To our parents
Preface

Immersions and submersions, which are special tools in Differential Geometry, also play a fundamental role in Riemannian Geometry, especially when the involved manifolds carry an additional structure (of contact, Hermitian, quaternionic type, etc.). Even if submersions are, in a certain sense, a counterpart of immersions, the corresponding theories are quite different, also from a historical point of view.

The theory of isometric immersions, started with the work of Gauss on surfaces in the Euclidean 3-space, is classical and widely explained in many books, whereas the theory of Riemannian submersion goes back to four decades ago, when B. O’Neill and A. Gray, independently, formulated the basis of such theory, which has hugely been developed in the last two decades. Nowadays several works are still in progress. For instance, a new point of view on Riemannian submersions appears in a paper by H. Karcher in 1999.

Obviously the content of this book is not exhaustive, anyway, the results presented are enough to solve problems concerning many areas, like Theoretical Physics and the theory of Einstein, Einstein-Weyl spaces. This theory falls into a more general context, extensively treated in the Besse’s book *Einstein Manifolds* and, more recently, in *Surveys in differential geometry: essays on Einstein manifolds*, edited by C. Le Brun and M. Wang.

At present, there is no work entirely devoted to a systematic exposition of the basic tools on the theory of Riemannian submersions and its developments. This is the aim of the present book, which is directed to graduate students as well as to researchers interested in Differential Geometry and Theoretical Physics. For this reason, many examples are explained in details and calculations, straightforward for specialists, are not omitted. Examples appear in each chapter. They often involve well-known defin-
itions and concepts, which are presented in the subsequent chapters and can be found through the index. Anyway, the reader should be familiar with the basic concepts of Lie groups, principal bundles and Riemannian Geometry. We refer to the books of Kobayashi and Nomizu [172], Helgason [139], Yano and Kon [350] for a wide treatment of these subjects.

Concerning the content of each chapter, which we are going to outline, we only remark that Chap. 8 is rather independent of the others, with the exception of Chap. 1.

In Chap. 1, where the basic tools of the mentioned theory are given, the main properties of the invariant tensors introduced by B. O'Neill are stated, together with explicit formulas relating the curvatures of the total space, the fibres and the base space. We also describe classical examples, like Hopf fibrations and generalized Hopf fibrations.

Chapter 2 essentially concerns with Riemannian submersions having totally geodesic fibres. In particular we state the classification theorem, due to R. Escobales and A. Ranjan, on the Riemannian submersions with totally geodesic fibres and the standard $m$-sphere as total space. For any $n \geq 1$, the Hopf projections provide examples of the above submersions and the representation theory of Clifford algebras is applied to obtain the uniqueness result, up to equivalence. Then, combining a theorem of R. Escobales with a result of J. Ucci, we classify the Riemannian submersions from $\mathbb{P}_{2n+1}(\mathbb{C})$ onto $\mathbb{P}_n(\mathbb{Q})$ with complete, complex and totally geodesic fibres.

Almost Hermitian submersions are investigated into details in Chap. 3. In particular, we discuss the transference of geometric properties from the total space to the fibres and to the base space. We also describe some results, due to D. L. Johnson, in order to illustrate the relationship between the existence of Kähler submersions and of holomorphic connections on principal bundles over Kähler manifolds. Furthermore, we examine almost Hermitian submersions having integrable horizontal distribution, as it happens for Kähler, almost Kähler and nearly Kähler submersions. Considering locally conformal Kähler submersions, we explain the results of J. C. Marrero and J. Rocha and give a large class of examples, involving generalized Hopf manifolds, also known as Vaisman manifolds. Finally, we discuss almost complex conformal submersions.

Riemannian submersions between Riemannian manifolds equipped with an additional structure of (almost) contact type, firstly studied by B. Watson and D. Chinea, independently, are the subject of Chap. 4. Riemannian submersions from an almost contact metric manifold with an $f$-structure onto an almost quaternionic manifold are also considered. In particular,
we show how 3-Sasakian manifolds can be related with quaternionic Kähler or hyperKähler manifolds. Submersions from a metric $f$-manifold onto an almost Hermitian manifold are also studied. Finally, we consider the concept of hyper $f$-structure due to G. Hernandez and examine into details Riemannian submersions with totally geodesic fibres from a $PS$-manifold to a hyperKähler one.

Chapter 5 deals with the problem of determining Einstein metrics and EinsteinWeyl structures on manifolds which can be considered as the total space or the base space of a submersion. In particular, we explain the method used by W. Ziller for describing the invariant Einstein metrics on the spheres and on the projective spaces. Following the papers of M. Wang, W. Ziller and Y. Sakane, we also discuss the existence of Einstein metrics on the total space of a principal torus bundle over the product of Kähler-Einstein manifolds and, more generally, over a C-space. Conditions on the total space of a principal bundle, due to H. Pedersen and A. Swann, are also stated. We point out the interrelation between Riemannian submersions and Hermitian or almost contact EinsteinWeyl structures. Since in the compact case Hermitian EinsteinWeyl spaces are just Kähler Einstein or Vaisman manifolds, this subject is closely related to Chap. 3. Following I. Vaisman, H. Pedersen, Y. Poon and A. Swann, we prove the existence of a Kähler structure on the leaves space of a compact Vaisman manifold by the vertical foliation. Finally, we state recent results of F. Narita relating Sasakian structures with EinsteinWeyl structures on manifolds which are the total space of a submersion.

Using the concept of $CR$-submanifold of an almost Hermitian manifold, due to A. Bejancu, in Chap. 6 we consider $CR$-submersions and present a result of S. Kobayashi and some generalizations. Then, we show how the theory of Riemannian submersions allows to link submanifolds of Sasakian manifolds to submanifolds of Kähler ones; particular emphasis on the results of H. Reckziegel is given. Finally, using the concept of reflection with respect to a submanifold introduced by B. Y. Chen and L. Vanhecke, we prove recent theorems of F. Narita concerning submersions such that the reflections with respect to the fibres are isometries.

Chapter 7 deals with semi-Riemannian submersions. Lorentzian submersions with totally geodesic fibres are studied. In particular, we prove a theorem of M. Magid stating that, if the total space is the anti-de Sitter space, then the base is holomorphically isometric to the complex hyperbolic space. We also present the classification of semi-Riemannian submersions with totally geodesic fibres from a real or complex pseudo-hyperbolic space.
onto a Riemannian manifold. These results, due to G. Baditoiu and the second author, are closely linked to the subject of Chap. 2. Finally, we investigate semi-Riemannian submersions with totally umbilical or minimal fibres.

Applications of Riemannian submersions in Physics is the tool of the last chapter. In the theory of Kaluza–Klein type, one starts with the hypothesis that the space-time has $(4+m)$-dimensions. Next, because of some dynamical mechanism, one supposes that the ground state of this system is partially compactified, i.e. of type $M^4 \times M^m$, where $M^4$ denotes a 4-dimensional Minkowski space and $M^m$ is a compact $m$-dimensional space. An interesting mechanism for space-time compactification is proposed in the form of a nonlinear sigma model. The general solutions of this model can be expressed in terms of harmonic maps satisfying the Einstein equations. A very general class of solutions is given by Riemannian submersions from the extra-dimensional space onto the space in which the scalar fields of the nonlinear sigma model take values.

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