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Stochastic Galerkin Method for Optimal Control Problem Governed by Random Elliptic PDE with State Constraints

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Abstract In this paper, we investigate a stochastic Galerkin approximation scheme for an optimal control problem governed by an elliptic PDE with random field in its coefficients. The optimal control minimizes the expectation of a cost functional with mean-state constraints. We firstly represent the stochastic elliptic PDE in term of the generalized polynomial chaos expansion and obtain the parameterized optimal control problems. By applying the Slater condition in the subdifferential calculus, we obtain the necessary and sufficient optimality conditions for the state-constrained stochastic optimal control problem in the first time in the literature. We then establish a stochastic Galerkin scheme to approximate the optimality system in the spatial space and the probability space. Then the a priori error estimates are derived for the state, the co-state and the control variables. A projection algorithm is proposed and analyzed. Numerical examples are presented to illustrate our theoretical results.

Keywords Stochastic optimal control · Stochastic Galerkin method · Optimal control problem with state constraints

1 Introduction

Numerical methods for optimal control problems governed by partial different equations have been a major research topic in applied mathematics and control theory. Since the milestone work of J.P Lions [31], a great deal of progress has been made in their numerical methods, which are too extensive to be mentioned here even very briefly. Galerkin approximation (in particular, finite element and spectral approximation) of optimal control

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problems plays a very important role in numerical methods for these problems, and has been much studied in the literature. There have been extensive studies on this aspect for optimal control governed by such as elliptic equations, parabolic equations, Stokes equations, and Navier-Stokes equations. They are again too extensive to be reviewed here even very briefly. Some of recent progress in this area has been summarized in [23,28,30–33, 36,40,47,49], and the references cited therein.

In many complex physical and engineering models, there is always a great amount of uncertainty involved, in such as the parameters, coefficients, forcing term, and boundary conditions. It is well known that these models are often represented by stochastic partial differential equations (SPDEs). In recent years, PDEs with random fields have been a subject of growing interest in the scientific community (see e.g. [2,43] for random elliptic PDEs). The reason is that such PDEs are conveniently used for modeling in many engineering areas, e.g., fluid flows in porous media, transport of pollutants in groundwater and oil recovery processes.

The Monte Carlo (MC) method is one of the most commonly used methods for simulating stochastic elliptic PDEs and dealing with the statistic characteristics of the solution (see e.g. [14,39]). Although MC method is straightforward to apply as it only requires repetitive executions of deterministic simulations, it is a rather computationally expensive method. Typically a large number of such executions are needed as the solution statistics converge relatively slowly, as tested and reported in [2].

Other alternatives to Monte Carlo method have been employed in the field of stochastic mechanics. A popular technique is the perturbation method, cf. [27]. Given certain smoothness conditions, the random functions and operators involved in the differential equation are expanded in a Taylor series about their respective mean values. Another approach is the Neumann expansion series method, e.g., [2], in which the inverse of the boundary value problem's stochastic operator is approximated by its Neumann series.

In recent years, stochastic Galerkin method for such random PDEs has received substantial attention, see [2], and has been applied to various stochastic problems, e.g.[3,43,52], with new extensions outlined e.g., in [17,18, 22,37]. These Galerkin methods can transform the random governing equations into a set of deterministic equations which can be readily discretized via standard numerical techniques. As reported in [2], it out-performances the Monte Carlo method for more accurate numerical simulations. Furthermore it paves the way for other methods like stochastic Collocation methods.

Naturally stochastic Galerkin method has recently been used to solve stochastic optimal control governed by random PDEs, see the very recent work of [15,19,21,29,45,46,48]. The work [21] dealt with optimal control governed by a random steady diffusion problem with deterministic Neumann boundary control, and obtained a priori error estimates for the stochastic finite element approximation. The existence of a local optimal solution was also demonstrated. In [42], numerical experiments were conducted with 'pure' stochastic control function as well as 'semi' stochastic control function for an optimal control problem constrained by stochastic steady diffusion problem. In [25] and [29], optimal control problems constrained by random elliptic PDEs with deterministic distributed control were introduced. The authors proved the existence of the optimal solution, establish the validity of the Lagrange multiplier rule and obtain stochastic optimality system. Then, they used the Wiener-Itô (W-I) chaos or the Karhunen-Loève (K-L) expansion as a main tool to convert stochastic optimality system to deterministic optimality system. Finally, the a priori error estimates for Galerkin approximation of the optimality system in both physical space and stochastic space were provided. In more recent work [15], the authors presented an effective gradient projection method for stochastic optimal control. The a priori error estimate of stochastic Galerkin method for optimal control problem governed by random parabolic PDE was obtained in [19]. Sun etc. in [46] presented the a priori error estimate of stochastic Galerkin method for optimal control problem governed by stochastic elliptic PDE with constrained control. Stochastic Galerkin method was used to analyze the constrained optimal control problem governed by an elliptic integro-differential PDE with stochastic coefficients in [45]. In [48], the authors discussed the use of stochastic collocation for the solution of optimal control problems, which are constrained by SPDE, and applied the method to develop a gradient descent algorithm as well as a sequential quadratic program (SQP) for the minimization of objective functions constrained by an SPDE.

Nevertheless, the development of stochastic optimal control problem constrained by stochastic PDEs can still be considered to be in its infancy, and there are many important open topics to be studied.

State-constrained optimal control is an important but difficult model in many applications and there has already existed much research on the numerical approximation of deterministic state constrained optimal control problem governed by PDEs in the literature. Casas in [8], firstly derived some optimality conditions and carried out important theoretical analysis for the model. For the standard finite element approximation of the control problem, the a priori error estimates were derived by Deckelnick and Hinze in [13], where non-classic techniques were developed to handle the delta-singularity of the co-stated equation. An augmented Lagrangian method was proposed to solve state and control constrained optimal control problems by Bergounioux and Kunisch in [5]. They also proposed another method: a primal-dual strategy to solve problem in [5]. Casas proved convergence of finite element approximations to optimal control problems for semi-linear elliptic equations with finitely many state constraints in [9]. Casas and Mateos extended these results in [10] to a less regular setting for the states, and proved convergence of finite element approximations to semi-linear distributed and boundary control problems. In [24], the state-constrained control problem was approximated by a sequence of control-constrained control problems, and then the interior point method was applied to approximating the solutions. In recent years, Liu and Yang have developed L^2 error estimates for integral constraint of optimal control problem in [34], and Chen did Galerkin spectral approximation of H^1 norm constraint of the state for elliptic optimal control problems in [11]. As far as we are aware there has no known research on numerical methods of state-constrained stochastic optimal control governed by random PDEs in the literature, which is an important gap to fill in this field.

In this work, we present a stochastic Galerkin approximation scheme for a state-constrained optimal control problem governed by an elliptic PDE with random field in its coefficients. By applying the Slater condition in the sub-differential calculus, we are able to obtain the necessary and sufficient optimality conditions for the problem in the first time in the literature. We then establish a stochastic Galerkin scheme to approximate the optimality system in the spatial space and the probability space. Then a priori error estimates are derived for the state, the co-state and the control variables. A projection algorithm is proposed and analyzed, and numerical examples are presented to illustrate our theoretical results. As far as we are aware, this is the first systematical study in numerical methods for state-constrained optimal control governed by random PDEs, and some new techniques are needed in analysis of its scheme and approximation errors.

The plan of this paper is as follows: In Section 2, we introduce the model control problem in suitable spaces. In Section 3, we set up its weak formulation by applying a finite dimensional representation assumption, and derive its optimality conditions. In Sections 4-5 we present a stochastic Galerkin scheme for the control problem and derive a priori error estimates for the state and control variables. In Section 6 a projection algorithm is proposed and its convergence is established, and some numerical tests are presented to illustrate our theoretical analysis.

2 Model control problem

2.1 Function spaces and notations

Let $D \subset \mathbb{R}^d$ $(1 \le d \le 3)$ be a convex bounded polygonal spatial domain with its boundary ∂D. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space, where Ω is a set of samples, $\mathscr{F}$ is a σ-algebra of events and $\mathscr{P} : \mathscr{F} \to [0,1]$ is a probability measure. Denote by $B(D)$ the Borel σ-algebra generated by the open subset of D.

Throughout this paper, we use the standard notations (e.g., see [1]) for Sobolev spaces on D. For examples, $L^2(D)$ and $H^1(D)$ are Hilbert spaces with norms $\|\cdot\|_{L^2(D)}$ and $\|\cdot\|_{H^1(D)}$, respectively; $H_0^1(D)$ is the subspace of $H^1(D)$ whose trace on ∂D is zero . With these standard Sobolev spaces, we adopt the definition of stochastic

Sobolev spaces (see e.g. [3,25,29]). For nonnegative integers s and $1 \leq q < +\infty$, let $L^q(\Omega;W^{s,q}(D))$ contain stochastic functions, $v:\ D\times\Omega \to \mathbb{R}$, that are measurable with respect to the product σ-algebra $\mathscr{F}\otimes B(D)$ and equipped with the averaged norms

$$\|v\|_{L^q(\Omega;W^{s,q}(D))} = \Big(E[\|v\|^q_{W^{s,q}(D)}]\Big)^{1/q} = \Big(E[\sum_{|\alpha|\leq s}\int_D |\partial^\alpha v|^q dx]\Big)^{1/q},$$

and

$$\|v\|_{L^\infty\Omega;W^{s,\infty}(D))} = \max_{|\alpha|\leq s}\big(\operatorname{ess\,sup}_{D\times\Omega}|\partial^\alpha v|\big),$$

where E is the expected value, $\partial^\alpha v$ is partial derivative in weak sense defined in [1]. Observe that if $v \in L^q(\Omega;W^{s,q}(D))$, then $v(\cdot,\omega)\in W^{s,q}(D)$ almost surely (a.s.) and $\partial^\alpha v(x,\cdot)\in L^q(\Omega)$ a.e. on D for $|\alpha|\leq s$.

Especially, when $s=0$, $q=2$, the above space is just

$$L^2(\Omega;L^2(D)) = \{v : D\times\Omega\to\mathbb{R} \mid \|v\|_{L^2(\Omega;L^2(D))} < \infty\},$$

with norm

$$\|v\|^2_{L^2(\Omega;L^2(D))} = E[\|v\|^2_{L^2(D)}].$$

Similarly, we can define spaces $L^2(\Omega;H^1(D))$ and $L^2(\Omega;H^1_0(D))$. Note that these stochastic Sobolev spaces are Hilbert spaces.

2.2 Stochastic optimal control problem governed by stochastic elliptic equation with constrained state

We will consider the following control problem governed by the stochastic elliptic equation with constrained state:

$$\min_{u\in L^2(D),y(u)\in K} J(y(u),u) = \min_{u\in L^2(D),y(u)\in K} E\Big[\frac{1}{2}\int_D |y-y_d|^2 dx + \frac{\alpha}{2}\int_D |u|^2 dx\Big] \tag{2.1}$$

subject to

$$A[y,v] = [u,v],\ \ \forall\, v\in V. \tag{2.2}$$

where J is a cost functional, $y:\bar{D}\times\Omega\to\mathbb{R}$ is the state variable, $y_d:\bar{D}\times\Omega\to\mathbb{R}$ is a given target state, $u: D\to\mathbb{R}$ is a deterministic control, $\bar{D}$ is the closure of D, α is a positive constant measuring the importance between two terms in J, the bilinear forms:

$$A[y,v] = E\Big[\int_D a\nabla y\cdot\nabla v dx\Big],\ \ \forall\, y,\ v\in L^2(\Omega;H^1(D)), \tag{2.3}$$

$$[u,v] = E\Big[\int_D uv dx\Big],\ \ \forall\, u\in U,\ v\in L^2(\Omega;L^2(D)), \tag{2.4}$$

$$(u,w) = \int_D uw dx,\ \ \forall\, u,\ w\in U, \tag{2.5}$$

here $a: D\times\Omega\to\mathbb{R}$ is a stochastic function with continuous and bounded covariance function, the operator ∇ means derivatives with respect to the spatial variable $x\in D$ only. K is a closed convex subset in the state space $L^2(\Omega;L^2(D))$:

$$K = \{y \mid y\in L^2(\Omega;H^1_0(D)\cap H^2(D)),\ E[\int_D y(x,w)dx]\geq 0\}.$$

Although the objective functional J in (2.1) contains stochastic function y subject to (2.2), its outcome is deterministic by using the expectation E. If we denote by $B(D)$ the Borel σ-algebra generated by the open subsets of D, then a is assumed measurable with respect to the σ-algebras $(\mathscr{F} \otimes B(D))$. To ensure regularity of the solution y, we assume that a is a second-order random field, there are positive constants a_{min} and a_{max} such that

$$a_{min} \leq a(x,\omega) \leq a_{max}, \qquad \text{a. e. } (x,\,\omega) \in D \times \Omega. \tag{2.6}$$

Then, with this assumption (2.6), by the theory of optimal control problem [31], the existence of an optimal solution for (2.1)-(2.2) can be proved as in [29]).

In the following, we take the state space $V = L^2(\Omega; H_0^1(D))$ and the control space $U = L^2(D)$. In addition, C will denote general constants.

Since the coefficient a is not known exactly, we should consider a perturbation of the weak formulation (2.2) and the size of the corresponding perturbation in the solution. Here, we can derive the following result similar to Corollary 2.1 in [3].

Lemma 2.1 *Let $1 < p < +\infty$ with $1/p + 1/q = 1$. Consider the Hilbert space $L^2(\Omega; H_0^1(D))$, perturbed coefficient $\hat{a}$ satisfying $0 < a_{min} \leq \hat{a} \leq a_{max} < \infty$, a.e. on $D \times \Omega$. Let $\hat{y}$ solve*

$$E\Big[\int_D \hat{a}\nabla\hat{y}\cdot\nabla v dx\Big] = E\Big[\int_D uvdx\Big],\ \ \forall\, v \in V. \tag{2.7}$$

Besides this, assume that the solution y belongs to space $L^{2q}(\Omega; W^{1,2q}(D))$. Then

$$\|y-\hat{y}\|_{L^2(\Omega;H_0^1(D))} \leq \frac{C}{a_{min}}\|a-\hat{a}\|_{L^{2p}(\Omega;L^{2p}(D))}\|y\|_{L^{2q}(\Omega;W^{1,2q}(D))}, \tag{2.8}$$

where $C > 0$ is the Poincaré constant for the domain D.

3 Finite dimensional representation of stochastic fields

3.1 Notations for finite expansion

In most of the existing models, the source of randomness is assumed to be expressed by a finite number of random variables that are mutually independent. Also whenever we apply numerical methods such as the finite element method to solve a problem, we always assume that we have finite expansions of input data. For those reasons, following the theory of Babuska [3], Wiener [50], as well as Xiu and Karniadakis [51], we can employ the following finite-dimensional noise assumption

Assumption 3.1 (finite dimensional noise) *Any general second-order random process $X(\omega)$, $\omega \in \Omega$ can be represented in terms of a prescribed finite number of random variables $\xi = \xi(\omega) = (\xi_1(\omega), \cdots, \xi_N(\omega))$ with independent components $\xi_i(\omega)$, $i = 1, \cdots, N \in \mathbb{N}$. Let $\Gamma_i = \xi_i(\Omega) \in \mathbb{R}$ be a bounded interval for $i = 1, \cdots, N$ and $\rho_i : \Gamma_i \to [0,1]$ be the probability density functions of the random variables $\xi_i(\omega)$, $\omega \in \Omega$. Then we can use the joint probability density function $\rho(\xi) = \prod_{i=1}^{N} \rho_i(\xi_i)$ for random vector ξ with the support $\Gamma = \prod_{i=1}^{N} \Gamma_i \subset \mathbb{R}^N$. On Γ, we have the probability measure $\rho(\xi)d\xi$.*

The preceding assumptions enable a parametrization of the problem in ξ in place of the random events ω. As an example, we can use a finite-term expansion of the stochastic coefficient a based on N random variables (cf. [42]) :

$$a(x,\xi) = \sum_{i=1}^{S} \alpha_i(x) L_i(\xi),\ \ x \in D,\ \xi \in \Gamma, \tag{3.1}$$

where $\alpha_i(x) : D \to \mathbb{R}$ and $L_i : \Gamma \to \mathbb{R}$. As discussed in [3,51], assumption (3.1) is valid in its own right in practical applications: if a is represented by a truncated Karhunen-Loève (KL) expansion [3], then $S = N+1$ with $L_i = \xi_{i-1}$ and $\xi_0 = 1$; if a generalized polynomial chaos expansion [51] is used, L_i is an $N-$variate polynomial of order up to p and $S = (N+p)!/(N!p!)$.

As commented in [51], the above finite-term expansion allows us to conduct numerical formulations in the finite dimensional (N-dimensional) random space Γ. Let us denote $L^2_\rho(\Gamma)$ as the probabilistic Hilbert space [35], in which the random processes based upon the random variables ξ reside. The inner product of this Hilbert space is given by

$$(X,Y)_{L^2_\rho(\Gamma)} = \int_\Gamma X(\xi)Y(\xi)\rho(\xi)d\xi, \ \forall X,\ Y \in L^2_\rho(\Gamma),$$

where we have exploited independence of the random variables to allow us to write the measure as product of measures in each stochastic direction. We similarly define the expectation of a random process $X \in L^2_\rho(\Gamma)$ as

$$E[X(\xi)] = \int_\Gamma X(\xi)\rho(\xi)d\xi,$$

and we refer to the expectation of the powers $E[X^i(\xi)]$ as the i^{th} moment of the random process.

Additionally, we define the mapping $f : (x,\xi) \in D\times\Gamma \to \mathbb{R}$ to be a set of random processes, which are indexed by the spatial position $x \in D$. Such a set of processes is referred to as a random field [26] and can also be interpreted as a function-valued random variable, because for every $\xi \in \Gamma$ the realization $f(\cdot,\xi) : D \to \mathbb{R}$ is a real valued function on D.

For a vector-space W on D, let the class $L^2_\rho(\Gamma;W)$ denote the space of random fields whose realizations lie in W for a.e (almost every) $\xi \in \Gamma$. If W is a Banach space, a norm on $L^2_\rho(\Gamma;W)$ is induced by

$$||f(x,\xi)||^2_{L^2_\rho(\Gamma;W)} = E[\|f(x,\xi)\|^2_W],$$

for example, on $L^2_\rho(\Gamma;L^2(D))$ we have

$$||f(x,\xi)||^2_{L^2_\rho(\Gamma;L^2(D))} = E[\|f(x,\xi)\|^2_{L^2(D)}] = \int_\Gamma\int_D (f(x,\xi))^2\rho(\xi)dxd\xi,$$

which denotes the expected value of the $L^2(D)$-norm of the function $f(x,\xi)$. Similarly, we have the norm

$$||f(x,\xi)||^2_{L^2_\rho(\Gamma;H^1(D))} = E[\|f(x,\xi)\|^2_{H^1(D)}] = \int_\Gamma\int_D \{(f(x,\xi))^2 + |\nabla f(x,\xi)|^2\}\rho(\xi)dxd\xi.$$

We now give a Banach space that will be used as the solution space for the stochastic optimality system of equations, cf. [16]. Here, a Banach space $C_\rho(\Gamma;H)$ comprises all continuous functions $f : \Gamma \to H$ with a norm $\|f\|_{C_\rho(\Gamma;H)} \equiv \sup_{\xi\in\Gamma}\|f(\cdot,\xi)\|_H$, where H is a Hilbert space. Similarly, $C^p_\rho(\Gamma;H)$ is a Banach space with a norm

$$\|f\|_{C^p_\rho(\Gamma;H)} = \|f\|_{C_\rho(\Gamma;H)} + \sum_{j=1}^{N}\sum_{k=1}^{p_j}\|\partial^k_{\xi_j} f\|_{C_\rho(\Gamma;H)},$$

where $p = (p_1, p_2, \cdots, p_N)$.

3.2 Finite dimensional representation of the control problem

By the above assumption and the Doob-Dynkin lemma (cf. [38]), we have that the solution y corresponding to (2.2), can be described by just a finite number of random variables, i.e., $y(x,\omega) = y(x,(\xi_1(\omega),\cdots,\xi_N(\omega)))$. The number N has to be large enough so that the approximation error is sufficient small.

Here, we will take the deterministic state space $V_\rho = L^2_\rho(\Gamma;H^1_0(D))$. Corresponding to equations (2.3)-(2.4), we have notations:

$$A[y,v] = \int_\Gamma \int_D a\nabla y\cdot\nabla v\rho dxd\xi,\ \ \forall\, y,\ v\in V_\rho, \tag{3.2}$$

and

$$[u,v] = \int_\Gamma \int_D uv\rho dxd\xi,\ \ \forall\, u\in U,\ \ v\in V_\rho. \tag{3.3}$$

Similarly, we have the finite dimensional presentation for the weak formulation of optimal control problem (2.1)-(2.2), which can be rewritten as:

$$\min_{u\in L^2(D),y\in K_\rho} J(y(u),u) = \min_{u\in L^2(D),y\in K_\rho} E\big[\frac{1}{2}\int_D |y-y_d|^2dx + \frac{\alpha}{2}\int_D |u|^2dx\big] \tag{3.4}$$

subject to

$$A[y,v] = [u,v],\ \ \forall\, v\in V_\rho. \tag{3.5}$$

Under the assumption (2.6), the existence of solutions to (3.4)-(3.5) can be proved similarly [25,31].

We take notation $\mathbb{S}: L^2(D)\to L^2_\rho(\Gamma;L^2(D))$ to denote the operator which assigns $u\in L^2(D)$ to the solution $y(u)\in L^2_\rho(\Gamma;H^1_0(D))\hookrightarrow L^2_\rho(\Gamma;L^2(D))$ of the state equation.

For simplicity, we denote $\|v\|_{L^2_\rho(\Gamma;L^2(D))}$, $\|v\|_{L^2_\rho(\Gamma;H^1_0(D))}$, $\|v\|_{L^2_\rho(\Gamma;H^2(D))}$ by $\|v\|_{0,\rho}$, $\|v\|_{1,\rho}$, $\|v\|_{2,\rho}$, respectively.

Following from [8,9,46], let

$$K_\rho = \{y \mid y(x,\xi)\in L^2_\rho(\Gamma;H^1_0(D)\cap H^2(D)),\ E[\int_D y(x,\xi)dx]\geq 0\}, \tag{3.6}$$

we have the following optimal control conditions of (3.4)-(3.5).

Theorem 1 *The pair $(y,u)\in K_\rho\times U$ is the solution of the optimal control problem* (3.4)-(3.5) *iff there is a pair $(p,\lambda)\in V_\rho\times L^2_\rho(\Gamma;L^2(D))$, such that (y,p,λ,u) satisfies the following optimality system:*

$$A[y,v] = [u,v],\ \ \forall\, v\in V_\rho, \tag{3.7a}$$

$$A[p,q] = [y-y_d,q] + E[\langle\lambda,q\rangle],\ \ \forall\, q\in V_\rho, \tag{3.7b}$$

$$E[\langle\lambda, v-y\rangle]\leq 0,\ \ \forall\, v\in K_\rho, \tag{3.7c}$$

$$E[p+\alpha u] = 0, \tag{3.7d}$$

where V^*_ρ is the dual space of V_ρ, $\langle\ ,\ \rangle$ is the dual product on $V^*_\rho\times V_\rho$.

Proof Define $\mathbb{I}_K: V \to \mathbb{R} \cup \{+\infty\}$ to be the indicator of the convex set K_ρ:

$$\mathbb{I}_K(y) = \begin{cases} 0, & y \in K_\rho, \\ +\infty, & y \overline{\in} K_\rho. \end{cases}$$

The functional $\hat{\mathbb{J}} : L^2(D) \to \mathbb{R}$ is defined by $\hat{\mathbb{J}}(u) := J(u, y(u))$. It is an essential fact that $\exists \tilde{u} > 0$ such that $y(\tilde{u}) = \mathbb{S}(\tilde{u}) > 0$ belongs to the interior of K_ρ in $L^2_\rho(\Gamma; L^2(D))$ topology. Here we view that K_ρ is a convex set in $L^2_\rho(\Gamma; L^2(D))$, so the Slater condition (i.e. K_ρ has an interior point in $L^2_\rho(\Gamma; L^2(D))$ topology) is satisfied. By the properties of subdifferential calculus, the convexity of $\mathbb{I}_k$ and the strict convexity of $\hat{\mathbb{J}}$, the pair $(y, u) = (\mathbb{S}u, u)$ is a solution to (3.4)-(3.5) if and only if

$$0 \in \partial(\hat{\mathbb{J}}(u) + \mathbb{I}_K(\mathbb{S}u)).$$

Using the Moreau-Rockafellar formulas (see, e.g. [41]), we have

$$0 \in \partial \hat{\mathbb{J}}(u) + \mathbb{S}^* \cdot \partial \mathbb{I}_K(\mathbb{S}u),$$

which is equivalent to the existence of $\lambda \in \partial \mathbb{I}_K(\mathbb{S}u) \in L^2(\Gamma; L^2(D))$ such that

$$0 \in \partial \hat{\mathbb{J}}(u) + \mathbb{S}^* \lambda. \tag{3.8}$$

Here $\mathbb{S}^*$ denotes the adjoint operator of the operator $\mathbb{S}$. As we know, λ is in the sub-differential set $\partial \mathbb{I}_k(\mathbb{S}u)$ of $\mathbb{I}_K$ at $y = \mathbb{S}u$ iff

$$E[\langle \lambda, w - y \rangle] \leq 0, \quad \forall\, w \in K_\rho \subset L^2_\rho(\Gamma; L^2(D)). \tag{3.9}$$

As $\hat{\mathbb{J}}$ is differentiable at u, so we can infer that

$$(\partial \hat{\mathbb{J}}(u), v) = (\mathbb{J}'(u), v) = [y - y_d, \mathbb{S}v] + \alpha[u, v].$$

Thus (3.8) can be expressed as

$$[y - y_d, \mathbb{S}v] + [\alpha u, v] + [\mathbb{S}^* \lambda, v] = 0, \ \forall\, v \in U. \tag{3.10}$$

Then we define $p \in V_\rho$, such that

$$A[p, q] = [y - y_d, q] + E[\langle \lambda, q \rangle], \qquad \forall\, q \in V_\rho. \tag{3.11}$$

Setting $q = \mathbb{S}v$ in (3.11), we have

$$A[p, \mathbb{S}v] = [y - y_d, \mathbb{S}v] + E[\langle \lambda, \mathbb{S}v \rangle], \qquad \forall\, v \in U. \tag{3.12}$$

From (3.10) and (3.12), we obtain

$$A[p, \mathbb{S}v] - E[\langle \lambda, \mathbb{S}v \rangle] + [\alpha u, v] + [\mathbb{S}^* \lambda, v] = 0, \ \forall\, v \in U, \tag{3.13}$$

that is

$$A[p, \mathbb{S}v] - [\mathbb{S}^* \lambda, v] + [\alpha u, v] + [\mathbb{S}^* \lambda, v] = 0, \ \forall\, v \in U. \tag{3.14}$$

Then we have

$$[p, -\nabla \cdot (a \nabla \mathbb{S}v)] + [\alpha u, v] = 0, \ \forall\, v \in U. \tag{3.15}$$

By the definition of $\mathbb{S}$, we have $y = \mathbb{S}v$, $-\nabla \cdot (a \nabla y) = v$, and thus $v = -\nabla \cdot (a \nabla \mathbb{S}v)$.
Then we obtain

$$[p, v] + [\alpha u, v] = 0, \ \forall\, v \in U. \tag{3.16}$$

Consequently

$$E[p+\alpha u]=0. \tag{3.17}$$

Then (3.7b)-(3.7d) follow from (3.9), (3.11), and (3.17).

Finally, we prove the uniqueness of the solution (y,p,λ,u) of (3.7a)-(3.7d). Assume that there exist two solutions: (y_1,p_1,λ_1,u_1) and (y_2,p_2,λ_2,u_2). Then we have

$$A[y_1-y_2,v]=[u_1-u_2,v],\ \ \forall\, v\in V_\rho, \tag{3.18}$$

$$A[p_1-p_2,q]=[y_1-y_2,q]+E[\langle\lambda_1-\lambda_2,q\rangle],\ \ \forall\, q\in V_\rho. \tag{3.19}$$

Taking $v=p_1-p_2$ in (3.18) and $q=y_1-y_2$ in (3.19), from (3.7d), we obtain

$$A[y_1-y_2,p_1-p_2]=-\frac{1}{\alpha}[p_1-p_2,p_1-p_2],$$
$$A[p_1-p_2,y_1-y_2]=[y_1-y_2,y_1-y_2]+E[\langle\lambda_1-\lambda_2,y_1-y_2\rangle].$$

Then we have

$$[y_1-y_2,y_1-y_2]+\frac{1}{\alpha}[p_1-p_2,p_1-p_2]+E[\langle\lambda_1-\lambda_2,y_1-y_2\rangle]=0.$$

Since $E[\langle\lambda_1,y_2-y_1\rangle]\le 0$ and $E[\langle\lambda_2,y_1-y_2\rangle]\le 0$, hence

$$\|y_1-y_2\|_{2,\rho}^2+\frac{1}{\alpha}\|p_1-p_2\|_{2,\rho}^2\le 0.$$

Thus $y_1=y_2$ and $p_1=p_2$. Furthermore, $\lambda_1=\lambda_2$. □

We remark that, the control problem is convex due to the linearity of the state equation. Therefore the above first order optimality conditions are sufficient for this problem.

In the following we will present the expression and property of λ by applying the following Lemma.

Lemma 1 *Let $\mathbb{Q}$ from $L^2_\rho(\Gamma;L^2(D))$ onto the convex set K_ρ be the projection operator such that*

$$\|v-\mathbb{Q}v\|_{2,\rho}=\min_{w\in K_\rho}\|v-w\|_{2,\rho}. \tag{3.20}$$

Then $\mathbb{Q}v$ satisfies (3.20) if and only if, for any $w\in K_\rho$

$$[\mathbb{Q}v-v,w-\mathbb{Q}v]\ge 0, \tag{3.21}$$

and

$$\mathbb{Q}v=v-\min\{\bar{v},0\}, \tag{3.22}$$

where $\bar{v}$ is the mean of v over $\Gamma\times D$ given by

$$\bar{v}=\frac{1}{|D\times\Gamma|}\int_\Gamma\int_D v\rho\,dxd\xi.$$

Proof Let $j(w)=\|w-v\|_{2,\rho}^2$ and the equivalence between (3.20) and (3.21) follows from the Lions's lemma: $j'(\mathbb{Q}v)(w-\mathbb{Q}v)\ge 0,\ \ \forall w\in K_\rho$. We now prove (3.22).
If $\bar{v}\ge 0$, i.e., $v\in K$, such that $\mathbb{Q}v=v$, then $[\mathbb{Q}v-v,w-\mathbb{Q}v]=0$, (3.21) holds;
If $\bar{v}<0$, then $[\mathbb{Q}v-v,w-\mathbb{Q}v]=-[\bar{v},w-(v-\bar{v})]=-\bar{v}\int_\Gamma\int_D w\rho\,dxd\xi\ge 0$. Hence (3.21) still holds.
This means that $\mathbb{Q}v$ defined by (3.22) is the project operator. □

Theorem 2 *Let* $(y,p,\lambda,u) \in K_\rho \times V_\rho \times L^2_\rho(\Gamma;L^2(D)) \times U$ *be the optimal solution to the continuous systems* (3.7a)-(3.7d)*, respectively. Then the multiplier* λ *satisfies the following equality*

$$\lambda = \min\{\overline{\lambda} + \bar{y}, 0\}. \tag{3.23}$$

As a consequence, λ *is a non-positive constant in the whole domain* $D \times \Gamma$ *and*

$$\lambda \bar{y} = 0. \tag{3.24}$$

Proof It follows from (3.7c) that

$$[(\lambda + y) - y, v - y] = [\lambda, v - y] \le 0,\ \forall\, v \in K_\rho.$$

That is

$$[y - (\lambda + y), v - y] \ge 0,\ \forall\, v \in K_\rho.$$

By Lemma 1, y is the projection of $\lambda + y$ in K , that is

$$[\mathbb{Q}(\lambda + y) - (\lambda + y), v - \mathbb{Q}(\lambda + y)] \ge 0,\ \forall\, v \in K_\rho,$$

we have

$$y = \mathbb{Q}(\lambda + y) = \lambda + y - \min\{\overline{\lambda + y}, 0\}.$$

This infers

$$\lambda = \min\{\bar{\lambda} + \bar{y}, 0\}.$$

and λ is a constant.
One the other hand, since $y \in K_\rho$ which means $\bar{y} \ge 0$, then we have:
If $\bar{y} > 0$, as $\lambda = \min\{\bar{\lambda} + \bar{y}, 0\}$, we have $\lambda \ne \bar{\lambda} + \bar{y}$, then $\lambda = 0$.
Otherwise, if $\bar{y} = 0$, clearly $\lambda \bar{y} = 0$. As $\lambda = \min\{\bar{\lambda} + \bar{y}, 0\}$, then $\lambda \le 0$.
So the conclusion (3.24) holds. □

4 Stochastic Galerkin method

4.1 Finite element spaces on D and Γ

To present the discretization of the optimality system (3.4)-(3.5), a stochastic Galerkin scheme will be formulated. We adopt finite element spaces defined on $D \times \Gamma$ by [4,25].

First of all, we consider finite element spaces defined on spatial domain $D \subset \mathbb{R}^d$. Let $\{T_h\}_{h>0}$ be a family of regular triangulation of D such that $\bar{D} = \cup_{\tau \in T_h} \bar{\tau}$. Let $h_s = \max_{\tau \in T_h} h_\tau$, where h_τ denotes the diameter of the element τ. Here regular triangulation of D [33] means: there is a positive constant C such that for all $\tau \in T_h$,

$$C^{-1} h_\tau^2 \le |\ \tau\ | \le C h_\tau^2,$$

where $|\ \tau\ |$ is the area of τ. Consider two finite element spaces $V_{h_s} \subset H_0^1(D)$ and $W_{h_s} \subset L^2(D)$, consisting of piecewise linear continuous functions on $\{T_h\}$ and piecewise constant functions on $\{T_h\}$, respectively. We assume that V_{h_s} and W_{h_s} satisfy the following approximation properties [12]:

(i) For any $\phi \in H^2(D) \cap H_0^1(D)$, there exists $\phi_{h_s} \in V_{h_s}$, such that

$$\inf_{\phi_{h_s} \in V_{h_s}} \|\phi - \phi_{h_s}\|_{H_0^1(D)} \le C h_s \|\phi\|_{H^2(D)}, \tag{4.1}$$

where $C > 0$ is a constant independent of ϕ and h_s.

(ii) For any $\phi \in H_0^1(D)$, there exists $\phi_{h_s} \in W_{h_s}$, such that

$$\inf_{\phi_{h_s} \in W_{h_s}} \|\phi - \phi_{h_s}\|_{L^2(D)} \le Ch_s \|\phi\|_{H_0^1(D)}, \tag{4.2}$$

where $C > 0$ is a constant independent of ϕ and h_s.

Next, we consider a finite dimensional space defined on $\Gamma \subset \mathbb{R}^N$ ([3]). Let Γ be partitioned into a finite number of disjoint boxes $B_i^N \subset \mathbb{R}^N$, that is, for a finite index set I, we have

$$\Gamma = \bigcup_{i \in I} B_i^N = \bigcup_{i \in I} \prod_{j=1}^{N} (a_i^j, b_i^j),$$

where $B_k^N \cap B_l^N = \emptyset$ for $k \neq l \in I$ and $(a_i^j, b_i^j) \subset \Gamma_j$. A maximum grid size parameter $0 < h_r < 1$ is denoted by

$$h_r = \max\{|b_i^j - a_i^j|/2 : 1 \le j \le N \,\text{and}\, i \in I\}.$$

Let $S_{h_r} \subset L_\rho^2(\Gamma)$ be the finite element space of piecewise polynomials with degree at most p_j on each direction ξ_j. Thus if $\psi_{h_r} \in S_{h_r}$, then $\psi_{h_r}|_{B_i^N} \in \text{span}\{\prod_{j=1}^{N} \xi_j^{n_j} : n_j \in \mathbb{N} \,\text{and}\, n_j \le p_j\}$. Letting the multi-index $p = (p_1, \cdots, p_N)$, we have (cf. [7]) the following property: for all $\psi \in C^{p+1}(\Gamma)$,

$$\inf_{\psi_{h_r} \in S_{h_r}} \|\psi - \psi_{h_r}\|_{L_\rho^2(\Gamma)} \le h_r^\gamma \sum_{j=1}^{N} \frac{\|\partial_{\xi_j}^{p_j+1} \psi\|_{L_\rho^2(\Gamma)}}{(p_j+1)!}, \tag{4.3}$$

where $\gamma = \min_{1 \le j \le N} \{p_j + 1\}$.

4.2 Tensor product finite element spaces on $D \times \Gamma$

Combining spaces V_{h_s}, W_{h_s} and S_{h_r} together, we now define a tensor product finite element space on $D \times \Gamma$.

We will use $V_h = V_{h_s} \times S_{h_r}$ for the sate variable y and co-state variable p, $U_h = W_{h_s}$ for the control variable u and let $K_h = V_h \cap K_\rho$ be the finite element space of the convex set K_ρ.

We define the $H_0^1(D)$-projection operator R_{h_s}: $H_0^1(D) \to V_{h_s}$ by

$$(R_{h_s}\phi, \phi_{h_s})_{H_0^1(D)} = (\phi, \phi_{h_s})_{H_0^1(D)}, \quad \forall\, \phi_{h_s} \in V_{h_s},\ \forall\, \phi \in H_0^1(D), \tag{4.4}$$

the $L^2(D)$-projection operator Π_{h_s}: $L^2(D) \to W_{h_s}$ by

$$(\Pi_{h_s}\phi, \phi_{h_s})_{L^2(D)} = (\phi, \phi_{h_s})_{L^2(D)}, \quad \forall\, \phi_{h_s} \in W_{h_s},\ \forall\, \phi \in L^2(D). \tag{4.5}$$

Similarly, let the $L_\rho^2(\Gamma)$-projection operator Π_{h_r}: $L_\rho^2(\Gamma) \to S_{h_r}$ by

$$(\Pi_{h_r}\psi, \psi_{h_r})_{L_\rho^2(\Gamma)} = (\psi, \psi_{h_r})_{L_\rho^2(\Gamma)}, \quad \forall\, \psi_{h_r} \in S_{h_r},\ \forall\, \psi \in L_\rho^2(\Gamma). \tag{4.6}$$

It follows from (4.1) that for all $\phi \in H^2(D) \cap H_0^1(D)$

$$\|\phi - R_{h_s}\phi\|_{H_0^1(D)} \le Ch_s \|\phi\|_{H^2(D)}, \tag{4.7}$$

and from (4.2) that for all $\phi \in H^1(D)$

$$\|\phi - \Pi_{h_s}\phi\|_{L^2(D)} \leq Ch_s\|\phi\|_{H^1(D)}. \tag{4.8}$$

Similarly, by (4.3) we obtain that for all $\psi \in C_\rho^{p+1}(\Gamma)$

$$\|\psi - \Pi_{h_r}\psi\|_{L^2_\rho(\Gamma)} \leq h_r^\gamma \sum_{j=1}^{N} \frac{\|\partial_{\xi_j}^{p_j+1}\psi\|_{L^2_\rho(\Gamma)}}{(p_j+1)!}. \tag{4.9}$$

Using the inequalities (4.7) and (4.9), we have the following approximation property (cf. [3], Proposition 3.1): for all $y \in C_\rho^{p+1}\big(\Gamma; H^2(D)\cap H_0^1(D)\big)$

$$\inf_{y_h\in Y_h}\|y-y_h\|_{1,\rho} \leq C\big\{h_s\|y\|_{2,\rho} + h_r^\gamma\sum_{j=1}^{N}\frac{\|\partial_{\xi_j}^{p_j+1}y\|_{1,\rho}}{(p_j+1)!}\big\}, \tag{4.10}$$

where positive constant C is independent of h_s, h_r, N and p.

In order to obtain the separate error estimates in D and Γ, we define a projection operator P_h which maps onto the tensor product space $W_{h_s}\times S_{h_r}$. It is defined as follows

$$P_h\varphi = \Pi_{h_s}\Pi_{h_r}\varphi = \Pi_{h_r}\Pi_{h_s}\varphi, \quad \forall\, \varphi \in\, L^2_\rho(\Gamma; L^2(D)). \tag{4.11}$$

Furthermore, we use the following decomposition

$$\varphi - P_h\varphi = (\varphi - \Pi_{h_s}\varphi) + \Pi_{h_s}(I - \Pi_{h_r})\varphi, \quad \forall\, \varphi \in\, L^2_\rho(\Gamma; L^2(D)). \tag{4.12}$$

To derive the error estimates, we need assumption and lemmas on the regularity as follows.

Assumption 4.1 *Let y, p, u satisfy the following regularity condition*

$$y,\, p \in C_\rho^{p+1}\big(\Gamma; H^2(D)\cap H_0^1(D)\big),\; u \in H^1(D). \tag{4.13}$$

Lemma 4.1 [20] *Let $u \in L^2(D)$. Then for any $\xi \in \Gamma$, $y(\cdot,\xi) \in H^2(D)$ and there exists $C > 0$ such that*

$$\|y(\cdot,\xi)\|_{H^2(D)} \leq C\|u\|_{L^2(D)}. \tag{4.14}$$

Similar to Lemma 3.7 in [21], the following Lemma follows from an inductive argument after taking derivatives with respect to ξ_j of (3.5) and using Greens formulas.

Lemma 4.2 *Let $u \in L^2(D)$ and $\varphi_j \in L^\infty(D)$. Then for all $j = 1,2,\cdots,N$ and for any $\xi \in \Gamma$, there exists $C > 0$ such that*

$$\frac{\|\partial_{\xi_j}^{p_j+1}y(\cdot,\xi)\|_{H_0^1(D)}}{(p_j+1)!} \leq C\|\varphi_j\|_{L^\infty(D)}^{p_j+1}\|u\|_{L^2(D)}. \tag{4.15}$$

4.3 Galerkin approximation scheme

Let us discrete the constrained set of the state $K^h = \{y_h| \ y_h \in V^h \bigcap K_\rho, \ E[\int_D y_h dx] \geq 0\}$. Then, we can use Galerkin finite element scheme to approximate the optimal control problem (3.4)-(3.5), which can be formulated as follows:

$$\min_{u_h \in U^h, y_h \in K^h} J_h(u_h) = \min_{u_h \in U^h, y_h \in K^h} E\Big[\frac{1}{2}\int_D |y_h - y_d|^2 dx + \frac{\alpha}{2}\int_D |u_h|^2 dx\Big] \tag{4.16}$$

subject to

$$A[y_h, v_h] = [u_h, v_h], \ \ \forall \ v_h \in V^h \ . \tag{4.17}$$

Define discrete operator $S_h : U^h \to V^h$ such that:

$$A[S_h u_h, v_h] = [u_h, v_h], \ \ u_h \in U_h, \ \forall v_h \in V^h.$$

Consequently, the conditions $\{u_h \in U^h, y_h \in K^h\}$ can be rewritten as $\{u_h \in U^h, E[\int_D S_h u_h dx] \geq 0\}$.
Then the discrete optimal control problem (4.16)-(4.17) is equivalent to the following optimal control problems of constrained control type:

$$\min_{u_h \in U^h_{ad}} J_h(u_h, y_h) = \min_{u_h \in U^h_{ad}} E\Big[\frac{1}{2}\int_D |y_h - y_d|^2 dx + \frac{\alpha}{2}\int_D |u_h|^2 dx\Big] \tag{4.18}$$

subject to

$$A[y_h, v_h] = [u_h, v_h], \ \ \forall \ v_h \in V_h, \tag{4.19}$$

where the discrete constrained set for the control is:

$$U^h_{ad} := \{v_h| \ v_h \in U^h : E[\int_D S_h v_h dx] \geq 0\} \subseteq U. \tag{4.20}$$

Then we have the following discrete optimal control conditions of (4.18)-(4.19).

Theorem 3 *The pair $(y_h, u_h) \in K^h \times U^h$ is the optimal solution to*(4.18)-(4.19)*, if and only if there exist $p_h \in V^h$ and $\lambda_h \in R_- \cup \{0\}$ such that*

$$A[y_h, v_h] = [u_h, v_h], \qquad \forall \ v_h \in V^h, \tag{4.21a}$$

$$A[p_h, q_h] = [y_h - y_d, q_h] + E[\langle \lambda_h, q_h \rangle], \ \ \forall \ q_h \in V^h; \tag{4.21b}$$

$$E[\langle \lambda_h, v_h - y_h \rangle] \leqslant 0, \qquad \forall \ v_h \in K^h, \tag{4.21c}$$

$$E[\mathbb{P} p_h + \alpha u_h] = 0. \qquad in \ D. \tag{4.21d}$$

Here $\mathbb{P}$ is the L^2-projection from $L^2_\rho(\Gamma; H^1_0(D))$ onto closed subspace $L^2_\rho(\Gamma; U^h)$. Furthermore, the solution $(y_h, u_h, p_h, \lambda_h)$ of (4.21a) - (4.21d) is unique.

Proof Here we only outline the proof. Similarly to the proof of Theorem 1, we can show there exists λ_h such that

$$0 \in \partial \hat{\mathbb{J}}_h(u_h) + \mathbb{S}_h^* \lambda_h, \tag{4.22}$$

$$E[\langle \lambda_h, v_h - y_h \rangle] \leqslant 0, \quad \forall\, v_h \in K^h, \tag{4.23}$$

and

$$[y_h - y_d, \mathbb{S}_h v_h] + [\alpha u_h, v_h] + [\mathbb{S}_h^* \lambda_h, v_h] = 0, \;\; \forall\, v_h \in U^h. \tag{4.24}$$

We define $p_h \in V^h$ such that

$$A[q_h, p_h] = [y_h - y_d, q_h] + E[\langle \lambda_h, q_h \rangle], \quad \forall q_h \in V^h. \tag{4.25}$$

Noting that $\mathbb{S}_h$ is a bijection from finite dimension space onto another finite dimension space, so there is $w_h \in U^h$ such that $\mathbb{S}_h w_h = q_h$, i.e.

$$A[\mathbb{S}_h w_h, v_h] = [w_h, v_h], \;\; \forall\, v_h \in V^h.$$

Furthermore we have

$$[w_h, p_h] = A[\mathbb{S}_h w_h, p_h] = [y_h - y_d + \lambda_h, \mathbb{S}_h w_h] = [\mathbb{S}_h^*(y_h - y_d + \lambda_h), w_h] = [-\alpha u_h, w_h].$$

Therefore we obtain

$$[p_h + \alpha u_h, w_h] = 0. \tag{4.26}$$

Since q_h is an arbitrary element in V^h and $\mathbb{S}_h$ is a bijection, the above equality (4.26) holds for any $w_h \in U^h$. This fact shows αu_h is a unique projection of p_h from $L^2_\rho(\Gamma; H^1_0(D))$ onto closed subspace $L^2_\rho(\Gamma; U^h)$. So we have

$$E[\mathbb{P} p_h + \alpha u_h] = 0, \;\; in\ D. \tag{4.27}$$

Then (4.21a)-(4.21d) follow from (4.23), (4.25), and (4.27).

Similarly, we can prove that the solution of (4.21a)-(4.21d) is unique and λ_h is a non-positive constant . □

Remark. If U^h is a piecewise constant space, we note that $\mathbb{P}$ is such that

$$\mathbb{P} v|_\tau = \frac{1}{|\tau|} \int_\tau v, \qquad \forall \tau \in T_U^h,$$

where $|\tau|$ is the measure of τ, which implies

$$\alpha \bar{u}_h = \overline{\mathbb{P}(-p_h)} = -\bar{p}_h. \tag{4.28}$$

If U^h is a piecewise linear element space, we have well-known result

$$\|v - \mathbb{P} v\|_0 \leqslant c h_U \|v\|_1. \tag{4.29}$$

We will use above facts to deduce convergence results of approximation in the following section.

We know that it is very useful to examine the structure of the matrix systems of the above finite element problems (4.21a)-(4.21d) for developing efficient numerical algorithms. Without loss of generality, we consider a particular case that the space S_{h_r} has no partition of Γ, i.e. only the polynomial degree is increased. Here, we use the tensor finite element space $S_{h_r} = \bigotimes_{n=1}^N Z_n^{p_n}$, where we use the global polynomial subspaces $Z_n^{p_n} = \{v : \Gamma_n \to \mathbb{R} : \; v \in \text{span}(1, y_n, \cdots, y_n^{p_n})\}, n = 1, \cdots, N$.

Let $\{\varphi_i(x)\}$ be the a basis of the space V_{h_s}, $\{\psi_j(\xi)\}$ be a basis of the space S_{h_r}. Then the solutions of the discrete optimality system of equations (4.21a)-(4.21d) are given by

$$\begin{cases} y_h = \sum_{i,j} y_{ij}\varphi_i(x)\psi_j(\xi), \\ p_h = \sum_{i,j} p_{ij}\varphi_i(x)\psi_j(\xi), \\ u_h = \sum_{i} u_i\varphi_i(x). \end{cases} \tag{4.30}$$

We take the state equation in (4.21a)-(4.21d) as an example. Using test function $v_h = \varphi_l(x)\psi_k(\xi)$, we have

$$\sum_{i,j}\Big(\int_\Gamma \rho(\xi)\psi_k(\xi)\psi_j(\xi)(a\nabla\varphi_i,\nabla\varphi_l)_{L^2(D)}d\xi\Big)y_{ij} = \sum_i\Big(\int_\Gamma \rho(\xi)\psi_k(\xi)(\varphi_i,\varphi_l)_{L^2(D)}d\xi\Big)u_i,\ \ \forall\, k,\ l, \tag{4.31}$$

which can be written as

$$\sum_{i,j}\Big(\int_\Gamma \rho(\xi)\psi_k(\xi)\psi_j(\xi)K_{i,l}(\xi)d\xi\Big)y_{ij} = \sum_i\Big(\int_\Gamma \rho(\xi)\psi_k(\xi)M_{i,l}d\xi\Big)u_i,\ \ \forall\, k,\ l, \tag{4.32}$$

where $K_{i,l}(\xi) = (a(\cdot,\xi)\nabla\varphi_i,\nabla\varphi_l)_{L^2(D)}$ and $M_{i,l} = (\varphi_i,\varphi_l)_{L^2(D)}$.

If the diffusion coefficient a is expanded by finite terms (3.1), i.e. $a(x,\xi) = \sum_{t=1}^{S}\alpha_t(x)L_t(\xi)$, we have a corresponding expression for the stiffness matrix

$$K_{i,l}(\xi) \equiv \int_D\Big(\sum_{t=1}^{S}\alpha_t(x)L_t(\xi)\Big)\nabla\varphi_i(x)\cdot\nabla\varphi_l(x)\,dx. \tag{4.33}$$

Since $\psi_k \in S_{h_r} = \bigotimes_{n=1}^{N} Z_n^{p_n}$, it is enough to take it as the product $\psi_k(\xi) = \prod_{r=1}^{N}\psi_{kr}(\xi_r)$, where $\psi_{kr} : \Gamma_r \to \mathbb{R}$ is a basis function of the subspace

$$Z_r^{p_r} = \mathrm{span}\{1,\xi_r,\cdots,\xi_r^{p_r}\} = \mathrm{span}\{\psi_{kr} : kr = 1,\cdots,p_r+1\}.$$

Putting this choice of ψ_k into (4.32), we obtain

$$\sum_{i,j}\Big(\int_\Gamma \prod_{r=1}^{N}\rho_r(\xi_r)\psi_{kr}(\xi_r)\psi_{jr}(\xi_r)K_{i,l}(\xi)d\xi\Big)y_{ij} = \sum_i\Big(\int_\Gamma \prod_{r=1}^{N}\rho_r(\xi_r)\psi_{kr}(\xi_r)M_{i,l}d\xi\Big)u_i,\ \ \forall\, k,\ l. \tag{4.34}$$

Following (4.33), we derive the coefficients of y_{ij} as

$$\sum_{t=1}^{S}K_{i,l}^t\int_\Gamma L_t(\xi)\prod_{r=1}^{N}\rho_r(\xi_r)\psi_{kr}(\xi_r)\psi_{jr}(\xi_r)d\xi, \tag{4.35}$$

where

$$K_{i,l}^t = \int_D \alpha_t(x)\nabla\varphi_i(x)\cdot\nabla\varphi_l(x)\,dx. \tag{4.36}$$

Similarly we can obtain explicit formulas for the other equations. Then we have to solve the linear systems to determine y_{ij}, p_{ij} and u_i that are coefficients of solutions of the discrete optimality system of equations (4.21a)-(4.21d), which will be carried out in Section 6.

Remark 4.1: For ease of exposition, we have chosen the basis $\{\varphi_i(x)\}$ to be the basis of the space K_h, see e.g. (4.30). Actually, the base functions of the space V_{h_s} and the space K_h can be chosen differently. In practice, the control variable of a constrained control problem normally has lower regularity than that of the state variable. Due to the limited regularity of the optimal control in general, there will be no advantage in considering higher-order finite element spaces than the piecewise constant space for the control. We therefore only consider the piecewise constant finite element space for the approximation of the control, though piecewise linear continuous finite element spaces will be used to approximate the state and the co-state.

5 A priori error estimates

In this section, we will derive the a priori error estimates for the approximation solutions.

In order to obtain error estimates of the approximation solutions, we introduce an auxiliary system with auxiliary state $y(u_h) \in V_\rho$ and co-state $p(u_h) \in V_\rho$, which is defined by the following system:

$$A[y(u_h), w] = [u_h, w], \qquad \forall w \in V_\rho, \tag{5.1a}$$

$$A[p(u_h), q] = [y(u_h) - y_d, q] + E[\langle \lambda_h, q \rangle], \quad \forall q \in V_\rho. \tag{5.1b}$$

Obviously, we note that according to Theorem 1 and the requirement of boundary, $y(u_h)$ and $p(u_h)$ are also bounded in $L^2_\rho(\Gamma; H^2(D))$. In order to obtain the a priori error estimates, we first present some Lemmas.

Lemma 2 *Let* $(y, p, u) \in K_\rho \times V_\rho \times U$ *and* $(y_h, p_h, u_h) \in K^h \times V^h \times U^h$ *be the solutions of optimality condition of continuous system* (3.7a)-(3.7d) *and discretized optimality conditions* (4.21a)-(4.21d) *respectively. Let* $p(u_h)$ *be the solution of the auxiliary system defined above. Then we have the following estimate:*

$$\|p - p(u_h)\|_{1,\rho} \leqslant C(\|u - u_h\|_0 + \|p_h - p(u_h)\|_{0,\rho} + \|p_h - \mathbb{P}(p_h)\|_{0,\rho}). \tag{5.2}$$

where $\|\cdot\|_0 = \|\cdot\|_{L^2(D)}, \|\cdot\|_{0,\rho} = \|\cdot\|_{L^2_\rho(\Gamma;L^2(D))}$.

Proof Select $\varphi \in L^2_\rho(\Gamma; C_0^\infty(D))$ such that $\bar{\varphi} = 1$ and $\|\varphi\|_{1,\rho} \leqslant C$. For brevity, we denote $\widetilde{C} = \overline{p - p(u_h)}$, and we note that $\widetilde{C}\varphi \in L^2_\rho(\Gamma; C_0^\infty(D)) \subset L^2_\rho(\Gamma; H^1_0(D))$. From (3.7b) and (5.1b), we have

$$A[p - p(u_h), q] = [y - y(u_h), q] + E[\langle \lambda - \lambda_h, q \rangle]. \tag{5.3}$$

Letting $q = \widetilde{C}\varphi$ and $q = p - p(u_h)$ respectively in (5.3), we obtain

$$A[p - p(u_h) - \widetilde{C}\varphi, p - p(u_h)] = [y - y(u_h) + \lambda - \lambda_h, p - p(u_h) - \widetilde{C}\varphi].$$

Note that $\int_\Gamma \int_D [p - p(u_h) - \widetilde{C}\varphi] \rho dx d\xi = 0$ and $\lambda - \lambda_h$ is a constant. Hence

$$A[p - p(u_h), p - p(u_h)] = A[\widetilde{C}\varphi, p - p(u_h)] + [y - y(u_h), p - p(u_h) - \widetilde{C}\varphi].$$

Therefore, we have

$$\begin{aligned} c\|p - p(u_h)\|^2_{1,\rho} &\leqslant A[\widetilde{C}\varphi, p - p(u_h)] + [y - y(u_h), p - p(u_h) - \widetilde{C}\varphi] \\ &\leqslant C(\varepsilon)\widetilde{C}^2\|\varphi\|^2_{1,\rho} + \varepsilon\|p - p(u_h)\|^2_{1,\rho} + C(\varepsilon)\|y - y(u_h)\|^2_{0,\rho}, \end{aligned}$$

which implies

$$\|p - p(u_h)\|^2_{1,\rho} \leqslant C\widetilde{C}^2 + C\|y(u) - y(u_h)\|^2_{0,\rho} \leqslant C\widetilde{C}^2 + C\|u - u_h\|^2_0. \tag{5.4}$$

Since D is a bounded domain, from (3.7d) and (4.21d), we note that

$$\begin{aligned}\widetilde{C} &\leqslant \int_\Gamma \int_D |p-p(u_h)|\rho dx d\xi \leqslant C\|p-p(u_h)\|_{0,\rho} \\ &= C\|p+\alpha u-\alpha u+\alpha u_h-\alpha u_h-\mathbb{P}(p_h)+\mathbb{P}(p_h)-p_h+p_h-p(u_h)\|_{0,\rho} \\ &\leqslant C(\|-\alpha u+\alpha u_h\|_0+\|p_h-p(u_h)\|_{0,\rho}+\|p_h-\mathbb{P}(p_h)\|_{0,\rho}) \\ &\leqslant C(\|u-u_h\|_0+\|p_h-p(u_h)\|_{0,\rho}+\|p_h-\mathbb{P}(p_h)\|_{0,\rho}).\end{aligned} \tag{5.5}$$

Then (5.2) follows from (5.4) and (5.5). □

Next we estimate the approximation λ_h of the multiplier λ.

Lemma 3 *Let the discretized solution* $(y_h, p_h, u_h) \in K^h \times V^h \times U^h$ *be defined as above. Let* λ *and* λ_h *be the solutions of multiplier of the continuous and discretized systems respectively. We have the following estimates:*

$$|\lambda-\lambda_h| \leqslant C(\|u-u_h\|_0+\|p_h-p(u_h)\|_{0,\rho}+\|p_h-\mathbb{P}(p_h)\|_{0,\rho}). \tag{5.6}$$

Proof Select $\varphi \in L^2_\rho(\Gamma; C_0^\infty(D)) \subset L^2_\rho(\Gamma; H_0^1(D))$ such that $\bar{\varphi}=1$ and $\|\varphi\|_{1,\rho} \leqslant C$. From the continuous and auxiliary systems, we obtain

$$E[\langle \lambda-\lambda_h, (\lambda-\lambda_h)\varphi\rangle] = A[(\lambda-\lambda_h)\varphi, p-p(u_h)] + [y(u_h)-y, (\lambda-\lambda_h)\varphi],$$

which implies

$$\begin{aligned}|\lambda-\lambda_h| &\leqslant C(\|p-p(u_h)\|_{1,\rho}+\|y-y(u_h)\|_{0,\rho}) \\ &\leqslant C(\|u-u_h\|_0+\|p_h-p(u_h)\|_{0,\rho}+\|p_h-\mathbb{P}(p_h)\|_{0,\rho}).\end{aligned} \tag{5.7}$$

Here we have used the estimate (5.2) and the fact $\|y-y(u_h)\|_{0,\rho} \leqslant \|y-y(u_h)\|_{1,\rho} \leqslant C\|u-u_h\|_0$. Therefore the result (5.6) holds. □

Lemma 4 *Let* $(y,p,u) \in K_\rho \times V_\rho \times U$ *and* $(y_h, p_h, u_h) \in K^h \times V^h \times U^h$ *be the solutions of optimality condition of continuous system* (3.7a)-(3.7d) *and discretized optimality conditions* (4.21a)-(4.21d) *respectively. Let* $(y(u_h), p(u_h))$ *be the solution pair of the auxiliary system* (5.1a)-(5.1b). *Then we have the following estimate:*

$$\begin{aligned}\|u-u_h\|_0 &\leqslant C(\|y_h-y(u_h)\|_{0,\rho}+\|p_h-p(u_h)\|_{0,\rho}+\|\mathbb{P}(p_h)-p_h\|_{0,\rho}) \\ &\leqslant C\Big(h_s^2+\|p_h-p(u_h)\|_{0,\rho}+h_s\|p\|_{1,\rho}+h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big).\end{aligned} \tag{5.8}$$

Proof By direct calculation, and using the continuous and auxiliary systems, we have

$$\begin{aligned}\mathbb{J}'(u)(u-u_h) &= [y-y_d, y'(u)(u-u_h)] + [\alpha u, u-u_h] \\ &= [y-y_d+\lambda, y'(u)(u-u_h)] + [\alpha u, u-u_h] - [\lambda, y'(u)(u-u_h)] \\ &= A[p, y'(u)(u-u_h)] + [\alpha u, u-u_h] - [\lambda, y(u)-y(u_h)] \\ &= [u-u_h, p] + [\alpha u, u-u_h] - [\lambda, y(u)-y(u_h)] \\ &= -[\lambda, y(u)-y(u_h)].\end{aligned} \tag{5.9}$$

Similarly,

$$\mathbb{J}'(u_h)(u-u_h) = [p(u_h)+\alpha u_h, u-u_h] - [\lambda_h, y-y(u_h)], \tag{5.10}$$

As the optimal control problem (2.1)-(2.2) is quadratic control of a linear system, $\mathbb{J}(u)$ is strictly convex. It follows from the properties of convex function, we have

$$c\|u-u_h\|_0^2 \leq \mathbb{J}'(u)(u-u_h)-\mathbb{J}'(u_h)(u-u_h).$$

From (5.9)-(5.10), we have

$$\mathbb{J}'(u)(u-u_h)-\mathbb{J}'(u_h)(u-u_h) = -[p(u_h)+\alpha u_h, u-u_h]-[\lambda-\lambda_h, y-y(u_h)].$$

Then we obtain

$$\begin{aligned}
c\|u-u_h\|_0^2 &\leqslant \mathbb{J}'(u)(u-u_h)-\mathbb{J}'(u_h)(u-u_h)\\
&= -[p(u_h)+\alpha u_h, u-u_h]-[\lambda-\lambda_h, y-y(u_h)]\\
&= [p_h-p(u_h), u-u_h]-\langle\lambda-\lambda_h, y-y_h\rangle-\langle\lambda-\lambda_h, y_h-y(u_h)\rangle+[\mathbb{P}(p_h)-p_h, u-u_h]\\
&\leqslant [p_h-p(u_h), u-u_h]-\langle\lambda-\lambda_h, y_h-y(u_h)\rangle+[\mathbb{P}(p_h)-p_h, u-u_h]\\
&\leqslant C(\varepsilon)\|p_h-p(u_h)\|_{0,\rho}^2+\varepsilon\|u-u_h\|_0^2+\varepsilon\|\lambda-\lambda_h\|_{0,\rho}^2+C(\varepsilon)\|y_h-y(u_h)\|_{0,\rho}^2+C(\varepsilon)\|\mathbb{P}(p_h)-p_h\|_{0,\rho}^2\\
&\leqslant C\|p_h-p(u_h)\|_{0,\rho}^2+C\varepsilon\|u-u_h\|_0^2+C\varepsilon\|p_h-p(u_h)\|_{0,\rho}^2+C\|y_h-y(u_h)\|_{0,\rho}^2+C\|\mathbb{P}(p_h)-p_h\|_{0,\rho}^2\\
&\leqslant C\|p_h-p(u_h)\|_{0,\rho}^2+C\varepsilon\|u-u_h\|_0^2+C\|y_h-y(u_h))\|_{0,\rho}^2+C\|\mathbb{P}(p_h)-p_h\|_{0,\rho}^2,
\end{aligned} \tag{5.11}$$

here we have used the complementary condition, conform condition $K^h \subset K$, and the estimate (5.6). Note that y_h is the Stochastic Galerkin finite element solution of $y(u_h)$, from the Aubin-Nitsche technique [12] together with regularity assumption of D, we have the finite element estimate

$$\|y_h-y(u_h)\|_{0,\rho} \leqslant Ch_s^2\|y(u_h)\|_{2,\rho} \leqslant Ch_s^2\|u_h\|_0 \leqslant Ch_s^2. \tag{5.12}$$

By the property of L^2 projection operator

$$\|\mathbb{P}(p_h)-p_h\|_{0,\rho} \leqslant C\Big(h_s\|p\|_{1,\rho}+h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big), \tag{5.13}$$

then the estimate (5.8) follows from (5.11), (5.12) and (5.13). □

Finally, we can obtain the a priori estimates of the Stochastic Galerkin approximations by using the above lemmas.

Theorem 4 *Let* $(y,p,\lambda,u) \in K_\rho \times V_\rho \times R_- \times U$ *and* $(y_h,p_h,\lambda_h,u_h) \in K^h \times V^h \times R_- \times U^h$ *be the solutions of continuous optimality condition* (3.7a)-(3.7d) *and discretized optimality condition* (4.21a)-(4.21d) *respectively. Let* $p(u_h)$ *be the solution of the auxiliary system* (5.1b). *We have the following estimate:*

$$\|p_h-p(u_h)\|_{1,\rho} \leqslant C\Big(h_s+h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big). \tag{5.14}$$

Therefore, we obtain the a priori error estimate for the approximation solutions:

$$|\lambda-\lambda_h|+\|u-u_h\|_0+\|y-y_h\|_{1,\rho}+\|p-p_h\|_{1,\rho} \leqslant C\Big(h_s+h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big). \tag{5.15}$$

Proof We denote $p(u_h)_I \in V^h \subset V_\rho$ to be the L^2 projection of $p(u_h) \in V_\rho$. Then

$$\begin{aligned} c\|p(u_h)-p_h)\|_{1,\rho}^2 &\leqslant A[p(u_h)-p_h, p(u_h)-p_h] \\ &= A[p(u_h)_I - p_h, p(u_h)-p_h] + A[p(u_h)-p(u_h)_I, p(u_h)-p_h] \\ &= [y(u_h)-y_h, p(u_h)_I - p_h] + A[p(u_h)-p(u_h)_I, p(u_h)-p_h] \\ &\leqslant \|y(u_h)-y_h\|_{0,\rho}^2 + C(\varepsilon)\|p(u_h)-p(u_h)_I\|_{1,\rho}^2 + \varepsilon\|p(u_h)-p_h)\|_{1,\rho}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|p(u_h)-p_h)\|_{1,\rho}^2 &\leqslant C(\|y(u_h)-y_h\|_{1,\rho}^2 + \|p(u_h)-p(u_h)_I\|_{1,\rho}^2) \\ &\leqslant Ch_s^2(\|y(u_h)\|_{2,\rho}^2 + \|p(u_h)\|_{2,\rho}^2) + C\Big(h_s^2\|p\|_{1,\rho}^2 + h_r^{2\gamma}\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{L^2_\rho(\Gamma;L^2(D))}}{(p_j+1)!}\big)^2\Big), \end{aligned}$$

Due to the regularity of the boundary ∂D, we have

$$\|p(u_h)\|_{2,\rho} \leqslant C\|y(u_h)-y_d+\lambda_h\|_{0,\rho} \leqslant C.$$

Therefore we obtain

$$\|p(u_h)-p_h\|_{0,\rho} \leqslant \|p(u_h)-p_h\|_{1,\rho} \leqslant C\Big(h_s\|p\|_{1,\rho} + h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big). \tag{5.16}$$

It follows from (5.8) and (5.16) that we have

$$\|u-u_h\|_0 \leqslant C\Big(h_s + h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big). \tag{5.17}$$

From (3.7a) and (5.1a) we have

$$A[y-y(u_h), w] = [u-u_h, w], \forall w \in V_\rho. \tag{5.18}$$

Letting $w = y - y(u_h)$ in (5.18), from (2.6) we have

$$c\|y-y(u_h)\|_{1,\rho}^2 \le A[y-y(u_h), y-y(u_h)] = [u-u_h, y-y(u_h)] \le C\|u-u_h\|_0\|y-y(u_h)\|_{1,\rho}.$$

Then

$$\|y-y(u_h)\|_{1,\rho} \le C\|u-u_h\|_0. \tag{5.19}$$

From (5.17) and (5.19), we obtain

$$\|y-y_h\|_{1,\rho} \leqslant \|y-y(u_h)\|_{1,\rho} + \|y(u_h)-y_h\|_{1,\rho} \leqslant C\|u-u_h\|_0 + Ch\|y(u_h)\|_{2,\rho} \leqslant Ch_s. \tag{5.20}$$

Further, it follows from (5.16) and Lemma 2 that we have

$$\begin{aligned} \|p-p_h\|_{1,\rho} &\leqslant \|p-p(u_h)\|_{1,\rho} + \|p_h-p(u_h)\|_{1,\rho} \\ &\leqslant C(h_s + \|p_h-p(u_h)\|_{1,\rho} + \|u-u_h\|_0 + \|p_h-\mathbb{P}(p_h)\|_{0,\rho}) \quad (5.21) \\ &\leqslant C\Big(h_s + h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big). \quad (5.22) \end{aligned}$$

Then it follows from (5.16), (5.17) and Lemma 3 that we have

$$|\lambda-\lambda_h| \leqslant C(\|u-u_h\|_0 + \|p_h-p(u_h)\|_{0,\rho} + \|p_h-\mathbb{P}(p_h)\|_{0,\rho}) \leqslant C\Big(h_s + h_r^\gamma\sum_{j=1}^N\big(\frac{\|\partial_{\xi_j}^{p_j+1}p\|_{0,\rho}}{(p_j+1)!}\big)\Big). \tag{5.23}$$

Thus (5.15) follows from (5.17)-(5.23). □

6 Numerical examples

6.1 A projection algorithm

To present our algorithm we reduce the control problem into an optimization on a convex subset U_{ad} of $L^2(D)$ and use the projection method to discuss algorithm. We consider the following optimal problem:

$$\min_{u\in U_{ad}} \mathbb{J}(u)$$

where $\mathbb{J}$ is the reduced functional as before and the constraint set $U_{ad} = \{u \in L^2(D) : E[\overline{\mathbb{S}u}] \geqslant 0\}$. For easy of exposition, we state our method for the continuous form. For discretized problems (4.18)-(4.19), the method is similar.

Our scheme is as follows:

$$\begin{cases} u_{n+\frac{1}{2}} = u_n - \rho\mathbb{J}'(u_n) = u_n - \rho E[\alpha u_n + \tilde{p}_n], \\ u_{n+1} = \mathbb{P}(u_{n+\frac{1}{2}}) := u_{n+\frac{1}{2}} - \rho E[\mathbb{S}^*\lambda_n], \end{cases} \tag{6.1}$$

where

$$\begin{aligned} A[y_n, v] &= [u_n, v], & y_n &\in L^2_\rho(\Gamma, H^1_0(D)), \\ A[\tilde{p}_n, q] &= [y_n - y_d, q], & \tilde{p}_n &\in L^2_\rho(\Gamma; H^1_0(D)). \end{aligned}$$

Next we will discuss how to select λ_n in each step such that $u_{n+1} \in U_{ad}$, i.e. $E[\mathbb{S}u_{n+1}] = E[y_{n+1}] \geqslant 0$.
Note that

$$A[y_{n+1}, v] = [u_{n+1}, v] = [u_n - \rho E[\alpha u_n + \tilde{p}_n] - \rho E[\mathbb{S}^*\lambda_n], v].$$

Therefore if we define $y_{n+\frac{1}{2}}$ by solving the following equation,

$$A[y_{n+\frac{1}{2}}, v] = [u_{n+\frac{1}{2}}, v] = [u_n - \rho E[\alpha u_n + \tilde{p}_n], v], \qquad y_{n+\frac{1}{2}} \in L^2_\rho(\Gamma; H^1_0(D)),$$

then we have $A[y_{n+1} - y_{n+\frac{1}{2}}, v] = [-\rho E(S^*\lambda_n), v] = E[-\rho\lambda_n S^*(1), v]$, and thus

$$S^{-1}(y_{n+1} - y_{n+\frac{1}{2}}) = -\rho\lambda_n S^*(1),$$

$$y_{n+1} - y_{n+\frac{1}{2}} = -\rho\lambda_n SS^*(1).$$

Therefore we have

$$E[y_{n+1}] = E[y_{n+\frac{1}{2}} - \rho\lambda_n \overline{SS^*(1)}].$$

Hence if we select

$$\lambda_n = \min\{\frac{E[y_{n+\frac{1}{2}}]}{\rho\overline{\mathbb{SS}^*(1)}}, 0\}, \tag{6.2}$$

we can assure $E[y_{n+1}] \geqslant 0$ in each step. Furthermore the constant $\overline{\mathbb{SS}^*(1)}$ can be computed as follows: Let y^* be the solution of

$$A[q, p^*] = [1, q], \quad \forall q \in L^2_\rho(\Gamma; H^1_0(D)),$$

$$A[y^*, v] = [p^*, q], \quad \forall q \in L^2_\rho(\Gamma; H^1_0(D)),$$

Table 1: **Algorithm**

Step 1: Select $u_0 \in U_{ad}$, and calculate y_0 by solving the state equation.
Step 2: Calculate $\tilde{p}_n$ by solving the equation

$$A[\tilde{p}_n, v] = [y_n - y_d, v], \qquad \tilde{p}_n \in L^2_\rho(\Gamma; H^1_0(D)).$$

Step 3: Set $u_{n+\frac{1}{2}} = u_n - \rho E[\alpha u_n + \tilde{p}_n]$. Calculate $y_{n+\frac{1}{2}}$ by solving the equation

$$A[y_{n+\frac{1}{2}}, v] = [u_{n+\frac{1}{2}}, v], \qquad y_{n+\frac{1}{2}} \in L^2_\rho(\Gamma; H^1_0(D)).$$

Step 4: Set $\lambda_n = \min\{\frac{\bar{y}_{n+\frac{1}{2}}}{\rho\overline{\mathbb{S}\mathbb{S}^*(1)}}, 0\}$, and let $u_{n+1} = u_{n+\frac{1}{2}} - \rho E[\mathbb{S}^*\lambda_n]$.
Calculate y_{n+1} by solving the equation

$$A[y_{n+1}, v] = [u_{n+1}, v], \qquad y_{n+1} \in L^2_\rho(\Gamma; H^1_0(D)).$$

Step 5: Stop if stopping criterion is satisfied, e.g., $| u_{n+1} - u_n | < \varepsilon$,where ε ia a given parameter. Otherwise set $n = n+1$ go to **Step 2**.

then

$$\overline{\mathbb{S}\mathbb{S}^*(1)} = \frac{1}{|D \times \Gamma|} \int_\Gamma \int_D y^* \rho(\xi) dx d\xi.$$

Before discussing convergence of our method, we summarize our algorithm in Table 1.

In order to present the proof for the convergence of our algorithm, we need the following lemma. This lemma indicates that actually $\mathbb{P}$ is the projection operator from Hilbert space $L^2(D)$ onto its non-empty closed convex subset U_{ad}.

Lemma 5 *The operator $\mathbb{P}$ defined in (6.1) is the projection operator from Hilbert space $L^2(D)$ onto its non-empty closed convex subset U_{ad}. Further, for any $u, v \in U_{ad}$,*

$$\|\mathbb{P}u - \mathbb{P}v\|_0 \leqslant \|u - v\|_0. \tag{6.3}$$

Proof For any $v \in U_{ad}$,

$$\begin{aligned}(u_{n+\frac{1}{2}} - \mathbb{P}(u_{n+\frac{1}{2}}), v - \mathbb{P}(u_{n+\frac{1}{2}})) &= (u_{n+\frac{1}{2}} - (u_{n+\frac{1}{2}} - \rho E[\mathbb{S}^*\lambda_n]), v - (u_{n+\frac{1}{2}} - \rho E[\mathbb{S}^*\lambda_n])) \\ &= \rho E[< \lambda_n, \mathbb{S}v - \mathbb{S}u_{n+\frac{1}{2}} + \lambda_n \rho \mathbb{S}\mathbb{S}^*(1) >] \\ &= \rho E[< \lambda_n, \mathbb{S}v - y_{n+\frac{1}{2}} + \lambda_n \rho \mathbb{S}\mathbb{S}^*(1) >].\end{aligned}$$

Noting that $E[< \lambda_n, -y_{n+\frac{1}{2}} + \lambda_n \rho \mathbb{S}\mathbb{S}^*(1) >] = 0$, and $E[\mathbb{S}v] \geqslant 0, \forall v \in U_{ad}$,
we have

$$(u_{n+\frac{1}{2}} - \mathbb{P}(u_{n+\frac{1}{2}}), v - \mathbb{P}(u_{n+\frac{1}{2}})) = \rho E[< \lambda_n, \mathbb{S}v - y_{n+\frac{1}{2}} + \lambda_n \rho \mathbb{S}\mathbb{S}^*(1) >] = \rho E[< \lambda_n, \mathbb{S}v >] \leqslant 0, \forall\, v \in U_{ad},$$

which implies $\mathbb{P}$ is the projection operator.
Furthermore, we have

$$\begin{aligned}\|\mathbb{P}u - \mathbb{P}v\|_0^2 &= (\mathbb{P}u - \mathbb{P}v, \mathbb{P}u - \mathbb{P}v) \\ &= (\mathbb{P}u - \mathbb{P}v, v - \mathbb{P}v) + (\mathbb{P}u - \mathbb{P}v, \mathbb{P}u - u) + (\mathbb{P}u - \mathbb{P}v, u - v) \\ &\leqslant (\mathbb{P}u - \mathbb{P}v, u - v).\end{aligned}$$

Then we have

$$\|\mathbb{P}u - \mathbb{P}v\|_0 \leqslant \|u - v\|_0.$$

This completes the proof. □

Obviously, the objective functional $\mathbb{J}'$ is Lipschtiz and uniformly monotone, i.e. there are positive constant c, C such that

$$\begin{aligned} \|\mathbb{J}'(u) - \mathbb{J}'(v)\|_0 &\leqslant C\|u-v\|_0, \quad \forall u, v \in L^2(D), \\ (\mathbb{J}'(u) - \mathbb{J}'(v), u-v) &\geqslant c\|u-v\|_0^2, \quad \forall u, v \in L^2(D). \end{aligned}$$

Following Lemma 5, we have the following convergence results.

Theorem 5 *There are* $0 < \delta < 1, \varepsilon > 0$ *such that*

$$\|u - u_n\|_0 \leqslant \delta^n \|u - u_0\|_0, \quad n = 0, 1, 2, \cdots, \tag{6.4}$$

provided $\rho < \varepsilon$.

Proof Firstly we show that $u = \mathbb{P}(u - \rho E[\alpha u + \tilde{p})]$, where $\mathbb{P}$ is defined in (6.1)and $\tilde{p}$ satisfies the equation

$$A[\tilde{p}, v] = [y - y_d, v], \qquad \tilde{p} \in L^2_\rho(\Gamma; H^1_0(D)).$$

Due to the optimality conditions (3.7c) and (3.7d), we conclude that for any $v \in U_{ad}$,

$$(u - \rho E[\alpha u + \tilde{p}] - u, v - u) = -\rho(E[-p + \tilde{p}], v - u) = \rho(E[\mathbb{S}^* \lambda], v - u) = \rho E[< \lambda, \mathbb{S}v - \mathbb{S}u >] \leqslant 0,$$

which means $u = \mathbb{P}(u - \rho E[\alpha u + \tilde{p}])$.
Furthermore we note that $\mathbb{J}'(u_n) = E[\alpha u_n + \tilde{p}_n]$ and $\mathbb{J}'(u) = E[\alpha u + \tilde{p}]$. From the iteration scheme, we deduce that

$$u_{n+1} - u = \mathbb{P}(u_n - \rho E[\alpha u_n + \tilde{p}_n]) - \mathbb{P}(u - \rho E[\alpha u + \tilde{p}]) = \mathbb{P}(u_n - \rho \mathbb{J}'(u_n)) - \mathbb{P}(u - \rho \mathbb{J}'(u)).$$

By the character of the projection operator $\mathbb{P}$, we have

$$\begin{aligned} \|u_{n+1} - u\|_0^2 &= \|\mathbb{P}(u_n - \rho \mathbb{J}'(u_n)) - \mathbb{P}(u - \rho \mathbb{J}'(u))\|_0^2 \\ &\leqslant \|u - u_n - \rho(\mathbb{J}'(u) - \mathbb{J}'(u_n))\|_0^2 \\ &= \|u - u_n\|_0^2 + \rho^2 \|\mathbb{J}'(u) - \mathbb{J}'(u_n)\|_0^2 - 2\rho(\mathbb{J}'(u) - \mathbb{J}'(u_n), u - u_n) \\ &\leqslant \|u - u_n\|_0^2 + C\rho^2 \|u - u_n\|_0^2 - c\rho \|u - u_n\|_0^2 \\ &\leqslant (1 - c\rho(1 - \frac{C\rho}{c})) \|u - u_n\|_0^2. \end{aligned} \tag{6.5}$$

Choose ρ such that

$$0 \leqslant 1 - c\rho(1 - \frac{C\rho}{c}) \leqslant \delta. \tag{6.6}$$

Then

$$\|u_{n+1} - u\|_0^2 \leqslant \delta \|u_n - u\|_0^2 \leqslant \delta^{n+1} \|u_0 - u\|_0^2. \tag{6.7}$$

This completes the proof. □

6.2 Numerical experiments

In this section, we present three numerical examples to illustrate our analytical results above. In these examples, for the state and co-state y, p, we take the Legendre polynomial in every direction of the random space Γ_i (whose order is 3) and use the piecewise linear finite element in the space D. For the control u, we use the piecewise constant finite element in the space D.

Example 1 We take space domain $D = [-1,1]\times[-1,1]$, and stochastic domain $\Gamma_i = [-1,1]$, $a(x,\xi) = 10 + \sum_{i=1}^{N} \xi_i$. We consider the following model problem:

$$\min_{u\in L^2(D), E[\int_D y dx]\geq 0} \mathbb{J}(u) = \min_{u\in L^2(D), E[\int_D y dx]\geq 0} \left(\frac{1}{2}\int_\Gamma \int_D (y-y_d)^2 \rho(\xi) dx d\xi + \frac{\alpha}{2}\int_D u^2 dx\right) \tag{6.8}$$

subject to

$$\begin{aligned} -\nabla\cdot(a(x,\xi)\nabla y(x,\xi)) &= f+u, \quad x\in D,\ \xi\in\Gamma, \\ y(x,\xi) &= 0, \quad x\in\partial D,\ \xi\in\Gamma. \end{aligned}$$

The target state $y_d = (1-2\pi^2)\sin(\pi x_1)\sin(\pi x_2) - 1$. The objective is to minimize the expectation of a cost functional and the constraint for the state is $E[\int_D y dx] \geq 0$. We assume each probability density function on Γ_i is uniform, i.e. $\rho_i(\xi_i) = \frac{1}{2}$, $i = 1,\cdots,N$. Thus, the joint probability density function $\rho(\xi)$ of random variable $\xi = (\xi_1,\cdots,\xi_N)$ is $\frac{1}{2^N}$. Let $\alpha = 1$ and the solutions for this problem are as follows:

$$\begin{aligned} p &= \frac{1}{a(x,\xi)}\sin(\pi x_1)\sin(\pi x_2), \\ y &= \sin(\pi x_1)\sin(\pi x_2), \\ u &= -\frac{E[p]}{\alpha}, \\ f &= 2\pi^2\sin(\pi x_1)\sin(\pi x_2)a(x,\xi) - u, \\ \lambda &= -1. \end{aligned}$$

In this example, we compute for two cases. First case for $N = 5$ are shown in Table 2. For simplicity, Γ_i has no partition and we use the same mesh in D for the state, the co-state and the control. Here h is the mesh size for the discretized optimal condition.

Table 2: The error for optimal control problem

h_s	$\|u_h-u\|_0$	$\|y_h-y\|_{0,\rho}$	$\|y_h-y\|_{1,\rho}$	$\|p_h-p\|_{0,\rho}$	$\|p_h-p\|_{1,\rho}$	$\lvert\lambda-\lambda_h\rvert$
1/4	0.2010	0.3918	1.1302	0.0404	0.1180	2.6156e-04
1/8	0.1026	0.1013	0.5654	0.0105	0.0595	6.3795e-05
1/16	0.0512	0.0252	0.2812	0.0026	0.0297	1.6151e-05
1/32	0.0249	0.0060	0.1413	6.1837e-04	0.0148	3.9977e-06
1/64	0.0108	0.0012	0.0711	1.1611e-04	0.0075	9.7267e-07

From Table 2, we can see the approximation errors of u, y, p, λ are at least linearly decreased as the meshes decrease.

The second case is for $N=2$ in which the stochastic space could be divided into quadrilateral mesh. We use the same degree of piecewise polynomials for each direction ξ_j which is labeled as p. Since the control is independent of random, we only compute the state and co-state to show the change of error with the change of h_r. The result is shown in Table 3 in which the mesh size of physical space is $h_s=\frac{1}{64}$. From Table 3, we find that the numerical results are consistent with our theoretical results in Theorem 4.

Table 3: The error for optimal control problem

p	h_r	$\|y_h-y\|_{1,\rho}$	$\|p_h-p\|_{1,\rho}$
1	1	0.3674	0.0442
1	1/2	0.0875	0.0103
1	1/4	0.0224	0.0025
1	1/8	0.0059	6.4103e-04
2	1/2	0.0443	0.0053
2	1/4	0.0056	6.5432e-04
2	1/8	7.1795e-04	8.2825e-05

Example 2 We take space domain $D=[-1,1]\times[-1,1]$, and stochastic domain $\Gamma_i=[-1,1]$, $a(x,\xi)=20+\sum_{i=1}^{N}\frac{3\sin(i\pi(x_1-x_2))}{(i\pi)^2}\xi_i$. We consider the following model problem:

$$\min_{u\in L^2(D),E[\int_D ydx]\geq 0}\mathbb{J}(u)=\min_{u\in L^2(D),E[\int_D ydx]\geq 0}\left(\frac{1}{2}\int_\Gamma\int_D(y-y_d)^2\rho(\xi)dxd\xi+\frac{\alpha}{2}\int_D u^2dx\right) \tag{6.9}$$

subject to

$$\begin{aligned}-\nabla\cdot(a(x,\xi)\nabla y(x,\xi))&=u,\quad x\in D,\ \xi\in\Gamma,\\ y(x,\xi)&=0,\quad x\in\partial D,\ \xi\in\Gamma.\end{aligned}$$

The target state $y_d=1$. The objective is to minimize the expectation of a cost functional, and the deterministic control is constrained by the condition $E[\int_D ydx]\geq 0$. We assume that each probability density function on Γ_i is uniform, i.e. $\rho_i(\xi_i)=\frac{1}{2}$, $i=1,\cdots,N$. Thus, the joint probability density function $\rho(\xi)$ of random variable $\xi=(\xi_1,\cdots,\xi_N)$ is $\frac{1}{2^N}$. Here we present two cases for this problem with $N=5,10$. Since the exact solution for this problem could not be given, we contrast the objective function value with the change of α.

For simplicity, Γ_i has no partition and we use the same mesh in D for the state, the co-state and the control. The results for $N=5,10$ are shown in Table 4. Since the exact solution is unknown, following the idea in [29] and [46], we let the parameter α approximating zero, so that the relative errors of y_h to y_d should approximate zero and the functional values should be getting smaller. From Table 4 we can see that $\frac{E[\|y_h-y_d\|^2]}{\|y_d\|^2}$ and the objective function values $J_h(y_h,u_h)$ are smaller, respectively, as the value α becomes smaller. And the mean value for y and u in $N=10$ with $\alpha=0.1$ and $\alpha=0.0001$ are shown in Fig. 1-2.

Table 4: The result for optimal control problem with order = 3 and h = 1/16

N	α	$\frac{E[\|y_h-y_d\|^2]}{\|y_d\|^2}$	$J_h(y_h,u_h)$
5	0.1	0.9990	1.9986
5	0.01	0.9905	1.9863
5	0.001	0.9137	1.8741
5	0.0001	0.5556	1.2955
10	0.1	0.9998	2.0001
10	0.01	0.9983	1.9977
10	0.001	0.9834	1.9773
10	0.0001	0.8582	1.8012

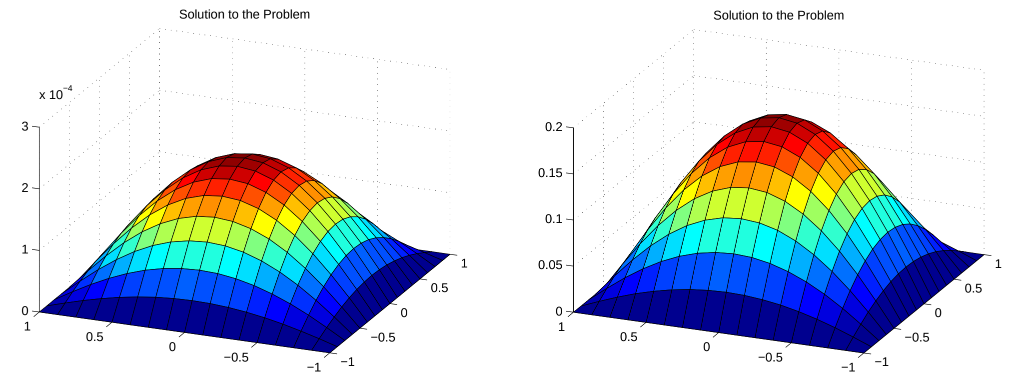

Fig. 1: Mean of y for N=10, $\alpha = 0.1$(left) and $\alpha = 0.0001$(right) for Example 2.

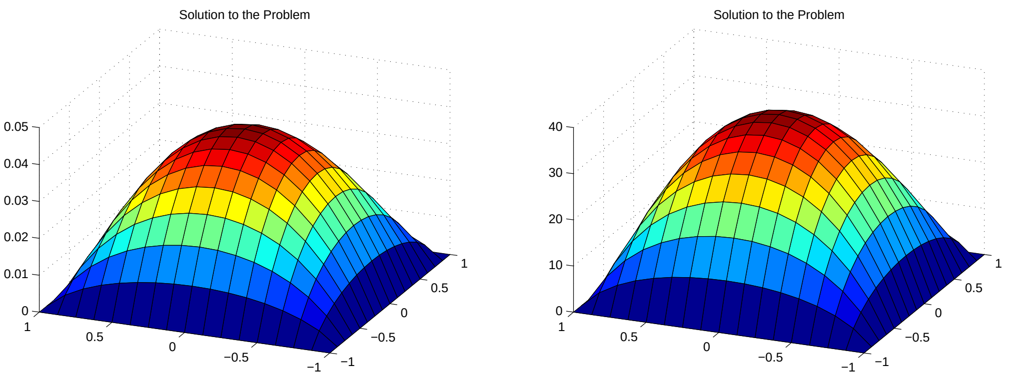

Fig. 2: Value of u for N=10, $\alpha = 0.1$(left) and $\alpha = 0.0001$(right) for Example 2.

Example 3 We take space domain $D=[-1,1]\times[-1,1]$, and stochastic domain $\Gamma_i=[-1,1]$, the random coefficient $a(x,\xi)$ is in the form of the KL(Karhunen-Loeve) expansion [6]:

$$a(x,\xi)=\mu(x)+\sigma\sqrt{3}\sum_{i=1}^{N}\sqrt{\lambda_i}\phi_i(x)\xi_i,$$

where $x\in D$, and $(\lambda_i,\phi_i)_{i=1}^N$ are the eigenpairs of the integral operator associated with $C[a]$. $C[a]$ is the covariance function

$$C[a](x,\bar{x})=\sigma^2 exp(-\frac{|x_1-\bar{x}_1|}{l_1}-\frac{|x_2-\bar{x}_2|}{l_2}),$$

where σ denotes the standard deviation and l_1,l_2 are correlation lengths.

We consider the following model problem:

$$\min_{u\in L^2(D),E[\int_D ydx]\geq 0}\mathbb{J}(u)=\min_{u\in L^2(D),E[\int_D ydx]\geq 0}(\frac{1}{2}\int_\Gamma\int_D(y-y_d)^2\rho(\xi)dxd\xi+\frac{\alpha}{2}\int_D u^2dx) \tag{6.10}$$

subject to

$$\begin{aligned}-\nabla\cdot(a(x,\xi)\nabla y(x,\xi))&=u,\quad x\in D,\ \ \xi\in\Gamma,\\ y(x,\xi)&=0,\quad x\in\partial D,\ \ \xi\in\Gamma.\end{aligned}$$

The target state $y_d=x_1^2+x_2^2$. $\mu(x)=1$, $\sigma=0.3$, $l_1=2$, $l_2=2$. The objective is to minimize the expectation of a cost functional, and the deterministic control is constrained by the condition $E[\int_D ydx]\geq 0$. We assume each probability density function on Γ_i is uniform, i.e. $\rho_i(\xi_i)=\frac{1}{2}$, $i=1,\cdots,N$. Thus, the joint probability density function $\rho(\xi)$ of random variable $\xi=(\xi_1,\cdots,\xi_N)$ is $\frac{1}{2^N}$. We present one case for this problem with $N=20$. Since the exact solution for this problem is un-known, we contrast the objective function value with the change of α.

For simplicity, Γ_i has no partition and we use the same mesh in D for the state, the co-state and the control. The results for $N=20$ are shown in Table 5. From Table 5 we can see that $\frac{E[\|y_h-y_d\|^2]}{\|y_d\|^2}$ and the objective function values $J_h(y_h,u_h)$ are smaller, respectively, as the value α becomes smaller. These simulation results are similar to that in [46]. The mean value for y and u for $N=20$ with different α are shown in Fig. 3-4.

Table 5: The result for optimal control problem with order = 3 and h = 1/16

N	α	$\frac{E[\|y_h-y_d\|^2]}{\|y_d\|^2}$	$J_h(y_h,u_h)$
20	1	0.9973	1.2282
20	0.1	0.9755	1.2085
20	0.01	0.8761	1.1094

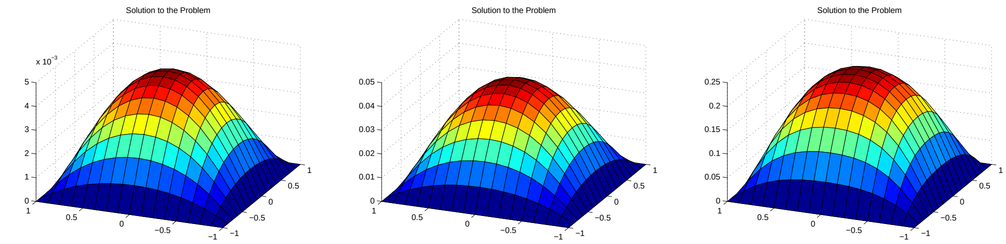

Fig. 3: Mean of y for N=20, $\alpha = 1$(left), $\alpha = 0.1$(middle) and $\alpha = 0.01$(right) for Example 3.

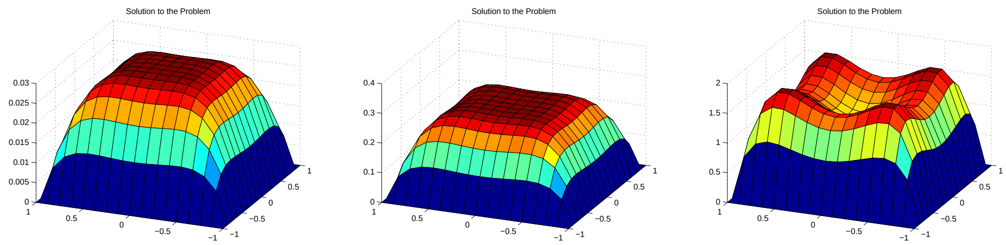

Fig. 4: Mean of y for N=20, $\alpha = 1$(left), $\alpha = 0.1$(middle) and $\alpha = 0.01$(right) for Example 3.

7 Conclusions

We have investigated a stochastic Galerkin approximation scheme for a model optimal control problem governed by an elliptic PDE with random field in its coefficients and state-meant constraints. We successfully obtain the necessary and sufficient optimality conditions for the state-constrained stochastic optimal control problem in the first time in the literature. We further establish a Stochastic Galerkin scheme to approximate the optimality system in the spatial space and the probability space. We have developed an efficient projection algorithm for solving the problem, and the numerical results presented have showed consistency with our theoretical the a priori error estimates derived.

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