Existence of Traveling Waves for a Class of Nonlocal Nonlinear Equations with Bell Shaped Kernels

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Abstract

In this article we are concerned with the existence of traveling wave solutions of a general class of nonlocal wave equations: $u_{tt} - a^2 u_{xx} = (\beta * u^p)_{xx}$, $p > 1$. Members of the class arise as mathematical models for the propagation of waves in a wide variety of situations. We assume that the kernel β is a bell-shaped function satisfying some mild differentiability and growth conditions. Taking advantage of growth properties of bell-shaped functions, we give a simple proof for the existence of bell-shaped traveling wave solutions.

Keywords: Solitary waves, Bell-shaped functions, Nonlocal wave equations, Variational methods.

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1. Introduction

The present paper is concerned with the existence of traveling wave solutions $u(x, t) = u(x - ct)$ of a general class of nonlocal nonlinear wave equations of the form

$$u_{tt} - a^2 u_{xx} = (\beta * u^p)_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

where a is some constant, $c \neq 0$ is the wave velocity, $p > 1$ is an integer, and

$$(\beta * v)(x) = \int_{\mathbb{R}} \beta(x - y) v(y) dy$$

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denotes the convolution in the x variable for a given bell-shaped kernel function β. As the name suggests, a function is bell-shaped if it is even, non-negative, and nonincreasing on $[0,\infty)$.

The problem (1.1) is closely related to the well known and much investigated question of solitary wave pulses propagating in an infinite chain of beads. The equation governing propagation of wave pulses turns out to be of the form

$$r_n'' = V'(r_{n+1}) - 2V'(r_n) + V'(r_{n-1}), \quad n \in \mathbb{Z}$$

where $r_n(t)$ denotes the distance between the centers of the $(n-1)$-th and n-th balls. In the continuum limit, for traveling waves, one obtains the stationary differential-difference equation

$$r'' = \Delta_{disc} V'(r), \tag{1.2}$$

with the discrete Laplacian $\Delta_{disc} w(x) = w(x+1) - 2w(x) + w(x-1)$. Letting Λ be the triangular kernel defined as $\Lambda(x) = (1-|x|)\chi_{[-1,1]}(x)$ with the characteristic function $\chi_{[-1,1]}$, it then follows that (1.2) can be written as

$$r'' = (\Lambda * V'(r))''.$$

We note that when $V(r) = \frac{1}{p+1} r^{p+1}$ this is the equation for a traveling wave solution of (1.1) for the particular kernel Λ. In this sense, our problem generalizes the chain problem (1.2) to bell shaped kernels.

Equation (1.1) is a particular case of the nonlocal double dispersive equation

$$u_{tt} - Lu_{xx} = (\beta * g(u))_{xx}, \tag{1.3}$$

where L is a general constant coefficient pseudodifferential operator (Fourier multiplier), studied in [1]. The case $a = 0$ with a general nonlinearity $f(u)$ was first proposed in [2] as a general equation governing the propagation of nonlinear strain waves in a one-dimensional, nonlocally elastic medium. Local well posedness of the Cauchy problems are characterized by the smoothness of β, or equivalently the decay condition on the Fourier transform

$$|\widehat{\beta}(\xi)| \le c_1^2 (1+\xi^2)^{-r/2}.$$

This decay condition means that the convolution operator has a smoothing effect of order r. It follows from [1] and [2] that the Cauchy problem for (1.1) with $a \neq 0$ is locally well posed for $r \geq 1$, while in the case of $a = 0$, further smoothness as $r \geq 2$ is required.

A traveling wave $u = u(x - ct)$ of (1.1) will satisfy

$$(c^2 - 1)u = \beta * u^p,$$

when u and u' vanish at infinity. The usual method of proving existence of nontrivial solutions to such stationary equations is via calculus of variations where one obtains a solution as an optimizer u_0 of a suitable energy functional. To pass from an optimizing sequence $\{u_n\}$ to an actual optimizer u_0 often requires some compactness argument. In our case and similar problems, one difficulty rises from compactness criteria in $L^p(\mathbb{R})$; Sobolev embeddings are not compact unless one has tail control in the sense of the Kolmogorov compactness criteria. In terms of the optimizing sequence $\{u_n\}$, this tail control means

$$\lim_{R\to\infty} \int_{|x|\geq R} |u_n(x)|^p dx = 0,$$

uniformly in n. Another difficulty is due to the translation invariance of the problem; if $\{u_n\}$ is a minimizing sequence, then so is any shift $\{\widetilde{u}_n\}$ with $\widetilde{u}_n(x) = u_n(x - x_n)$. This, in particular, means that to achieve tail control, one should first align the functions u_n , i.e. choose the correct shifts x_n so that the sequence $\{\widetilde{u}_n\}$ is uniformly small for $|x| \geq R$. The Concentration Compactness Principle of Lions [7] settles these two issues at once, yet it involves heavy techniques. Moreover it does not give any extra information on the shape of the traveling wave.

On the other hand, symmetries of the problem may offer an easier alternative. Restricting the optimization problem to bell shaped functions, whenever possible, leads to a more direct approach. The decay of a bell-shaped function can be easily estimated by its L^p norm, which in turn will yield tail control. In other words, bell-shaped functions are already aligned in the sense we mentioned above. Moreover, this approach will yield a bell-shaped traveling wave, as is in the case of many examples where explicit solutions are known.

Traveling waves for (1.3) were studied in [4] where their existence was proved using the Concentration Compactness Principle, for kernels β that satisfy the elipticity condition

$$c_2^2(1+\xi^2)^{-r/2} \leq \widehat{\beta}(\xi) \leq c_1^2(1+\xi^2)^{-r/2}. \tag{1.4}$$

Their results imply, in particular, existence of traveling waves for (1.1). In the present work we are able to extend their existence result to kernels that are not necessarily elliptic.

Existence of traveling waves for (1.2) was first proved by Friesecke and Wattis in 1994 [5] using the Concentration Compactness Principle. In 2005, English and Pego investigated decay properties of the solitary waves [3]. In a series of papers, Stefanov and Kevrekides studied traveling waves for Hertzian chains with precompression using bell-shaped functions [8], [9]. In [8], they consider the simpler precompression-free case, namely the equation

$$u_{tt} = \Delta_{disc}(u^p). \tag{1.5}$$

Starting with a variational approach, they prove the existence of a maximizer using growth estimates related to bell-shapedness. In [9], the authors summarise an alternative proof for (1.5) relying again on growth estimates but this time employing the Lebesgue Dominated Convergence Theorem rather than compactness. We recall that (1.5) can be written as $u_{tt} = (\Lambda * u^p)_{xx}$, where Λ is the triangular kernel introduced above.

In this paper we adapt the approach in [8], [9] to give a simple proof for the existence of traveling waves for (1.1). Our proof works for bell shaped kernels satisfying mild conditions, but we do not require the ellipticity condition (1.4).

The plan of the paper is as follows: In Section 2, we give the necessary preliminaries about bell-shaped rearrangements. In Section 3, we define a related maximization problem and prove the existence of a maximizer. In Section 4, via the Euler Lagrange equation and the regularity result, we prove that maximizers are indeed smooth bell-shaped traveling waves and present some examples of kernels for which our results apply.

In what follows, $W^{k,p}(\mathbb{R})$ denotes the Sobolev space of k times weakly differentiable functions with derivatives in $L^p(\mathbb{R})$, $W^{\infty,p}(\mathbb{R}) = \cap_{k\in\mathbb{N}} W^{k,p}(\mathbb{R})$.

2. Rearrangement and Bell Shaped Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if it is nonnegative, even and non-increasing on $[0, \infty)$. In this section we will consider bell-shaped rearrangements. We follow the presentation in [9] and refer to [6] for further information.

For a measurable function $f:\mathbb{R}\to\mathbb{R}$, the distribution function of f is defined as

$$d_f(s) = m(\{x\in\mathbb{R} : |f(x)| > s\})$$

where m stands for the Lebesgue Measure. It follows that for a C^1 function φ one has the identity

$$\int_{-\infty}^{\infty} \varphi(|f(x)|)dx = \int_0^{\infty} \varphi'(s)d_f(s)ds,$$

whenever the first integral exists. The non-increasing rearrangement of f is defined on $\mathbb{R}_+$ as,

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\};$$

and its bell-shaped rearrangement $f^\#$ on $\mathbb{R}$, as $f^\#(t) = f^*(2|t|)$. Clearly f^* is a nonnegative and nonincreasing measurable function and $f^\#$ is bell shaped. Moreover, for a bell shaped function we have $f^\# = f$. Both rearrangements have the same distribution function as f, namely $d_f = d_{f^*} = d_{f^\#}$. The integral identity above then implies:

Lemma 2.1. *Suppose $\phi\in C^1$. Then,*

$$\int_{-\infty}^{\infty} \varphi(|f(x)|)dx = \int_0^{\infty} \phi(|f^*(x)|)dx = \int_{-\infty}^{\infty} \phi(f^\#(x))dx,$$

whenever the first integral is finite. In particular for $f\in L^p(\mathbb{R})$, $1\le p\le\infty$, we have

$$\|f\|_{L^p(\mathbb{R})} = \|f^*\|_{L^p(\mathbb{R}_+)} = \|f^\#\|_{L^p(\mathbb{R})}.$$

Lemma 2.2. ***(Riesz's Rearrangement Inequality)*** *Let f, g and h be nonnegative measurable functions on a real line, vanishing at infinity. Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x-y)h(y)dxdy \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^\#(x)g^\#(x-y)h^\#(y)dxdy.$$

We will also make use of Young's inequality:

Lemma 2.3. ***(Young's Inequality)*** *Suppose $f\in L^p(\mathbb{R})$, $g\in L^q(\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$, $1\le p,q,r\le\infty$. Then*

$$\|f*g\|_r \le \|f\|_p\|g\|_q.$$

3. Setting the Variational Problem and Constructing a Maximizer

Throughout this paper we will assume that the the kernel $\beta \in W^{1,\frac{p+1}{2}}$ is bell shaped, and satisfies the decay condition at infinity, $\beta(x) = O(|x|^{-s})$ for some $s > 1$. Then the Sobolev embedding theorem implies that $\beta \in L^\infty$; while the growth condition gives $\beta \in L^1$.

A traveling wave solution $u = u(x - ct)$ of (1.1) satisfies the equation

$$(c^2 - a^2)u'' = (\beta * u^p)''.$$

Assuming that u, u' vanish at infinity we can integrate twice to get,

$$(c^2 - a^2)u = \beta * u^p. \tag{3.6}$$

When u is nonnegative, letting $v = u^{\frac{1}{p}}$, (3.6) can be expressed as

$$(c^2 - a^2)v^{\frac{1}{p}} = \beta * v. \tag{3.7}$$

Considering the related Euler Lagrange equation, this leads to the functional $J(v) = \langle \beta * v, v\rangle$ and the constraint $\|v\|_{L^{1+\frac{1}{p}}} = 1$. We thus define the constrained optimization problem

$$J_{max} = \sup\{J(v) : \|v\|_{L^q} = 1\}, \tag{3.8}$$

where $q = 1 + \frac{1}{p}$ and $\langle \beta * v, v\rangle$ denotes the (L^{q^*}, L^q) duality with $\frac{1}{q} + \frac{1}{q^*} = 1$. We note that $q = \frac{p+1}{p}$, and $q^* = p + 1$.

Our aim in this section is to prove that there is a bell-shaped maximizer of (3.8). This will be done through the following steps.

For v with $\|v\|_{L^q} = 1$, since $\frac{2}{p+1} = \frac{1}{p+1} - \frac{p}{p+1} + 1 = \frac{1}{q^*} - \frac{1}{q} + 1$, by Young's inequality, we have

$$\begin{aligned} J(v) &= \langle \beta * v, v\rangle \le \|\beta * v\|_{L^{q^*}} \|v\|_{L^q} \\ &\le \|\beta\|_{L^{\frac{p+1}{2}}} \|v\|^2_{L^q} = \|\beta\|_{L^{\frac{p+1}{2}}}. \end{aligned} \tag{3.9}$$

This shows that J_{max} exists. We next consider the more restricted optimization problem,

$$J^{\#}_{max} = \sup\{J(v) : \|v\|_{L^q} = 1, v \text{ is bell-shaped}\}.$$

Clearly $J^{\#}_{max} \leq J_{max}$. Conversely, let $v \in L^q$ with $\|v\|_{L^q} = 1$. Then for the bell-shaped rearrangement $v^{\#} = |v|^{\#}$ we have $\|v^{\#}\|_{L^q} = 1$. By Riesz's rearrangement inequality

$$J(w) = \langle \beta * v, v\rangle \leq \langle \beta * |v|, |v|\rangle \leq \langle \beta * v^{\#}, v^{\#}\rangle = J(v^{\#}).$$

Hence $J_{max} \leq J^{\#}_{max}$, so that $J_{max} = J^{\#}_{max}$. Based on this, in the first maximization problem, we can reduce the set of allowable v to the set of bell-shaped functions. We next obtain some growth estimates for bell-shaped functions [9].

Lemma 3.1. *Suppose that $v \in L^q$ is bell-shaped. Then for $x \neq 0$,*

$$v(x) \leq 2^{-\frac{1}{q}} \|v\|_{L^q} |x|^{-\frac{1}{q}}.$$

Proof. By bell-shapedness, for $|y| \leq |x|$ we have $v(y) \geq v(x)$. Then

$$2|x|v(x)^q \leq \int_{|y|\leq|x|} |v(y)|^q dy \leq \|v\|^q_{L^q},$$

which implies the required estimate. □

Lemma 3.2. *There is some C so that for all bell-shaped v with $\|v\|_{L^q} = 1$,*

$$\beta * v(x) \leq \Theta(x) = C \begin{cases} 1 & for \quad |x| < 2 \\ (|x|-1)^{\frac{-1}{q}} & for \quad |x| \geq 2 \end{cases}$$

Proof. Since

$$\|\beta * v\|_{L^\infty} \leq \|\beta\|_{L^{q^*}} \|v\|_{L^q} \leq \|\beta\|^{\frac{1}{2}}_{L^\infty} \|\beta\|^{\frac{1}{2}}_{L^{\frac{p+1}{2}}} \|v\|_{L^q} \leq \|\beta\|^{\frac{1}{2}}_{L^\infty} \|\beta\|^{\frac{1}{2}}_{L^{\frac{p+1}{2}}},$$

$\beta * v$ is bounded. For the case $|x| \geq 2$, as $\beta * v$ is even it suffices to consider $x \geq 2$.

$$\beta * v(x) = \int_{x-1}^{x+1} \beta(x-y)v(y)dy + \int_{|y-x|>1} \beta(x-y)v(y)dy = I + II$$

By Lemma 3.1, $v(x) \leq |x|^{-\frac{1}{q}}$; since $0 < x-1$, so we have

$$I = \int_{x-1}^{x+1} \beta(x-y)v(y)dy \leq v(x-1) \int \beta(x-y)dy \leq C\|\beta\|_{L^1}(|x|-1)^{\frac{-1}{q}}.$$

For II, we first show that $|x|^{\frac{1}{q}}\beta \in L^{q^*}$. But

$$(|x|^{\frac{1}{q}}\beta(x))^{q^*} \leq \|\beta\|^{q^*}_{L^\infty} |x|^{-q^*(s-\frac{1}{q})} = O(|x|^{-q^*(s-1)-1}).$$

Since $-q^*(s-1)-1<-1$, and β is bounded, this shows that $|x|^{\frac{1}{q}}\beta \in L^{q^*}$. Then,

$$\begin{aligned} II &= \int_{|x|>1} |x-y|^{\frac{-1}{q}} |x-y|^{\frac{1}{q}} \beta(x-y)v(y)dy \\ &\leq (x-1)^{\frac{-1}{q}} \int_{-\infty}^{\infty} |x-y|^{\frac{1}{q}} \beta(x-y)v(y)dy \\ &\leq (x-1)^{\frac{-1}{q}} \| |x|^{\frac{1}{q}} \beta\|_{L^{q^*}} \|v\|_{L^q} \leq (x-1)^{\frac{-1}{q}} \| |x|^{\frac{1}{q}} \beta\|_{L^{q^*}}. \end{aligned}$$

Adding up I and II gives the estimate for $|x| \geq 2$. □

We now pick a bell shaped maximizing sequence $\{v^n\}$, that is $\lim J(v^n) = J_{max}$ and $\|v^n\|_{L^q} = 1$. The next lemma provides the crucial step where we pass from the maximizing sequence to the actual maximizer.

Lemma 3.3. *The constrained variational problem (3.8) has a bell shaped maximizer.*

Proof. We start with some bell-shaped maximizing sequence $\{v^n\}$. Since $\|v^n\|_{L^q} = 1$, by Alaoglu's theorem there is a weakly convergent subsequence such that $v^{n_k} \rightharpoonup v$ for some $v \in L^q$. Then for each $x \in \mathbb{R}$,

$$\lim \beta * v^{n_k}(x) = \lim \langle \beta(x-\cdot, v^{n_k}\rangle = \langle \beta(x-\cdot), v\rangle = \beta * v(x).$$

In other words, $\beta * v^{n_k}(x)$ converges pointwise to $\beta * v(x)$. By Lemma 3.3, we have $|\beta * v^n(x)|^{q^*} \leq \Theta(x)^{q^*}$, so for $|x| \geq 2$

$$|\Theta(x)|^{q^*} \leq C(|x|-1)^{\frac{-q^*}{q}}.$$

Recall that $q = 1 + \frac{1}{p}$ with $p > 1$. Then $q < 2 < q^*$, thus $\frac{q^*}{q} > 1$. This shows that $|\Theta|^{q^*} \in L^1$, and by the Lebesgue Dominated Convergence Theorem $\beta * v^{n_k}$ converges to $\beta * v$ in L^{q^*}. Then

$$|J(v^{n_k}) - J(v)| \leq |\langle \beta * v^{n_k} - \beta * v, , v^{n_k}\rangle| + |\langle \beta * v, v^{n_k} - v\rangle|.$$

The first term is bounded by $\|\beta * v^{n_k} - \beta * v\|_{L^{q^*}}$ hence converges to zero. The second term also converges to zero since $v^{\ n_k} \rightharpoonup v$. Adding up we get

$$J_{\max} = \lim J(v^{n_k}) = J(v).$$

By the lower semicontinuity of weak convergence, we have $\|v\|_{L^q} \leq 1$. As $J(v) = J_{\max} \neq 0$, we have $v \neq 0$. Letting $\lambda = \|v\|_{L^q}$ then $\|\lambda^{-1}v\|_{L^q} = 1$, hence $\lambda^{-2}J_{\max} = \lambda^{-2}J(v) =$

$J(\lambda^{-1}v) \leq J_{\max}$, which shows that $\|v\|_{L^q} = 1$ and v is a maximizer. Finally, if v is not bell-shaped, then $v^{\#}$ will be a bell-shaped maximizer. $\square$

4. Proof of the Main Result

We have obtained a maximizer v, the next step is to derive the corresponding Euler-Lagrange equation. Consider perturbations of v in the form $v + t\varphi$, where φ is a fixed nonnegative $C_0^\infty(\mathbb{R})$ function. Clearly, $\|v + t\varphi\|_{L^q}^{-1}(v + t\varphi)$ will satisfy the constraint. Thus

$$g(t) = J(\|v + t\varphi\|_{L^q}^{-1}(v + t\varphi)) = \|v + t\varphi\|_{L^q}^{-2} J(v + t\varphi)$$

will have a maximum at $t = 0$. Setting $g'(0) = 0$, gives

$$2\int (\beta * v - J_{\max} v^{q-1})\varphi dx = 0.$$

Since this holds for all nonnegative test functions $\varphi \geq 0$, we get the inequality Euler Lagrange equation

$$\beta * v - J_{\max} v^{q-1} \leq 0.$$

Multiplying by v and integrating over $\mathbb{R}$ gives

$$J_{\max} = J(v) \leq J_{\max} \int v^q dx = J_{\max},$$

so that we have the Euler Lagrange equation $\beta * v = J_{\max} v^{q-1}$. With $u = v^{\frac{1}{p}}$, as $q = 1 + \frac{1}{p}$, this implies $u \in L^{p+1}$ and

$$J_{\max} u = \beta * u^p.$$

Then for any $c^2 > a^2$, $u_c = (\sqrt{\frac{c^2 - a^2}{J_{\max}}})^{\frac{2}{p-1}} u$ will satisfy the traveling wave equation (3.6). Clearly u_c and $\beta * u_c^p$ are in L^{p+1}.

Finally, we will consider the regularity of these traveling waves. Since $\beta \in W^{1,\frac{p+1}{2}}$ we have $\beta' \in L^{\frac{p+1}{2}}$. By Young's inequality, both $\beta * u^p$ and $D_x(\beta * u^p) = \beta' * u^p$ are in L^{p+1}; that is $\beta * u^p \in W^{1,p+1}$; so $u_c = c^{-2}\beta * u_c^p \in W^{1,p+1}$. Then we will have $D_x^2(\beta * u_c^p) = \beta' * (u_c^p)' \in L^{p+1}$, hence $u_c = c^{-2}\beta * u_c^p \in W^{2,p+1}$. Repeating the same steps, this boot-strap argument shows that $u_c \in W^{\infty,p+1}$, in other words $u_c \in C^\infty$ and all its derivatives decay at infinity. We can then integrate (3.6) and obtain the main result:

Theorem 4.1. *Suppose $\beta \in W^{1,\frac{p+1}{2}}$ is bell shaped and $\beta(x) = O(|x|^{-s})$ as $|x| \to \infty$ for some $s > 1$. Then for any $c^2 > a^2$, the equation (1.1) has a bell shaped traveling wave solution $u \in W^{\infty,p+1}$ with velocity c.*

Remark 4.2. *It is possible to have an alternative proof using the Kolmogorov compactness criteria. The condition $\beta \in W^{1,\frac{p+1}{2}}$ implies that the sequence $\{\beta * v^{n_k}\}$ is bounded in W^{1,q^*}. The uniform decay estimates on $\beta * v^{n_k}$ will then enable using Kolmogorov's compactness criteria to obtain a subsequence that converges in L^{q^*}. This alternative proof is in fact closer in spirit to the one provided in [8].*

Remark 4.3. *Replacing the nonlinear term u^p by $|u|^{p-1}u$, or $|u|^p$, the approach above can be extended to all real values of $p > 1$. The whole proof works except one has to pay attention to regularity. When p is not an integer $f(x) = |x|^{p-1}x$ or $f(x) = |x|^p$ is only differentiable k times with the greatest integer $k \leq p$; we then have $u_c \in W^{k,p+1}$.*

Remark 4.4. *As the equation for traveling waves is the same, our approach also shows that the nonlocal unidirectional wave equation*

$$u_t - au_x = (\beta * u^p)_x$$

has bell-shaped traveling wave solutions with velocity $c > a$. We note that the BBM equation is an example for this form.

We now give some examples of kernels. Clearly they are all bell-shaped, in $W^{1,1} \cap W^{1,\infty}$ and $O(x^{-s})$ for some $s > 1$.

1. The exponential kernel $\beta(x) = \frac{1}{2}e^{-|x|}$.
2. The Gaussian kernel $\beta(x) = e^{-x^2}$.
3. $\beta(x) = \frac{1}{1+x^2}$.
4. The triangular kernel $\Lambda(x) = (1 - |x|)\chi_{[-1,1]}$.
5. $\beta(x) = (1 - x^2)(1 - |x|)\chi_{[-1,1]}$.

Note that the exponential kernel is in fact the Green's function for the operator $1-D_x^2$ with decay conditions at infinity. The corresponding equation (1.1) can also be written as the Improved Boussinesq type equation

$$u_{tt} - u_{ttxx} = (u^p)_{xx}.$$

The triangular kernel leads to the differential-difference equation considered in [8] .The other kernels will yield integro-differential equations. Clearly examples 4 and 5 can be generalized to any sufficiently smooth bell-shaped kernel in with compact support. We also note that except for the exponential kernel, none of the others satisfies the ellipticity condition (1.4) so that the results in [4] will not provide the existence of traveling wave solutions.

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