditary for the backwards direction. A polyhedral complex is a *hereditary* complex if for all nonempty $\sigma \in \Delta$, $\text{st}(\sigma)$ is strongly connected. The star of σ in Δ is defined to be the smallest subcomplex of Δ containing all faces that contain σ. A complex is said to be *strongly connected* when the graph of it is connected. [BR92]

2.3 Dimension of Splines Spaces

The fundamental problems regarding spline spaces is what is the dimension of them, and what is the basis for the spline space. The problem was first discussed by Strange, in [Str73] and [Str74] who was interested in its applications to PDE via the finite element method.

Let $\Delta \subset \mathbb{R}^n$. When $n = 1$, all polynomials of $S^r_m(\Delta)$ are univariate, and when $n = 2$, all polynomials of the spline space are bivariate. Alfeld wrote a paper [Alf86] that gives formulas for the univariate case, some bivariate and trivariate cases.

The univarite case is solved, the bivariate case has been worked on a while and there are many results. When it comes to the trivariate case, things become much more difficult, and there aren't many results.

In the univariate case, each $p \in S^r_m$ is defined on an closed interval of $\mathbb{R}$, and any two faces of Δ will intersect at a point which may be the left or right point of the interval. The following Proposition gives the formula for the dimension for a spline space when the splines are univariate.

Proposition 3. (Chapter 2 of [Alf86]) Let $S^r_m(\Delta)$ where $\Delta \subseteq \mathbb{R}$, then

$$\dim(S^r_m) = m + 1 + (N - 1)(m - r)$$

where N is the number of 1-dimensional faces of Δ (intervals).

In general, we find the the dimension of a spline space depends on the degree d, the smoothness conditions denoted by r, and even the 'geometry' of the space, or how it is embedded, but counter-intuitively, it can depend on the location of the vertices. [Alf00] Indeed, the dimension of a spline spaces can change after an arbitrarily small peturbation of a vertex. It was first noted in [MS75] that the dimension depends on the geometry of the triangulation.

When jumping from the univariate case to the bivariate case, there is a complication. The dimension of spline space can depend on the geometry of the triangulation. For example, the triangulation of a rectangle with two crossing diagonals has a dimension of $C^1_2(\Delta)$ one higher than the combinatorically identical situation where the central point is the not the intersection of the diagonals. The issue here is that each edge connected to the central point has the same slope as the edge attached to the central point that is opposite to it. This hints to a fact that the dimension of a spline space might jump when the vertices are altered by a small amount. [Bil88]

We find that some researchers like to restrict to easier situations to work with Δ's. For example, Farin works with Δ that are *unconstricted triangulations*, which is when a triangulation has a minimal amount of edges incident to boundary vertices. [Far06]

The bivariate case is much harder. Let's take a look at the case when the spline space consists of continuous piecewise linear functions, also known as the spline space denoted by $S^0_1(\Delta)$. It isn't hard to show that the dimension of this spline spaces is precisely the number of vertices of Δ. Indeed, a basis for the spline space is given by functions for each vertex, where on that vertex the function is 1, and is 0 on the other vertices. The case $S^0_m(\Delta)$ was done by Billera in [Bil89], which consists in continuous piece-wise functions of degree up to m, we find that

$$\dim S^0_m(\Delta) = \sum_{j=0}^{d} f_j \binom{m-1}{j}$$

for any m, and f_j is the number of j-dimensional faces of Δ.

Further, in [Alf86], Alfeld gives the dimension for spline spaces in the trivarite case when the splines were first and second-order differentiable.

Back on to bivariate splines, looking at spline spaces whose functions are continuous and differentiable, we've got Strang's Conjecture where Δ is a planar 2-manifold. The lower bound has been proved by Billera in [Bil88].

Conjecture 4 (Strang's Conjecture). For a generic embedding of a planar 2-manifold,

$$\dim S^1_m(\Delta) = \binom{m+2}{2} f_2 - (2m+1)f_1^\circ + 3f_0^\circ$$

where f_2 is the number of triangles of Δ, f_1° and f_0° the number of interior edges and vertices, respecitively.

This was only hypothesized for generic embeddings, so that we miss out on defective arrangements. Let Δ be a simplical complex on n vertices. A property holds for *generic embeddings* of $\Delta \subset \mathbb{R}^2$, or *generically*, when the property is true for any embedding of Δ in $\mathbb{R}^2$, where vertices are located at (x_i, y_i) for $i = 1, \dots, n$ and there is a nonzero real polynomial p of $2n$ variables such that $p(x_1, y_1, \dots, x_n, y_n) \neq 0$ [Bil88]. In this way, the conjecture ignores cases where the embedding can make a difference in the dimension of the spline space.

The most popular ways to find the dimension of a spline space include the Bézeir Bernstein approach, which was first used in [Alf00]. This approach uses the Bézeir-Bernstein form of a bivariate polynomial. Using this one can find a minimal determining set of points whose cardinality gives an upperbound for the dimension of the spline space. This is used to find the dimension of spline spaces with bivariate splines in [Alf00]

In [KS14], the Kolesnikov and Sorokina use algebraic geometric methods similar to those of Billera that we've seen above, and Bernstein-Bezier techniques to find the dimensions of spline spaces consisting of C^1 smooth splines on the Alfled split of a simplex. An Alfeld split can be built from a non-degenerate simplex, and they show that the space of splines over a particular Alfeld split can be identified with the space of splines over another simplex, which they call the Alfeld pyramid.

The Strang Conjecture was proved for cases $m \geq 5$ in the paper [MS75] by Morgan and Scott. The way this was proved also gives a basis for any subspace $S^1_m(\Delta)$ where $m \geq 5$. The result by Morgan and Scott was shown to be a lower bound for the dimension of $S^1_m(\Delta)$ for all $m \geq 2$, and also lower bound for the dimension of $S^r_m(\Delta)$ for $m \geq r + 1$.

In Alfeld and Schumaker's paper [AS90], then find the dimension of spline spaces S^r_m when $r \geq 1$ and $m = 3r + 1$, and we find that the dimension of the spline space depends on the different numbers of slopes of edges incident to a vertex.

Morgan and Scott working out some dimensions of splines spaces in the bivariate case, giving upper and lower bounds for spline spaces S_m^r where $d \geq 5$ and $r > 1$ including the case when $r = 1$ and $d < 5$ for arbitrary partitions for Δ. They also showed that the dimension of $S_m^1(\Delta)$ is given by Strange's conjecture plus the number of rectangles triangulated by crossing diagonals. The proof of this gives a way to find a basis for $S_m^1(\Delta)$. [MS75]

Alfeld and Schumaker tried to extend these results in [AS87] by giving formulas for the dimensions of spline spaces $S_m^r(\Delta)$ where $d \geq 4r+1$. They were also able to prove the existence of locally supported basis functions for $S_m^r(\Delta)$ when $r \geq 1$ and $d \geq 4r + 1$, and they were also able to find bases of locally supported functions for cases $r = 2$ and $r = 3$. In another paper, with Alfeld, Piper, and Schumaker, they were able to find explicit basis for C^1 quartic bivariate splines. [APS87]

There have been several efforts to find when $C^r(\Delta)$ is free. In [BR92], Billera/Rose show that we can get a reduced basis for the spline space when $C^r(\Delta)$ is free, and this freeness can be obtained when Δ is a manifold with boundary. In [SS97], Schenck and Stillman give a complex for a spline space and find the cohomology of it. Let $\hat{\Delta}$ be Δ that is joined with the origin in $\mathbb{R}^3$. They find that $C^r(\hat{\Delta})$ is free if and only if Δ is genus zero and $S_m^r(\Delta)$ has the expected dimension for $k = r + 1$. They define a complex R/J, and find that $H_1(R/J)$ is zero exactly when $C^r(\hat{\Delta})$ is free. They also give a simple non-freeness condition.

Alfeld and Shumaker extended the result from Morgan and Scott in [AS90], showing that it is equal to the $\dim S_m^r(\Delta)$ for all $m \geq 4r + 1$.

The trivarite case is harder and there aren't as many results for that case just yet. In [Alf86], Alfeld gives the dimension for spline spaces in the trivarite case when the splines were C^1 and C^2. Further, in [AS08], Alfeld and Schumaker gave bounds for trivariate spline spaces by giving upper and lower bounds for how a dimension of a spline space can change when a tetrahedron is added to Δ. To do this, they analyzed all kinds of different ways tetrahedrons can intersect and what this does to the dimension of the spline space. This can be used to give upper and lower bounds for any trivariate spline space.

2.3.1 Homology and Cohomology Theories of S^r_m

Homological methods gives a way to do a lot of linear algebra like computations in a very neat and organized fashion. Look at Chapter 3 for a more categorical approach to homology and cohomology, which will be useful later when discussing Čech cohomology. Homological methods have proved useful for finding the dimensions of splines space and we go over the results of Billera's and Schenck's work. Indeed, Billera had first used homological methods with spline spaces in [Bil88] where he developed a homology theory for $S^r_k(\Delta)$ using ideals defined by splines that vanish on certain faces of Δ, and proved the lower bound for the dimension of spline spaces for some cases of the Strang's Conjecture.

Schenck and Stillman also created a complex for spline spaces in [Sch97] and [SS97]. In [Sch97], Schenck creates a complex R/J of graded modules on a d-dimensional simplicial complex. The homology modules of this complex consist of splines of smoothness r on Δ. He also gives bounds on the dimensions of the homology modules $H_i(R/J)$. In the next paper with Stillman, [SS97], they discuss how the space of bivariate splines $S^r_k(\Delta)$ for all k relates to the freeness of $C^r(\Delta)$. They also found that $C^r(\hat{\Delta})$ is free if and only if the topology of Δ is genus zero and $S^r_m(\Delta)$ has the expected dimension for $k = r + 1$, when $\hat{\Delta}$ is Δ joined with the origin in $\mathbb{R}^3$. They work with the same complex defined in [Sch97], and find some basic properties of $H_1(R/J)$. They find that $H_1(R/J)$ measures the deviation of the Hilbert series of $C^r(\hat{\Delta})$ from the generic series, and that $H_1(R/J)$ is zero exactly when $C^r(\hat{\Delta})$ is free.

In another paper by Billera and Rose [BR92], aiming to find the dimension of $S^r_k(\Delta)$ as an $\mathbb{R}$-vector space where Δ is some triangulated region of $\mathbb{R}^d$, they started working on finding when $C^r(\Delta)$ is a free $\mathbb{R}$-module, where $C^r(\Delta)$ is the set of piecewise polynomial functions on Δ that are continuously differentiable up to order r.

Now $C^r(\Delta)$ is a $\mathbb{R}$-module and also a $\mathbb{R}$-algebra. The elements of the spline spaces $S^r_k(\Delta)$ are contained inside $C^r(\Delta)$, so that understanding $C^r(\Delta)$ will provide more information about how the spline spaces S^r_k are related to each other over various values for k. In [BR92], Billera and Rose looked for conditions on Δ, r, and d so that the freeness of $C^r(\Delta)$ will be independent of the embedding of Δ in $\mathbb{R}^d$.

One of the results is that when $d = 2$, $C^r(\Delta)$ is free if and only if Δ is a manifold. They find that when $d > 2$ and $r > 0$, the freeness of $C^r(\Delta)$ is dependent on the embedding of Δ into $\mathbb{R}^d$. Further, $C^r(\Delta)$ is free if and only if $C^r(\text{star}(\sigma))$ is free for all faces $\sigma \in \Delta$.

Schenck had done similar work. To compute the dimension of $S^r_k(\Delta)$, he looked at $C^r(\hat{\Delta})$, where $\hat{\Delta}$ is the joint of Δ with the origin in $\mathbb{R}^{d+1}$. Then $C^r(\hat{\Delta})$ is a graded $R[x_1, \ldots, x_{d+1}]$-module that is finitely generated. The relationship between $C^r(\hat{\Delta})$ and $S^r_d(\Delta)$ is quite clear, because $C^r(\hat{\Delta})_k$ is the homogenizations of the elements of $S^r_k(\Delta)$. In this way, the dimension of $S^r_k(\Delta)$ can be studied by looking at the Hilbert series of $C^r(\hat{\Delta})$. [Sch97]

In both papers, [SS97] authored by Stillman and Schenck, and [Sch97] by Schenck, a short sequence of complexes based around Δ is established. First, J is a complex of ideals on Δ, ideals that vanish on particuliar faces of Δ, and R is the constant complex on Δ, so that $R(\sigma) = R[x_1, \ldots, x_{d+1}]$ for all $\sigma \in \Delta$, then there is a short exact sequence

$$0 \to J \to R \to R/J \to 0.$$

Any short exact seqence gives way to a long exact sequence by taking the homology $H_*(-)$ of each of the complexes. So we get the long sequence:

$$0 \to H_2(R) \to H_2(R/J) \to H_1(J) \to H_1(R) \to H_1(R/J) \to H_0(J) \to 0$$

The main result of Schenck and Stillman's paper [SS97] is $C^r(\hat{\Delta})$ is free if and only if $|\Delta|$ has genus zero and $S^r_k(\Delta)$ has the expected dimension for $k = r + 1$. In particular, $C^1(\hat{\Delta})$ is free if Δ is generically embedded in the plane. Later in the paper, we also find that $C^r(\hat{\Delta})$ is free if and only if

$H_1(R/J)$ is zero.

In this case, the Hilbert Series of $C^r(\hat{\Delta})$ is determined by local data of Δ. Also when $d = 2$, $C^r(\hat{\Delta})$ can only be free if Δ is a topoligical disk.

In another paper by Schenck, [Sch97], it is shown that $C^r(\hat{\Delta}) \cong H_d(R/J)$. They restrict to when Δ is a topological d-ball, there is a spectral sequence relating $C^r(\hat{\Delta})$ to the modules $H_i(R/J)$ when $i < d$.

2.4 Another Approach using Commutative Algebra

This next paper introduced a new computational method in spline theory for computing bases for splines spaces, and also for computing the dimensions for spline spaces. This method done by Clarke and Foucart used commutative algebra to work with spline spaces. Unlike other papers, these methods are not dependent on bivariate, trivariate splines, or even on simplicial partitions. This work gave way to new code written in SAGE which is used to give another perspecitve to Rose's freeness conjecture for spline spaces, verify dimensions formulas that have already been proposed, and also to give new dimension formulas for triangulations with 'hanging vertices'. [CS13]

For our sakes, we are interested in the implementation of commutative algebra for working with spline spaces, and even the SAGE code that is written that gives informations about spline spaces, including but not limited to, the bases and Hilbert series for the spline spaces.

The spline spaces are defined in [CS13] like this. Let $\Omega \subseteq \mathbb{R}^n$ be a domain with fixed subdivision $\Sigma = \{\sigma_1, \ldots \sigma_s\}$ of Ω by closed sets. A spline over Ω with respect to Σ is an assignment of an n-variable polynomial, g_j, to each region σ_j, so that there is a piecewise function $\Omega \to \mathbb{R}$ whose value on σ_j is g_j. [CS13]

Now, we discuss the smoothness conditions for such splines. Let a point $p \in \Omega$, we can look at the splines on which any element of a given set $\mathcal{D}_p$, of linear differential operators has a well-defined value. A choice of each point of Ω of linear differential operators is called a family of smoothness conditions when it satisfies this property:

- Give a real-valued function h on Ω, whenever Dh is well defined for all $p \in \Omega$ and all operators $D \in \mathcal{D}_p$, then so is $D(f \cdot h)$ for any $f \in \mathbb{R}[x_1, \ldots, x_n]$. [CS13]

Example 5. Consider splines that are C^r functions on Ω. The operators at any given point are those in the span of [CS13]:

$$\{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}\Big|_p, \ \alpha_1 + \ldots \alpha_n \leq r\}$$

We work with the ring $\mathbb{R}[x_1, \ldots, x_n]$ of n-variable polynomials denoted by R, and the set of spline functions which satisfy a family of smoothness conditions by $\mathcal{S}$. The set $\mathcal{S}$ is an R-algebra. By adding formal variables $y_1, \ldots, y_s$ to the ring R, $\mathcal{S}$ can be described entirely of certain ideals in $R[y_1, \ldots, y_s]$. [CS13]

Let $\mathcal{I}(\sigma_j \cap \sigma_k) = \{f \in R \mid f(p) = 0 \text{ for all } p \in \sigma_j \cap \sigma_k\}$ the set of polynomials that vanish on the intersection of σ_j and σ_k. These are the 'contact ideals' if $\mathcal{S}$ where a quasi-spline sheaf.

The next Theorem shows that the smoothness conditions that determine a spline space are equivalent to giving ideals $\mathcal{J}_{kj}$.

Theorem 6. [CS13, Theorem 3.1] The set of splines satisfying a family of smoothness conditions forms an R-algebra and there exists ideals $\mathcal{J}_{jk} \subseteq \mathcal{I}(\sigma_j \cap \sigma_k)$ such that a s-tuple of polynomials $G = (g_1, \ldots, g_s)$ is in this ring if and only if

$$g_i - g_j \in \mathcal{J}_{jk} \text{ for all } 1 \leq j, k \leq s \tag{IC}$$

Conversely, given any family of ideals, $\mathcal{J}_{kj}$ such that $\mathcal{J}_{kj} \subseteq \mathcal{I}(\sigma_j \cap \sigma_k)$, there is a corresponding family of smoothness conditions whose splines are determined by the equations (IC)

We find that $\mathcal{S}$ can be identifed with an ideal of the ring $R[y_1, \ldots, y_s]/\langle y_1, \ldots, y_s\rangle^2$.

Theorem 7. [CS13, Theorem 3.2] The R-algebra $\mathcal{S}$, as an R-module, is isomorphic to the module

$$M = \bigcap_{jk} (\mathcal{J}_{kj} \cdot \langle y_j + y_k \rangle + \langle y_1, \ldots, \hat{y}_j, \ldots, \hat{j}_k, \ldots, y_s \rangle) \subseteq R[y_1, \ldots, y_s]/\langle y_1, \ldots, y_s \rangle^2, \tag{M}$$

where ˆ indicates omission of the variable.

Now we can work on trying to find the generators of $\mathcal{S}$ as a R-module. That is, we want elements $G_1 = (g_{11}, g_{12}, \ldots, g_{1s}), G_2 = (g_{21}, g_{22}, \ldots, g_{2s}), \ldots, G_\ell = (g_{\ell 1}, g_{\ell 2}, \ldots, g_{\ell s})$ inside of $\mathcal{S}$ such that any spline $G \in \mathcal{S}$ can be written like

$$G = c_1 G_1 + c_2 G_2 + \ldots c_\ell G_\ell \text{ for polynomials } c_1, \ldots, c_\ell \in R.$$

Generators of $\mathcal{S}$ appear as the coefficient vectors of the y_i-linear terms of a generating set for $\tilde{M}$, where $M = \tilde{M}/\langle y_1, \ldots, y_s \rangle^2$. [CS13]

Lemma 8. [CS13, Lemma 4.1] Let B denote any generating set for the ideal $\tilde{M}$. For each element $b \in B$, let b_1 denote the y-linear term. Then the image of the set

$$\{b_1, b \in B\}$$

under the map $\tilde{M} \twoheadrightarrow M \xrightarrow{\sim} \mathcal{S}$ generates $\mathcal{S}$ as an R-module.

Essentially, Theorem 6 and Lemma 8 together imply that we can find a set of generators for $\mathcal{S}$ by computing a Gröbner basis for the intersection

$$\tilde{M} = \langle y_1, \ldots, y_s \rangle^2 + \bigcap_{jk} (\mathcal{J}_{kj} \cdot \langle y_j + y_k \rangle + \langle y_1, \ldots, \hat{y}_j, \ldots, \hat{j}_k, \ldots, y_s \rangle) \subseteq R[y_1, \ldots, y_s] \tag{M'}$$

where ˆ indicates omission of the variable.

Clarke and Foucart were able to write code in Sage, using the ideas above, that finds generators for $\mathcal{S}$ when smoothness conditions are specified (see [CS13]). I use this code to work out some examples of spline sheaves, which helped to disprove some hypotheses we had that will be addressed in the next chapter.

Further in the paper [CS13], another way of computing the Hilbert series for $\mathcal{S}$ is introduced, as the question now becomes that of finding the Hilbert series for an ideal. Clarke and Foucart also wrote code to compute the Hilbert series as well. Last, there is more code written to address whether a spline space $\mathcal{S}$ is free module or not.

3. Spline Sheaf Preliminaries

3.1 Ring Theory

Any ring we work with will be commutative and with unity and we assume that the two binary operations on a ring are addition and multiplication. The definitions of a ring and ring homomorphism can be found in [AM69] or [Lan93].

Definition 9. Let I be an ideal of R. Then I is a subset of R and satisfies the following properties:

- If $r, s \in I$, then $r + s \in I$.
- If $r \in R$ and $s \in I$, then $r \cdot s \in I$.

Let M be an *R-module*. This means that M is an abelian group and R is a ring, and we have an action on M with R such that for all $a, b \in R$ and $x, y \in M$, $(a+b)x = ax+bx$ and $a(x+y) = ax+ay$. Usually this is called a left R-module, but R is commutative so the notion of left and right modules coincide.

Definition 10. Let M, M' and M'' be R-modules, then the sequence

$$0 \to M \xrightarrow{\phi} M' \xrightarrow{\psi} M'' \to 0$$

is an exact sequence if the following is true:

- ϕ is an injective map.
- ψ is a surjective map
- im $(\phi) = \ker(\psi)$

Before moving on, we can show that, when I_1 and I_2 are ideals of a ring R, then the sequence is exact:

$$0 \to R/I_1 \cap I_2 \to R/I_1 \oplus R/I_2 \to R/I_1 + I_2 \to 0.$$

This will be useful later on when working with a complex we find for a quasi-spline sheaf and we need to prove exactness for it in certain cases. So the next couple results are simply to prove this result.

Proposition 11. Let I be an ideal of R. Then the sequence of maps:

$$0 \to I \to R \to R/I \to 0$$

where each map is the natural map, is an exact sequence.

Proof. Clearly the first map, ϕ_1 is injective, and the second map, ϕ_2 is surjective. It should also be clear that im $(\phi_1) \subseteq \ker(\phi_2)$. Let $a \in \ker(\phi_2)$. This means that $\phi_2(a) = 0$, so that $a \in I$. Then we have that $\phi_1(a) = a$, and $a \in$ im (ϕ_1). □

Let I_1 and I_2 be ideals of R, then we have an exact sequence

$$0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to I_1 + I_2 \to 0$$

where the first map sends $a \mapsto a \oplus a$ and the second map sends $a_1 \oplus a_2 \to a_1 - a_2$.

This is an exact sequence because it should be clear that the first map, call it ϕ_1, is injective and the second map, ϕ_2, is surjective. To see that im $(\phi_1) = \ker(\phi_2)$, it should be clear from the definition of the maps that im $(\phi_1) \subseteq \ker(\phi_2)$. To see the other containment, let $a_1 \oplus a_2 \in \ker(\phi_2)$. This means that $a_1 - a_2 = 0$ in $I_1 + I_2$, which implies that $a_1 = a_2$. Now $a_1 \in I_1$ and $a_2 \in I_2$, so that $a_1 \in I_1 \cap I_2$ and $\phi_1(a_1) = a_1 \oplus a_2$.

Next, it should be clear that the sequence of maps

$$0 \to R \to R \oplus R \to R \to 0$$

where the maps are the same as the sequence before, gives an exact sequence of R-modules. This will be claimed without proof, as the proof is very similar to the last claim.

Proposition 12 (Five-Lemma). [Lan93, p.169] Let the following be a complex of R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{p} & B & \xrightarrow{q} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{s} & B' & \xrightarrow{t} & C' & \longrightarrow & 0 \end{array}$$

if the rows are exact sequences, and g and f are isomorphisms, then h is an isomorphism as well.

Proposition 13. Let I_1 and I_2 be ideals of R. Then the sequence of maps:

$$0 \to R/I_1 \cap I_2 \to R/I_1 \oplus R/I_2 \to R/I_1 + I_2 \to 0$$

is exact.

Proof. First, we have a large commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1 \cap I_2 & \longrightarrow & I_1 \oplus I_2 & \longrightarrow & I_1 + I_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R/I_1 \cap I_2 & \longrightarrow & R/I_1 \oplus R/I_2 & \longrightarrow & R/I_1 + I_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

We can compress this to the diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
(I_1 \oplus I_2)/I_1 \cap I_2 & \longrightarrow & I_1 + I_2 \\
\downarrow & & \downarrow \\
(R \oplus R)/R & \longrightarrow & R \\
\downarrow & & \downarrow \\
(R/I_1 \oplus R/I_2)\,/(R/I_1 \cap I_2) & \longrightarrow & R/I_1 + I_2 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

where the columns are exact sequence and the first two rows are isomorphisms. Then by the Five-Lemma above, we know that the third row is an isomorphism, which means that the sequence in question is exact. □

Theorem 14. **Chinese Remainder Theorem**: Let $a_1, \ldots a_n$ be ideals of a ring A such that $a_i + a_j = A$ for all $i \neq j$. Given elements $x_1, \ldots, x_n \in A$, there exists $x \in A$ such that $x \equiv x_i (\text{mod } a_i)$ for all i.[Lan93]

Corollary 15. Let $a_1, \ldots, a_n$ be ideals of A. Assume that $a_i + a_j = A$ for $i \neq j$. Let

$$f : A \to \prod_{i=1}^{n} A/a_i = (A/a_1) \times \cdots \times (A/a_n)$$

be the map of A into the product induced by the canonical map of A onto A/a_i for each factor. Then the kernel of f is $\bigcap_{i=1}^{n} a_i$, and f is surjective, thus giving an isomorphism

$$A/\bigcap a_i \xrightarrow{\cong} \bigcap A/a_i$$

[Lan93]

Last, we define what an algebra over a ring is.

Definition 16 (**algebra**). Let A be a R-algebra. This means that A and R are both rings, and A acts on R via a ring homomorphism $f : R \to A$ which defines an operation on A on R with the map $(r, a) \mapsto f(r)a$ for all $r \in R$ and $a \in A$. [Lan93]

3.2 Sheaf Theory

Next, we introduce some basic sheaf theory. My research is on a generalization of the spline space to a sheaf, so we need to be familiar with these basics for sheaves. Our main object will be a sheaf of subalgebras on a topological space. When we try to prove things about this sheaf, we can prove things locally, or just on open sets of the topological space. This makes things easier, because on certain open sets the sheaves are just algebras, instead of sheaves of algebras.

Definition 17. [Har77, II.1] Let X be a topological space. A *presheaf* $\mathcal{F}$ of rings on X consists of the data:

- for every open set $U \subseteq X$, a ring $\mathcal{F}(U)$, and
- for every inclusion $V \subseteq U$ of open subsets of X, a morphism of rings $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, subject to the conditions:
  - $\mathcal{F}(\emptyset) = 0$ where $\emptyset$ is the empty set,
  - ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
  - if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Definition 18. [Har77, II.1] A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following:

- if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s\big|_{V_i} = 0$ for all i, then $s = 0$.

- if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each i, j, $s_i\big|_{V_i \cap V_j} = s_j\big|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s\big|_{V_i} = s_i$ for each i.

The first condition here implies that the $'s'$ we get from the second condition is unique.

Definition 19. [Har77, II.1] If $\mathcal{F}$ is a presheaf on X, and if P is a point of X, we define the stalk $\mathcal{F}_P$, of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P, via the restriction maps ρ.

An element of a stalk $\mathcal{F}_P$ can be written as $\langle U, s\rangle$, wehre U is an open neighborhood of P, and s is an element of $\mathcal{F}(U)$. Two such pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ define the same element of $\mathcal{F}_P$ if and only if there is an open neighborhood W of P with $W \subseteq U \cap V$, such that $s\big|_W = t\big|_W$. In this way, we can talk about the elements of $\mathcal{F}_P$ as germs of sections of $\mathcal{F}$ at the point P. [[Har77] II.1]

Definition 20. [Har77, II.1] If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a morphism $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U, such that wherever $V \subseteq U$ is an inclusion, the diagram below commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \Big\downarrow \rho_{UV} & & \Big\downarrow \phi'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

This definition of a morphism of sheaves induces a map on the stalks, denoted by $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$. A morphism will be an isomorphism if the morphisim has a two-sided inverse.

The next proposition can make working with sheaves easier than one might think. The stalks of a sheaf, a sheaf of rings, looks like a local ring. So we prove exactness of a sequence of sheaves can checking exactness on the stalks, so that a sufficient argument on local rings would work.

Proposition 21. [Har77, II.1] If $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then ϕ is an isomorphism if and only if the induced map on the stalk $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for every $P \in X$.

Definition 22 (restriction sheaf). [Har77, II.1] If U is an open set of X, which is a topological subspace of X with the induced topology, then $\mathcal{F}\big|_U$ is the *restriction* of $\mathcal{F}$ onto U, and $\mathcal{F}\big|_U(V) = \mathcal{F}(U \cap V)$ for any open set $V \subseteq X$ so that $\mathcal{F}\big|_Z$ is a sheaf on X. Note that as long as $P \in U$, the stalk of $\mathcal{F}\big|_U$ at P is the same as $\mathcal{F}_P$.

Definition 23 (subsheaf). [Har77, II.1] A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subring of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It also follows that for every point P, the stalk $\mathcal{F}'_P$ is a subring of $\mathcal{F}_P$.

Definition 24 (kernel, image, cokernel). [Har77, II.1] If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, we can define the *presheaf kernel* of ϕ, the *presheaf image* of ϕ, and the *presheaf cokernel* of ϕ to be presheaves given by the maps $U \mapsto \ker(\phi(U))$, $U \mapsto \operatorname{im}(\phi(U))$, and $U \mapsto \operatorname{coker}(\phi(U))$ respectively.

In this way, $\ker(\phi)$ is a subsheaf of $\mathcal{F}$, and $\operatorname{im}(\phi)$ can be identified as a subsheaf of $\mathcal{G}$.

If we are given a morphism of sheaves instead of presheaves, the presheaf kernel will automatically be a sheaf, but the presheaf image and cokernel are usually not, so we use sheafification to find the most appropriate sheaves to represent the image and cokernel which can be found in [Har77, II.1 Definition 1.2].

Remark 25. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. In [Har77, II.1 Exercise 1.2] we find that the stalk of the kernel is the same as the kernel of the map on the stalks of the sheaves, that is: $(\ker\phi)_P = \ker(\phi_P)$ for each point P. Further, the same things happens to the image sheaf: $(\operatorname{im}\phi)_P = \operatorname{im}(\phi_P)$ for all points P.

Definition 26 (exact sequence). [Har77, II.1] A sequence

$$\ldots \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \to \ldots$$

of sheaves and morphisms is exact if at each stage $\ker(\phi^i) = \operatorname{im}(\phi^{i-1})$.

Proposition 27. The exact sequence of sheaves and morphisms

$$\ldots \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \to \ldots$$

is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of rings. ([Har77, II.1, Ex 1.2])

First, exactness of that sequence is equivalent to showing that the appropriate kernels and images in 26 are isomorphic. Showing that the kernel and images are isomorphic is equivalent to showing that they are isomorphic on the stalks 21. But by 25, showing they are isomorphic on the stalks is equivalent to showing exact of the sequence of sheaves on their stalks 26.

Proposition 28. The exact sequence of sheaves and morphisms

$$\ldots \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \to \ldots$$

is exact if and only if for every $P \in X$, there is an open set U such that $x \in U$ and

$$\ldots \mathcal{F}^{i-1}\big|_U \xrightarrow{\phi^{i-1}} \mathcal{F}^i\big|_U \xrightarrow{\phi^i} \mathcal{F}^{i+1}\big|_U \to \ldots$$

is an exact sequence.

This proposition is true because both conditions are equivalent to showing the exactness on all of the stalks of the sheaves (27), and the stalk of $\mathcal{F}$ at $P \in U$ is isomorphic to the stalk of the restricted sheaf $\mathcal{F}\big|_U$ (22).

The proposition above means that if we can show a sequence remains exact for any ring, that will be enough to show that the sequecnce remains exact for any sheaves of rings.

3.3 Homology and Cohomology

Finding a complex for a spline space, or spline sheaf, and proving that you've got a resolution makes it so studying the spline space can be done by studying the complex. In this section, we introduce how complexes, homology, and cohomology work. Then we look into sheaf cohomology and one example which is Čech Cohomology.

In order to make any of these work, we will need to dive into some category theory, in particular, we need to work with objects in an abelian category.

Definition 29. An *abelian category* is a category C such that: for every two objects A and B in C,Hom(A, B) has a structure of an abelian group; the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is teh cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism. [Har77, III.1]

Some examples of abelian categories include the category of abelian groups, the category of modules over a ring, and the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. [Har77, III.1]

Now we can review some homological algebra.

A complex $\hat{A}$ in an abelian category C is a collection of objects A^i, $i \in \mathbb{Z}$, and morphisms $d^i : A^i \to A^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i. In most cases we work with, the objects A^i are only specified up to a certain range, like $i \geq 0$, so we say that $A^i = 0$ for all other i. A morphism of complexes $f : \hat{A} \to \hat{B}$ is a set of morphisms $f^i : A^i \to B^i$ for each i, which commute with the boundary maps d^i. [Har77, III.1]

Then the i^{th} cohomology object $h^i(\hat{A})$ of the complex $\hat{A}$ is defined to be $\ker(d^i)/\mathrm{im}\,(d^{i-1})$. If $f : \hat{A} \to \hat{B}$ is a morphism of complexes then there is an induced map $h^i(f) : h^i(\hat{A}) \to h^i(\hat{B})$. If $0 \to \hat{A} \to \hat{B} \to \hat{C} \to 0$ is a short exact sequence of complexes, then there are natural maps

$\delta^i : h^i(\hat{C}) \to h^{i+1}(\hat{A})$ giving rise to a long exact sequence

$$\cdots \to h^i(\hat{A}) \to h^i(\hat{B}) \to h^i(\hat{C}) \xrightarrow{\delta^i} h^{i+1}(\hat{A}) \to \dots$$

[Har77, III.1]

Two morphisms of complexes $f, g : \hat{A} \to \hat{B}$ are homotopic, written like $f \sim g$, if there is a set of morphisms $k^i : A^i \to B^{i-1}$ for each i such that $f - g = dk + kd$. The collection of morpshisms $(k^i)_i$ is called the homotopy operator. If $f \sim g$, then f and g induced the *same* morphism $h^i(\hat{A}) \to h^i(\hat{B})$ on the cohomology objects, for each i. [Har77, III.1]

Let $F : C \to C'$ be a covariant functor from one abelian category to another. We say F is additive if for any two objects $A, B \in C$, the induced map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(FA, FB)$ is a homomorphism of abelian groups. We say F is left exact if it is additive and for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in C, the sequence

$$0 \to FA' \to FA \to FA''$$

is exact in C'. If we can write 0 on the right side instead of the left, we say that F is *right exact*. If F is both left exact and right exact, we say that F is *exact*. [Har77, III.1]

Next, we discuss resolutions and derived functors. The idea is that the original object we are building a resolution for, can literally be replaced by the resolution, so that studying the object is the same as studying the resolution. Functors will not always give an exact sequence when being applied to an exact sequence. For example, the covariant Hom functor is left-exact. The derived functors are ways to continue an exact sequence after applying a functor like that, into a longer exact sequence.

We say an object $I \in C$ is injective if the functor $\mathrm{Hom}(\cdot, I)$ is exact. An *injective resolution* of an object $A \in \mathcal{C}$ is a complex $\hat{I}$, where $I^i = 0$ for $i < 0$, with a morphism $\epsilon : A \to I^0$, such that I^i is an injective object of $\mathcal{C}$ for each $i \geq 0$, and the sequence

$$0 \to A \xrightarrow{\epsilon} I^0 \to I^1 \to \dots$$

is exact. [Har77, III.1]

If every object of $\mathcal{C}$ is isomorphic to a subobject of an injective object of $\mathcal{C}$, we say that $\mathcal{C}$ has *enough injectives*. In this case, every object of $\mathcal{C}$ has an injective resolution. [Har77, III.1]

Let $\mathcal{C}$ be an abelian category with enough injectives, and let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant left exact functor. Then we construct *right derived functors* R^iF, $i \geq 0$, of F as such: for each object $A \in \mathcal{C}$, we choose just one injective resolution $\hat{I}$ of A. Then we define $R^iF(A) = h^i(F(\hat{I}))$. [Har77, III.1]

This next Theorem verifies that derived functors exist.

Theorem 30. [Har77, Theorem 1.1A] Let $\mathcal{C}$ be an abelian category with enough injectives, and let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant left exact functor to another abelian category $\mathcal{C}'$. Then

1. For each $i \geq 0$, R^iF is an additive functor from $\mathcal{C}$ to $\mathcal{C}'$. Furthermore, it is independent (up to natural isomorphism of functors) of the choices of injective resolutions made.

2. There is a natural isomorphism $F \cong R^0F$.

3. For each short exact sequence $0 \to A' \to A \to A'' \to 0$, and for each $i \geq 0$, there is a natural morphism $\delta^i : R^iF(A'') \to R^{i+1}F(A')$, such that we obtain a long exact sequence

$$\dots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \dots$$

4. Given a morphism of the exact sequence of 3 above, to another $0 \to B' \to B \to B'' \to 0$,

then δ's give a commutative diagram

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^{i+1}F(B') \end{array}$$

5. For each injective object I of C, and for each $i \geq 0$, we have $R^iF(I) = 0$.

Next, we can look into Cohomology theories for Sheaves. First, we find that the category of sheaves of O_X-modules where (X, O_X) is a ringed space, is an abelian category with enough injectives.

Proposition 31. [Har77, Proposition 2.2] Let (X, O_X) be a ringed space. Then the category of sheaves of O_X-modules has enough injectives.

Letting O_X be the constant sheaf of rings $\mathbb{Z}$, then the category of O_X-modules is the same as the category of abelian groups over the topological space X.

Definition 32. [Har77, III.2] Let X be a topological space. Let $\Gamma(X, \cdot)$ be the global section functor from abelian groups over X to abelian groups. We define the *cohomology functors* $H^i(X, \cdot)$ to be the right derived functors of $\Gamma(X, \cdot)$. For any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are the cohomology groups of $\mathcal{F}$.

Next, we introduce the Čech cohomology for a sheaf. In particular, we construct Čech cohomology groups for a sheaf of abelian groups on a topological space X with respect to an open covering of X. We find that if X is a Noetherian seperated scheme, the sheaf is quasi-coherent, and the open covering is of open affine subschemes, then the Čech cohomology groups are precisely the cohomology groups for the sheaf identified above. Therefore, taking Čech cohomology gives a practical way to compute the cohomology of a quasi-coherent sheaf on a scheme.

Definition 33. [Har77, III.4] Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of X. Let I have a well-ordering. For a finite set of indices $i_0, \ldots, i_p \in I$ we write the intersection $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$ as $U_{i_0,\ldots,i_p}$.

Now let $\mathcal{F}$ be a sheaf of abelian groups on X. The complex $C^\circ(\mathcal{U}, \mathcal{F})$ as follows: for each $p \geq 0$, let

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

Thus an element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is determined by giving an element $\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots i_p})$ for each $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements of I. Then we defined the coboundary map $d : C^p \to C^{p+1}$ as follows:

$$(d\alpha)_{i_0 \ldots i_p} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}} \big|_{U_{i_0 \cdots i_{p+1}}}.$$

where $\hat{i}_k$ means to omit i_k. Note that $\alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_p}$ is an element of $\mathcal{F}(U_{i_0,\ldots,\hat{i}_k,\ldots,i_p})$, we can restrict to $U_{i_0 \ldots i_{p+1}}$ to get an element of $\mathcal{F}(U_{i_0,\ldots i_{p+1}})$. You can easily check that $d^2 = 0$.

Definition 34. [Har77, III.4] Let X be a topological space and let $\mathcal{U}$ be an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ of X, we defined the p^{th} *Čech cohomology group* of $\mathcal{F}$, with respect to the covering $\mathcal{U}$, to be

$$H^p(\mathcal{U}, \mathcal{F}) = h^p(C^\circ(\mathcal{U}, \mathcal{F}))$$

When it comes what we've done with quasi-spline sheaves, we try to use an altered version of Čech cohomology, where instead of using open subschemes, we use closed subschemes.

3.4 Schemes

Next, we need to discuss a bit of scheme theory. We've addressed sheaves, which cover our 'sheaf of splines', but we also look into what is called the 'quasi-spline scheme', which includes the topology that the splines are defined on together with the sheaf of splines.

Let A be a commutative ring with unity. We define Spec A to be the set of prime ideals of A.

Let $p \subseteq A$ be a *prime ideal*. That is to say, p is an ideal of A, and if $x, y \in A$ and $xy \in p$, then it is the case that $x \in p$ or $y \in p$.

Let $a \subseteq A$ be an ideal of A, we define $V(a) \subseteq \text{Spec } A$ to be the set of all prime ideals that contain a.

We define the topology on Spec A by taking sets of the form $V(a)$ to be the closed subsets. Notice that $V(A) = \emptyset$, $V((0)) = \text{Spec } A$, and the lemma below shows that all intersections and finite unions of sets of that form remain of that form. Therefore, these closed sets make up a topology on Spec A. This topology is usually known as the Zariski Topology and the next Lemma shows how this satisfies the other axioms for a topology.

Lemma 35 ([Har77] II.2 Lemma 2.1).

- If a and b are ideals of A, then $V(ab) = V(a) \cup V(b)$.
- If $\{a_i\}$ is any set of ideals of A, then $V(\sum_i a_i) = \bigcap V(a_i)$.
- if a and b are any two ideals, $V(a) \subseteq V(b)$ if and only if $\sqrt{a} \supseteq \sqrt{b}$.

Here, $\sqrt{a}$ means the radical ideal of a, which is: $\sqrt{a} = \{x \ : \ x^n \in a \text{ for some } n \in \mathbb{N}\}$.

Now we can define the sheaf of rings on A, denoted by $\mathcal{O}_{\text{Spec } A}$. For any prime $p \in \text{Spec } A$, let A_p denote the localization of A at p. Let $U \subseteq \text{Spec } A$ be an open set, then we define $\mathcal{O}_{\text{Spec } A}(U)$ to be the set of functions $s : U \mapsto \coprod_{p \in U} A_p$ such that

- $s(p) \in A_p$ for each $p \in \text{Spec } A$
- for each $p \in \text{Spec } A$, there is a open set V where $p \in V$ and $V \subseteq U$, and elements $a, f \in A$, such that for each $q \in V$, where $f \notin q$, and $s(q) = a/f \in A_q$.

It should be clear that sums and products of these functions will give back these kinds of functions, and that $1 \in A_p$ gives the identity. In this way, $\mathcal{O}_{\text{Spec } A}(U)$ is a commutative ring with identity. Further notice, that for open sets $V \subseteq U$, that natural restriction map $\mathcal{O}_{\text{Spec } A}(U) \to \mathcal{O}_{\text{Spec } A}(V)$ is a homomorphism of rings. From this, it's clear that $\mathcal{O}_{\text{Spec } A}$ is a presheaf, and from the definition and its local characteristics it can shown to be a sheaf. [[Har77] II.2 p. 70]

Definition 36 ([Har77] II.2). Let A be a ring. The *spectrum* of A is the pair of the topological space Spec A and the sheaf of rings $\mathcal{O}_{\text{Spec } A}$.

Definition 37 ([Har77] II.2). A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A *morphism* of ringed spaces from $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y. Notice that for a open set $U \subset Y$, $f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$.

Definition 38 ([Har77] II.2). A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for every point $P \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. A *morphism* of locally ringed spaces is a morphism $(f, f^\#)$ or ringed spaces, such that for each point $p \in X$, the induced map of local rings $f_P^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a *local homomorphism* of local rings.

To see how the local homomorphism $f_P^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is induced check [[Har77] II.2 p.72-73].

Last, we require that $f_P^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism. That is to say, if A and B are local rings with ring homomorphism $\rho : A \to B$, with m_A and m_b being their respective maximal ideals, then ρ is a local homomorphism if $\rho^{-1}(m_B) = m_A$. [[Har77] p.73]

Note that a morphism of locally ringed spaces will be an isomorphism if this morphism has a two-sided inverse. In this way, the morphism $(f, f^\#)$ is an isomorphism if and only if f is a homeomorphism and $f^\#$ is an isomorphism of sheaves. [[Har77] II.2]

Definition 39 ([Har77] II.2 p.74). An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of some ring. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U, along with the restricted sheaf $\mathcal{O}_X\big|_U$ is an affine scheme. We call the sheaf $\mathcal{O}_X$ the *structure sheaf* of the scheme $(X, \mathcal{O}_X)$. A *morphism* of schemes is a morphism of locally ringed spaces, and isomorphism is a morphism with a two-sided inverse.

Next, we can construct schemes from graded rings. Check [[Har77] II.2 p.76] for the construction.

Example 40 ([Har77] II.2 Ex 2.5.1). Let A be a ring, then the *projective n-space* over A to the be scheme $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$.

Definition 41 ([Har77] II.2). Let S be a fixed scheme. A *scheme over S* is a scheme X, together with a morphism $X \to S$. If X and Y are schemes over S, a morphism of X to Y as schemes over S, is a morphism $f : X \to Y$ which is compatible with the given morphisms to S. This makes a category of schemes over S.

Before moving on, we need to discuss how to take a spec of a sheaf of algebras, which is called the global spec of an algebra.

Definition 42 (*global spec of sheaf of algebras*). [Har77, II.6 Exercise 5.17] Let Y be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. There is an unique scheme X, and a morphism $f : X \to Y$ such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$, and for every inclusion $U \to V$ of open affines of Y, the inclusion morphism $f^{-1}(U) \to f^{-1}(V)$ corresponds with a restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$. The scheme X is $\operatorname{Spec} \mathcal{A}$.

Definition 43 ([Har77] II.3). A scheme X is *locally noetherian* if it can be covered by open affine subsets $\operatorname{Spec} A_i$ where each A_i is a noetherian ring. We say that X is *noetherian* if it is locally noetherian and quasi-compact.

Last, we want to introduce open subschemes and closed subschemes.

Definition 44 ([Har77] II.2 Exercise 2.2). An *open subscheme* of a scheme X is a scheme U, whose topological space is an open subset X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X\big|_U$ of the structure sheaf of X.

Definition 45 ([Har77] II.3). A *closed immersion* is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of $sp(Y)$ onto a closed subset of $sp(X)$, and the induced map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ and $f' : Y' \to X'$ are equivalent if there is an isomorphism $i : Y' \to Y$ such taht $f' = f \circ i$.

Remark 46 ([Har77] II.3 Example 3.2.3). Let Y be a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and that there is an ideal $a \subseteq A$, such that Y is the image of the closed immersion $\operatorname{Spec} A/a \to \operatorname{Spec} A$.

Next, we introduce a sheaf of modules.

Definition 47 ([Har77] II.5). Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A *morphism* $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Now, we introduce a little more about sheaves that will be relevant for defining quasi-splines sheaves.

Definition 48 ([Har77] II.5). Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if X can be covered by open affine subschems $U_i = \operatorname{Spec} A_i$, such that for each i there is an A_i-module M_i with $\mathcal{F}\big|_{U_i} \cong \tilde{M}_i$. The sheaf $\mathcal{F}$ is coherent if $\mathcal{F}$ is quasi-coherent and each M_i can be given as a finitely generated A_i-module.

Definition 49 ([Har77] II.5). Let $(X, \mathcal{O}_X)$ be a scheme. A *sheaf of ideals* on X is a sheaf of modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. In another words, for every open set $U \subseteq X$, $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$.

4. Quasi-Spline Sheaves and Contact Subschemes

Given a base scheme Y, a quasi-coherent sheaf of $\mathcal{O}_Y$-subalgebras $\mathcal{S}$ of the $\mathcal{O}_Y$-algebra $\mathcal{O}_Y^{\prod s}$ is called a sheaf of **s-sheeted quasi splines** over Y. We call the global spec

$$X = \underline{\text{Spec}}\, \mathcal{S}$$

an **s-sheeted quasi-spline scheme** over Y. [Cla15]

We have two maps homomorphisms locally given as $\pi^* : \mathcal{O}_Y \to \mathcal{S}$, $g \mapsto (g, \ldots, g)$ and $\sigma_i^* : \mathcal{S} \to \mathcal{O}_Y$, $(g_1, \ldots, g_s) \mapsto g_i$. Each of which defines a morphism and s-sections

$$\begin{array}{c} X \\ \pi \downarrow \uparrow \sigma_i \\ Y \end{array}$$

σ_i's that are closed immersions, and whose image is called the i^{th} **sheet** of X, we denote by X_i. The quasi-coherent sheaf of ideals defined the i^{th} sheet is locally given by $\{(g_1, \ldots, g_s) \ : \ g_i = 0\}$.

To any quasi-spline scheme X, there are subschemes $K_{ij} \subseteq Y$ locally defined by the quasi-coherent ideals

$$\mathcal{J}_{ij} = \langle g_i - g_j \mid (g_1, \ldots, g_s) \in \mathcal{S} \rangle.$$

The closed subschemes K_{ij} is the same as $X_i \cap X_j$. We refer to them as the **contact subschemes** of X.

The following are a couple examples of quasi-spline sheaves. To build a quasi-spline sheaf, we can specifiy ideals, which we call later as *ideal-difference conditions*, and then build a quasi-spline sheaf from those ideals like in this next example:

Example 50. Let $Y = \text{Spec}\,\mathbb{R}[x]$. The sheaf associated to the $\mathbb{R}[x]$

$$S = \{(g_1, g_2) \mid g_1 - g_2 \in (x^2)\} \subseteq (\mathbb{R}[x])^2$$

is a sheaf of quasi-splines of 2 sheets. You can think of this as the spline space of splines with continuous first derivatives over the subdivision $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$. This example was given in [Cla15]

In that last example, the contact ideal is equal to (x^2), which geometrically correspondes to the origin in the real line with an first order infinitesimal thickening.

Example 51. Let $Y = \text{Spec}\,\mathbb{R}[x]$. The sheaf associated to the $\mathbb{R}[x]$

$$S = \{(g_1, g_2) \mid g_1 - g_2 \in (x^2 + 1)\} \subseteq (\mathbb{R}[x])^2$$

is a sheaf of quasi-splines of 2 sheets. This example was given in [Cla15]. Unlike the example above, this one does not have an easy geometric description.

The contact ideals are not the only ideals that can determine a quasi-spline sheaf in this way. There are a couple of examples that are given after the proof of that Lemma. Take for example, the 3-sheeted quasi-spline sheaf

$$\mathcal{S} = \{(g_1, g_2, g_3) \in \mathcal{O}_Y^3 \mid g_1 - g_2 \in (x^2),\ g_1 - g_3 \in (x^2),\ \text{and } g_2 - g_3 \in (x)\}$$

Here, the contact ideals are all equal to (x^2), there is no contact ideal equal to (x).

We can specify ideal sheaves $I_{ij} \subseteq \mathcal{O}_Y$ for each $1 \leq i < j \leq s$ and determine a s-sheeted quasi-spline sheaf $\mathcal{S}$ using these ideals as below, and we say that $\mathcal{S}$ is given by *ideal-difference conditions* I_{ij}'s.

Definition 52 (Ideal-Difference Conditions). Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf over a scheme

Y. We say that $\mathcal{S}$ is determined by *ideal-difference conditions* $(I_{ij})_{ij}$, where I_{ij} are ideal sheaves over Y for each i, j such that $1 \leq i < j \leq s$, when $\mathcal{S}$ is the kernel of the map

$$\mathcal{O}_Y^s \to \bigoplus_{ij} \mathcal{O}_Y / I_{ij}.$$

or in another words, $\mathcal{S}$ is locally defined as:

$$\mathcal{S} = \{(g_1, \ldots, g_s) \in \mathcal{O}_Y^s \ : \ g_i - g_j \in I_{ij}\}$$

Let D_{ij} be the corresponding closed subscheme of Y whose ideal sheaf is I_{ij}.

Now we can describe how smoothness conditions can translate to information given by ideals. We can require particular smoothness conditions on the D_{ij}'s for any section $g = (g_1, \ldots, g_s) \in \mathcal{S}$. Let $b \in \text{domain}(g) \cap \Omega$, then $g \in C^r$ at b if for every i, j such that $b \in D_{ij}$, $g_i - g_j$ vanishes on the r^{th} order infinitesimal neighborhood of b in Y. This is the same thing as g_i and g_j have the same value on D_{ij}, and so do their i^{th} derivatives for $0 < i \leq r$. This is the same as requiring that that contact subscheme is contains the $(r+1)^{th}$ order neighborhood of the reduction of the D_{ij}, which is to say:

$$K_{ij} \supseteq (\sqrt{D_{ij}})^{(r+1)}.$$

This next section we go over research on splines using moduli theory done in Clarke's paper [Cla15]. This way, we can study sheaves of splines that depend on parameters. We can look into the moduli space of quasi-spline sheaves, and the moduli space of ideal-difference conditions.

Typically when studying spline spaces, we are given a geometric object where the splines are defined with their smoothness conditions, then we ask what the dimension of the entire vector space of splines is, or other things concerned the splines. Working in such generality, we can, in a sense, reverse the question. Instead of working with splines on a given region, a region cut into pieces via smooth conditions for the splines, we ask what is the most optimal subdivision of the given region.

[Cla15]

This question can be rephrased as how to minimize a functional on $C^r(\Omega)$ where Ω is some domain. The points of the moduli space cooresponds to a subdivision of a region of Ω with a spline sheaf over that. The moduli spaces of spline schemes can be thought as the set of all subdivisions of a region Ω. As these spaces become more understood, techniques for optimal subdivisions of Ω could be found. Existence of these moduli spaces has been proved in certain cases [Cla15].

Before discussing the moduli space of quasi-spline sheaves we have to define a family of quasi-spline sheaves. Let Z be a Y-scheme. Then we can define a Z-**family of quasi-splines sheaves over** Y as a

- a sheaf of quasi-splines $\mathcal{S}$ over Y such that
- for any morphism $f : Z' \to Z$, the pullback $\pi_B^* \mathcal{S}$ is a sheaf of quasi-splines over $Z' \times_Z B$.

The way this is defined eliminates situations of sheaves $\mathcal{S} \subset \mathcal{O}_B^s$ whose inclusion map $\mathcal{S} \to \mathcal{O}_B^s$ fails to be an inclusion after fixing the values of the parameters. [Cla15]

Let T be a locally Noetherian scheme and $Y \subset \mathbb{P}_T^n$ be a closed subscheme, with Y is flat over T. Let $\mathcal{M}$ be the moduli space of quasi-spline sheaves. The moduli space $\mathcal{M}$ represents the contravariant functor

$$\mathcal{QS}^{(s)}(Y/T)(Z) = \{Z\text{-families of quasi spline sheaves } \mathcal{S} \subseteq \mathcal{O}_{Z \times_T Y}^s\}$$

in the category of locally Noetherian schemes [Cla15, Theorem 3.10]. A family of quasi-spline sheaves is defined precisely in the way to make $\mathcal{QS}^{(s)}(Y/T)$ functorial. [Cla15]

The construction of the moduli space of quasi-spline sheaves $QS^{(s)}(Y/T)$ it was assumed that Y is flat. This means that the Hilbert polynomials of $\mathcal{S}$ is locally independent of the point in $QS^{(s)}(Y/T)$. [Cla15]

5. Contact Ideals and Ideal-Difference Conditions

My research focuses on the contact ideals of a spline sheaf. At first, we hypothesized that every quasi-spline sheaf is determined by is contact ideals. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with contact ideals $\mathcal{J}_{ij}$'s, then we hypothesized that the sequence below is exact:

$$0 \to \mathcal{S} \to \mathcal{O}_Y^s \to \bigoplus_{ij} \mathcal{O}_Y / \mathcal{J}_{ij}$$

which is the same as saying that $\mathcal{S}$ be can described locally as such:

$$\mathcal{S} = \{(g_1, \ldots, g_s) \ : \ g_i - g_j \in \mathcal{J}_{ij}\}$$

But we find that this isn't true. You can think of quasi-spline sheaves that are not determined by its contact ideals, as subsheaves of quasi-spline sheaves that are.

Next, we tried to find how contact ideals change when the ideal-difference conditions are changed slightly. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with contact ideals $\mathcal{J}_{ij}$'s, then we define $\mathcal{S}'$ to be the s-sheeted quasi-spline sheaf

$$\mathcal{S}' = \{g = (g_1, \ldots, g_s) \ : \ g \in \mathcal{S} \text{ and } g_a - g_b \in I_P\}$$

where $a, b \in \{1, \ldots, s\}$, and I_P is an ideal sheaf of $\mathcal{O}_Y$. Geometrically, $\mathcal{S}'$ is the quasi-spline sheaf resulting from pinching the sheets X_a and X_b of Spec $\mathcal{S}$ to a closed subscheme P of Y, where I_P is the ideal sheaf of P. We hypothesized that the contact ideals for $\mathcal{S}'$, denoted by $\mathcal{J}'_{ij}$, would be:

$$\mathcal{J}'_{ab} = \mathcal{J}_{ab} \cap \mathcal{J}_P$$

$$\mathcal{J}'_{ai} = \mathcal{J}_{ai} \cap (\mathcal{J}_{bi} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{aj} = \mathcal{J}_{aj} \cap (\mathcal{J}_{bj} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{bi} = \mathcal{J}_{bi} \cap (\mathcal{J}_{ai} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{bj} = \mathcal{J}_{bj} \cap (\mathcal{J}_{aj} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{ij} = \mathcal{J}_{ij} \cap (\mathcal{J}'_{ai} + \mathcal{J}'_{bi}) \cap (\mathcal{J}'_{aj} + \mathcal{J}'_{bj}).$$

To prove this, we try to show that we can restrict to the case where $\mathcal{S}$ is 4-sheeted. The idea here is that, the new parts that will be added to the contact ideals will be contained inside of $X_a \cup X_b$, or inside of P. Therefore, to find the contact ideal $\mathcal{J}_{ij}$, we only need to see how things change for the 4-sheets X_a, X_b, X_i, and X_j. This is not true all the time though.

These formulas seem to work for the majority of cases when $Y = \mathbb{A}^n_{\mathbb{Q}}$ for some n. But we find another counterexample showing these formulas do not always work.

This prompted me to further my inquiry into what makes contact ideals special as compared to any other set of ideals. Through the rest of my research I assume that the spline sheaves I'm working with are determined by their contact ideals. I find conditions for when a set of ideals are the contact ideals for a quasi-spline sheaf.

First, if $\mathcal{J}_{ij}$'s are contact ideals for some quasi-spline sheaf $\mathcal{S}$, then $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ for all i, j, k. We find necessary and sufficient conditions for a set of ideals $(\mathcal{J}_{ij})_{1\leq i<j\leq s}$ to be the contact ideals for some quasi-spline sheaf $\mathcal{S}$ that is determined by ideal-difference conditions. The conditions are as such: let $J = (I_{ij})_{ij}$ be ideal-difference conditions for a spline sheaf $\mathcal{S}$. We say that $J' = (\mathcal{J}_{ij})_{ij}$ are contact ideals for $\mathcal{S}$ if and only if

- J' satisfies the conditions $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ for all i, j, k
- $\mathcal{J}_{ij} \subseteq I_{ij}$ for all i, j
- if there is another $J'' = (\mathcal{J}'_{ij})_{ij}$ that satisfies the inequalities above and $\mathcal{J}'_{ij} \subseteq I_{ij}$ for all i, j,

then $\mathcal{J}'_{ij} \subseteq \mathcal{J}_{ij}$ for all i, j.

This can be used to obtain formulas for certain contact ideals when the majority of contact ideals are known.

We also look into ways to obtain new sets of ideals that satisfy the conditions $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ from old sets that satisfy the same inequalities. Indeed, the counterexample for the pinching formulas show that the formulas given do not always give a set of ideals that satisfy those containments. I hope this effort will give better insight into what the *true* pinch formulas may be.

Last, we try to work out a complex for a quasi-spline sheaf using the contact subschemes and the intersections of contact subschemes. We assume the quasi-spline sheaf is determined by its contact ideals, so that the first step of the sequence is exact. This complex looks very similar to a Čech complex; but instead of open subschemes, we use closed subschemes. So far, we know this complex is exact when $s = 3$, and it seems plausible that it is true for $s = 4$, but I suspect that it is not always exact for any s.

In the case that this complex is a resolution for $\mathcal{S}$, then we can use this to as an alternative way to compute the dimension of $\mathcal{S}$. We give a particular example of this when $Y = \mathbb{A}^1_{\Bbbk}$ where $\Bbbk$ is a infinite field.

5.1 Spline Sheaves Determined by Contact Subschemes

Earlier, we found that a spline space is determined by the intersections of its maximal faces. If there is a smallest affine form ℓ_{ij} that for each i, j that contains $\sigma_i \cap \sigma_j$, then we can write:

$$S^r_m(\Delta) = \{F\big|_{\sigma_i} - F\big|_{\sigma_j} \in (\ell^{r+1}) \mid F : \Delta \to \mathbb{R}, F \in C^r(\Delta),\ F\big|_{\sigma_i} \text{ is a polynomial of degree } \geq m \text{ for all } i\}$$

If this result held for quasi-spline sheaves, that is, if the contact ideals of a quasi-spline sheaf determine it, then we can a morphism:

$$\mathcal{M} \to \prod_{i,j} \mathrm{Hilb}$$

from the morphism of quasi-spline sheaves, $\mathcal{M}$, and a product of Hilbert schemes, where the map on the functors is $\mathcal{S} \to \{K_{ij}\}_{ij}$ where K_{ij}'s are the contact subschemes of $\mathcal{S}$. We could prove this by using the Valuative Criterion of Flatness to show that $X_i \cap X_j$ has a locally constant Hilbert polynomial. The X_i's are the sheets of the quasi-spline scheme Spec $(\mathcal{S})$, and the X_i's always have locally constant Hilbert polynomial, so we can use an exact sequence to show that K_{ij} must also have locally constant Hilbert polynomial. Then there [Cla15, Lemma 4.7] to show that K_{ij} is flat over the base scheme Y. This gives the morphism $\mathcal{M} \to \prod_{ij} \mathrm{Hilb}$.

There is no an analogous result we have for quasi-spline sheaves. That is, it isn't always the case that the contact subschemes determine the quasi-spline sheaf in the way that

$$S \cong \ker\left(\mathcal{O}_Y^s \to \bigoplus_{i,j} \mathcal{O}_Y/\mathcal{J}_{ij}\right)$$

in terms of structure sheaves of contact subschemes.

The following is an example of two different quasi-spline sheaves that have the same contact ideals, but are different sheaves of algebras. If all quasi-spline sheaves are determined by their contact ideals, these two quasi-spline sheaves must be the same.

Example 53. Let Y be $\mathbb{A}^2_{\Bbbk}$ for some infinite field $\Bbbk$. Let $\mathcal{S}$ be a 3-sheeted quasi-spline sheaf over Y that is generated by $(1,1,1)$ and $(0,x,y)$, so that $\mathcal{S}$ also a $\Bbbk[x,y]$-module generated by $(1,1,1)$, and tuples of the form $(0,x^n,y^n)$ for $n \geq 1$. Note here that $\mathcal{S}$ has contact ideals: $I_{12} = (x)$, $I_{13} = (y)$, and $I_{23} = (x-y)$ and we let K_{ij}'s be the contact subschemes corresponding to I_{ij}.

We now argue that $\mathcal{S} \subsetneq \ker(\mathcal{O}_Y^3 \to \bigoplus_{ij} \mathcal{O}_{K_{ij}})$. The kernel is also a 3-sheeted quasi-spline sheaf with the same contact ideals as $\mathcal{S}$. Now $(xy,0,0) \in \ker(\mathcal{O}_Y^3 \to \bigoplus_{ij} \mathcal{O}_{K_{ij}})$ but we claim that

$(xy, 0, 0) \notin \mathcal{S}$.

To see this, assume that $(xy, 0, 0) \in \mathcal{S}$, then

$$(xy, 0, 0) = p_0(1, 1, 1) + p_1(0, x, y) + \cdots + p_n(0, x^n, y^n)$$

for $p_i \in \Bbbk[x, y]$ for all i and some $n \in \mathbb{N}$, so that

$$(xy, 0, 0) = p_0(1, 1, 1) + p(0, x, y) = (p_0, p_0 + px, p_0 + py)$$

where $p \in \Bbbk[x, y]$. Then $p_0 = xy$, so that $(xy, 0, 0) = (xy, xy + px, xy + py)$, but there is no single p that can make this happen. Therefore $(xy, 0, 0) \notin \mathcal{S}$. This means that there are two different quasi-spline sheaves that have the same contact ideals.

5.2 Ideal-Difference Conditions for a Quasi-Spline Sheaf

There is more than one way to describe a particular quasi-spline sheaf. This section is about ideal difference-conditions, and how they compare with the contact ideals of a quasi-spline sheaf.

Definition 54 (Ideal-Difference Conditions). Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf over a scheme Y. We say that $\mathcal{S}$ is determined by *ideal-difference conditions* $(I_{ij})_{ij}$, where I_{ij} are ideal sheaves over Y for each i, j such that $1 \leq i < j \leq s$, when $\mathcal{S}$ is the kernel of the map

$$\mathcal{O}_Y^s \to \bigoplus_{ij} \mathcal{O}_Y / I_{ij}.$$

This is the same as saying that locally speaking,

$$\mathcal{S} = \{(g_1, \ldots, g_s) \ : \ g_i - g_j \in I_{ij}\}$$

Let D_{ij} be the corresponding closed subscheme for I_{ij}.

Remark 55. Note that by definition of contact ideals, the contact ideals of a quasi-spline sheaf also give ideal difference-conditions for the quasi-spline sheaf.

Lemma 56. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf given by ideal difference conditions $(I_{ij})_{ij}$ and contact ideals $\mathcal{J}_{ij}$. Then $\mathcal{J}_{ij} \subseteq I_{ij}$ for all i, j.

Proof. First, the inclusion of ideal sheaves can be written as an exact sequence, and by using Prop 28 proving the claim locally when the ideal sheaves are ideals of a ring will prove the Lemma. So let $p \in Y$ where Y is the base scheme for $\mathcal{S}$, then we can find an open set U such that $p \in U$ where $\mathcal{S}(U)$ is an algebra, $\mathcal{O}_Y(U)$ is a ring, $I_{ij}(U)$ and $\mathcal{J}_{ij}(U)$ are ideals because $\mathcal{S}$ is quasi-coherent.

Let $c \in \mathcal{J}_{ij}$. there exists a section in $\mathcal{S}$, that locally looks like $(g_1, \ldots, g_s)$ and $g_i - g_j = c$. Now $\mathcal{S}$ is determined by the ideals I_{ij}'s, so that $(g_1, \ldots, g_s)$ satisfies the properties that $g_i - g_j \in I_{ij}$ for all i, j. This implies that $c = g_i - g_j \in I_{ij}$, which gives the containment we need. □

Geometrically, this means that $K_{ij} \supseteq D_{ij}$ for all i, j.

The following is an example where the ideal-difference conditions for a quasi-spline sheaf are not the contact ideals.

Example 57. Let $Y = \operatorname{Spec} \mathbb{R}[x]$. The sheaf associated to the $\mathbb{R}[x]$-module,

$$\{(g_1, g_2, g_3) \in \mathbb{R}[x]^3 \ : \ g_1 - g_2 \in (x^2),\ g_1 - g_3 \in (x^2),\ \text{and } g_2 - g_3 \in (x)\}$$

is a sheaf of quasi-splines. Notice that $g_1 = g_2$ at the origin and their first derivatives at the origin are equal too, and the same goes for g_1 and g_3. Therefore, the same must hold for g_2 and g_3, so the contact ideal here is (x^2), not (x).

For the rest of this chapter, we will only work with quasi-spline sheaves that are determined by their contact ideals. Next, we try to find a way to calculate the contact ideals of a quasi-spline sheaf when only ideal-difference conditions are given. This is important because it is so much easier to give just any ideal-difference conditions for a quasi-spline sheaf. The contact ideals are the canonical example of ideals that give ideal-difference conditions for a quasi-spline sheaf, and give special geometric information about the quasi-spline sheaf.

Later, we will give equivalent conditions for ideal-difference conditions to actually be the contact ideals: $(I_{ij})_{ij}$ are the contact ideals if and only if $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ for all i, j, k. See Corollary 65.

5.3 Necessary and Sufficient Conditions for Contact Ideals

We start this section with another suggested conjecture of contact ideals:

Definition 58. Let $J = (\mathcal{J}_{ij})_{ij}$ be a tuple of ideal sheaves. We say that J satisfies *the IDC inequalities* if

$$\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj} \text{ for all } i, j, k$$

The IDC stands for ideal difference-conditions. Then we've got some more notation:

Definition 59. Let $J = (J_{ij})_{ij}$ and $J' = (J'_{ij})_{ij}$. Then we say $J' \subseteq J$ if for all i, j, $J'_{ij} \subseteq J_{ij}$. We also say $J' \subsetneq J$ if $J' \subseteq J$ and there exists a i, j such that $J'_{ij} \subsetneq J_{ij}$.

Definition 60. Let $J = (J_{ij})_{1 \leq i < j \leq s}$. Let S_J be the quasi-spline sheaf that is given by ideal-difference conditions J_{ij}'s.

Lemma 61. Let $J = (J_{ij})_{ij}$ and $J' = (J'_{ij})_{ij}$ be ideals such that $J \subseteq J'$. Then $S_J \subseteq S_{J'}$.

Proof. First assume that $J \subseteq J'$. Then let $(g_1, \ldots, g_s) \in S_J$. Then since $J \subseteq J'$, we know that $J_{ij} \subseteq J'_{ij}$ for all i, j. Then $g_i - g_j \in J'_{ij}$ for all i, j so that $(g_1, \ldots, g_s) \in S_{J'}$.

□

Conjecture 62. Let $J = (J_{ij})_{ij}$ be ideal-difference conditions for a spline sheaf $\mathcal{S}$. We say that $\mathcal{J} = (\mathcal{J}_{ij})_{ij}$ are contact ideals for $\mathcal{S}$ if and only if

- $\mathcal{J}$ satisfies the IDC inequalities
- $\mathcal{J} \subseteq J$
- if there is another J'' that satisfies the IDC inequalities above and $J'' \subseteq J$, then $J'' \subseteq \mathcal{J}$.

We can prove this. It should be clear that contact ideals for a quasi-spline sheaf will satisfy all those properties.

Proposition 63. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with ideal-difference conditions $I = (I_{ij})_{ij}$. Let J be a set of ideals that satisfies the conditions above. Then J are the contact ideals for $\mathcal{S}$.

Proof. Note that $J \subseteq I$, this implies that $S_J \subseteq S_I \cong \mathcal{S}$ by Lemma 61.

Let $\mathcal{J}$ be the contact ideals of $\mathcal{S}$, so that $\mathcal{S} \cong S_{\mathcal{J}}$. By the properties of J, we know that $\mathcal{J} \subseteq J$, which implies that $\mathcal{S} \cong S_{\mathcal{J}} \subseteq S_J$.

After proving both containments, its clear that $\mathcal{S} \cong S_J$. □

Lemma 64. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with ideal-difference conditions $I = (I_{ij})_{ij}$, and let $(\mathcal{J}_{ij})_{ij \neq kl}$ be contact ideals for $\mathcal{S}$. Say that we do not know what the contact ideal $\mathcal{J}_{kl}$ is. Then we can show

$$\mathcal{J}_{kl} = I_{kl} \cap \left(\bigcap_j \mathcal{J}_{kj} + \mathcal{J}_{jl} \right)$$

Proof. Let $J = \{(\mathcal{J}_{ij})_{ij \neq kl}, \mathcal{J}_{kl}\}$. Let J'_{kl} be the true contact ideal for K_{kl} of $\mathcal{S}$ and let $J' = \{(\mathcal{J}_{ij})_{ij}, \mathcal{J}'_{kl}\}$. Then we know that $\mathcal{J}'_{kl} \subseteq \mathcal{J}_{kl}$.

Next, we can show J satisfies the inequalities contacts should. Indeed, $\mathcal{J}_{kl} \subseteq \mathcal{J}_{kj} + \mathcal{J}_{jl}$ for all j from the definition of $\mathcal{J}_{kl}$, and $\mathcal{J}_{jk} \subseteq \mathcal{J}_{kl} + \mathcal{J}_{jl}$ and $\mathcal{J}_{jl} \subseteq \mathcal{J}_{kl} + \mathcal{J}_{jk}$ for all k, because $\mathcal{J}_{jk} \subseteq \mathcal{J}'_{kl} + \mathcal{J}_{jl} \subseteq \mathcal{J}_{kl} + \mathcal{J}_{jl}$ and $\mathcal{J}_{jl} \subseteq \mathcal{J}'_{kl} + \mathcal{J}_{jk} \subseteq \mathcal{J}_{kl} + \mathcal{J}_{jk}$ since $\mathcal{J}'_{kl}$ and $(\mathcal{J}_{ij})_{ij \neq kl}$ are contact ideals. Then J satisfies the inequalities because all other containments for ideals that don't involve the ideal $\mathcal{J}_{kl}$ are satisfied. Then from the Proposition 63, S_J must be the same as $\mathcal{S}$. □

Using this Lemma, once we do a pinch, we only need to know that ideals $\mathcal{J}_{ak}$ and $\mathcal{J}_{bk}$ for $k \in \{i, j\}$, because we can use the Lemma above to find $\mathcal{J}_{ij}$.

Corollary 65. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf over Y given by ideal-difference conditions $(I_{ij})_{ij}$. Then the set of ideals $(I_{ij})_{ij}$ are the contact ideals of $\mathcal{S}$ if and only if $I_{ij} \subseteq I_{ik} + I_{kj}$ for all i, j and $k \neq i, j$.

Proof. First, we'll do the forward direction. Assume that $(I_{ij})_{ij}$ are the contact ideals for $\mathcal{S}$. Let i, j be given and let $c \in I_{ij}$. Then there exists $g = (g_1, \ldots, g_s) \in \mathcal{S}$ such that $g_i - g_j = c$. Now let $k \neq i, j$ be given. Then $c = g_i - g_j = (g_i - g_k) - (g_k - g_j) \in I_{ik} + I_{kj}$. This means that $I_{ij} \subseteq I_{ik} + I_{kj}$, which completes the forward direction.

Second, we do the backwards direction. It should be clear that the I_{ij}'s satisfy the all conditions in 84 because it satisfies the containment conditions and also serves as ideal-difference conditions. This implies from Proposition 63 that the I_{ij}'s are contact ideals.

□

Remark 66. The only contact ideals one will need for finding $\mathcal{J}_{ij}$ using the Lemma above are $\mathcal{J}_{ik}$ and $\mathcal{J}_{kj}$ for all k. When $s = 3$, $\mathcal{J}_{12}$ requires knowledge of all other contact ideals. When $s = 4$, we need all contact ideals except $\mathcal{J}_{12}$ and $\mathcal{J}_{34}$. I think for $s = 5$, there are 3 contact ideals we can use the Lemma for once we know all the others.

After realizing that the pinch formulas fail and do not always give contact ideals, I have tried to find operations amongst sets of ideals that send contact ideals to other contact ideals. We've already shown that contact ideals are those ideal difference-conditions that satisfy the IDC inequalities (Corollary 65).

We aim to prove the following about contact ideals, as described above: Let $J = (J_{ij})_{ij}$ be a set ideals that satisfy the IDC inequalities.

- Let J_1 and J_2 satisfy the IDC inequalities. Then $J_1 + J_2$ satisfy the IDC inequalities.

- Let $c \in R$ where $\mathcal{S}$ is over the ring R. Then $c \cdot J$ satisfy the IDC inequalities.
- Let $\phi : R \to R'$, then $\phi(J)^e$ satisfy the IDC inequalities.
- Let $\psi : R'' \to R$, then $\psi^{-1}(J)$ are satisfy the IDC inequalities when ψ is surjective.

I conjecture that $\psi^{-1}(J)$ always satisfies the IDC inequalities, but it is only clear to me that its true when ψ is a quotient map.

It may be difficult to see that there would be only one tuple of ideals that represents the contact ideals, or just one *largest* tuple of ideals that satisfies the IDC inequalities and is contained in the ideal-difference conditions. The next Proposition will make this more clear.

Proposition 67. Let $J_1 = (J_{ij}^{(1)})_{ij}$ and $J_2 = (J_{ij}^{(2)})_{ij}$ be tuples of ideals that satisfies the inequalities. Then tuple of ideals $J_1 + J_2 = (J_{ij}^{(1)} + J_{ij}^{(2)})_{ij}$ also satisfies the IDC inequalities.

Proof. We will just one of the inequalities. Let i, j, k be given. Since J_1 and J_2 satisfies the IDC inequalities, we know that $J_{ij}^{(1)} \subseteq J_{ik}^{(1)} + J_{kj}^{(1)}$ and $J_{ij}^{(2)} \subseteq J_{ik}^{(2)} + J_{kj}^{(2)}$. Therefore,

$$J_{ij}^{(1)} + J_{ij}^{(1)} \subseteq (J_{ik}^{(1)} + J_{ik}^{(2)}) + (J_{kj}^{(1)} + J_{kj}^{(2)})$$

□

Corollary 68. Let J_1 and J_2 both be contact ideals for the spline sheaf $\mathcal{S}$ with ideal-difference conditions J in the sense above. Then $J_1 + J_2$ is also contact ideals for $\mathcal{S}$.

Proof. Note that if $J_{ij}^{(1)} \subset J_{ij}$ and $J_{ij}^{(2)} \subseteq J_{ij}$, then $J_{ij}^{(1)} + J_{ij}^{(2)} \subseteq J_{ij}$. □

Proposition 69. Let J be set of ideals that satisfy the inequalities such that $J_{ij} \subseteq R$. Let $c \in R$, then $c \cdot J$ also satisfies the IDC inequalities.

Proof. Let $g' \in c \cdot J_{ij}$, this implies $g' = cg$ for $g \in J_{ij}$. We just need to show that $g \in c \cdot J_{ik} + c \cdot J_{kj}$. We know that $J_{ij} \subseteq J_{ik} + J_{kj}$ so that $g = g_1 + g_2$ where $g_1 \in J_{ik}$ and $g_2 \in J_{kj}$. This means that $g' = c(g_1 + g_2) = cg_1 + cg_2 \in c \cdot J_{ik} + c \cdot J_{kj}$. □

Definition 70. Let I be an ideal of R and let $\phi : R \to R'$ be a ring homomorphism. Then $\phi(I)^e$ is the ideal generated by all elements in $\phi(I)$. If $\phi(I)$ is already an ideal, then $\phi(I)^e = \phi(I)$.

Proposition 71. Let $\phi : R \to R'$, and let J be ideals of R that satisfy the IDC inequalities. Then $\phi(J)^e$ satisfies the IDC inequalities in R'.

Proof. Let $c \in \phi(J_{ij})^e$. Then $c = \sum_{i=1}^{n} d_i g_i$ where $d_i \in R'$ and $g_i \in \phi(J_{ij})$. Now $J_{ij} \subseteq J_{ik} + J_{kj}$ implies that $\phi(J_{ij}) \subseteq \phi(J_{ik} + J_{kj}) = \phi(J_{ik}) + \phi(J_{kj})$. So for each g_i, there exists $h_i \in \phi(J_{ik})$, $h'_i \in \phi(J_{kj})$ and $g_i = h_i + h'_i$. Therefore, $c = \sum_i d_i g_i = \sum d_i(h_i + h'_i) = \sum d_i h_i + \sum d_i h'_i \in \phi(J_{ik})^e + \phi(J_{kj})^e$. □

Proposition 72. Let $\psi : R'' \to R$ be a surjective ring homomorphism, and let J be ideals of R that satisfy the IDC inequalities. Then $\psi^{-1}(J)$ satisfies the IDC inequalities.

Proof. Let $b_{ij} \in \psi^{-1}(J_{ij})$ where $\psi(b_{ij}) = c_{ij} \in J_{ij}$, and $c_{ij} = c_{ik} + c_{kj} \in J_{ik} + J_{kj}$. Choose $b_{ik} = c_2$ where $\psi(c_2) = c_{ik}$. Then we claim that $b_{ij} - b_{ik} \in \psi^{-1}(J_{kj})$.

To see this, $\psi(b_{ij} - b_{ik}) = \psi(c_1 - c_2) = \psi(c_1) - \psi(c_2) = c_{ij} - c_{ik} = c_{kj} \in J_{kj}$.

Then $b_{ij} = b_{ik} + (b_{ij} - b_{ik})$ where $b_{ij} \in \psi^{-1}(J_{ij})$, $b_{ik} \in \psi^{-1}(J_{ik})$, and $b_{ij} - b_{ik} \in \psi^{-1}(J_{kj})$. □

5.4 Pinch Operation to Find Contact Ideals

This section, we try to find a recursive procedure to find the contact ideals of a quasi-spline sheaf using its ideal difference conditions.

We start by asking the question, "What happens to the contact ideals of a quasi-spline sheaf if one more ideal-difference condition is added to $\mathcal{S}$?" This is to say, given a quasi spline sheaf $\mathcal{S}$ given by ideal difference conditions $(\Im_{ij})_{ij}$, what are the contact ideals of the quasi-spline sheaf $\mathcal{S}'$ locally described by

$$\mathcal{S}' = \{(g_1, \ldots, g_s) \ : (g_1, \ldots, g_s) \in \mathcal{S} \text{ and } g_a - g_b \in \mathcal{J}_P\}.$$

where $\mathcal{J}_P$ is an ideal sheaf. Geometrically speaking, there are two closed subschemes X_a and X_b of X such that $X_a \cap X_b = K_{ab}$. Adding the ideal difference condition $g_a - g_b \in \mathcal{J}_P$ means we are pinching X_a and X_b to the closed subscheme P of Y, so that the contact subscheme K_{ab} gets larger by P.

Solving this problem would give the recursive step in the algorithm to find the contact ideals of any quasi-spline sheaf. Indeed, starting with the quasi-spline sheaf with empty contact subschemes, we add one ideal-difference condition at a time, computing the new contacts after each time. We add every ideal difference condition that give the quasi-spline sheaf whose contact ideal we want to find. The last contact ideals we compute will be the contact ideal for the quasi-spline sheaf.

Definition 73. (**pinch**) Let $\mathcal{S}$ be a s-sheeted quasi-spline scheme over a scheme Y, with contact ideals $(\mathcal{J}_{ij})_{ij}$. Let $\mathcal{S}'$ be a s-sheeted quasi-spline scheme over a scheme Y that results from imposing one new ideal-difference condition on $\mathcal{S}$, say $\mathcal{J}_P$, for a closed subscheme P of Y, so that $\mathcal{S}'$ can be locally described as

$$\mathcal{S}' = \{(g_i)_i \ : \ g_i - g_j \in \mathcal{J}_{ij} \text{ for all } 1 \le i < j \le s \text{ and } g_a - g_b \in \mathcal{J}_P\}.$$

We can also say that $\mathcal{S}'$ is the result of pinching the a^{th} and b^{th} sheets of $\mathcal{S}$ to $\mathcal{J}_P$.

Before moving on to show how contact ideals change after pinching, ideal-difference will be discussed more to make the following proofs easier.

This first Lemma gives ideal-difference conditions for the image of $\pi : \mathcal{S} \to O_Y^p$, that forgets $s-p$ coordinates, none of which are a or b. This alone simplifies the problem to finding the contact ideals of s-sheeted quasi-spline sheaves for $s \le 4$.

Notation. Let $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, s\}$. Let $\mathcal{S}_I$ denote the image of the map $\mathcal{S} \to O_Y^p$ that projects on coordinates $i_1, \ldots, i_p$.

To try to show what the contact ideals of $\mathcal{S}'$ are after pinching, we thought we could restrict to a subset of sheets. If we want to find the contact ideal $\mathcal{J}'_{ij}$ and the pinch occured amongst the sheets X_a and X_b, then we thought we could restrict to the sheets $X_a \cup X_b \cup X_i \cup X_j$ and solve the problem there. But its not quite clear what the ideal-difference conditions for those 4-tuples should be. Geometrically, we think any new part of K'_{ij} should be inside of $P \subset X_a \cup X_b$, so that finding what K'_{ij} on the 4 sheets should give us exactly what K'_{ij} is on the original quasi-spline sheaf. In the same way, it should appear that the ideal difference-conditions for the 4-sheets should be the same for $\mathcal{S}'$.

Next, we suggest what the ideal difference-conditions for $\mathcal{S}'$ should be when we've restricted onto those 4-sheets. Let $I \subseteq \{1, \dots, s\}$ such that $a, b \in I$. Then

$$\mathcal{S}'_I = \{(g_i)_{i \in I} \ : \ g_i - g_j \in \mathcal{J}_{ij} \text{ for } i, j \in I \text{ and } g_a - g_b \in \mathcal{J}_P\}.$$

This guess seems to make sense because looking back at the definition of the 'pinch', this is exactly what the ideal difference-conditions should be for $\mathcal{S}'_I$ if we started with the coordinates I in the first place. If this guess is wrong, it seems like there should be something nonintuitive going on, like K'_{ij} gets something new outside of P. Something like, the pinching causes the space to *wrinkle* in a sort of way, causing the sheets X_i and X_j to intersect outside of $P \subseteq X_a \cup X_b$.

We find that this guess for $\mathcal{S}'_I$ is incorrect, and we've got a counterexample. The code given in [CS13] was used to find and show that this example is a counterexample.

Example 74. Let $\mathcal{S}$ be a 4 sheeted quasi-spline sheaf over $\mathbb{A}^3_{\mathbb{R}}$ in variables x, y, z, determined by ideal-difference conditions given by:

- $I_{01} = (x)$
- $I_{02} = (y)$
- $I_{03} = (x + z)^2$
- $I_{12} = (y)$

- $I_{13} = (x^2 - z^2)$

- $I_{23} = (z^2)$

We can use Clarke/Foucart code to find that the contact ideals are:

- $\mathcal{J}_{01} = (x^2y + xyz)$

- $\mathcal{J}_{02} = (yz^2, x^2y + 2xyz)$

- $\mathcal{J}_{03} = (x^2y + 2xyz + yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

- $\mathcal{J}_{12} = (yz^2, x^2y)$

- $\mathcal{J}_{13} = (x^2y - yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

- $\mathcal{J}_{23} = (yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

The generators of $\mathcal{S}$ are:

- $(0, 0, 0, x^3z^2 + x^2z^3 - xz^4 - z^5)$,

- $(0, 0, x^3y + x^2yz, x^3y + x^2yz - xyz^2 - yz^3)$,

- $(0, x^2y + xyz, 1/2x^2y + xyz, 1/2x^2y + xyz + 1/2yz^2)$,

- $(0, 0, yz^2, 0)$,

- $(1, 1, 1, 1)$

Now let $\mathcal{S}'$ be the resulting spline sheaf after pinching shets X_0 and X_1 of $\mathcal{S}$ to $\mathcal{J}_P = (x^2)$. We find that contacts for $\mathcal{S}'$ are:

- $\mathcal{J}'_{01} = (x^3yz + x^2yz^2)$

- $\mathcal{J}'_{02} = (yz^2, x^3y + x^2yz)$

- $\mathcal{J}'_{03} = (x^3y + x^2yz - xyz^2 - yz^3, x^2yz^2 + 2xyz^3 + yz^4, x^3z^2 + x^2z^3 - xz^4 - z^5)$
- $\mathcal{J}'_{12} = (yz^2, x^3y + x^2yz)$
- $\mathcal{J}'_{13} = (x^3y + x^2yz - xyz^2 - yz^3, x^2yz^2 - yz^4, x^3z^2 + x^2z^3 - xz^4 - z^5)$
- $\mathcal{J}'_{23} = (yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

To see this, $\mathcal{S}'$ is generated by:

- $(0, x^3yz + x^2yz^2, 0, 1/2x^2yz^2 + xyz^3 + 1/2yz^4)$,
- $(0, 0, 0, x^3z^2 + x^2z^3 - xz^4 - z^5)$,,
- $(0, 0, x^3y + x^2yz, x^3y + x^2yz - xyz^2 - yz^3)$
- $(0, 0, yz^2, 0)$,
- $(1, 1, 1, 1)$

Now, we'll restrict to $\mathcal{S}'_I$ first, then find the contact ideals, where $I = \{0, 1, 2\}$. Then the contact ideals for $\mathcal{S}'_I$ are

- $\mathfrak{I}'_{01} = (x^3yz + x^2yz^2)$
- $\mathfrak{I}'_{02} = (yz^2, x^2yz, x^3y)$
- $\mathfrak{I}'_{12} = (yz^2, x^2yz, x^3y)$

To see this, $\mathcal{S}'_I$ is generated by:

- $(0, x^3yz + x^2yz^2, 0)$,
- $(0, 0, x^3y)$,
- $(0, 0, x^2yz)$,

- $(0, 0, yz^2)$,
- $(1, 1, 1)$

As you can see, the contact ideals for $\mathcal{S}'$ are not the same as the contact ideals for $\mathcal{S}'_I$.

We haven't spoken of what we thought are the contact ideals for $\mathcal{S}'$ after the pinch. This last example just goes to show that the problem is a bit more nonintuitive than we thought at first.

The following is the Conjecture we tried to prove, that gives formulas for the contact ideals of $\mathcal{S}'$. As far as computing example in Sage, these tend to work in the majority of examples we tested using Sage.

Conjecture 75. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf over Y with contact ideals $(\mathcal{J}_{ij})_{ij}$. Let $\mathcal{S}'$ be the s-sheeted quasi-spline sheaf over Y resulting from pinching a^{th} and b^{th} sheets of $\mathcal{S}$ to P, with contact ideals $(\mathcal{J}'_{ij})_{ij}$. Then for all i, j:

$$\mathcal{J}'_{ab} = \mathcal{J}_{ab} \cap \mathcal{J}_P$$

$$\mathcal{J}'_{ai} = \mathcal{J}_{ai} \cap (\mathcal{J}_{bi} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{aj} = \mathcal{J}_{aj} \cap (\mathcal{J}_{bj} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{bi} = \mathcal{J}_{bi} \cap (\mathcal{J}_{ai} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{bj} = \mathcal{J}_{bj} \cap (\mathcal{J}_{aj} + \mathcal{J}'_{ab})$$

$$\mathcal{J}'_{ij} = \mathcal{J}_{ij} \cap (\mathcal{J}'_{ai} + \mathcal{J}'_{aj}) \cap (\mathcal{J}'_{bi} + \mathcal{J}'_{bj}).$$

The next proposition shows that if we can prove the formulas for $\mathcal{J}'_{ak}$ and $\mathcal{J}'_{bk}$ for $k \in \{i, j\}$, then we get $\mathcal{J}'_{ij}$ for free.

Proposition 76. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf, with contact ideals $\mathcal{J}_{ij}$'s, and $\mathcal{S}'$ is the quasi-spline resulting from pinching X_a and X_b of $\mathcal{S}$ to P, and let $\mathcal{J}'_{ij}$'s be the contact ideals of $\mathcal{S}'$.

Say we know all contact ideals of $\mathcal{S}'$ except $\mathcal{J}'_{ij}$. Then:

$$\mathcal{J}'_{ij} = \mathcal{J}_{ij} \cap \bigcap_k (\mathcal{J}'_{ki} + \mathcal{J}'_{kj}).$$

Proof. This is a direct result of Lemma 64. □

Remark 77. When $s = 4$, the formula for $\mathcal{J}_{ij}$ is precisely what we predicted above.

Remark 78. If the Conjecture were true for certain cases, it would give a way of computing the contact ideals of a quasi-spline sheaf $\mathcal{S}$ whose ideal-difference conditions given by $(\mathfrak{I}_{ij})_{ij}$. The algorithm below describes how to find contact ideals of $\mathcal{S}$ given by ideal-difference conditions $(\mathfrak{I}_{ij})_{ij}$:

Let $\mathcal{S}_0 := \mathcal{O}_Y^s$ with contact ideals $\mathfrak{I}_{ij}^0 = (1)$ for all i, j;

n = 0;

for *each* $\mathfrak{I}_{ab} \in (\mathfrak{I}_{ij})_{ij}$ **do**

> $\mathcal{S}_{n+1} = \{(g_1, \ldots, g_s) : (g_1, \ldots, g_s) \in \mathcal{S}_n$ and $g_a - g_b \in \mathfrak{I}_{ab}\}$;
>
> Use Theorem to find contacts, $(\mathfrak{I}_{ij}^{n+1})_{ij}$ for $\mathcal{S}_{n+1}$ using contacts $(\mathfrak{I}_{ij}^{n})_{ij}$ of $\mathcal{S}_n$;
>
> ++n;

end

After running through every ideal-difference condiion $\mathfrak{I}_{ab}$, the last sheaf will be $\mathcal{S}$, as it satisfies the same ideal-difference conditions, along with its contacts ideals calculated.

Next, there is an example where the pinching formulas do not give contact ideals, and we can see this because the resulting ideals do not satisfy the conditions $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ for all i, j, k, like any contact ideals should.

Example 79. Let $\mathcal{S}$ be a 4 sheeted quasi-spline sheaf over $\mathbb{A}^3_{\mathbb{R}}$ in variables x, y, z, determined by ideal-difference conditions given by:

- $I_{01} = (x)$

- $I_{02} = (y)$

- $I_{03} = (x + z)^2$

- $I_{12} = (y)$

- $I_{13} = (x^2 - z^2)$

- $I_{23} = (z^2)$

We can use Clarke/Foucart code in [CS13] to find that the contact ideals are:

- $\mathcal{J}_{01} = (x^2y + xyz)$

- $\mathcal{J}_{02} = (yz^2, x^2y + 2xyz)$

- $\mathcal{J}_{03} = (x^2y + 2xyz + yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

- $\mathcal{J}_{12} = (yz^2, x^2y)$

- $\mathcal{J}_{13} = (x^2y - yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

- $\mathcal{J}_{23} = (yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

Then we pinch $J_p = (z^2)$, at X_0, X_1, and the contact ideals become

- $\mathcal{J}'_{01} = (x^2yz^2 + xyz^3)$

- $\mathcal{J}'_{02} = (yz^2, x^3y + x^2yz)$

- $\mathcal{J}'_{03} = (x^3y + x^2yz - xyz^2 - yz^3, x^2yz^2 + 2xyz^3 + yz^4, x^3z^2 + x^2z^3 - xz^4 - z^5)$

- $\mathcal{J}'_{12} = (yz^2, x^3y + x^2yz)$

- $\mathcal{J}'_{13} = (x^3y + x^2yz - xyz^2 - yz^3, x^2yz^2 - yz^4, x^3z^2 + x^2z^3 - xz^4 - z^5)$

- $\mathcal{J}'_{23} = (yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

generated by:

- $(0, 0, 0, x^3z^2 + x^2z^3 - xz^4 - z^5)$,
- $(0, x^2yz^2 + xyz^3, 0, 1/2x^2yz^2 + xyz^3 + 1/2yz^4)$,
- $(0, 0, x^3y + x^2yz, x^3y + x^2yz - xyz^2 - yz^3)$
- $(0, 0, yz^2, 0)$,
- $(1, 1, 1, 1)$

We find that the issue is, the pinch formulas do not give a set of ideals that satisfies the inequalities it should that contact ideals would. The computed pinch ideals are:

- $\mathfrak{I}'_{01} = (x^2yz^2 + xyz^3)$
- $\mathfrak{I}'_{02} = (yz^2, x^2yz, x^3y)$
- $\mathfrak{I}'_{03} = (x^3y + x^2yz - xyz^2 - yz^3, x^2yz^2 + 2xyz^3 + yz^4, x^3z^2 + x^2z^3 - xz^4 - z^5)$
- $\mathfrak{I}'_{12} = (yz^2, x^2yz, x^3y)$
- $\mathfrak{I}'_{13} = (x^3y + x^2yz - xyz^2 - yz^3, x^2yz^2 - yz^4, x^3z^2 + x^2z^3 - xz^4 - z^5)$
- $\mathfrak{I}'_{23} = (yz^2, x^3z^2 + x^2z^3 - xz^4 - z^5)$

Indeed, working with the ideals we get from the pinch formulas, we find that $\mathfrak{I}'_{02} = (yz^2, x^2yz, x^3y)$, and $\mathfrak{I}'_{02} \cap (\mathfrak{I}'_{03} + \mathfrak{I}'_{23}) = (yz^2, x^3y + x^2yz)$ so that $\mathfrak{I}'_{02}$ is not contained in $\mathfrak{I}'_{03} + \mathfrak{I}'_{23}$.

Also with, $\mathfrak{I}'_{12} = (yz^2, x^2yz, x^3y)$, $\mathfrak{I}'_{12} \cap (\mathfrak{I}'_{13} + \mathfrak{I}'_{23}) = (yz^2, x^3y + x^2yz)$ so that $\mathfrak{I}'_{12}$ is not contained in $\mathfrak{I}'_{13} + \mathfrak{I}'_{23}$.

Both of these facts goes against Corollary 65, so that the pinch formulas do not give the contact ideals. We can also verify this just by looking at the ideals computed using the code, the other $\mathcal{J}'$ ideals above.

Now, if we apply the pinch formulas on the $\Im_{ij}$ ideals and pinch on the same location so that $J_p = (z^2)$, then we get the actual contact ideals. So it seems that if we continue to reapply the formulas and recalculate the ideals, after some amount of iterations we should reach the contact idelas. We explore this idea further in the next section.

5.5 Closed Formulas for Contact Subschemes

First, we find easy formulas for contact ideals in terms of ideal-difference conditions given for $\mathcal{S}$ when $s = 3$. After that, we try to suggest other formulas that may be the contact ideals. A first one we had conjectured, but found a counterexample. Another one, I still believe may be the contact ideals. Again, to check to see if these are contact ideals for some quasi-spline sheaf, we only have to check the the IDC inequalities: $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ for all i, j, k.

Proposition 80. Let $\mathcal{S}$ be a 3-sheeted quasi-spline sheaf with ideal difference conditions $(I_{ij})_{ij}$. Then the contact ideals of $\mathcal{S}$, denoted by $(\mathcal{J}_{ij})_{ij}$ are

$$\mathcal{J}_{ij} = I_{ij} \cap (I_{ik} + I_{kj})$$

for all i, j, k.

Proof. Let $c \in \mathcal{J}_{ij}$. There exists a $(g_i)_i \in \mathcal{S}$ such that $g_i - g_j = c$. Then $g_i - g_j = (g_i - g_k) - (g_k - g_j) \in I_{ik} + I_{kj}$, and clearly $g_i - g_j \in I_{ij}$.

For the right containment, let $c \in I_{ij} \cap (I_{ik} + I_{kj})$. Then $c = a - b$ where $a \in I_{ik}$ and $b \in I_{kj}$. Then $(0, a, b) \in \mathcal{O}_Y^3$, is actually a section of $\mathcal{S}$. Indeed, 0 is the k^{th} coordinate, a is the i^{th} coordinate, and b is the j^{th} coordinate, so that section satisfies ideal-difference conditions for $\mathcal{S}$. □

I'd like to find a way to generalize this kind of argument. But things become much more difficult when the number of sheets increases to 4 or more. To illustrate how the case $s = 4$ is harder, we can guess at what the formula for $\mathcal{J}_{12}$ could be. Assume the formula was $\mathcal{J}_{12} = I_{12} \cap (\bigcap_k I_{1k} + I_{k2})$. Let

$c \in I_{12}$, then we can find a $g_3 \in I_{13}$ such that $c - g_3 \in I_{23}$ so that $(0, c, g_3)$ satisfies ideal difference-conditions involving I_{ij} where $i, j \neq 4$. We also find that $\mathcal{J}_{12} \subset I_{12} \cap (I_{14} + I_{24})$ so that we can find a $g_4 \in I_{14}$ such that $c - g_4 \in I_{24}$. But we don't know if $g_4 - g_3 \in I_{34}$ so $(0, c, g_3, g_4)$ isn't necessarily in $\mathcal{S}$.

We know that for any contact ideal $\mathcal{J}_{ij}$, we need $\mathcal{J}_{ij} \subseteq \mathcal{J}_{ik} + \mathcal{J}_{kj}$ for any i, j, k, and also that $\mathcal{J}_{ij} \subset \mathcal{J}_{ik_0} + \mathcal{J}_{k_1k_2} + \cdots + \mathcal{J}_{k_pj}$ for any $i, j, k_i \in \{1, \ldots, s\}$. Using these facts, we thought that these next formulas would give contact ideals.

Let $\mathcal{S}$ be a s-sheeted quasi spline sheaf with ideal difference conditions $J = (I_{ij})_{ij}$ and let $(\mathcal{J}_{ij})_{ij}$ be its contact ideals. The conjecture states that

$$\mathcal{J}_{ij} = I_{ij} \cap \left(\bigcap_{i,e_1,\ldots e_m,j} I_{ie_1} + I_{e_1e_2} + \cdots + I_{e_mj} \right) \text{ for all } i, j, k \text{ such that } 1 \leq i, j, k \leq s. \tag{5.1}$$

where the big intersection spans over sequences containing elements of $\{1, \ldots, s\}$ that start at i, and end at j.

But there is a counterexample to this. This was easy to find when $Y = \operatorname{Spec} \mathbb{Z}[x]$. This example was found using sage. There were many others.

Example 81. Let $s = 4$. Let $I_{12} = (x)$, $I_{13} = (x)$, $I_{14} = (x - 2)$, $I_{23} = (x - 4)$, $I_{24} = (x)$, and $I_{34} = (x - 2)$. Now we will compute the conjecture ideals. So

$$\mathcal{J}_{12} = I_{12} \cap (I_{13} + I_{23}) \cap (I_{14} + I_{24}) = (x) \cap (x, x - 4) \cap (x - 2, x) = (x)$$

$$\mathcal{J}_{13} = I_{13} \cap (I_{12} + I_{23}) \cap (I_{14} + I_{34}) = (x) \cap (x, x - 4) \cap (x - 2) = (x^2 - 2x)$$

$$\mathcal{J}_{23} = I_{23} \cap (I_{12} + I_{13}) \cap (I_{24} + I_{34}) = (x - 4) \cap (x, x) \cap (x, x - 2) = (x^2 - 4x)$$

Some of the conjectured ideals, written as J_{ij}'s, are:

$$\mathcal{J}_{12} = (x) \quad \mathcal{J}_{13} = (x^2 - 2x) \quad \mathcal{J}_{23} = (x^2 - 4x)$$

Then $\mathcal{J}_{13} + \mathcal{J}_{23} = (x^2 - 2x, x^2 - 4x) = (x^2, 2x)$, so that $\mathcal{J}_{12} \not\subseteq \mathcal{J}_{13} + \mathcal{J}_{23}$ so that these couldn't possibly be the contact ideals for the quasi-spline sheaf determined by ideals I_{ij}'s (Corollary 65).

Next, we suggest another kind of formula that may give contact ideals. But before that, we need to know that these next formulas will give ideal difference-conditions for a quasi-spline sheaf.

Lemma 82. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with ideal difference conditions $(I_{ij})_{ij}$. Let

$$J_{ij} = I_{ij} \cap \left(\bigcap_k I_{ik} + I_{kj} \right).$$

Then $J = (J_{ij})_{ij}$ gives ideal-difference conditions for $\mathcal{S}$.

Proof. Let S_J be the quasi-spline sheaf that has ideal-difference conditions given by J_{ij}'s. Then we want to show that $\mathcal{S} = S_J$.

Showing that $\mathcal{S} \subseteq S_J$ is the same as showing $\mathcal{J}_{ij} \subseteq J_{ij}$ where $(\mathcal{J}_{ij})_{ij}$ are the contact ideals of $\mathcal{S}$. First, we know that $\mathcal{J}_{ij} \subseteq I_{ij}$ for all i, j because the I_{ij} give ideal difference-conditions for $\mathcal{S}$. Let i, j be given and let $c \in \mathcal{J}_{ij}$ so there exists $(g_i)_i \in \mathcal{S}$ such that $g_i - g_j = c$. Notice that for all k, $c = (g_i - g_k) + (g_k - g_j) \in I_{ik} + I_{kj}$ so that $\mathcal{J}_{ij} \subseteq J_{ij}$.

To see that $S_J \subseteq \mathcal{S}$, just notice that $J_{ij} \subseteq I_{ij}$. Indeed, for any element of S_J which necessarily satifies the ideal-difference conditions given by the J_{ij}'s must satisfy the ideal-difference conditions given by the I_{ij}'s, so that this is also an element of $\mathcal{S}$. □

Notation. Let I^1_{ij} be defined as:

$$I^1_{ij} = I_{ij} \cap \left(\bigcap_k (I_{ik} + I_{kj}) \right)$$

For any $n \geq 2$, let I_{ij}^n be defined as:

$$I_{ij}^n = I_{ij}^{n-1} \cap \left(\bigcap_k (I_{ik}^{n-1} + I_{kj}^{n-1}) \right)$$

We will now conjecture the contact ideals of $\mathcal{S}$ using these new ideals described above, depending on what s is. Let $(\mathcal{J}_{ij})_{ij}$ be the contact ideals of $\mathcal{S}$.

When $s = 2$, note that $\mathcal{J}_{ij} = I_{ij}$. When $s = 3$, we've proved that $\mathcal{J}_{ij} = I_{ij}^1$ in Prop 80

We've run many examples of quasi-spline sheaves over Sage, and it appears that when $m = s-2$, these ideals stabilize or have stabilized for as value $m < s - 2$, in the way that $I_{ij}^m = I_{ij}^{m+1}$, which is equivalent to saying that the I_{ij}^{s-2} are the contact ideals for these quasi-spline sheaves.

Proposition 83. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf given by ideal-difference conditions $(I_{ij})_{ij}$. Assume that there exists an $m \in \mathbb{N}$ such that

$$I_{ij}^m = I_{ij}^{m+1}$$

where I_{ij}^k is defined as above. Then I_{ij}^m are the contact ideals of $\mathcal{S}$.

Proof. Using Corollary 65 I_{ij}^m are the contact ideals if and only if $I_{ij}^m \subseteq I_{ik}^m + I_{kj}^m$ for all i, j, k. Let i, j be given. Then from the definition of I_{ij}^{m+1}, $I_{ij}^{m+1} \subseteq I_{ij}^m$ and $I_{ij}^{m+1} \subseteq I_{ik}^m + I_{kj}^m$ for all k. But $I_{ij}^{m+1} = I_{ij}^m$ so we get the containments we need to say that I_{ij}^m 's are contact ideals of $\mathcal{S}$. □

Last, I propose a conjecture for when these ideals are the contact ideals.

Conjecture 84. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf with $s \geq 3$, given by ideal difference conditions $(I_{ij})_{ij}$ and contact ideals $\mathcal{J}_{ij}$. Then

$$\mathcal{J}_{ij} = I_{ij}^{s-2}$$

Note that we have already proved that this true for $s = 3$.

5.6 Resolution of Quasi-Spline Sheaf

This section, we construct a complex for a quasi-spline sheaf using the structure sheaves for the contact subschemes, and intersections of contact subshcemes.

Proving that this is a resolution gives a way to compute right derived functors for $\mathcal{S}$ once we know the contact ideals for $\mathcal{S}$. In the case where X is affine, they can be calculated directly from the cohomology of the complex. When X is not affine, using an affine cover of Y, take the Čech cohomology of the complex $C^{\bullet}(\mathcal{S})$. The total complex can be calculated, and the right derived functors can be computed.

Last, we can compute the Hilbert series of S using the resolution above. This gives a alternative way to compute the dimension of spline spaces. This will be discussed more later.

Definition 85. **(the complex $C^{\bullet}(\mathcal{S})$)** Set a well-ordering of the index set I. For each $p \geq 0$, let

$$C^p(\mathcal{S}) = \bigoplus_{i_0 < \cdots < i_p} i_* \mathcal{O}_{X_{i_0,\dots,i_p}}$$

for each $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements of I and where $i : X_{i_0 \dots i_p} \hookrightarrow X$. Here $X_{i_0 \dots i_p} = X_{i_0} \cap X_{i_1} \cap \cdots \cap X_{i_p}$ where $X_i \cap X_j = K_{ij}$, and $\mathcal{O}_{X_{i_0 \dots i_p}}$ is the structure sheaf of $\mathcal{O}_{X_{i_0 \dots i_p}}$. Let the co-boundary map $d_p : C^p(\mathcal{S}) \to C^{p+1}(\mathcal{S})$ be defined

$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\hat{i_k},\dots i_{p+1}}\big|_{X_{i_0,\dots,i_{p+1}}}$$

where the notation $\hat{i_k}$ means to omit i_k. This works because $\alpha_{i_0,\dots,\hat{i_k},\dots i_{p+1}}$ is an section of $\mathcal{S}_{i_0,\dots,\hat{i_k},\dots i_{p+1}}$, which naturally maps into $\mathcal{S}_{i_0,\dots,i_{p+1}}$. It is easy to see that $d_{p+1} \circ d_p = 0$, so $C^{\bullet}(\mathcal{S})$ a complex.

In this section, we assume that any quasi-spline sheaf will be determined by its contact ideals so the first step of the sequence is assumed to be exact. Clearly when a quasi-spline sheaf is not determined by its contact ideals, this complex will not be a resolution. We try to prove this gives a resolution of S when it is determined by its contact ideals, or in another words, that the sequence of maps is an exact sequence. It is easy to show that the image of $C^{p-1}(\mathcal{S}) \to C^p(\mathcal{S})$ is contained in the kernel of the map $C^p(\mathcal{S}) \to C^{p+1}(\mathcal{S})$. The last step is to show it the other way, which proves to be difficult. The next two lemma s helps us to prove the reverse containment, but only in certain situations.

Conjecture 86. We conjecture that the following sequence

$$0 \to \mathcal{S} \to C^0(\mathcal{S}) \xrightarrow{d_0} C^1(\mathcal{S}) \xrightarrow{d_1} C^2(\mathcal{S}) \to \cdots \to C^{s-1}(\mathcal{S}) \to 0$$

is exact when $\mathcal{S}$ is determined by its contact ideals.

It is not accidental that this is reminiscent of a Čech complex, as it arises from a Mayer-Vietoris-type complex on X. This is a new contribution to the complexes already introduced to study spline functions by Billera [BR91] and later Schenck [SS97], and is closely related to the Čech construction used by Yuzvinsky [Yuz92].

The next couple Lemmas demonstrate a strategy we used to try to prove exactness of the complex, but this only goes so far, and we show just how far this can be used. At the end, we find that the resolution is exact when $\mathcal{S}$ is a 3-sheeted quasi-spline sheaf, and almost for 4-sheeted quasi-spline sheaves.

Lemma 87. Let

$$\alpha = \sum_{i_0<\cdots<i_p} \alpha_{i_0,\ldots,i_p}\big|_{X_{i_0}\cap\cdots\cap X_{i_p}} \in C^p$$

be closed, and $\alpha_{i_0,\ldots i_p} = 0$ whenever $i_0 < \ell$, Assume we can find,

$$\tilde{\alpha} = \sum_{i_1<\cdots<i_p} \tilde{\alpha}_{i_1,\ldots,i_p}\big|_{X_{i_1}\cap\cdots\cap X_{i_p}} \in C^{p-1}$$

such that

- $\tilde{\alpha}_{i_1,\ldots,i_p}\big|_{X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}} = \alpha_{\ell,i_1,\ldots,i_p}$ and
- $\tilde{\alpha}_{i_1,\ldots,i_p}\big|_{X_\lambda \cap X_{i_1} \cap \cdots \cap X_{i_p}} = 0$ for all $\lambda < \ell$

then

$$\alpha - d(\tilde{\alpha})$$

is closed, and $\alpha_{i_0,\ldots,i_p} = 0$ whenever $i_0 \leq \ell$.

Proof. The closure claim is immediate because α is assumed closed, and $d(\tilde{\alpha})$ is a boundary and so closed.

The vanishing of the coefficients with $i_0 \leq \ell$ fall into two cases. The difference $\alpha - d(\tilde{\alpha})$ can be simplified to

$$\sum_{i_0 < \cdots < i_p} \left(\alpha_{i_0,\ldots,i_p} - \tilde{\alpha}_{i_1,\ldots,i_p}\big|_{X_{i_0} \cap \cdots \cap X_{i_p}} \right)\big|_{X_{i_0} \cap \cdots \cap X_{i_p}}.$$

Indeed, $\tilde{\alpha}_{\ell,i_0,\ldots,\hat{i}_k,\ldots,i_p}\big|_{X_{i_0} \cap \cdots \cap X_{i_p}} = 0$ for $k \neq 0$ for both cases. When $i_0 < \ell$, this follows from the second property for $\tilde{\alpha}$. When $i_0 = \ell$, this follows because two indices are the same and $\alpha_{i_0,k,\ldots k} = 0$

If $i_0 < \ell$, then both terms vanish by hypothesis. If $i_0 = \ell$, then the definition of $\tilde{\alpha}$ guarentees the terms cancel with each other. □

The complicated part of this is proving the existence of $\tilde{\alpha}$.

Remark 88. Let C_1 and C_2 be two closed subschemes of X, then

$$0 \to \mathcal{O}_{C_1 \cup C_2} \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0$$

This can be proved by looking at the case when all of these are affine schemes. If I_1 and I_2 are the ideal sheaves of C_1 and C_2, and R is the affine scheme for X, then this can be proved by showing that the sequence

$$0 \to R/(I_1 \cap I_2) \to R/I_1 \oplus R/I_2 \to R/(I_1 + I_2) \to 0$$

is exact. This follows from Prop 13.

We will find that the existence of $\tilde{\alpha}$ is bound by the use of Remark [**?**] above. In that case, we can't have a section of $\tilde{\alpha}_{i_1,\dots,i_p}\big|_{X_\lambda\cap X_\ell\cap X_{i_1}\cap\cdots\cap X_{i_p}} = 0$ for all $\lambda < \ell$ when $\ell \geq 2$. Indeed, using Remark [**?**], we can only extend

Lemma 89. Let

$$\alpha = \sum_{i_0<\cdots<i_p} \alpha_{i_0,\dots,i_p}\big|_{X_{i_0}\cap\cdots\cap X_{i_p}} \in C^p$$

be closed, and $\alpha_{i_0,\dots i_p} = 0$ whenever $i_0 < \ell$ and $\ell < 3$, then we can find,

$$\tilde{\alpha} = \sum_{i_1<\cdots<i_p} \tilde{\alpha}_{i_0,\dots,i_p}\big|_{X_{i_1}\cap\cdots\cap X_{i_p}} \in C^{p-1}$$

such that

- $\tilde{\alpha}_{i_1,\dots,i_p}\big|_{X_\ell\cap X_{i_1}\cap\cdots\cap X_{i_p}} = \alpha_{\ell,i_1,\dots,i_p}$ and
- $\tilde{\alpha}_{i_1,\dots,i_p}\big|_{X_\lambda\cap X_{i_1}\cap\cdots\cap X_{i_p}} = 0$ for all $\lambda < \ell$.

Proof. Note the conditions on $\tilde{\alpha}$ can be expressed purely in terms of the first bullet point as $\tilde{\alpha}_{i_1,\dots,i_p}\big|_{X_\lambda\cap X_{i_1}\cap\cdots\cap X_{i_p}} = \alpha_{\lambda,i_1,\dots,i_p}$ for all $\lambda \leq \ell$. This is the point of view we take here.

Our hypothetical object $\tilde{\alpha}_{i_1,\dots,i_p}$ is a function on $X_{i_1}\cap\cdots\cap X_{i_p}$ and we are interested in its behavior on the closed subschemes $X_\lambda \cap X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}$ when $\lambda \leq \ell$. So we consider the map

$$\mathcal{O}_{X_{i_1}\cap\cdots\cap X_{i_p}} \to \bigoplus_{\lambda\leq\ell} \mathcal{O}_{X_\lambda\cap X_{i_1}\cap\cdots\cap X_{i_p}}$$

and the element $(\alpha_{\lambda,i_1,\dots,i_p})_{\lambda,i_1,\dots,i_p}$ in the codomain.

The possibility of applying Remark 88 is encouraged by the fact that each map $\mathcal{O}_{X_{i_1}\cap\cdots\cap X_{i_p}} \to \mathcal{O}_{X_\lambda\cap X_{i_1}\cap\cdots\cap X_{i_p}}$ is surjective. It remains to check the agreement of $\alpha_{\ell,i_1,\dots,i_p}$ and 0 on

$$X_\lambda \cap X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}.$$

for all $\lambda < \ell$

Consider the boundary term of α sitting in $\mathcal{O}_{X_\lambda \cap X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}}$

$$\left(\alpha_{\ell, i_1, \ldots, i_p} - \alpha_{\lambda, i_1, \ldots, i_p} + \sum_{i=1}^{p} (-1)^{k+1} \alpha_{\lambda, \ell, i_1, \ldots \hat{i_k}, \ldots, i_p} \right) \Big|_{X_\lambda \cap X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}} = 0.$$

By hypothesis, $\lambda < \ell$, so that all terms $\alpha_{\lambda, \ldots} = 0$ which means only one term is not equal to 0 in the sum above, we means we've got this left:

$$(\alpha_{\ell, i_1, \ldots, i_p}) \Big|_{X_\lambda \cap X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}} = 0.$$

which is precisely what we wanted.

Last, we need to use the Remark 88 to be sure $\tilde{\alpha}$ exists. Now $\lambda < \ell < 3$ which implies that $\lambda < 2$ so that λ can only take on one value, which is 1. Therefore, we need a section $\tilde{\alpha}_{i_1, \ldots, i_p}$ that satisfies the two conditions in the description of this Lemma. We know that

$$\alpha_{\ell, i_1, \ldots, i_p} \Big|_{X_\lambda \cap X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}} = 0$$

So, we can let $C_1 = X_\ell \cap X_{i_1} \cap \cdots \cap X_{i_p}$ with section $\alpha_{\ell, i_1, \ldots, i_p}$ defined on it, and $C_2 = X_\lambda \cap X_{i_1} \cap \cdots \cap X_{i_p}$ with the section 0, then $\alpha_{\ell, i_1, \ldots, i_p} = 0$ on $C_1 \cap C_2$, so we can use the Remark 88 to extend both of these sections to one on $C_1 \cup C_2$, and since $C_1 \cup C_2$ is a closed section of $X_{i_1} \cap \cdots \cap X_{i_p}$ we can find a section on the latter and this will be $\tilde{\alpha}_{i_1, \ldots, i_p}$. □

So long as $\tilde{\alpha}$ exists for each $\alpha \in C^p(\mathcal{S})$, the sequence will remain exact. The only limitation is that we require that $\ell < 3$. That is, let $\mathcal{S}$ be an s-sheeted quasi-spline sheaf, so long as there are $\alpha \in C^p(\mathcal{S})$ for any p such that $\alpha = \sum_{i_0 < \cdots < i_p} \alpha_{i_0, \ldots i_p}$ where $i_0 < 3$.

If there is a p such that $\alpha \in C^p(\mathcal{S})$ and $\alpha = \sum_{i_0 < \cdots < i_p} \alpha_{i_0, \ldots, i_p}$ and $i_0 \geq 3$ for some $\alpha_{i_0 \ldots i_p}$, then we have to have an alternate way to prove the existence of some $\tilde{\alpha}$.

Now, we prove its exactness for $s = 3$ and we get very close to proving existence for $s = 4$.

The next Lemma describes just how useful the the last couple Lemmas are in showing that the complex is exact.

Lemma 90. Let $\mathcal{S}$ be a s-sheeted quasi-spline sheaf that is determined by its contact ideals $(\mathcal{J}_{ij})_{ij}$. Then the sequence of maps as they are described above

$$C^{p-1}(\mathcal{S}) \xrightarrow{d_{p-1}} C^p(\mathcal{S}) \xrightarrow{d_p} C^{p+1}(\mathcal{S})$$

is exact if $s - p < 3$.

Proof. To prove exactness at this step we need to show that im $(d_{p-1}) = \ker(d_P)$. It should be clear that im $(d_{p-1}) \subseteq \ker(d_P)$.

To show the other direction, we choose $\alpha \in \ker(d_p)$, so $\alpha \in C^p(\mathcal{S})$ and $d(\alpha) = 0$. The existence of $\tilde{\alpha}$ in the above Lemmas will guarentee that $\alpha \in$ im (d_{p-1}).

The one stipulation for the existence of $\tilde{\alpha}$ is that $\ell < 3$. Now ℓ is described as i_0, for each section of α on $X_{i_0,\ldots,i_p}$. That is, $\alpha \in C^p(\mathcal{S})$ and $\alpha_{i_0\ldots i_p}$ is the section of α on $X_{i_0\ldots i_p}$ where $i_0 < i_1 < \cdots < i_p$, and $\tilde{\alpha}$ is guarenteed to exist if $i_0 < 3$.

There are two variables in involved, s and p. Consider some section $\alpha_{i_0\ldots i_p}$ of α on $X_{i_0\ldots i_p}$. Now each $i_k \in \{1, \ldots, s\}$, and clearly, the largest possible value for i_p is s. The largest possible value for $i_{p-1} = s - 1$, and continuing this, the largest possible value for $i_0 = s - p$. Going back to Lemma 89, the variable ℓ will be i_0 for each $\alpha_{i_0\ldots i_p}$, and if $i_0 \geq 3$, the existence of $\tilde{\alpha}$ cannot be guarenteed.

Last, we assume that $s - p < 3$, which guarentees that for every $\alpha_{i_0\ldots i_p}$, $i_p = s - p < 3$. Then using Lemma 89, there exists a $\tilde{\alpha}$ so that $d(\tilde{\alpha}) = \alpha$, so that the sequence is exact at that step. □

Proposition 91. Let $\mathcal{S}$ be a 3-sheeted quasi-spline sheaf that is determined by its contact ideals.

Then the following sequence is exact:

$$0 \to \mathcal{S} \to C^0(\mathcal{S}) \xrightarrow{d_0} C^1(\mathcal{S}) \xrightarrow{d_1} C^2(\mathcal{S}) \xrightarrow{d_2} 0.$$

Proof. We assume that $\mathcal{S}$ is determined by its contact ideals, so that the first step is exact. This means that we are only looking for exactness at the steps

$$C^{p-1}(\mathcal{S}) \xrightarrow{d_{p-1}} C^p(\mathcal{S}) \xrightarrow{d_p} C^{p+1}(\mathcal{S})$$

when $p \geq 1$. Now $s - p = 3 - p \leq 3 - 1 = 2$, so we can use Lemma 90 to prove exactness at every step. □

For $s = 4$, I can reduce the exactness of the sequence to showing that an element $(0, \ldots, 0, h_{34}) \in \ker d_1$ is also in the im (d_0).

Proposition 92. Let $\mathcal{S}$ be a 4-sheeted quasi-spline sheaf that is determined by its contact ideals. Then the following sequence is exact:

$$0 \to \mathcal{S} \to C^0(\mathcal{S}) \xrightarrow{d_0} C^1(\mathcal{S}) \xrightarrow{d_1} C^2(\mathcal{S}) \xrightarrow{d_2} C^3(\mathcal{S}) \xrightarrow{d_3} 0.$$

if we can assume that for every element $h = (0, \ldots, 0, h_{34}) \in \ker(d_1)$, $h \in \operatorname{im}(d_0)$.

Proof. We assume that $\mathcal{S}$ is determined by its contact ideals, so that the first step is exact. This means that we are only looking for exactness at the steps

$$C^{p-1}(\mathcal{S}) \xrightarrow{d_{p-1}} C^p(\mathcal{S}) \xrightarrow{d_p} C^{p+1}(\mathcal{S})$$

when $p \geq 1$.

First, lets assume that $p \geq 2$. Then $s - p = 4 - p \leq 4 - 2 = 2$, so we have exactness at every step $p \geq 2$.

For $p = 1$, $s - p = 3$ so we can't completely prove exactness at this step. First, we can consider $\alpha \in C^1(\mathcal{S})$, $\alpha \in \ker(d_1)$ and where $\alpha = (h_{12}, h_{13}, h_{14}, \dots)$. We can use Lemmas 87, and 89 to find a $\tilde{\alpha}$ so that $d_0(\tilde{\alpha}) - \alpha = (0, 0, 0, h_{23}, h_{24}, \dots)$ because $\ell = 1$ here. Next, we can use the same Lemmas to find a $\tilde{\alpha_1}$ so that $d_0(\tilde{\alpha_1}) - (d_0(\tilde{\alpha}) - \alpha) = (0, 0, 0, 0, 0, h_{34})$ because $\ell = 2$ here. Last, from the hypothesis of this Proposition, there exists a α_2 so that $d_0(\alpha_2) - (d_0(\tilde{\alpha_1}) - (d_0(\tilde{\alpha}) - \alpha)) = 0$, which shows that $\alpha \in \text{im}\,(d_0)$. □

I could imagine other ways to prove the existence of some $\tilde{\alpha}$. We had thought of a sort of, multiple-Chinese Remainder Theorem, something like this: let $C_1, \dots, C_n$ be closed subschemes of X, then the following sequence is exact

$$0 \to \mathcal{O}_{\bigcup_i C_i} \to \bigoplus_i \mathcal{O}_{C_i} \to \bigoplus_{i<j} \mathcal{O}_{C_i \cap C_j}$$

But if this is true, then every quasi-spline sheaf will be determined by its contact ideals. But we know this isn't true: Example 53.

5.6.1 Using the Resolution for Quasi-Spline Sheaves

Usually we are working with splines over a field, like $\mathbb{R}$ or $\mathbb{C}$, in the form of spline spaces. This means that the modules we are working with like $\mathcal{S}$ are vector spaces over a field $\mathbb{k}$. So the resolution of $\mathcal{S}$, a s-sheeted quasi-spline sheaf:

$$0 \to \mathcal{S} \to C^0(\mathcal{S}) \to C^1(\mathcal{S}) \to \cdots \to C^{s-1}(\mathcal{S}) \to 0,$$

each $C^p(\mathcal{S})$ is a $\Bbbk$-vector space, and these can be used to calculate the dimension of $\mathcal{S}$ over $\Bbbk$. Using the Rank Nullity Theorem for vector spaces over a field $\Bbbk$,

$$\dim_{\Bbbk}(\mathcal{S}) = \sum_{i=0}^{s-1}(-1)^i \dim_{\Bbbk}(C^i(\mathcal{S}))$$

It seems to me that it may be difficult to prove the formulas for $\dim_{\Bbbk} \mathcal{S}$, but certainly possible.

Recall that for any $s \in \mathcal{S}$, $s\big|_T \in \mathcal{P}_d$ where $\mathcal{P}_d$ is the $\binom{d+2}{2}$ linear space of polynomials in 2 variables up to degree d. This means that

$$\dim_{\Bbbk}(C^0(\mathcal{S})) = N \cdot \binom{d+2}{2}$$

where N is the total number of triangles.

In more generality, for the $\Bbbk$-module $\Bbbk[x_1, \ldots, x_n]$, where $\mathcal{P}^n_d$ is the $\Bbbk$ module of polynomials with degree $\geq d$,

$$\dim_{\Bbbk} \mathcal{P}^n_d = \binom{n+d}{n}$$

It may be the situation that K_{ij} is a line, something like $y - f(x) = 0$ where $f(x)$ is just a linear function in x. Then

$$\dim_{\Bbbk} \mathcal{O}_{K_{ij}} = \dim_{\Bbbk} (\mathcal{P}^n_d/(y - f(x))) = \dim_{\Bbbk} \mathcal{P}^{n-1}_d = \binom{n-1+d}{n-1}$$

In the situation that K_{ij} is a point, something like $\sum_i (x_i - p_i)$, then

$$\dim_{\Bbbk} \mathcal{O}_{K_{ij}} = \dim_{\Bbbk} (\mathcal{P}^n_d/(x_i - p_i)_i) = \dim_{\Bbbk} \Bbbk = 1$$

Now we will use the formula for the dimension of $\mathcal{S}$, which would work if the complex for $\mathcal{S}$ is a resolution, to rediscover the dimension of a spline space where each spline is univariate.

Let T_i's be the 1-simplexes. In this situation, the T_i's will be intervals over $\Bbbk$. They partition an interval, say $[a, b]$, where $T_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots N$, and $a = x_0 < x_1 < \cdots < x_N = b$. The space of splines is

$$\mathcal{S}_1^{r,d} = \{p \in C^r[a, b] \ : p \text{ is a polynomial of degree } d \text{ on each } [x_{i-1}, x_i]\}$$

All intervals are 1-simplices. The intersection of any of these will be 0-simplices, or points. So $I_{i,i+1} = (x_i - p_i)_i^{r+1}$, and only $I_{ij} \neq (1)$ when $i < j$ and $j = i + 1$.

This means that the I_{ij}'s are actually the contact ideals, even if $r > 0$.

Proposition 93. In this situation that the complex for $\mathcal{S}$ is a resolution,

$$\dim \mathcal{S}_1^{r,d} = d + 1 + (N - 1)(d - r)$$

Proof. This is all being done on a line, so that $T_i \cap T_j \cap T_k = \emptyset$. This means that

$$\dim \mathcal{S}_1^{r,d} = \dim C^0(\mathcal{S}_1^{r,d}) - \dim C^1(\mathcal{S}_1^{r,d})$$

using the resolution.

First note that $\dim C^0(\mathcal{S}_1^{r,d}) = N \cdot \binom{d+1}{1} = N(d + 1)$.

Second we will find $\dim C^1(\mathcal{S}_1^{r,d})$, but to do that we need to find the dimension of each $T_i \cap T_{i+1}$. Note that $T_i \cap T_{i+1}$ will be a point. Without loss of generality assume this point is the origin. We require that for $s \in \mathcal{S}$, define $s\big|_{T_i} = s_i$ for all i, then we require that

$$\frac{d^n(s_i - s_{i+1})}{dx^n} = 0$$

for all $n \leq r$ from the definition of $\mathcal{S}_1^{r,d}$. This means that we have to look at the $(r + 1)^{th}$ order neighborhood of the origin. So $I_{ij} = (x)^{r+1} = (x^{r+1})$ as an ideal of $\Bbbk[x]$. As a submodule of $k[x]$, this means that $I_{ij} = (x^{r+1}, x^{r+2}, \ldots, x^m)$. Then $\dim_{\Bbbk}(\mathcal{O}_{T_i \cap T_{i+1}}) = \dim_{\Bbbk} k[x]/I_{ij} = \dim_{\Bbbk}(1, x, \ldots, x^r) = r + 1$.

There are $N - 1$ of these intersections between the intervals. Therefore

$$\dim \mathcal{S}_1^{r,d} = N(d+1) - (N-1)(r+1) = (d+1) + (N-1)(d+1-r-1) = (d+1) + (N-1)(d-r)$$

□

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