UNIVERSAL FUNCTIONS FOR COMPOSITION OPERATORS WITH NON-AUTOMORPHIC SYMBOL

KARL-G. GROSSE-ERDMANN AND RAYMOND MORTINI

Abstract. For sequences (ϕ_n) of eventually injective holomorphic self-maps of planar domains Ω we present necessary and sufficient conditions for the existence of holomorphic functions f on Ω whose orbits under the action of (ϕ_n) are dense in $H(\Omega)$. It is deduced that finitely connected, but non-simply connected domains never admit such universal functions. On the other hand, when allowing arbitrary sequences of holomorphic self-maps (ϕ_n), then we show that the situation changes dramatically.

August 1, 2007

1. Introduction

Let Ω be a domain in $\mathbb{C}$ and let $H(\Omega)$ denote the space of holomorphic functions in Ω; as usual, this space is endowed with the topology of locally uniform convergence, under which it becomes a complete separable metric space. We are interested in the existence of dense orbits for composition operators on $H(\Omega)$. If ϕ is a holomorphic self-map of Ω then the composition operator with symbol ϕ is defined as $C_\phi(f) = f \circ \phi, f \in H(\Omega)$.

Birkhoff [8] showed that there exists an entire function f such that $\{f \circ \tau_n : n \in \mathbb{N}\}$ forms a dense set in $H(\mathbb{C})$, where the τ_n are the $\mathbb{C}$-automorphisms $z \mapsto z+n$. Such a function f is called universal. If, in a general domain Ω, one restricts f to functions bounded by 1, then one may still hope to be able to approximate all holomorphic functions that are bounded by 1. A first such result is due to Heins [20]. He constructed a Blaschke product f and a sequence (z_n) in the unit disk $\mathbb{D}$ such that any holomorphic function in $\mathbb{D}$ that is bounded by 1 can be locally uniformly approximated by functions of the form $f \circ \phi_n, n \in \mathbb{N}$, where the ϕ_n are the $\mathbb{D}$-automorphisms $z \mapsto \frac{z+z_n}{1+\overline{z}_n z}$. Such a function f is called $\mathscr{B}$-universal.

Universal functions, in one or several variables, have subsequently been investigated by Abe, Bernal, Bonilla, Calderón, Chee, Godefroy, León, Luh, Montes, Rezaei, Shapiro, Yousefi and Zappa (see [10], [29], [1], [14], [22], [7], [23], [21], [5], [28]). In recent years, $\mathscr{B}$-universality has been the object of intensive studies; we refer to papers by Aron, Bayart, Gauthier, Gorkin, Grivaux, León, Mortini and Xiao (see [16], [17],[12], [2],[3],[24], [15], [4]).

The investigation of universal functions has recently taken a new turn. Motivated by a concluding remark in [15], Bayart, Gorkin, Grivaux and Mortini [4] were the first to study systematically composition operators with non-automorphic symbols. In this generality they characterize universality and $\mathscr{B}$-universality on the unit disk $\mathbb{D}$. We shall here continue their study on arbitrary planar domains Ω.

As usual, $H^\infty(\Omega)$ denotes the space of bounded holomorphic functions on the domain Ω; the space is endowed with the supremum norm. The closed unit ball of $H^\infty(\Omega)$ is denoted by $\mathscr{B}(\Omega)$.

1991 *Mathematics Subject Classification.* 47B33; 30H05; 47A16.

Key words and phrases. Composition operators; bounded holomorphic functions; universal functions; hypercyclic functions; Ω-convex sets.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be a domain and (ϕ_n) a sequence of holomorphic self-maps of Ω.

(a) A function $f \in H(\Omega)$ is called *universal* for (ϕ_n) if the set $\{f \circ \phi_n : n \in \mathbb{N}\}$ is dense in $H(\Omega)$. The sequence (C_{ϕ_n}) is called *universal* if it admits a universal function.

(b) A function $f \in \mathscr{B}(\Omega)$ is called *$\mathscr{B}$-universal* for (ϕ_n) if the set $\{f \circ \phi_n : n \in \mathbb{N}\}$ is locally uniformly dense in $\mathscr{B}(\Omega)$. The sequence (C_{ϕ_n}) is called *$\mathscr{B}$-universal* if it admits a $\mathscr{B}$-universal function.

A major tool for showing the existence of universal functions will be Birkhoff's transitivity criterion (for example, see [19, Theorem 1]). A sequence (T_n) of continuous maps on a topological space X is called *topologically transitive* if, for every pair U and V of non-empty open sets in X, there exists some $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$. The transitivity criterion then says that, if X is a separable complete metric space, then there exists a dense set of points $x \in X$ (called *universal elements* for (T_n)) for which $\{T_n(x) : n \in \mathbb{N}\}$ is dense in X if and only if (T_n) is topologically transitive. In our setting, X will be either $H(\Omega)$ or $\mathscr{B}(\Omega)$, and $T_n = C_{\phi_n}$.

Let us introduce some more notation. For any set M in a topological space we denote by $\overline{M}$ its closure, by M° its interior and by ∂M its boundary. As usual, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ will denote the punctured plane and the punctured disk, respectively, while $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended plane. By an exhaustion of a domain Ω we mean a sequence (K_n) of compact subsets of Ω with $\bigcup_{n\geq 1} K_n = \Omega$ such that, for every $n \in \mathbb{N}$, $K_n \subseteq K_{n+1}^\circ$. Finally, we shall write $f_n \to f$ if a sequence (f_n) converges locally uniformly to f.

In Section 2 we give a sufficient condition for the existence of $\mathscr{B}$-universal functions. In Section 3 we study universal functions on arbitrary domains.

2. Universality in the ball of $H^\infty(\Omega)$

We assume throughout this section that $H^\infty(\Omega)$ is not trivial; that means that it contains non-constant functions.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{C}$ be a domain and (ϕ_n) a sequence of holomorphic self-maps of Ω. Suppose that (ϕ_n) satisfies*

(1) *there exists $B \in \mathscr{B}(\Omega)$, B non-constant, such that $B \circ \phi_n \to 1$;*

(2) *there exists a holomorphic self-map u of Ω such that $u \circ \phi_n \to z$.*

Then there exists a $\mathscr{B}$-universal function f for (ϕ_n).

Proof. We show that (C_{ϕ_n}) satisfies Birkhoff's transitivity criterion. So let $K \subseteq \Omega$ be compact, $0 < \epsilon < 1$ and $g, h \in \mathscr{B}(\Omega)$. We have to look for a function $f \in \mathscr{B}(\Omega)$ and an index n such that $|f - g| < \varepsilon$ as well as $|f \circ \phi_n - h| < \varepsilon$ on K. To this end, choose $0 < \eta < 1$ so that $B(K) \subseteq \{z \in \overline{\mathbb{D}} : |z - 1| > \eta\}$. By [18, Lemma 3.9] there exists a conformal automorphism ψ of the disk with fixed points 1 and -1 such that the image of $\{z \in \overline{\mathbb{D}} : |z - 1| > \eta\}$ under ψ is contained in $\{w \in \overline{\mathbb{D}} : |w + 1| < \varepsilon\}$. Hence $p := \psi \circ B$ has the property that $p \in \mathscr{B}(\Omega)$, $p \circ \phi_n \to 1$ and $|p(z) + 1| < \varepsilon$ on K. Now let

$$f = \left(\frac{1-p}{2}\right)^2 g + \left(\frac{1+p}{2}\right)^2 (h \circ u).$$

Then $f \in \mathscr{B}(\Omega)$, $f \circ \phi_n \to h$ and on K we have that f is very close to g; in fact,

$$|f - g| \leq \left|\left(\frac{1-p}{2}\right)^2 - 1\right| + \left|\frac{1+p}{2}\right|^2 \leq \left|\left(\frac{1-p}{2}\right) - 1\right| \left|\left(\frac{1-p}{2}\right) + 1\right| + |1+p|^2 \leq$$

$$\leq |p+1| + |1+p|^2 \leq 2\varepsilon.$$

Another proof works along the same lines as in [15]. First we note that condition (1) shows that for $z_0 \in \Omega$ the sequence $(w_n) := (\phi_n(z_0))$ admits an asymptotic interpolating subsequence of type 1. Actually it is known that, in a certain sense, both statements are equivalent (see [18, Proposition 4.1]). We can assume that (w_n) itself is such a sequence.

So, let (K_n) be an exhaustion of Ω, and let $\{p_j : j \in \mathbb{N}\}$ be a set of functions that is dense in $\mathscr{B}(\Omega)$. Since, for every j, we have $(p_j \circ u) \circ \phi_n \to p_j$ as $n \to \infty$, there exists a subsequence (n_j) such that

$$\sup_{K_j} |(p_j \circ u) \circ \phi_{n_j} - p_j| < 1/j,$$

and there are $f_j, g_j \in \mathscr{B}(\Omega)$ that satisfy conditions (4.1)-(4.5) of [15] for $z_j = \phi_{n_j}(z_0)$ and $\varepsilon_{jk} = 1/2^{j+k+1}$.

Hence, by the proofs of [15, Theorem 4.3] and [15, Theorem 4.4], the function

$$F = \sum_{j=1}^{\infty} (p_j \circ u) f_j g_1 \cdots g_{j-1}$$

is the universal function we are looking for. Note that

$$F \circ \phi_{n_j} - (p_j \circ u) \circ \phi_{n_j} \to 0.$$

□

If we specialize to the case of the unit disk, then we obtain the result given in [4] that a sequence of holomorphic self-maps (ϕ_n) in $\mathbb{D}$ with $\phi_n(0) \to 1$ admits a $\mathscr{B}$-universal function if and only if

$$\limsup_{n\to\infty} \frac{|\phi_n'(0)|}{1 - |\phi_n(0)|^2} = 1. \tag{2.1}$$

In fact, if $\phi_n(0) \to 1$ and (2.1) is satisfied, then we define b to be the Blaschke product associated with a thin subsequence $(\phi_{n_j}(0))$ of $(\phi_n(0))$. Then it is straightforward to check that $b \circ \phi_{n_{j_k}} \to \lambda z$ for some $\lambda, |\lambda| = 1$, see [4, 2.4]. Now let $u = \overline{\lambda} b$. Then u satisfies condition (2) in Theorem 2.1 for some subsequence of (ϕ_n). If we let B be the peak function $(1+z)/2$, then we see that Condition (1) is satisfied, too. For the converse see [4, 2.3].

Definition 2.2. A sequence of self-maps (ϕ_n) of a domain $\Omega \subseteq \mathbb{C}$ is said to be a *run-away sequence*, if for every compact set $K \subseteq \Omega$ there exists $n \in \mathbb{N}$ such that $\phi_n(K) \cap K = \emptyset$.

Bernal and Montes [7, Theorem 3.6] have shown that for planar domains not conformally isomorphic to $\mathbb{C}^*$ this run-away property is necessary and sufficient for a sequence of automorphisms to admit universal functions. In [15, Theorem 5.3] it was shown that for certain domains they also characterize $\mathscr{B}$-universality for invertible composition operators. Here we have the following statement.

Proposition 2.3. *Let $\Omega \subseteq \mathbb{C}$ be a domain and (ϕ_n) a sequence of holomorphic self-maps of Ω. Suppose that f is a $\mathscr{B}$-universal function for (ϕ_n). Then (ϕ_n) is a run-away sequence.*

Proof. Choose a subsequence (n_j) so that $f \circ \phi_{n_j} \to 1$. Suppose that (ϕ_n) is not a run-away sequence. Then there exists a compact set $K \subseteq \Omega$ such that $\phi_n(K) \cap K \neq \emptyset$ for every n. Choose $\xi_n \in K$ so that $\phi_n(\xi_n) \in K$. Passing, if necessary, to a subsequence of (n_j), we may assume that $\phi_{n_j}(\xi_{n_j})$ converges to some $w_0 \in K$. By uniform convergence on K we have $(f \circ \phi_{n_j})(\xi_{n_j}) \to f(w_0)$, but also $(f \circ \phi_{n_j})(\xi_{n_j}) \to 1$. Thus $f(w_0) = 1$, the supremum of f on Ω. The maximum principle implies that f is constant, a contradiction. □

In the case of non-automorphic symbols, this condition though is far away from being sufficient for $\mathscr{B}$-universality, as was already observed in [4] for the case of the unit disk (just take $\phi_n(z) = \alpha_n z + 1 - \alpha_n$, where $0 < \alpha_n < 1$ and $\alpha_n \to 0$; these ϕ_n do not satisfy (2.1)).

The following result complements [15, Corollary 3.7], where only automorphic symbols were considered.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{C}$ be a domain and (ϕ_n) a sequence of holomorphic self-maps of Ω for which there exists $B \in \mathscr{B}(\Omega)$, B non-constant, such that $B \circ \phi_n \to 1$. Suppose that $H^\infty(\Omega)$ is dense in $H(\Omega)$. Then the existence of a $\mathscr{B}$-universal function for (ϕ_n) implies the existence of an $H(\Omega)$-universal function.*

Proof. We show that (C_{ϕ_n}) satisfies Birkhoff's transitivity criterion on $H(\Omega)$. To this end, let u be a $\mathscr{B}$-universal function, $g, h \in H(\Omega)$, g bounded, and let $K \subseteq \Omega$ be compact. Fix $\varepsilon \in\ ]0,1[$. Choose $v \in \mathscr{B}(\Omega)$ and $N \in \mathbb{N}$ so large that $|Nv - h| < \varepsilon$ on K. Since u is $\mathscr{B}$-universal, there exist infinitely many n' so that $|u \circ \phi_{n'} - v| < \varepsilon/N$ on K. As in the proof of Theorem 2.1, we choose $p \in \mathscr{B}(\Omega)$ so that $p \circ \phi_n \to 1$ and

$$\sup_K |p+1| < \frac{\varepsilon}{\max(1, \sup_K |Nu|, \sup_K |g|)}.$$

Now let

$$f = \left(\frac{1-p}{2}\right)^2 g + \left(\frac{1+p}{2}\right)^2 Nu.$$

Then, on K, we have

$$|f - g| \leq |p+1|\ |g| + \left|\frac{1+p}{2}\right| N|u| \leq \varepsilon + \varepsilon = 2\varepsilon,$$

and for n' sufficiently large we have on K

$$|f \circ \phi_{n'} - h| \leq \left|\frac{1 - p \circ \phi_{n'}}{2}\right|^2 ||g||_\infty + N|u \circ \phi_{n'} - v| + |Nv - h| \leq 3\varepsilon.$$

□

In the following section we derive necessary conditions for the existence of universal functions in $H(\Omega)$. In view of Theorem 2.4, these conditions apply to $\mathscr{B}$-universality as well. At present, though, we do not know whether in finitely connected, but not simply connected domains, there exist $\mathscr{B}$-universal functions.

3. Universality in $H(\Omega)$

We turn to universality of holomorphic functions that are not necessarily bounded. We shall see that the degree of connectivity of the underlying domain plays an important role. For this reason our investigation will cover, in turn, simply connected, finitely connected and infinitely connected domains.

Definition 3.1.
(a) Let M be an open or compact subset of $\mathbb{C}$. Then a *hole* of M is a bounded component of $\widehat{\mathbb{C}} \setminus M$.
(b) A compact subset M of a domain Ω is called Ω-*convex* if every hole of M contains a point of $\mathbb{C} \setminus \Omega$.

We will often tacitly use the fact that if $\Omega \subseteq \mathbb{C}$ is a domain with N holes and ϕ is an injective holomorphic mapping on Ω, or if K is a compact set with N holes and ϕ is an injective holomorphic mapping on a neighbourhood of K, then the images $\phi(\Omega)$ and $\phi(K)$ also have N holes (see [26, p. 276]).

3.1. **Simply connected domains.** The simply connected case has already been addressed in [4, Theorem 5.2]. There the domain $\mathbb{D}$ is considered, which by Riemann's mapping theorem can be replaced by any simply connected domain other than $\mathbb{C}$ itself. In the following theorem a more general result is given that includes the domain $\mathbb{C}$.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, and let (ϕ_n) be a sequence of holomorphic self-maps of Ω. Then the following assertions are equivalent:*

(a) *The sequence of composition operators (C_{ϕ_n}) is universal.*
(b) *There exists a subsequence (n_j) such that for each compact subset K of Ω there is some $J \in \mathbb{N}$ such that $\phi_{n_j}(K) \cap K = \emptyset$ and $\phi_{n_j}|_K$ is injective for all $j \geq J$.*
(c) *For every compact subset K of Ω there is some $n \in \mathbb{N}$ such that $\phi_n(K) \cap K = \emptyset$ and $\phi_n|_K$ is injective.*

Proof. (a)$\Longrightarrow$(b). We fix an exhaustion (K_n) of compact subsets of Ω. Let f be a universal function. Then for every $j \in \mathbb{N}$ there exists an n_j such that

$$\sup_{z \in K_j} |f(\phi_{n_j}(z)) - (z + j)| < \tfrac{1}{j}. \tag{3.1}$$

We show that the sequence (n_j) satisfies condition (b). To see this, let $K \subseteq \Omega$ be compact and let $J_1 \in \mathbb{N}$ be such that $K \subseteq K_j^\circ$ for all $j \geq J_1$.

First, by (3.1) we have for $z \in K$ and $j \geq J_1$

$$|f(\phi_{n_j}(z))| \geq j - \sup_{\zeta \in K} |\zeta| - \tfrac{1}{j}.$$

Thus, if j is larger than a suitable $J_2 \in \mathbb{N}, J_2 \geq J_1$, then

$$\inf_{z \in K} |f(\phi_{n_j}(z))| > \sup_{\zeta \in K} |f(\zeta)|,$$

which implies that $\phi_{n_j}(K) \cap K = \emptyset$.

Secondly, let

$$\delta = \inf\{|z_0 - z| : z_0 \in K, z \in \partial K_{J_1}\},$$

which is clearly positive. Let $j \geq J_2, z_0 \in K$ and $z \in \partial K_j$. We then have by (3.1) that

$$\big|[f(\phi_{n_j}(z_0)) - f(\phi_{n_j}(z))] - [z_0 - z]\big| \leq 2 \sup_{\zeta \in K_j} |f(\phi_{n_j}(\zeta)) - (\zeta + j)| < \tfrac{2}{j}.$$

Hence there is some $J \geq J_2$ such that for $j \geq J$

$$\big|[f(\phi_{n_j}(z_0)) - f(\phi_{n_j}(z))] - [z_0 - z]\big| < \delta.$$

This implies that for $j \geq J, z_0 \in K$ and $z \in \partial K_j$,

$$\big|[f(\phi_{n_j}(z_0)) - f(\phi_{n_j}(z))] - [z_0 - z]\big| < |z_0 - z|,$$

where we have used that $K_{J_1} \subseteq K_j$. By Rouché's theorem, see [25], $f(\phi_{n_j}(z_0)) - f(\phi_{n_j}(z))$ and $z_0 - z$ have the same number of zeros in K_j°, which shows that $f(\phi_{n_j}(z)) \neq f(\phi_{n_j}(z_0))$ whenever $j \geq J, z_0 \in K, z \in K_j^\circ, z \neq z_0$. In particular, $f \circ \phi_{n_j}$ is injective on K for $j \geq J$, as is therefore ϕ_{n_j}.

(b)$\Longrightarrow$(c) is trivial.

(c)$\Longrightarrow$(a) Unlike the constructive proof in [4] we base the proof on Birkhoff's transitivity criterion, by which it suffices to show that the sequence (C_{ϕ_n}) is topologically transitive. Thus, let $K \subseteq \Omega$ be compact, $f, g \in H(\Omega)$ and $\varepsilon > 0$. Let K_1, K_2 be compact subsets of Ω with connected complement and such that $K \subseteq K_1 \subseteq K_2^\circ$ (here we have used that Ω is simply connected). By (c) there is some $n \in \mathbb{N}$ such that $\phi_n(K_2) \cap K_2 = \emptyset$ and $\phi_n|_{K_2}$ is injective. Thus the mapping $\psi = \phi_n^{-1} : \phi_n(K_2^\circ) \to K_2^\circ$ is a well-defined holomorphic mapping. Moreover, by the hole-invariance of injective holomorphic mappings, $K_1 \cup \phi_n(K_1)$

has connected complement. By Runge's theorem, applied to the mappings f on K_1 and $g \circ \psi$ on $\phi_n(K_1)$, we obtain a function $h \in H(\Omega)$ such that

$$|h(z) - f(z)| < \varepsilon \quad \text{for } z \in K_1,$$
$$|h(w) - g(\psi(w))| < \varepsilon \quad \text{for } w \in \phi_n(K_1),$$

hence also

$$|h(\phi_n(z)) - g(z)| < \varepsilon \quad \text{for } z \in K_1,$$

which had to be shown. □

Remark 3.3. An analysis of the proof shows that the implications $(a) \implies (b) \implies (c)$ of the theorem above remain true for all domains Ω in $\mathbb{C}$.

We also note that Theorem 3.2 contains [4, Theorem 5.2] as a special case. To see this note that if ϕ_n are holomorphic self-maps of $\mathbb{D}$ such that $\phi_n(0) \to 1$ then (ϕ_n) converges locally uniformly to 1 on $\mathbb{D}$, so that, for any compact set $K \subseteq \mathbb{D}$, $\phi_n(K) \cap K = \emptyset$ for sufficiently large n.

Finally, Theorem 3.2 allows us to give examples of universal composition operators on $H(\mathbb{C})$ that do not come out of simple variations of Birkhoff's theorem. In order to do this we must consider non-injective entire functions ϕ_n, because any injective function $\phi \in H(\mathbb{C})$ is an automorphism and therefore of the form $\phi(z) = az + b, a \neq 0$.

Example 3.4. (a) For $p \in \mathbb{N}$ let $\phi_n(z) = (z + n)^p$. Then (ϕ_n) satisfies condition (c) of the theorem; the injectivity follows from the fact that $z \mapsto z^p$ is injective in the cone $C = \{z \in \mathbb{C} : |\arg z| < 2\pi/p\}$ and that the translate $K+n$ of any compact set K eventually lies in C.

Hence there is some $f \in H(\mathbb{C})$ such that the functions $f((z+n)^p), n \in \mathbb{N}$, form a dense set in $H(\mathbb{C})$. In other words, the function $f \circ z^p$ is universal in the sense of Birkhoff. This can also be phrased in the following way. For every $p \in \mathbb{N}$ there is a Birkhoff-universal function $h \in H(\mathbb{C})$ of the form

$$h(z) = \sum_{k=0}^{\infty} a_k z^{kp}.$$

This is a special case of a much more general result recently obtained by Gharibyan, Luh and Müller [13, Theorem 4.1].

(b) Let $\phi_n(z) = e^{z/n} + n$. Then (ϕ_n) satisfies condition (c) of the theorem. Hence there is some $f \in H(\mathbb{C})$ such that the functions

$$f(e^{z/n} + n), n \in \mathbb{N}$$

form a dense set in $H(\mathbb{C})$.

3.2. **Non-simply connected domains: existence of universal functions.** Every simply connected domain supports an automorphism ϕ whose iterates $(\phi^{[n]})$ define a universal sequence of composition operators, see [7]. However, when passing to non-simply connected domains of finite connectivity, then, as observed by Bernal and Montes, no sequence (C_{ϕ_n}) with automorphic symbols can be universal; see [7, pp. 51–52, p. 55, Theorem 3.6]. It is therefore of interest to investigate if the situation changes when we allow non-automorphic symbols. We shall see that the answer depends on the geometry of the domain.

We first note that the observation of Bernal and Montes extends to non-automorphisms if the complement of Ω consists of at least two but at most finitely many points. This result has little to do with universality but follows from the fact that Ω supports only finitely many non-constant holomorphic self-maps. This fact is certainly known, but we have not been able to find a reference.

Proposition 3.5. *Let $z_1, \dots, z_N$ be $N \geq 2$ distinct points in $\mathbb{C}$. If $\Omega = \mathbb{C} \setminus \{z_1, \dots, z_N\}$, then there are at most finitely many non-constant holomorphic self-maps of Ω. Moreover, these are linear fractional transformations. In particular, there exists no sequence (ϕ_n) of holomorphic self-maps of Ω such that (C_{ϕ_n}) is universal.*

Proof. By Picard's theorem, any holomorphic function with an essential isolated singularity takes every value, with at most one exception, infinitely often. Thus any holomorphic self-map ϕ of Ω is a rational function. Now, ϕ being a self-map of Ω, we have that $\phi^{-1}\{z_1, \dots, z_N, \infty\} \subseteq \{z_1, \dots, z_N, \infty\}$. Since non-constant rational functions are surjective mappings on $\widehat{\mathbb{C}}$, our map ϕ acts as a permutation on $\{z_1, \dots, z_N, \infty\}$.

Let τ be the (unique) linear fractional map that satisfies $\tau(z_1) = \phi(z_1), \tau(z_2) = \phi(z_2)$ and $\tau(\infty) = \phi(\infty)$. Then $\psi := \tau^{-1} \circ \phi$ is a rational function that keeps z_1, z_2 and ∞ fixed. Moreover, if $\psi(w) = \infty$ then $\phi(w) = \tau(\infty) = \phi(\infty)$, hence $w = \infty$. This shows that ψ is a polynomial. Similarly, $\psi^{-1}(z_1) = \{z_1\}$ and $\psi^{-1}(z_2) = \{z_2\}$. Let d be the degree of ψ. Then both z_1 and z_2 have d preimages (counted with multiplicity). Hence the multiplicities of each of these preimages must be d. Therefore the degree of the derivative ψ' is at least $2(d-1)$, hence $d \geq 2(d-1)+1$. This implies that $d = 1$. Hence $\phi = \tau \circ \psi$ is a linear fractional map.

Since only finitely many permutations of the set $\{z_1, \dots, z_N, \infty\}$ are possible, we conclude that there can only be finitely many non-constant self-maps of Ω. □

In many interesting cases, however, universality can happen.

Proposition 3.6. *Let $\Omega \subseteq \mathbb{C}$ be a domain for which $H^\infty(\Omega)$ is dense in $H(\Omega)$. Then there exists a sequence (ϕ_n) of holomorphic self-maps of Ω such that (C_{ϕ_n}) is universal.*

Proof. By the separability of $H(\Omega)$ the assumption implies that there exists a sequence (ψ_n) of bounded functions on Ω that is dense in $H(\Omega)$. Then there are closed disks B_{r_n} of radius $r_n > 0$ with centre 0 that contain $\psi_n(\Omega)$. Moreover, let b be a boundary point of Ω, and let $B_{\rho_n}(a_n)$ be disks of radius $\rho_n > 0$ with centre a_n that lie in Ω, are pairwise disjoint and satisfy $a_n \to b$. Then there are maps $\tau_n(z) = \alpha_n z + \beta_n, \alpha_n \neq 0$, such that $\tau_n(B_{r_n}) = B_{\rho_n}(a_n)$ for all $n \in \mathbb{N}$. We finally define ϕ_n by

$$\phi_n = \tau_n \circ \psi_n.$$

Clearly the ϕ_n are holomorphic self-maps of Ω.

We now show that (C_{ϕ_n}) is topologically transitive. To see this, let K be a compact subset of Ω, $f, g \in H(\Omega)$ and $\varepsilon > 0$. By enlarging K we may assume that it is Ω-convex. It follows from the hypotheses that there is some $n \in \mathbb{N}$ such that $B_{\rho_n}(a_n) \cap K = \emptyset$ and $|g(z) - \psi_n(z)| < \varepsilon$ for all $z \in K$.

Since $K \cup B_{\rho_n}(a_n)$ is Ω-convex, we may apply Runge's theorem to the function f on K and the inverse τ_n^{-1} on $B_{\rho_n}(a_n)$ to obtain a function $h \in H(\Omega)$ such that

$$|h(z) - f(z)| < \varepsilon \quad \text{for all } z \in K,$$

$$|h(w) - \tau_n^{-1}(w)| < \varepsilon \quad \text{for all } w \in B_{\rho_n}(a_n). \tag{3.2}$$

Since $\tau_n(\psi_n(z)) \in B_{\rho_n}(a_n)$ for $z \in \Omega$, (3.2) implies that for all $z \in K$,

$$\begin{aligned} |h(\phi_n(z)) - g(z)| &\leq |h(\phi_n(z)) - \psi_n(z)| + |\psi_n(z) - g(z)| \\ &= |h(\tau_n(\psi_n(z))) - \tau_n^{-1}(\tau_n(\psi_n(z)))| + \varepsilon < 2\varepsilon. \end{aligned}$$

This implies that (C_{ϕ_n}) is topologically transitive, hence universal. □

Remark 3.7. The assumption of the proposition holds, in particular, for each domain Ω for which each component of $\mathbb{C} \setminus \Omega$ has interior points. In fact, choose a set P having exactly one point from the interior of each such component. Then by Runge's theorem the rational functions with poles from P form a dense set in $H(\Omega)$, and these functions belong to $H^\infty(\Omega)$. Further examples are provided by bounded finitely connected domains Ω for which each component of $\mathbb{C} \setminus \Omega$ is a continuum. According to a result of Gauthier and Melnikov [11], $H^\infty(\Omega)$ is dense in $H(\Omega)$ if and only if for every open set D in the Riemann sphere $\widehat{\mathbb{C}}$ that meets $\mathbb{C} \setminus \Omega$ we have that $\gamma(D \cap (\mathbb{C} \setminus \Omega)) > 0$, where γ denotes analytic capacity.

Problem 3.8. Characterize all domains (possibly only among the finitely connected ones) that support a universal sequence (C_{ϕ_n}) of composition operators.

3.3. **Non-simply connected domains: necessary conditions.** What is remarkable about the proof of Proposition 3.6 is that it is not so much the universal function f that behaves wildly but the symbols ϕ_n. This seems to be contrary to the usual idea of universality. Motivated by Theorem 3.2 we shall in the sequel impose a regularity condition on the ϕ_n, namely that they are eventually injective. On the one hand this will allow us to apply the Runge approximation theorem at a crucial point, and on the other hand this will eliminate the pathological kind of universality constructed in the proof of Proposition 3.6.

Definition 3.9. Let $\Omega \subseteq \mathbb{C}$ be a domain, and let ϕ_n be holomorphic self-maps of Ω. Then (ϕ_n) is called *eventually injective* if, for every compact subset K of Ω, there is some $N \in \mathbb{N}$ such that $\phi_n|_K$ is injective for all $n \geq N$.

By Remark 3.3, a sequence of composition operators (C_{ϕ_n}) on an arbitrary domain Ω can only be universal if the sequence (ϕ_n) has an eventually injective subsequence.

¿From now on we shall only consider eventually injective sequences (ϕ_n). Our first aim is to derive a necessary condition for the universality of (C_{ϕ_n}). This result will be crucial for all that follows.

We begin with two simple geometric lemmas.

Lemma 3.10. *Let K be an Ω-convex compact subset of a domain $\Omega \subseteq \mathbb{C}$. Then K has at most finitely many holes.*

Proof. Otherwise we can choose points $z_n \in \mathbb{C} \setminus \Omega, n \in \mathbb{N}$, lying in different holes of K. By compactness, a subsequence (z_{n_j}) converges to some $z \in \mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus K$. Hence the component of $\mathbb{C} \setminus K$ containing z also contains infinitely many z_n, a contradiction to the choice of the z_n. □

Let us point out though that a hole of K can contain infinitely many components of $\widehat{\mathbb{C}} \setminus \Omega$.

Lemma 3.11. *Let K and L be compact subsets of a domain $\Omega \subseteq \mathbb{C}$ with $K \subseteq L$. Let ϕ be a holomorphic self-map of Ω that is injective on some neighbourhood of L. If K and $\phi(L)$ are Ω-convex then so is $\phi(K)$.*

Proof. By Lemma 3.10, K has a finite number, N say, of holes. Suppose that $\phi(K)$ is not Ω-convex. Then it has a hole O that contains no point from $\mathbb{C} \setminus \Omega$. But then O cannot contain a point from $\mathbb{C} \setminus \phi(L)$ since, otherwise, it would contain a hole of $\phi(L)$, hence also a point from $\mathbb{C} \setminus \Omega$ because $\phi(L)$ is Ω-convex. Thus, $\phi(K) \cup O$ is a compact subset of $\phi(L)$ with $N-1$ holes. Therefore, $\phi^{-1}(\phi(K) \cup O)$ is a compact subset of L with $N-1$ holes. This implies that one hole of K lies in $\phi^{-1}(\phi(K) \cup O)$, hence in Ω. But that contradicts the fact that K is Ω-convex. □

We can now prove the main result of this subsection.

Theorem 3.12. *Let $\Omega \subseteq \mathbb{C}$ be a domain, and let (ϕ_n) be an eventually injective sequence of holomorphic self-maps of Ω. Suppose that (C_{ϕ_n}) is universal. Then, for every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$, there is some $n \geq N$ such that $\phi_n(K)$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$.*

Proof. By Theorem 3.2 and the hole-invariance principle, the result is true for simply connected domains. So let Ω be non-simply connected. In the sequel we follow the notation and terminology of Rudin [27, section 13]. Let $b \in \Omega \setminus K$. We shall construct a connected Ω-convex compact subset L of Ω that contains K and b in its interior and that is bounded by finitely many Jordan curves.

To this end, consider the $\frac{1}{2^n}$-lattice of all points $z = x + iy$ in $\mathbb{C}$ for which x and y are integer-multiples of $\frac{1}{2^n}$. Let L_n be the largest compact set inside $\Omega \cap \{|z| < n\}$ whose boundary lies on the $\frac{1}{2^n}$-lattice. If n is big enough then L_n will contain K and b in its interior. By maximality, L_n is Ω-convex. If L_n is not connected, we enlarge it by finitely many small tubes in such a way that the new set, call it L, remains Ω-convex and has a Jordan curve as outer boundary. By Lemma 3.10, L has only a finite number of holes. Denote the positively oriented outer boundary by γ_0. The p negatively oriented boundaries of the holes O_j constitute Jordan curves, denoted by $\gamma_1, \ldots, \gamma_p$. Note that the connectedness of L prevents that one hole of L surrounds another (see Figure 1), and that the outer boundary surrounds all the p holes. This ends the construction of L.

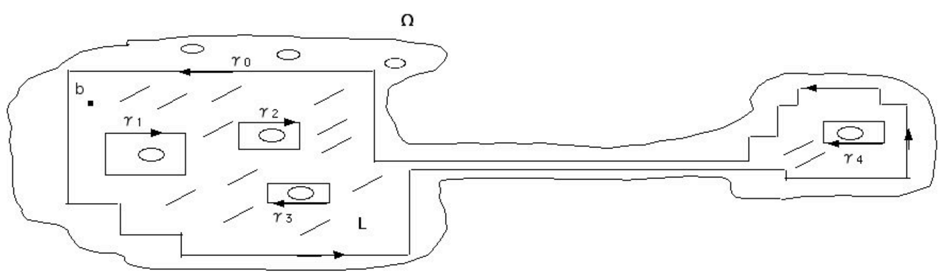

FIGURE 1. The set L

Now fix, in each hole O_j of L, $j = 1, \ldots, p$, a point $a_j \in \mathbb{C} \setminus \Omega$. Then our curves γ_j will have the following properties:

$$\begin{aligned} &\operatorname{ind}_{\gamma_j}(a_j) = -1, \quad \operatorname{ind}_{\gamma_j}(a_k) = 0, \quad \operatorname{ind}_{\gamma_j}(b) = 0, \quad j,k = 1, \ldots, p,\ j \neq k, \\ &\operatorname{ind}_{\gamma_0}(a_j) = 1, \quad \operatorname{ind}_{\gamma_0}(b) = 1, \quad j = 1, \ldots, p. \end{aligned}$$

We define, for any $m \in \mathbb{N}$,

$$g_m(z) = m \frac{(z-b)^{p+1}}{\prod_{j=1}^{p}(z-a_j)}.$$

We then have for $j = 1, \ldots, p$ and $m \in \mathbb{N}$ that

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{g_m'(z)}{g_m(z)} dz = 1,$$

and

$$\frac{1}{2\pi i} \int_{\gamma_0} \frac{g_m'(z)}{g_m(z)} dz = (p+1) - p = 1.$$

Let f be a universal function for (C_{ϕ_n}). Then, given any $m \in \mathbb{N}$, we can find subsequences $(n_k^{(m)})$ such that $f \circ \phi_{n_k^{(m)}} \to g_m$ and $(f \circ \phi_{n_k^{(m)}})' \to g_m'$. Thus $\frac{(f\circ\phi_{n_k^{(m)}})'}{f\circ\phi_{n_k^{(m)}}}$ converges to $\frac{g_m'}{g_m}$ locally uniformly on $\Omega \setminus \{b\}$. Since $\min_{z\in K} |g_m(z)| \to \infty$ as $m \to \infty$ (note that $b \notin K$), we conclude that there is a sequence (n_m) such that

$$\begin{aligned} &f \circ \phi_{n_m} - g_m \to 0 \quad \text{in } H(\Omega),\\ &\frac{(f \circ \phi_{n_m})'}{f \circ \phi_{n_m}} - \frac{g_m'}{g_m} \to 0 \quad \text{in } H(\Omega \setminus \{b\}),\\ &\min_{z\in K} |f(\phi_{n_m}(z))| > \max_{z\in K} |f(z)|. \end{aligned}$$

Now let $N \in \mathbb{N}$. Then there is some $n \geq N$ such that ϕ_n is injective on a neighbourhood of L and such that, for $j = 0, 1, \dots, p$,

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\phi_n(\gamma_j)} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_j} \frac{(f\circ\phi_n)'(z)}{(f\circ\phi_n)(z)}dz = 1, \qquad (3.3)\\ &\min_{z\in K} |f(\phi_n(z))| > \max_{z\in K} |f(z)|. \end{aligned}$$

The last equation obviously implies that

$$\phi_n(K) \cap K = \emptyset.$$

We now claim that $\phi_n(L)$ is Ω-convex. By Lemma 3.11 this implies that $\phi_n(K)$ is Ω-convex, which will finish the proof.

Since ϕ_n is injective on a neighbourhood of L, $\phi_n(L)$ is a compact set with exactly p holes. We assume that one of these holes, call it O, does not contain a point from $\mathbb{C} \setminus \Omega$. Since injective holomorphic functions map boundaries to boundaries, there is some $l \in \{0, 1, \dots, p\}$ such that the Jordan curve $\phi_n(\gamma_l)$ is the boundary of O. Moreover, since O contains no point from $\mathbb{C} \setminus \Omega$, we have that

$$\operatorname{ind}_{\phi_n(\gamma_l)}(\zeta) = 0 \quad \text{for } \zeta \notin \Omega.$$

Now, the compact set L is to the left of each curve $\gamma_j, j = 0, 1, \dots, p$. Since injective holomorphic mappings preserve orientation, $\phi_n(L)$ must also be to the left of the image curve $\phi_n(\gamma_l)$. This implies that $\phi_n(\gamma_l)$ is oriented negatively. Since f is holomorphic in a neighbourhood of $\overline{O}$, the integral

$$-\frac{1}{2\pi i}\int_{\phi_n(\gamma_l)} \frac{f'(z)}{f(z)}dz$$

equals the number of zeros of f in O; but, by (3.3), that integral has the value -1, a contradiction. Thus we can conclude that $\phi_n(L)$ is Ω-convex. □

For our application in Section 3.5 we need to strengthen the necessary condition expressed in Theorem 3.12 along the lines of Bernal and Montes' Lemmas 2.11 and 2.12 in [7]. Their argument should also work in our context but the proof given here seems geometrically simpler and does not require infinite connectivity.

Lemma 3.13. *Let $\Omega \subseteq \mathbb{C}$ be a domain, and let (ϕ_n) be an eventually injective sequence of holomorphic self-maps of Ω.*

Suppose that, for every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$, there is some $n \geq N$ such that $\phi_n(K)$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$.

Then, for every connected Ω-convex compact subset K of Ω with at least two holes and every $N \in \mathbb{N}$, there is some $n \geq N$ such that $\phi_n(K) \cup K$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$.

Proof. Let K be a connected Ω-convex compact subset of Ω with at least two holes, and let $N \in \mathbb{N}$. We fix an exhaustion (K_l) of Ω of connected Ω-convex compact sets, all containing K; see the proof of Theorem 3.12. Then, by assumption, there is a subsequence (n_l) such that, for all $l \in \mathbb{N}$, $n_l \geq N$, $\phi_{n_l}|_{K_{l+1}}$ is injective, $\phi_{n_l}(K_l)$ is Ω-convex and $\phi_{n_l}(K_l) \cap K_l = \emptyset$. Hence $\phi_{n_l}(K) \cap K = \emptyset$ and, by Lemma 3.11, $\phi_{n_l}(K)$ is Ω-convex, too. We claim that, for some $l \in \mathbb{N}$, $\phi_{n_l}(K) \cup K$ is Ω-convex.

We distinguish three cases. First, if, for some $l \in \mathbb{N}$, $\phi_{n_l}(K)$ lies in the unbounded component of $\mathbb{C} \setminus K$ and K lies in the unbounded component of $\mathbb{C} \setminus \phi_{n_l}(K)$ then it follows immediately that $\phi_{n_l}(K) \cup K$ is Ω-convex.

Secondly, infinitely many of the $\phi_{n_l}(K)$ might lie in holes of K. Since, by Lemma 3.10, K has only finitely many holes, infinitely many $\phi_{n_l}(K)$ must lie in some fixed hole O of K. By passing to a subsequence we may assume that all of them do. We choose some $l \in \mathbb{N}$ such that $\phi_{n_1}(K) \subseteq K_l$. Since $\phi_{n_l}(K_l) \cap K_l = \emptyset$, we have that $\phi_{n_1}(K)$ and $\phi_{n_l}(K)$ are disjoint subsets of O. Now there are three possibilities: If both of these sets lie in the unbounded component of the complement of the other then $\phi_{n_l}(K) \cup K$ is Ω-convex (as is $\phi_{n_1}(K) \cup K$); if $\phi_{n_1}(K)$ lies in a hole of $\phi_{n_l}(K)$ then $\phi_{n_1}(K) \cup K$ is Ω-convex because $\phi_{n_l}(K)$ has at least two holes; or if $\phi_{n_l}(K)$ lies in a hole of $\phi_{n_1}(K)$ then $\phi_{n_l}(K) \cup K$ is Ω-convex because $\phi_{n_1}(K)$ has at least two holes.

Finally, for infinitely many $l \in \mathbb{N}$, K might lie in holes of $\phi_{n_l}(K)$. Again we can assume that this is true for all l. We then choose some $l \in \mathbb{N}$ such that $\phi_{n_1}(K) \subseteq K_l$. Since $\phi_{n_l}(K_l) \cap K_l = \emptyset$ we have that $\phi_{n_1}(K)$ and $\phi_{n_l}(K)$ are disjoint sets. Since both these sets contain K in one of their holes, we must have that either $\phi_{n_1}(K)$ lies in a hole of $\phi_{n_l}(K)$ or $\phi_{n_l}(K)$ lies in a hole of $\phi_{n_1}(K)$. We then argue as above that either $\phi_{n_l}(K) \cup K$ or $\phi_{n_1}(K) \cup K$ is Ω-convex. □

Example 3.14. The lemma is not true for sets K with exactly one hole. Indeed, let $\Omega = \mathbb{C}^*$ and $\phi_n(z) = nz$. Then, for $K = \partial\mathbb{D}$ we have that $\phi_n(K) \cup K$ is not Ω-convex for $n \geq 2$ even though the assumptions of the lemma are satisfied.

3.4. **Finitely connected domains.** We have seen in Section 3.2 that some finitely connected domains can support universal functions even if they are not simply connected. We shall now show that this can only happen if the sequence (ϕ_n) is not eventually injective.

Theorem 3.15. *Let $\Omega \subseteq \mathbb{C}$ be a finitely connected domain that is not simply connected. Then (C_{ϕ_n}) is not universal for any eventually injective sequence of holomorphic self-maps (ϕ_n) of Ω.*

Proof. We first assume that Ω has exactly $p \geq 2$ holes. Suppose that (C_{ϕ_n}) is universal. We then consider a connected Ω-convex compact subset K of Ω that has p holes. By Theorem 3.12 there is some $n \in \mathbb{N}$ such that $\phi_n(K)$ is Ω-convex, ϕ_n is injective on a neighbourhood of K and $\phi_n(K) \cap K = \emptyset$. But since K and hence also $\phi_n(K)$ is connected, $\phi_n(K)$ must lie inside a component O of $\widehat{\mathbb{C}} \setminus K$.

Due to the invariance of the number of holes under injective holomorphic mappings, $\phi_n(K)$ has p holes. Since $\phi_n(K)$ is Ω-convex, each of these holes contains a hole of Ω; thus O contains p holes of Ω. Hence, Ω would contain at least $(p-1)+p = 2p-1$ holes of Ω; a number that is strictly bigger than p if $p \geq 2$. This contradicts the assumption on the number of holes of Ω.

In the case where Ω has exactly one hole we can assume, via conformal equivalence, that Ω is either $\mathbb{C}^*$ or $\mathbb{D}^*$ or an annulus, see [9, Theorem 10.2]. We shall treat the case of the punctured plane, the other two being very similar, and we proceed by a variant of the proof of Theorem 3.12.

Consider the compact annulus $K = \{z \in \mathbb{C} : 1/2 \le |z| \le 2\}$. Its boundary is given by the circle γ_0 of radius 2 around 0, with positive orientation, and the circle γ_1 of radius 1/2 around 0, with negative orientation. In addition we consider the functions

$$g_m(z) = m\frac{(z+1)^2}{z}, \quad m \in \mathbb{N}.$$

We then have that

$$\frac{1}{2\pi i}\int_{\gamma_0} \frac{g'_m(z)}{g_m(z)}dz = 1,$$
$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{g'_m(z)}{g_m(z)}dz = 1.$$

Suppose now that (C_{ϕ_n}) is universal, and let f be a corresponding universal function. As in the proof of Theorem 3.12 we obtain some $n \in \mathbb{N}$ such that ϕ_n is injective on a neighbourhood of K and

$$\frac{1}{2\pi i}\int_{\phi_n(\gamma_0)} \frac{f'(z)}{f(z)}dz = 1, \tag{3.4}$$

$$\frac{1}{2\pi i}\int_{\phi_n(\gamma_1)} \frac{f'(z)}{f(z)}dz = 1, \tag{3.5}$$

$$\min_{z\in\partial K\cup[\frac{1}{2},2]} |f(\phi_n(z))| > \sup_{z\in K} |f(z)|. \tag{3.6}$$

It then follows from (3.6) that

$$\phi_n(\partial K \cup [\tfrac{1}{2}, 2]) \cap K = \emptyset. \tag{3.7}$$

Due to injectivity, $\phi_n(K)$ has exactly one hole. We denote by Γ_0 and Γ_1 the outer and inner boundary of $\phi_n(K)$, respectively. As in the proof of Theorem 3.12 it follows that Γ_0 coincides either with $\phi_n(\gamma_0)$ or $\phi_n(\gamma_1)$, and its orientation is positive, while Γ_1 coincides with the remaining curve, and its orientation is negative. Hence, by (3.4) and (3.5),

$$\frac{1}{2\pi i}\int_{\Gamma_0} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f'(z)}{f(z)}dz = 1. \tag{3.8}$$

This also implies that $\phi_n(K)$ is Ω-convex. For, otherwise, f would be holomorphic in the hole of $\phi_n(K)$, so that $-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f'(z)}{f(z)}dz = -1$ would be the number of zeros of f in that hole, which is impossible.

Thus, using (3.7), it follows that the connected set $\phi_n(K)$ is contained either in $\{|z| < 1/2\}$ or in $\{|z| > 2\}$ and that the origin lies in the hole of $\phi_n(K)$ (see Figure 2).

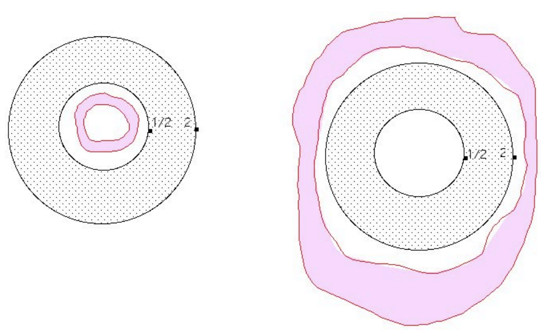

FIGURE 2. The set $\phi_n(K)$

We next choose $r > 0$ and $R > 0$ such that $K' = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ contains $K \cup \phi_n(K)$. Arguing as above we can find some $n' \in \mathbb{N}$ such that the set $\phi_{n'}(K')$ is bounded by a positively oriented outer curve Γ_0' and a negatively oriented inner curve Γ_1', that the origin lies in the hole of $\phi_{n'}(K')$, and that $\phi_{n'}(K')$ is contained either in $\{|z| < r\}$ or in $\{|z| > R\}$. Moreover,

$$\frac{1}{2\pi i}\int_{\Gamma_0'} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\Gamma_1'} \frac{f'(z)}{f(z)}dz = 1. \tag{3.9}$$

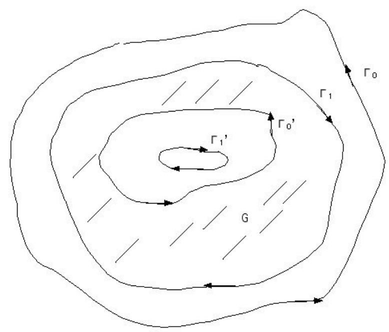

FIGURE 3. The four curves

In this way we have found four Jordan curves, $\Gamma_0, \Gamma_1, \Gamma_0', \Gamma_1'$, each of which surrounds 0. Also, either both Γ_0' and Γ_1' lie in the interior of both Γ_0 and Γ_1, or vice versa (see Figure 3). We assume that the first case holds, the second case being treated similarly.

Consider the domain G surrounded by the curves Γ_1 and Γ_0', and let Γ be the cycle $\Gamma_1 + \Gamma_0'$. Then the outer boundary Γ_1 of G is oriented negatively, the inner boundary Γ_0' of G positively. Since f is holomorphic in G,

$$\lambda := \frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)}dz$$

is the negative of the number of zeros of f in G. But it follows from (3.8) and (3.9) that $\lambda = 2$, a contradiction. Hence (C_{ϕ_n}) cannot be universal. □

3.5. **Infinitely connected domains.** We finally turn to domains of infinite connectivity. Theorem 3.12 and Lemma 3.13 lead us to the desired characterization of universality for sequences of composition operators (C_{ϕ_n}), provided that (ϕ_n) is eventually injective.

Theorem 3.16. *Let $\Omega \subseteq \mathbb{C}$ be a domain of infinite connectivity, and let (ϕ_n) be an eventually injective sequence of holomorphic self-maps of Ω. Then the following assertions are equivalent:*

(a) *The sequence (C_{ϕ_n}) is universal;*
(b) *For every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$ there is some $n \geq N$ such that $\phi_n(K)$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$.*

Proof. The necessity was shown in Theorem 3.12. We now show sufficiency.

Assume that (b) holds. It suffices to show that the sequence (C_{ϕ_n}) is topologically transitive. To this end, let K be a compact subset of Ω, $\varepsilon > 0$, and $f, g \in H(\Omega)$. By making K larger, if necessary, we can assume that it is connected, Ω-convex and has at least two holes. By Lemma 3.13 and the hypothesis there is some $n \in \mathbb{N}$ such that ϕ_n is injective on a neighbourhood of K, $\phi_n(K) \cup K$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$. It follows that the function $g \circ \phi_n^{-1}$ is holomorphic on $\phi_n(K)$, as is f on K. By Runge's theorem there exists a function $h \in H(\Omega)$ such that

$$\begin{aligned} |h(z) - f(z)| &< \varepsilon \quad \text{for } z \in K, \\ |h(w) - g(\phi_n^{-1}(w))| &< \varepsilon \quad \text{for } w \in \phi_n(K), \end{aligned}$$

hence also

$$|h(\phi_n(z)) - g(z)| < \varepsilon \quad \text{for } z \in K.$$

This implies that (C_{ϕ_n}) is topologically transitive. □

We consider an example along the lines of [15, Example 1].

Example 3.17. Let ψ be the self-mapping of $\mathbb{D}$ given by

$$\psi(z) = \frac{z}{4} + \frac{3}{4}.$$

Then $\psi(\mathbb{D})$ is the disk of radius 1/4 around 3/4. Let $\psi^{[n]}$ denote the nth iterate of ψ. Then the sets

$$K_n = \psi^{[n]}(\{z : |z| \le \tfrac{1}{2}\}), \quad n \in \mathbb{N}_0,$$

are pairwise disjoint subsets of $\mathbb{D}$ that accumulate at 1 and the restriction $\phi = \psi|_\Omega$ of ψ to the domain

$$\Omega = \mathbb{D} \setminus \bigcup_{n \ge 0} K_n$$

is a self-map. Clearly, the $\phi^{[n]}$ are injective non-automorphisms on Ω that satisfy condition (b) of Theorem 3.16. Hence $(C_{\phi^{[n]}})$ is universal. In other words, the operator C_ϕ is hypercyclic; see Definition 3.20 below.

3.6. **Heredity.** When we reconsider Proposition 3.6 in the light of Theorem 3.15 we arrive at an interesting observation. If Ω is a domain for which $H^\infty(\Omega)$ is dense in $H(\Omega)$ then a universal sequence (C_{ϕ_n}) exists. By Remark 3.3 the sequence (ϕ_n) must have an eventually injective subsequence (ϕ_{n_j}). But if Ω is, in addition, finitely connected and not simply connected then, by Theorem 3.15, the subsequence $(C_{\phi_{n_j}})$ cannot be universal. Now, in the theory of universality a sequence of operators is called *hereditarily universal* (or *hereditarily hypercyclic*) if each of its subsequences is universal, see [6].

Corollary 3.18. *Let $\Omega \subseteq \mathbb{C}$ be a finitely connected, not simply connected domain for which $H^\infty(\Omega)$ is dense in $H(\Omega)$. Then no universal sequence (C_{ϕ_n}) of composition operators on Ω is hereditarily universal.*

This applies, for example, to annuli. In such a domain, then, not even a subsequence of (C_{ϕ_n}) can be hereditarily universal. This implies that (C_{ϕ_n}) is another example of a universal sequence that does not satisfy the so-called Hypercyclicity Criterion; see [6, Theorem 2.2].

What we have observed here is rather pathological. Let us note that arbitrary universal sequences (C_{ϕ_n}) on simply connected domains and universal sequences (C_{ϕ_n}) with eventually injective symbols on infinitely connected domains do have hereditarily universal subsequences. The first statement is obvious from Theorem 3.2, while the second follows from Theorem 3.16 by looking at an exhaustion of Ω by Ω-convex compact subsets.

3.7. **Injective self-maps, and hypercyclicity.** In this final section we formulate our results in two important special cases.

In their pioneering work, Bernal and Montes [7] have treated the case when the maps ϕ_n are automorphisms. The notion of a run-away sequence, see Definition 2.2 above, turned out to be crucial there: If Ω is not conformally equivalent to $\mathbb{C}^*$ then a sequence (C_{ϕ_n}) of composition operators with automorphic symbols is universal if and only if (ϕ_n) is run-away, see [7, Theorem 3.6]. They also note that in the finitely connected but not simply connected case no such sequence of composition operators can be universal.

The following special case of Theorems 3.2, 3.15 and 3.16 extends their results to injective sequences (ϕ_n).

Theorem 3.19. *Let (ϕ_n) be a sequence of injective holomorphic self-maps of a domain $\Omega \subseteq \mathbb{C}$.*

(a) *If Ω is simply connected then (C_{ϕ_n}) is universal if and only if (ϕ_n) is run-away.*

(b) *If Ω is finitely connected but not simply connected then (C_{ϕ_n}) is never universal.*

(c) *If Ω is infinitely connected then (C_{ϕ_n}) is universal if and only if, for every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$ there is some $n \geq N$ such that $\phi_n(K)$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$.*

Our second special case concerns iterates $\phi^{[n]}$ of a single self-map ϕ. In this case we also speak of hypercyclicity; see [19].

Definition 3.20. Let ϕ be a holomorphic self-map of a domain $\Omega \subseteq \mathbb{C}$. Then a function $f \in H(\Omega)$ is called *hypercyclic* for C_ϕ if the set $\{f \circ \phi^{[n]} : n \in \mathbb{N}\}$ is dense in $H(\Omega)$. The operator C_ϕ is called *hypercyclic* if it admits a hypercyclic function.

Several authors have recently observed that C_ϕ can only be hypercyclic if ϕ is injective; see [5, Corollary 3.2], [28, Proposition 2.1] and [15]. Thus we obtain a characterization of all hypercyclic composition operators C_ϕ. With this complete solution of a natural problem we conclude our paper.

Theorem 3.21. *Let ϕ be a holomorphic self-map of a domain $\Omega \subseteq \mathbb{C}$.*

(a) *If Ω is simply connected then C_ϕ is hypercyclic if and only if ϕ is injective and $(\phi^{[n]})$ is run-away.*

(b) *If Ω is finitely connected but not simply connected then C_ϕ is never hypercyclic.*

(c) *If Ω is infinitely connected then C_ϕ is hypercyclic if and only if ϕ is injective and, for every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$ there is some $n \geq N$ such that $\phi^{[n]}(K)$ is Ω-convex and $\phi^{[n]}(K) \cap K = \emptyset$.*

Acknowledgements

We thank Paul M. Gauthier for having communicated to us his result with Melnikov on the denseness of $H^\infty(\Omega)$ in $H(\Omega)$ (see Remark 3.7).

The research for this paper was undertaken while the first author visited the Département de Mathématiques of the Université Paul Verlaine, Metz, in February/March 2007. He wishes to thank the department for its kind hospitality.

References

[1] Y. Abe, Universal holomorphic functions in several variables, *Analysis* 17 (1997), 71–77.

[2] R. Aron and P. Gorkin, An infinite dimensional vector space of universal functions for H^∞ of the ball, *Canad. Math. Bull.* 50 (2007), 172–181.

[3] F. Bayart and P. Gorkin, How to get universal inner functions, Math. Ann. 337 (2007), 875–886.

[4] F. Bayart, P. Gorkin, S. Grivaux, and R. Mortini, Bounded universal functions for sequences of holomorphic self-maps of the disk, preprint 2006.

[5] L. Bernal-González, A. Bonilla and M. C. Calderón-Moreno, Compositional hypercyclicity equals supercyclicity, *Houston J. Math.* 33 (2007), 581–591.
[6] L. Bernal-González and K.-G. Grosse-Erdmann, The hypercyclicity criterion for sequences of operators, *Studia Math.* 157 (2003), 17–32.
[7] L. Bernal-González and A. Montes-Rodríguez, Universal functions for composition operators, *Complex Variables Theory Appl.* 27 (1995), 47–56.
[8] G. D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* 189 (1929), 473–475.
[9] R. B. Burckel, *An introduction to classical complex analysis, Vol. 1,* Academic Press, New York, 1979.
[10] P. S. Chee, Universal functions in several complex variables, *J. Austral. Math. Soc. Ser. A* 28 (1979), 189–196.
[11] P. M. Gauthier and M. S. Melnikov, $H^\infty(\Omega)$ and $A(\Omega)$ approximation, preprint 2007.
[12] P. M. Gauthier and J. Xiao, The existence of universal inner functions on the unit ball of $\mathbb{C}^n$, *Canad. Math. Bull.* 48 (2005), 409–413.
[13] T. L. Gharibyan, W. Luh, and J. Müller, Approximation by lacunary polynomials and applications to universal functions, *Analysis (Munich)* 23 (2003), 199–214.
[14] G. Godefroy and J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, *J. Funct. Anal.* 98 (1991), 229–269.
[15] P. Gorkin, F. León-Saavedra, and R. Mortini, Bounded universal functions in one and several complex variables, *Math. Z*, to appear.
[16] P. Gorkin and R. Mortini, Universal Blaschke products, *Math. Proc. Cambridge Philos. Soc.* 136 (2004), 175–184.
[17] P. Gorkin and R. Mortini, Universal singular inner functions, *Canad. Math. Bull.* 47 (2004), 17–21.
[18] P. Gorkin and R. Mortini, Asymptotic interpolating sequences in uniform algebras, *J. London Math. Soc. (2)* 67 (2003), 481 – 498.
[19] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc. (N.S.)* 36 (1999), 345–381.
[20] M. Heins, A universal Blaschke product, *Arch. Math.* 6 (1955), 41–44.
[21] F. León-Saavedra, Universal functions on the unit ball and the polydisk, *Function spaces* (Proc. Conf., Edwardsville, IL, 1998), 233–238, Amer. Math. Soc., Providence, RI, 1999.
[22] W. Luh, Universal functions and conformal mappings, *Serdica* 19 (1993), 161–166.
[23] A. Montes-Rodríguez, A note on Birkhoff open sets, *Complex Variables Theory Appl.* 30 (1996), 193–198.
[24] R. Mortini, Infinite dimensional universal subspaces generated by Blaschke products, *Proc. Amer. Math. Soc.* 135 (2007) 1795–1801.
[25] R. Narasimhan, *Complex analysis in one variable*, Birkhäuser, Boston, MA, 1985.
[26] R. Remmert, *Funktionentheorie II*, Springer-Verlag, Berlin, New York, 1991.
[27] W. Rudin, *Real and complex analysis*, third edition, McGraw-Hill, New York, 1987.
[28] B. Yousefi, H. Rezaei, Hypercyclic property of weighted composition operators, *Proc. Amer. Math. Soc.*, to appear.
[29] P. Zappa, On universal holomorphic functions, *Boll. Un. Mat. Ital. A* (7) 2 (1988), 345–352.

Karl-Goswin Grosse-Erdmann, Fakultät für Mathematik und Informatik, Fernuniversität Hagen, D-58084 Hagen, Germany
E-mail address: `kg.grosse-erdmann@FernUni-Hagen.de`

Raymond Mortini, Département de Mathématiques, Université Paul Verlaine Metz, Ile du Saulcy, F-57045 Metz, France
E-mail address: `mortini@poncelet.univ-metz.fr`