Remarks on $\lambda$-commuting operators (The research of geometric structures in quantum information based on Operator Theory and related topics)

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Remarks on $\lambda$-commuting operators

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Abstract

In this paper, we study properties of $\lambda$-commuting operators. We give spectral and local spectral relations between $\lambda$-commuting operators. Moreover, we show that the operators $\lambda$-commuting with a unilateral shift are representable as weighted composition operators. We also provide the polar decomposition of the product of $(\lambda, \mu)$-commuting operators where $\lambda$ and $\mu$ are real numbers with $\lambda \mu > 0$. Finally, we find the restriction of $\mu$ for the product of $(\lambda, \mu)$-commuting quasihyponormal operators to be quasihyponormal.

1 Introduction

This paper is part of a paper submitted for possible publication in some journal. Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{le}(T)$, and $r(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the left essential spectrum, and the spectral radius of $T$, respectively.

We say that operators $S$ and $T$ in $\mathcal{L}(\mathcal{H})$ are $\lambda$-commuting if $ST = \lambda TS$, where $\lambda$ is a complex number. In [3], S. Brown showed that every operator $\lambda$-commuting with a nonzero compact operator has a nontrivial hyperinvariant subspace, as one of the generalizations of the famous Lomonosov’s theorem (see [10]). Since then, many mathematicians have been interested in $\lambda$-commuting operators.

Different classes of operators can be specified depending on the restriction on $\lambda$ (see [11]). An operator $T \in \mathcal{L}(\mathcal{H})$ is called normal if $T^*T = TT^*$. We say that $T \in \mathcal{L}(\mathcal{H})$ is hyponormal if $T^*T \geq TT^*$. In [12], J. Yang and H. Du showed that if $S$ and $T$ are $\lambda$-commuting normal operators with $ST \neq 0$, then $|\lambda| = 1$. Moreover, M. Cho, J. Lee, and T. Yamazaki proved in [4] that if $S$ and $T$ are $\lambda$-commuting operators such that both $S^*$ and $T$ are hyponormal and $ST \neq 0$, then $|\lambda| \leq 1$.

For $\lambda, \mu \in \mathbb{C}$, operators $S, T \in \mathcal{L}(\mathcal{H})$ are said to be $(\lambda, \mu)$-commuting if $ST = \lambda TS$ and $S^*T = \mu TS^*$. By Fuglede-Putnam Theorem, if $A, B \in \mathcal{L}(\mathcal{H})$ are normal and $AX = XB$ for some $X \in \mathcal{L}(\mathcal{H})$, then $A^*X = XB^*$ (see [7]). Hence, if $S$ is

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normal and $\lambda$-commuting with $T$, then $S$ and $T$ are $(\lambda, \overline{\lambda})$-commuting. For a simple example, given any fixed complex constant $\lambda$ with $|\lambda| \leq 1$, suppose $D$ is a diagonal operator given by $De_n = \lambda^n e_n$ for $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Then, every weighted shift $W$ on $\mathcal{H}$ given by $We_n = \alpha_n e_{n+1}$ for $n \geq 0$ satisfies $DW = \lambda WD$. Since $D$ is normal, the operators $D$ and $W$ are $(\lambda, \overline{\lambda})$-commuting by Fuglede-Putnam Theorem; we also observe that $W$ and $D$ are $(\lambda^{-1}, \lambda)$-commuting.

For another example, the $2 \times 2$ matrices $S = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ are $(\frac{1}{3}, 3)$-commuting.

In this paper, we study properties of $\lambda$-commuting operators. We give spectral and local spectral relations between $\lambda$-commuting operators. Moreover, we show that the operators $\lambda$-commuting with a unilateral shift are representable as weighted composition operators. We also provide the polar decomposition of the product of $(\lambda, \mu)$-commuting operators where $\lambda$ and $\mu$ are real numbers with $\lambda \mu > 0$. Finally, we find the restriction of $\mu$ for the product of $(\lambda, \mu)$-commuting quasihyponormal operators to be quasihyponormal.

2 Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the single-valued extension property (or SVEP) if for every open set $G$ in $\mathbb{C}$ and every analytic function $f : G \to \mathcal{H}$ with $(T - z)f(z) \equiv 0$ on $G$, we have $f(z) \equiv 0$ on $G$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the set $\rho_T(x)$, called the local resolvent of $T$ at $x$, consists of elements $z_0$ in $\mathbb{C}$ such that there exists an $\mathcal{H}$-valued analytic function $f(z)$ defined in a neighborhood of $z_0$ which verifies $(T - z)f(z) \equiv x$. The local spectrum of $T$ at $x$ is given by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Moreover, we define the local spectral subspace of $T$ as $H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$, where $F$ is a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have Dunford's property $(C)$ if $H_T(F)$ is closed for each closed subset $F$ of $\mathbb{C}$. We say that $T \in \mathcal{L}(\mathcal{H})$ is said to have Bishop's property $(\beta)$ if for every open subset $G$ of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of $G$, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of $G$. The following implications are well known (see [2], [5], or [9] for more details):

Bishop's property $(\beta) \Rightarrow$ Dunford's property $(C) \Rightarrow$ SVEP.
3  Main results

In this section, we give several properties of $\lambda$-commuting operators. We first consider the product of $\lambda$-commuting operators. We say that $T \in \mathcal{L}(\mathcal{H})$ is quasinilpotent if $\sigma(T) = \{0\}$.

**Theorem 3.1.** Let $S$ and $T$ be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. Then the following statements hold:
(i) $r(ST) \leq r(S)r(T)$ and $r(TS) \leq r(S)r(T)$.
(ii) If $|\lambda| \neq 1$, then $ST$ and $TS$ are quasinilpotent.

Recall that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is called normaloid if $\|T\| = r(T)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to class $A$ if $|T^2| \geq |T|^2$. Every operator which belongs to class $A$ is normaloid, and hyponormal operators belong to class $A$ (see [6]).

**Corollary 3.2.** Let $S$ and $T$ be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$ and $ST$ belongs to class $A$. If $S$ or $T$ is quasinilpotent, then $ST = TS = 0$.

We next provide spectral properties of $\lambda$-commuting operators.

**Theorem 3.3.** Suppose that $S, T \in \mathcal{L}(\mathcal{H})$ satisfy $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. For $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$, the following assertions hold:
(i) either $0 \in \sigma_\Delta(T)$ or else $\lambda \sigma_\Delta(S) \subset \sigma_\Delta(S)$;
(ii) either $0 \in \sigma_\Delta(S)$ or else $\sigma_\Delta(T) \subset \lambda \sigma_\Delta(T)$.

**Remark.** One can derive that $T \ker(S - \mu) \subset \ker(S - \lambda \mu)$ and $S \ker(T - \mu) \subset \ker(\lambda T - \mu)$ for each $\mu \in \mathbb{C}$. Hence, $\ker(S)$ and $\ker(T)$ are common invariant subspaces for $S$ and $T$.

**Corollary 3.4.** Let $S$ and $T$ be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. Then the following assertions hold:
(i) If $0 \not\in \sigma_{ap}(T)$, then $\sigma_{ap}(S) = \{0\}$ or $|\lambda| \leq 1$.
(ii) If $0 \not\in \sigma_{ap}(S)$, then $\sigma_{ap}(T) = \{0\}$ or $|\lambda| \geq 1$.

Hence, if $0 \not\in \sigma_{ap}(S) \cup \sigma_{ap}(T)$, then $|\lambda| = 1$.

When $\lambda$ is a root of unity, the inclusions in Theorem 3.3 become equalities, as follows:

**Corollary 3.5.** Let $S, T \in \mathcal{L}(\mathcal{H})$ satisfy that $ST = \lambda TS$ where $\lambda$ is a root of unity. Then the following statements hold for $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$:
(i) If $0 \not\in \sigma_\Delta(T)$, then $\sigma_\Delta(S) = \lambda \sigma_\Delta(S)$;
(ii) If $0 \not\in \sigma_\Delta(S)$, then $\sigma_\Delta(T) = \lambda \sigma_\Delta(T)$.
Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be an $m$-isometry if $\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}T^{*j}T^{j} = 0$, where $m$ is a positive integer. In [1], it turned out that every $m$-isometry has approximate point spectrum contained in the unit circle.

**Corollary 3.6.** Suppose that $S$ and $T$ are operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. If $|\lambda| \neq 1$ and $S$ is an $m$-isometry for some positive integer $m$, then $0 \in \sigma_p(T)$.

We now consider local spectral properties of $\lambda$-commuting operators.

**Proposition 3.7.** Let $S, T \in \mathcal{L}(\mathcal{H})$. If $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$, then the following statements hold:

(i) $\sigma_S(Tx) \subset \lambda \sigma_S(x)$ and $\lambda \sigma_T(Sx) \subset \sigma_T(x)$ for all $x \in \mathcal{H}$.

(ii) $TH_S(F) \subset H_S(\lambda F)$ for any subset $F$ of $\mathbb{C}$.

(iii) If $\lambda \neq 0$, then $SH_T(\lambda F) \subset H_T(F)$ for any subset $F$ of $\mathbb{C}$.

**Corollary 3.8.** Suppose that $S, T \in \mathcal{L}(\mathcal{H})$ are $\lambda$-commuting where $\lambda$ is a root of unity with order $k$. If $\lambda$ is a root of unity with order $k$ and $S$ has Dunford's property $(C)$, then $H_S(F)$ is a common invariant subspace of $S$ and $T^k$, where $F$ is any closed subset of $\mathbb{C}$.

Combining Corollary 3.8 with [12], we obtain the following corollary.

**Corollary 3.9.** Assume that $S, T \in \mathcal{L}(\mathcal{H})$ are $\lambda$-commuting. If $S \in \mathcal{L}(\mathcal{H})$ is hyponormal and $\sigma(ST)$ consists of $k$ distinct nonzero elements, then $H_S(F)$ is a common invariant subspace of $S$ and $T^k$.

For an operator $T \in \mathcal{L}(\mathcal{H})$, we define the quasi-nilpotent part of $T$, denoted by $H_0(T)$, as $H_0(T) := \{x \in \mathcal{H} : \lim_{n \to \infty} \|T^n x\|^\frac{1}{n} = 0\}$ (see [2] and [9] for more details).

**Proposition 3.10.** Let $S, T \in \mathcal{L}(\mathcal{H})$. If $ST = \lambda TS$ for some $\lambda \in \mathbb{C} \setminus \{0\}$, then $H_0(S)$ is invariant for $T$.

Let $H^2 = H^2(\mathbb{D})$ be the canonical Hardy space of the open unit disk $\mathbb{D}$, and let $H^\infty$ be the space of bounded functions in $H^2$. For an analytic map $\varphi$ from $\mathbb{D}$ into itself and $u \in \mathbb{D}$, the weighted composition operator $W_{f, \varphi} : H^2 \to H^2$ is defined by $W_{u, \varphi} h = u \cdot (h \circ \varphi)$. In particular, $C_{\varphi} := W_{1, \varphi}$ is said to be a composition operator. In the following theorem, we assert that if $|\lambda| = 1$, then the operators $\lambda$-commuting with the unilateral shift $U$ on $H^2$ given by $(Uf)(z) = zf(z)$ are representable as weighted composition operators.
Theorem 3.11. Let $U$ be the unilateral shift on $H^2$ given by $(Uf)(z) = zf(z)$. Assume that $S \in \mathcal{L}(H^2)$ and $\lambda \in \partial \mathbb{D}$. Then $SU = \lambda US$ if and only if $S = W_{u,\lambda z}$ for some $u \in H^\infty$.

For a bounded sequence $\{\alpha_n\}_{n=0}^\infty$ in $\mathbb{C}$, a weighted shift on $\mathcal{H}$ with weights $\{\alpha_n\}$ is an operator $T$ such that $Te_n = \alpha_ne_{n+1}$ for $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ denotes an orthonormal basis for $\mathcal{H}$.

Proposition 3.12. Let $S$ and $T$ be weighted shifts in $\mathcal{L}(\mathcal{H})$ with weights $\{\alpha_n\}$ and $\{\beta_n\}$, respectively, and let $\lambda \in \mathbb{C}$. Then $ST = \lambda TS$ if and only if $\alpha_{n+1}\beta_n = \lambda \beta_{n+1}\alpha_n$ for all $n$.

In the following example, we consider the case when $S$ is the Bergman shift determined by the weights $\{\sqrt{n+1}/n+2\}_{n=0}^\infty$.

Example 3.13. If $S$ is the Bergman shift, then its weights form an increasing sequence. Then $S$ is hyponormal. Suppose that $T$ is any weighted shift with positive weights $\{\beta_n\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. By Proposition 3.12, it follows that $ST = \lambda TS$ if and only if $\beta_{n+1} = \frac{n+2}{\lambda\sqrt{(n+1)(n+3)}}\beta_n$ for $n \geq 0$, that is, $\beta_n = \frac{1}{\lambda^n}\sqrt{\frac{2(n+1)}{n+2}}\beta_0$ for $n \geq 0$.

For a positive integer $n > 1$, define $J_r$ and $J_l$ on $\oplus_1^n \mathcal{H}$ by

$$J_r = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \quad \text{and} \quad J_l = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proposition 3.14. Let $T \in \mathcal{L}(\oplus_1^n \mathcal{H})$. For a complex number $\lambda$, the following statements hold:

(i) $TJ_r = \lambda J_r T$ if and only if

$$T = \begin{pmatrix} T_1 & 0 & \cdots & 0 & 0 \\ T_2 & \lambda T_1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ T_{n-1} & \lambda T_{n-2} & \ddots & \ddots & \ddots \\ T_n & \lambda T_{n-1} & \cdots & \cdots & \lambda^{n-2} T_1 & \lambda^{n-1} T_1 \end{pmatrix}.$$
where \( \{T_j\}_{j=1}^{n} \subset \mathcal{L}(\mathcal{H}) \).

(ii) \( T J_l = \lambda J_l T \) if and only if

\[
T = \begin{pmatrix}
\lambda^{n-1} T_n & \lambda^{n-2} T_{n-1} & \cdots & \cdots & \lambda T_2 & T_1 \\
0 & \lambda^{n-2} T_n & \cdots & \cdots & \lambda T_3 & T_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda T_{n-1} & T_{n-2} \\
0 & 0 & \cdots & \lambda T_n & T_{n-1} \\
0 & 0 & \cdots & 0 & T_n
\end{pmatrix}
\]

where \( \{T_j\}_{j=1}^{n} \subset \mathcal{L}(\mathcal{H}) \).

We next consider \((\lambda, \mu)\)-commuting operators. To obtain the polar decomposition of the product of \((\lambda, \mu)\)-commuting operators, we show that their partial isometric parts and positive parts satisfy the following extended commuting relationships.

**Lemma 3.15.** Let \( S, T \in \mathcal{L}(\mathcal{H}) \) be \((\lambda, \mu)\)-commuting where \( \lambda \) and \( \mu \) are real numbers with \( \lambda \mu > 0 \). If \( S = U_S |S| \) and \( T = U_T |T| \) denote the polar decompositions, then the following statements hold:

(i) \( |T|S = (\lambda^{-1} \mu)^{\frac{1}{2}} S |T| \) and \( |S|T = (\lambda \mu)^{\frac{1}{2}} T |S| \);

(ii) \( |S|U_T = (\lambda \mu)^{\frac{1}{2}} U_T |S| \) and \( |T|U_S = (\lambda^{-1} \mu)^{\frac{1}{2}} U_S |T| \);

(iii) \( |S||T| = |T||S|, |S^*||T| = |T||S^*|, \) and \( |S||T^*| = |T^*||S| \);

(iv) \( U_S U_T = U_T U_S \) and \( U_S U_T = -U_T U_S \) if \( \lambda \) and \( \mu \) are positive, and \( U_S U_T = -U_T U_S \) and \( U_S^* U_T = -U_T U_S^* \) if \( \lambda \) and \( \mu \) are negative.

**Theorem 3.16.** Assume that \( S, T \in \mathcal{L}(\mathcal{H}) \) are \((\lambda, \mu)\)-commuting where \( \lambda \) and \( \mu \) are real numbers with \( \lambda \mu > 0 \). If \( ST = U_{ST} |ST| \) is the polar decomposition, then

\[
U_{ST} = U_S U_T \quad \text{and} \quad |ST| = (\lambda \mu)^{\frac{1}{2}} |S||T|.
\]

In addition, if \( TS = U_{TS} |TS| \) is the polar decomposition, then

\[
U_{TS} = U_T U_S \quad \text{and} \quad |TS| = (\lambda^{-1} \mu)^{\frac{1}{2}} |S||T|.
\]

For an operator \( T \in \mathcal{L}(\mathcal{H}) \) with polar decomposition \( T = U|T| \), the Aluthge transform \( \tilde{T} \) of \( T \) is defined by \( \tilde{T} = |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}} \). In [8], the authors showed several connections between operators and their Aluthge transforms.

**Corollary 3.17.** If \( S, T \in \mathcal{L}(\mathcal{H}) \) are \((\lambda, \mu)\)-commuting operators where \( \lambda \) and \( \mu \) are real numbers with \( \lambda \mu > 0 \), then the following statements hold:

(i) \( \tilde{S} \) and \( \tilde{T} \) are \((\lambda, \mu)\)-commuting and \( \tilde{ST} = |\mu|^{\frac{1}{2}} \tilde{S} \tilde{T} = \lambda |\mu|^{\frac{1}{2}} \tilde{T} \tilde{S} \).

(ii) \( \tilde{S} \) and \( T \) are \((\lambda, \mu)\)-commuting.

(iii) \( S \) and \( \tilde{T} \) are \((\lambda, \mu)\)-commuting.
Corollary 3.18. Let $S, T \in \mathcal{L}(\mathcal{H})$ be $\lambda$-commuting for some nonzero real number $\lambda$. If $S$ is hyponormal and $T$ is normal, then the following statements are equivalent:

(i) $ST$ is hyponormal.
(ii) $\sigma(ST) \neq \{0\}$.
(iii) $\lambda = \pm 1$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be quasinormal if $T^*T$ commutes with $T$.

Corollary 3.19. Let $S, T \in \mathcal{L}(\mathcal{H})$ be $(\lambda, \mu)$-commuting quasinormal operators such that $ST \neq 0$, where $\lambda$ and $\mu$ are real numbers with $\lambda \mu > 0$. Then $ST$ is quasinormal if and only if $\mu = \pm 1$. In particular, if $ST$ is quasinormal and one of $S$ and $T$ is normal, then $\lambda = \mu = \pm 1$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called quasihyponormal if $T^*(T^*T - TT^*)T \geq 0$, or $\|T^2x\| \geq \|T^*Tx\|$ for all $x \in \mathcal{H}$. In the following theorem, we show that if $|\mu| \leq 1$, then the product of two $(\lambda, \mu)$-commuting quasihyponormal operators is again quasihyponormal.

Theorem 3.20. Let $S$ and $T$ be quasihyponormal operators in $\mathcal{L}(\mathcal{H})$ that are $(\lambda, \mu)$-commuting. If $|\mu| \geq 1$, then $ST$ is quasihyponormal. Furthermore, if $\lambda \neq 0$ and $|\mu| \geq 1$, then $TS$ is quasihyponormal.

An operator $T$ in $\mathcal{L}(\mathcal{H})$ is said to be nilpotent if $T^n = 0$ for some positive integer $n$; in this case, the smallest positive integer $n$ with $T^n = 0$ is referred to as the order of $T$.

Corollary 3.21. Let $S$ and $T$ be quasihyponormal operators in $\mathcal{L}(\mathcal{H})$ that are $(\lambda, \mu)$-commuting and $ST \neq 0$. If $|\lambda| \neq 1$ and $|\mu| \geq 1$, then $ST$ is nilpotent of order 2 and one of $S$ and $T$ has a nontrivial invariant subspace.

Corollary 3.22. Let $S \in \mathcal{L}(\mathcal{H})$ be normal and $T \in \mathcal{L}(\mathcal{H})$ be quasihyponormal with $ST \neq 0$. If $ST = \lambda TS$ for some $|\lambda| \geq 1$, then both $ST$ and $TS$ are quasihyponormal; in particular, if $|\lambda| > 1$, then $ST$ and $TS$ are nilpotent of order 2.
References


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