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Supersymmetry restoration in superstring perturbation theory

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Abstract: Superstring perturbation theory based on the 1PI effective theory approach has been useful for addressing the problem of mass renormalization and vacuum shift. We derive Ward identities associated with space-time supersymmetry transformation in this approach. This leads to a proof of the equality of renormalized masses of bosons and fermions and identities relating fermionic amplitudes to bosonic amplitudes after taking into account the effect of mass renormalization. This also relates unbroken supersymmetry to a given order in perturbation theory to absence of tadpoles of massless scalars to higher order. The results are valid at the perturbative vacuum as well as in the shifted vacuum when the latter describes the correct ground state of the theory. We apply this to SO(32) heterotic string theory on Calabi-Yau 3-folds where a one loop Fayet-Iliopoulos term apparently breaks supersymmetry at one loop, but analysis of the low energy effective field theory indicates that there is a nearby vacuum where supersymmetry is restored. We explicitly prove that the perturbative amplitudes of this theory around the shifted vacuum indeed satisfy the Ward identities associated with unbroken supersymmetry. We also test the general arguments by explicitly verifying the equality of bosonic and fermionic masses at one loop order in the shifted vacuum, and the appearance of two loop dilaton tadpole in the perturbative vacuum where supersymmetry is expected to be broken.

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1 Introduction and summary

In conventional string perturbation theory we set the external states to be 'on-shell' from the beginning by setting the squared momentum k^2 to be equal to $-m^2$ where m is the *tree level* mass of the state. Therefore in this approach we cannot directly compute the renormalized mass of a state at loop level. For this reason this approach is most efficient for dealing with states whose masses are not renormalized in perturbation theory — massless gauge particles and BPS states. If we attempt to apply this approach to states whose masses *are renormalized* in perturbation theory, we encounter certain infrared divergences in the amplitude that can be taken as a signal of mass renormalization, but this does not lead to a systematic procedure for computing the renormalized mass or the S-matrix involving these states.

A related problem in this approach is that the background is fixed to be a solution to the classical equations of motion from the beginning. Therefore we cannot deal with the situation where the classical vacuum is destabilized at the loop level, but there is a nearby stable vacuum. We can detect the instability in perturbation theory — again in the form of certain infrared divergences associated with tadpoles of massless fields, but there is no systematic procedure for computing physical amplitudes by quantizing the theory around the nearby vacuum that does not solve the classical equations of motion.

Ordinary quantum field theories have a systematic procedure for dealing with both these issues. Therefore one might be tempted to resolve these problems by invoking a field theory for strings. While this is possible for bosonic string theory [1, 2], despite many attempts [3–12] there is not yet a fully consistent string field theory for heterotic of type II strings that works at loop level. Furthermore by working with string field theory we lose one of the main advantages of string theory — while in field theory a given amplitude is obtained as a sum of many Feynman diagrams, in string theory at each loop order there is a single term to compute — given by the integral of certain quantity over the moduli space of punctured Riemann surfaces of a given genus. For example, amplitudes computed from bosonic string field theory expresses the result as a sum over many terms, with each term describing the integral over only a small subspace of the moduli space of Riemann surfaces. When we add up the contribution from all the Feynman diagrams of string field theory, we recover the integral over the full moduli space [2, 13].

In [14, 15] (which in turn is based on results of [16–20]) we suggested a way of overcoming this problem by using the formalism of one particle irreducible (1PI) effective theory that combines some of the good features of quantum field theories with some of the good features of string theory. In conventional quantum field theory the 1PI effective action can be used to deal with the problems of mass renormalization and vacuum shift. This is defined as the generating function of off-shell 1PI amplitudes, and is free from all infrared divergences associated with mass renormalization or massless tadpoles.[1] Given the 1PI action we are supposed to first find its local extremum, and then find plane wave solutions to the *classical linearized equations of motion* derived from the action. If these occur at momenta k then the values of $-k^2$ give the renormalized squared masses of the theory. Once we have determined them, the *tree level S-matrix* computed from the 1PI action gives us the full renormalized S-matrix of the original quantum field theory.

In quantum field theory most of the complexities of computing loop amplitudes lie in the computation of the 1PI effective action as it involves sum over many Feynman diagrams. It turns out that in string theory it is possible to directly write down the expression for the 1PI amplitudes without going through the route of first having to formulate a field theory of strings. A 1PI amplitude takes the form of an integral of certain correlation function of the underlying world-sheet superconformal field theory (SCFT) over a subspace of the moduli space of Riemann surfaces called the 1PI subspace. By construction these 1PI subspaces do not include separating type degenerations of the moduli space — the sources of the usual infrared divergences of ordinary string perturbation theory associated with mass renormalization and massless tadpoles. Once the 1PI theory is available we can follow the usual route of quantum field theory, i.e. first find the correct vacuum by solving the classical equations of motion of this 1PI theory, then determine the renormalized mass by examining the solutions to the linearized equations of motion around the shifted vacuum and then compute the tree level S-matrix of the 1PI theory to determine the renormalized S-matrix of the full string theory.

The construction of 1PI action requires extra data [23] compared to what is used in conventional superstring perturbation theory. These data involve a choice of local coordinate system around every puncture and locations of picture changing operators (PCO's) on the Riemann surface.[2] In conventional perturbation theory, where we use BRST invariant and conformally invariant vertex operators as external states, these data do not affect the result, but since the construction of the 1PI theory requires 1PI amplitudes with off-shell external states, the result depends on these additional data. However it was shown in [14, 15] following earlier work of [46] that different 1PI theories associated with two different choices of local coordinates around the punctures and/or different PCO locations are related to each other by field redefinition. Since field redefinition does not affect the renormalized masses or tree level S-matrix elements, we see that the physical quantities computed from the 1PI effective theory do not suffer from any ambiguity.

[1] Infrared divergences associated with internal lines in the loop going on-shell are tamed by the usual $i\epsilon$ prescription [21, 22].

[2] It is expected that the choice of PCO's can be traded in for the choice of local superconformal coordinate system around the punctures in the formalism of [24–45], but the details of this have not been worked out fully.

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It was also found in [14, 15] that the 1PI effective theory constructed this way possesses an infinite dimensional gauge invariance. This includes the general coordinate invariance, local supersymmetry etc. but also includes gauge invariances associated with massive fields. Furthermore since the 1PI action includes the effect of loop corrections, the gauge transformation laws are full quantum corrected gauge transformation laws of the theory. In specific background some of these gauge symmetries leave the background invariant. These then represent 'global symmetries' of the theory, e.g space-time translation invariance, global supersymmetry etc.

Since the full string theory amplitudes are computed from tree amplitudes of the 1PI theory, once we have identified a gauge symmetry of the 1PI theory, there is no further scope for any anomaly — the effect of anomalies would have already been captured in the 1PI action. Therefore we should be able to use the gauge symmetries (and their global counterpart) to derive appropriate Ward identities for the physical amplitudes. This will be one of the main goals of the present paper. Even though our main focus will be on Ward identities associated with local and global supersymmetries, much of our analysis carries through for other symmetries as well.

Supersymmetry Ward identities in string theory have been studied extensively — most recently in [36, 37]. The difference between the results of [36, 37] and the ones discussed here is that while the analysis of [36, 37] had to be restricted to amplitudes in the perturbative vacuum, for external states which do not suffer any mass renormalization, our analysis is valid for general external states and also in vacuum which may not be the perturbative vacuum but a nearby vacuum obtained by condensation of some scalar fields. Indeed one of our goals will be to show how supersymmetry Ward identities can be used to prove the degeneracy between *renormalized* masses of bosons and fermions in the quantum theory. Another goal will be to show how in some SO(32) heterotic string compactification where supersymmetry is broken at the perturbative vacuum at one loop order, it is restored in a nearby vacuum by condensation of a scalar field. In particular we shall verify the bose-fermi degeneracy at one loop and vanishing of tadpoles at two loops in the shifted vacuum, even though at the perturbative vacuum we do not have these properties.

It is worth re-emphasizing that even though the construction of the 1PI action is very similar to that of string field theory, at least in this paper the motivation is quite opposite to that of string field theory. String field theory provides us with a triangulation of the moduli space with different regions of the moduli space coming from different Feynman diagrams, and the main motivation for string field theory stems from the hope that one can use it to study non-perturbative aspects of string theory. Instead here our main motivation is to use 1PI action to study perturbative amplitudes of string theory. With this in mind we try to bring the expression for the S-matrix elements and Ward identities computed from the 1PI action to a form that is close to the usual Polyakov prescription. We find that while the final result is very similar to what we get according to the usual Polyakov prescription, it comes with an in built subtraction procedure that removes all the usual infrared divergences associated with tadpoles and mass renormalization.

Given that much of the analysis is technical, we shall try to summarize our main results here.

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1. Our first main result is to arrive at a prescription for computing S-matrix elements in string theory involving states that undergo mass renormalization. We find that the prescription reduces to the usual Polyakov prescription with two important differences. The usual Polyakov prescription suffers from possible infrared divergences from separating type degeneration where momentum conservation forces the momentum flowing through the long tube connecting the two Riemann surfaces to be either zero or equal to one of the momenta carried by external states. These are the divergences associated with tadpoles and mass renormalization [36, 37]. Our approach leads to the result that in a finite neighborhood of any such separating type degeneration, determined by the choice of local coordinate system used to define the 1PI amplitudes, we have a definite subtraction that removes the possible divergences associated with tadpoles and mass renormalization. The subtraction involving tadpole like degenerations is compensated by having to sum over arbitrary number of insertions of certain external state — not necessarily BRST invariant — which is determined by a recursive procedure. Both the number of insertions and the required number of recursions are of course bounded when we work to some fixed order in the string coupling g_s. These recursion relations are given in (2.27), (2.28) with $|\psi_k\rangle$ denoting the external state to be inserted if we want results up to order g_s^k. Similarly the subtraction associated with degenerations corresponding to mass renormalization is compensated by having to replace the usual BRST invariant external states by a new set of states — not in general BRST invariant — which are determined by a recursive procedure. These recursion relations can be found in (2.52)–(2.54) with $|\phi_n\rangle$ denoting the state that replaces the external state if we want result up to order g_s^n. The subtraction procedure for both kinds of divergences has been explained in section 3.2.

2. This procedure is also valid in situations in which there is more than one possible choice of vacuum at some given order in g_s. The existence of multiple vacua is reflected the existence of multiple solutions to the recursion relation (2.27), (2.28) and our result for S-matrix elements holds in any of the vacua. The results in different vacua differ by the choice of the state $|\psi_k\rangle$ that we have to insert into the amplitude arbitrary number of times.

3. Our second main result is a rederivation of the Ward identities associated with global and local (super-)symmetry when the external states undergo mass renormalization and/or the vacuum is shifted from the usual perturbative vacuum. We find that the Ward identities take forms which are identical to those given in [36, 37] except for the kind of modifications already mentioned in point 1. We also derive the equality of *renormalized* bosonic and fermionic masses to all orders in vacua with unbroken supersymmetry. This result could not be proven with the usual form of the Ward identities that are valid only in the absence of mass renormalization.

4. Refs. [36, 37] described how unbroken supersymmetry to a given order in perturbation theory implies vanishing of tadpoles to one higher order. We rederive this result, but our analysis also includes the cases where the vacuum under consideration is not

the perturbative vacuum but related to it by a shift of order g_s of the string field. This applies in particular to the case of SO(32) heterotic string theory on Calabi-Yau 3-folds where often a Fayet-Iliopoulos term breaks supersymmetry in the perturbative vacuum at one loop order [47–49] but supersymmetry can be restored in a new vacuum obtained by shifting the string field.

5. We apply our general method to obtain some explicit results in SO(32) heterotic string theory on Calabi-Yau manifolds. In particular we explicitly compute the one loop result for masses of a set of scalars and fermions related by supersymmetry and show that in the shifted vacuum they are equal as predicted by supersymmetry even though this equality is absent at the perturbative vacuum. This agrees with the general results quoted in point 3 above. We also show by explicit calculation that the two loop diaton tadpole is non-vanishing at the perturbative vacuum. This is in agreement with general arguments [47–50] and explicit results [37, 51, 52]. We also show that the dilaton tadpole at the shifted vacuum vanishes to this order (i.e. order g_s^4).

The rest of the paper is organized as follows. In section 2 we briefly review the results of [14, 15] that we shall use in our analysis. In section 3 we discuss the construction of the S-matrix elements from the 1PI theory and compare this with the usual prescription for computing S-matrix in string theory. We show that while the final results are very similar, there are subtle differences which precisely remove the divergences associated with massless tadpoles and mass renormalization that we encounter in the usual perturbation theory. In section 4 we study the Ward identities associated with global and local (super-)symmetries. In particular we show how as a consequence of global supersymmetry we have equality of renormalized masses of bosons and fermions and relation between different S-matrix elements involving external bosons and fermions. Local (super-)symmetry on the other hand can be used to show how pure gauge states decouple from the S-matrix. In section 5 we discuss the relationship between existence of global supersymmetry and tadpoles of massless fields and show that global supersymmetry to a given order implies vanishing of tadpoles to one higher order in perturbation theory. This result is in the same spirit as in [36, 37] but we use 1PI effective theory instead of on-shell amplitudes to derive the results. In sections 6–10 we apply these general methods to a specific class of theories — SO(32) heterotic string theory compactified on Calabi-Yau 3-folds. Section 6 contains a review of SO(32) heterotic string theory on Calabi-Yau 3-folds. In section 7 we show that in 1PI effective theory at one loop order we have multiple vacua, and while supersymmetry is broken at the perturbative vacuum, it is restored at the shifted vacuum. In section 8 we verify one of the consequences of the supersymmetry restoration by checking that the degeneracy between scalar and fermion mass for a particular multiplet, that was broken at the perturbative vacuum, is restored at the shifted vacuum. In section 9 we use the method described in section 5 to compute the two loop dilaton tadpole at the perturbative vacuum, and find a non-zero value confirming that supersymmetry is indeed broken there. On the other hand we show in section 10 that to the same order, the dilaton tadpole at the shifted vacuum vanishes. In appendix A we give a glossary of the various symbols that have been used frequently in the rest of the paper. Appendices B–F contain various technical details

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which have been left out from the main text. In appendix G we check the consistency of the results of sections 6–10 with the predictions of supersymmetric low energy effective action.

Throughout this paper we shall work in $\alpha' = 1$ unit.

We shall conclude this introduction with one final comment. In the definition of the 1PI amplitudes given in [14, 15, 19] we have used local coordinate system at punctures / PCO locations in a way that is gluing compatible at separating type degenerations. For discussing various aspects of mass renormalization and vacuum shift this is sufficient. Indeed it was shown in [14–17] that once the choice of local coordinates satisfies this condition, the renormalized masses and S-matrix are independent of any other details of the choice of local coordinates and PCO locations. However it may happen that for analyzing other aspects of perturbation theory, e.g. in dealing with the standard infra-red divergences in loop amplitudes, we may need gluing compatibility at non-separating type degenerations as well. This can be easily accommodated in our formalism by restricting the possible choice of local coordinate system and PCO locations in an appropriate manner.

2 Review

We begin by reviewing our notation and some of the main results of [14, 15]. Since the analysis in the heterotic and type II string theories are similar, we shall carry out our discussion mostly in the context of heterotic string theory and point out briefly where the analysis in type II string theories differs.

2.1 World-sheet theory

The world-sheet theory for any heterotic string compactification at string tree level contains a matter superconformal field theory with central charge (26,15), and a ghost system of total central charge $(-26, -15)$ containing anti-commuting b, c, $\bar{b}$, $\bar{c}$ ghosts and commuting β, γ ghosts. The (β, γ) system can be bosonized as [53]

$$\gamma = \eta\, e^{\phi}, \quad \beta = \partial\xi\, e^{-\phi}, \quad \delta(\gamma) = e^{-\phi}, \quad \delta(\beta) = e^{\phi}\,, \tag{2.1}$$

where ξ, η are fermions and ϕ is a scalar with background charge. We assign (ghost number, picture number, GSO) quantum numbers to various fields as follows:

$$\begin{aligned} &c, \bar{c} : (1, 0, +), \qquad b, \bar{b} : (-1, 0, +), \qquad \gamma : (1, 0, -), \qquad \beta : (-1, 0, -), \\ &\xi : (-1, 1, +), \qquad \eta : (1, -1, +), \qquad e^{q\phi} : (0, q, (-1)^q)\,. \end{aligned} \tag{2.2}$$

The operator products of these fields take the form

$$c(z)b(w) = (z-w)^{-1} + \cdots, \quad \xi(z)\eta(w) = (z-w)^{-1} + \cdots, \quad e^{q_1\phi(z)} e^{q_2\phi(w)} = (z-w)^{-q_1 q_2} e^{(q_1+q_2)\phi(w)} + \cdots, \tag{2.3}$$

where $\cdots$ denote less singular terms. We denote by $\bar{T}_m(\bar{z})$ the anti-holomorphic part of the matter stress tensor, by $T_m(z)$ the holomorphic part of the matter stress tensor, by $T_{\beta,\gamma}(z)$ the stress tensor of the (β, γ) system and by $T_F(z)$ the world-sheet supersymmetry current in the matter sector. In terms of these the BRST charge is given by

$$Q_B = \oint dz j_B(z) + \oint d\bar{z} \bar{j}_B(\bar{z})\,, \tag{2.4}$$

where

$$\bar{j}_B(\bar{z}) = \bar{c}(\bar{z})\bar{T}_m(\bar{z}) + \bar{b}(\bar{z})\bar{c}(\bar{z})\bar{\partial}\bar{c}(\bar{z}), \tag{2.5}$$

$$j_B(z) = c(z)(T_m(z) + T_{\beta,\gamma}(z)) + \gamma(z)T_F(z) + b(z)c(z)\partial c(z) - \frac{1}{4}\gamma(z)^2 b(z), \tag{2.6}$$

and $\oint$ is normalized so that $\oint dz/z = 1$, $\oint d\bar{z}/\bar{z} = 1$. The picture changing operator (PCO) $\mathcal{X}$ is given by [53, 54]

$$\mathcal{X}(z) = \{Q_B, \xi(z)\} = c\partial\xi + e^{\phi}T_F - \frac{1}{4}\partial\eta e^{2\phi}b - \frac{1}{4}\partial\left(\eta e^{2\phi}b\right). \tag{2.7}$$

This is a BRST invariant dimension zero primary operator and carries picture number 1. Finally to get the signs of various correlation functions we need to describe our normalization condition for the SL$(2,C)$ invariant vacuum $|0\rangle$. We choose this to be

$$\langle 0|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1\xi_0 e^{-2\phi(z)}|0\rangle = V, \tag{2.8}$$

where V is the volume of space-time (also equal to $(2\pi)^D\delta^{(D)}(0)$ where the argument of the delta function represents zero value of D-component momentum vector). In subsequent discussions we shall set the space-time volume V to unity. Also for most of our analysis we shall work in the small Hilbert space [53] containing states annihilated by η_0, and include the ξ_0 factor in the definition of the inner product to write

$$\langle 0|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1 e^{-2\phi(z)}|0\rangle = 1. \tag{2.9}$$

For type II string theories the world-sheet theory of matter sector has central charge $(15,15)$. The ghost system now also includes left-moving $(\bar{\beta},\bar{\gamma})$ system so that the total central charges of the ghost system now is $(-15,-15)$. The left-moving BRST current $\bar{j}_B(\bar{z})$ now contains extra terms as in (2.6) and we have left-handed PCO $\bar{\mathcal{X}}(\bar{z})$ given by an expression identical to (2.7) with all right-handed fields replaced by their left-handed counterpart.

We denote by $\mathcal{H}_T$ the Hilbert space of GSO even states of the matter-ghost CFT with arbitrary ghost and picture numbers, with coefficients taking values in the grassmann algebra, satisfying the constraints

$$|s\rangle \in \mathcal{H}_T \quad \text{iff} \quad b_0^-|s\rangle = 0, \quad L_0^-|s\rangle = 0, \quad \eta_0|s\rangle = 0, \quad \bar{\eta}_0|s\rangle = 0\,(\text{in type II}), \tag{2.10}$$

where $\bar{L}_n$ and L_n denote total Virasoro generators in the left and right-moving sectors of the world-sheet theory, and

$$b_0^\pm \equiv (b_0 \pm \bar{b}_0), \quad L_0^\pm \equiv (L_0 \pm \bar{L}_0), \quad c_0^\pm \equiv \frac{1}{2}(c_0 \pm \bar{c}_0). \tag{2.11}$$

In the heterotic theory $\mathcal{H}_T$ decomposes into a direct sum of the Neveu-Schwarz (NS) sector $\mathcal{H}_{NS}$ and Ramond (R) sector $\mathcal{H}_R$. In the type II string theories the corresponding decomposition is $\mathcal{H}_T = \mathcal{H}_{NSNS} \oplus \mathcal{H}_{NSR} \oplus \mathcal{H}_{RNS} \oplus \mathcal{H}_{RR}$. We shall denote by $\{|\varphi_r\rangle\}$ and $\{|\varphi^r\rangle\}$ a set of dual basis of $\mathcal{H}_T$ satisfying

$$\langle\varphi^r|c_0^-|\varphi_s\rangle = \delta^r{}_s \qquad \Leftrightarrow \qquad \langle\varphi_s|c_0^-|\varphi^r\rangle = \delta^r{}_s, \tag{2.12}$$

and the completeness relation

$$|\varphi_r\rangle\langle\varphi^r| = |\varphi^r\rangle\langle\varphi_r| = b_0^- \, . \tag{2.13}$$

In heterotic string theory the basis states $|\varphi_r\rangle$ in $\mathcal{H}_{NS}$ are grassmann even for even ghost number and grassmann odd for odd ghost number. In $\mathcal{H}_R$ the situation is opposite. Similar results hold for type II string theories as well. During the intermediate stages of calculation we shall also make use of GSO odd operators. The grassmann parity of GSO odd operators are opposite of that of GSO even operators for given ghost and picture numbers.

In the heterotic string theory we define

$$\mathcal{G}|s\rangle = \begin{cases} |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{NS} \\ \mathcal{X}_0\,|s\rangle & \text{if } |s\rangle \in \mathcal{H}_R \end{cases} , \tag{2.14}$$

while in type II string theories we define

$$\mathcal{G}|s\rangle = \begin{cases} |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{NSNS} \\ \mathcal{X}_0\,|s\rangle & \text{if } |s\rangle \in \mathcal{H}_{NSR} \\ \bar{\mathcal{X}}_0\,|s\rangle & \text{if } |s\rangle \in \mathcal{H}_{RNS} \\ \mathcal{X}_0\bar{\mathcal{X}}_0\,|s\rangle & \text{if } |s\rangle \in \mathcal{H}_{RR} \\ , \end{cases} \tag{2.15}$$

where

$$\mathcal{X}_0 \equiv \oint \frac{dz}{z}\,\mathcal{X}(z), \quad \bar{\mathcal{X}}_0 \equiv \oint \frac{d\bar{z}}{\bar{z}}\,\bar{\mathcal{X}}(\bar{z}). \tag{2.16}$$

Note that

$$[Q_B, \mathcal{G}] = 0\, . \tag{2.17}$$

2.2 1PI effective string field theory

Let $\widehat{\mathcal{H}}_T$ and $\widetilde{\mathcal{H}}_T$ denote the subspaces of $\mathcal{H}_T$ with the following restriction on the picture numbers:

	heterotic	type II
$\widehat{\mathcal{H}}_T$:	$-1, -1/2$	$(-1,-1), (-1.-1/2), (-1/2,-1), (-1/2,-1/2)$
$\widetilde{\mathcal{H}}_T$:	$-1, -3/2$	$(-1,-1), (-1.-3/2), (-3/2,-1), (-3/2,-3/2)$.

(2.18)

This means that $\widehat{\mathcal{H}}_T$ denotes the subspace of $\mathcal{H}_T$ containing NS sector states of picture number -1 and R sector states of picture number $-1/2$. In type II string theories $\widehat{\mathcal{H}}_T$ will contain states of picture numbers $(-1,-1)$, $(-1,-1/2)$, $(-1/2,-1)$ and $(-1/2,-1/2)$. Similar interpretation applies to $\widetilde{\mathcal{H}}_T$. In [14, 15] we introduced a multilinear function $\{A_1 \cdots A_N\}$ of $|A_1\rangle, \cdots |A_N\rangle \in \widehat{\mathcal{H}}_T$ taking values in the grassmann algebra, and another multilinear function $[A_2 \cdots A_N]$ of $|A_2\rangle, \cdots |A_N\rangle \in \widehat{\mathcal{H}}_T$ taking values in $\widetilde{\mathcal{H}}_T$. Physically $\{A_1 \cdots A_N\}$ denotes contribution to the off-shell amplitude with external states $|A_1\rangle, \cdots |A_N\rangle$ from the '1PI region of the moduli space'. $[\cdots]$ is related to $\{\cdots\}$ via

$$\langle A_1|c_0^-|[A_2 \cdots A_N]\rangle = \{A_1 \cdots A_N\} \tag{2.19}$$

for all $|A_1\rangle \in \widehat{\mathcal{H}}_T$. These multilinear functions satisfy

$$\{A_1 A_2 \cdots A_{i-1} A_{i+1} A_i A_{i+2} \cdots A_N\} = (-1)^{\gamma_i \gamma_{i+1}} \{A_1 A_2 \cdots A_N\}, \tag{2.20}$$

$$[A_1 \cdots A_{i-1} A_{i+1} A_i A_{i+2} \cdots A_N] = (-1)^{\gamma_i \gamma_{i+1}} [A_1 \cdots A_N], \tag{2.21}$$

where γ_i is the grassmannality of $|A_i\rangle$. They also satisfy

$$\begin{aligned} &\sum_{i=1}^{N} (-1)^{\gamma_1 + \cdots \gamma_{i-1}} \{A_1 \cdots A_{i-1} (Q_B A_i) A_{i+1} \cdots A_N\} \\ &\quad = -\frac{1}{2} \sum_{\substack{\ell, k \geq 0 \\ \ell + k = N}} \sum_{\substack{\{i_a; a=1, \cdots \ell\}, \{j_b; b=1, \cdots k\} \\ \{i_a\} \cup \{j_b\} = \{1, \cdots N\}}} \sigma(\{i_a\}, \{j_b\}) \{A_{i_1} \cdots A_{i_\ell} \mathcal{G}[A_{j_1} \cdots A_{j_k}]\} \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &Q_B [A_1 \cdots A_N] + \sum_{i=1}^{N} (-1)^{\gamma_1 + \cdots \gamma_{i-1}} [A_1 \cdots A_{i-1} (Q_B A_i) A_{i+1} \cdots A_N] \\ &\quad = -\sum_{\substack{\ell, k \geq 0 \\ \ell + k = N}} \sum_{\substack{\{i_a; a=1, \cdots \ell\}, \{j_b; b=1, \cdots k\} \\ \{i_a\} \cup \{j_b\} = \{1, \cdots N\}}} \sigma(\{i_a\}, \{j_b\}) \, [A_{i_1} \cdots A_{i_\ell} \mathcal{G} \, [A_{j_1} \cdots A_{j_k}]] \end{aligned} \tag{2.23}$$

where $\sigma(\{i_a\}, \{j_b\})$ is the sign that one picks up while rearranging $b_0^-, A_1, \cdots A_N$ to $A_{i_1}, \cdots A_{i_\ell}, b_0^-, A_{j_1}, \cdots A_{j_k}$. Finally we also have a relation

$$\{A_1 \cdots A_k \mathcal{G}[\widetilde{A}_1 \cdots \widetilde{A}_\ell]\} = (-1)^{\gamma + \tilde{\gamma} + \gamma \tilde{\gamma}} \{\widetilde{A}_1 \cdots \widetilde{A}_\ell \mathcal{G}[A_1 \cdots A_k]\}, \tag{2.24}$$

where γ and $\tilde{\gamma}$ are the total grassmannalities of $A_1, \cdots A_k$ and $\widetilde{A}_1, \cdots \widetilde{A}_\ell$ respectively.

Using these functions we constructed the equations of motion of gauge invariant 1PI effective string field theory. The string field $|\Psi\rangle$ is taken to be a state in $\widehat{\mathcal{H}}_T$ of ghost number 2. In the heterotic string theory $|\Psi\rangle$ can be decomposed as $|\Psi_{NS}\rangle + |\Psi_R\rangle$ where $|\Psi_{NS}\rangle$ is a state of picture number -1 and ghost number 2 in $\mathcal{H}_{NS}$ with coefficients multiplying the basis states given by even elements of the grassmann algebra, and $|\Psi_R\rangle$ is a state in $\mathcal{H}_R$ of picture number $-1/2$ and ghost number 2 with coefficients multiplying the basis states given by odd elements of the grassmann algebra. In the type II string theory the string field is decomposed as $|\Psi_{NSNS}\rangle + |\Psi_{NSR}\rangle + |\Psi_{RNS}\rangle + |\Psi_{RR}\rangle$ with similar restriction on picture number and ghost number. It follows from the discussion below (2.13) that in all theories all components of $|\Psi\rangle$ are grassmann even. The equations of motion of the 1PI effective string field theory takes the form

$$|\mathcal{E}\rangle = 0, \qquad |\mathcal{E}\rangle \equiv Q_B |\Psi\rangle + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \mathcal{G}[\Psi^{n-1}]. \tag{2.25}$$

This can be shown to be invariant under the gauge transformation

$$|\delta\Psi\rangle = Q_B |\Lambda\rangle + \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{G}[\Psi^n \Lambda], \tag{2.26}$$

where $|\Lambda\rangle$ is an arbitrary grassmann odd state in $\widehat{\mathcal{H}}_T$ carrying ghost number 1.

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Since [ ] — the $n = 1$ term on the right hand side of (2.25) — gets non-zero contribution from Riemann surfaces of genus ≥ 1, $|\Psi\rangle = 0$ is not a solution to the equations of motion (2.25). In [14] we described a systematic procedure for finding the vacuum solution $|\Psi_{\text{vac}}\rangle$ — a solution to (2.25) in the NS sector carrying zero momentum. This solution is constructed iteratively as a power series in the string coupling g_s. If $|\Psi_k\rangle$ denotes the solution to order g_s^k then it is obtained iteratively by solving[3]

$$|\Psi_{k+1}\rangle = -\frac{b_0^+}{L_0^+}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(1-\mathbf{P})\mathcal{G}[\Psi_k^{n-1}] + |\psi_{k+1}\rangle\,, \tag{2.27}$$

where $\mathbf{P}$ the projection operator into zero momentum $L_0^+ = 0$ states and $|\psi_{k+1}\rangle$ satisfies

$$\mathbf{P}|\psi_{k+1}\rangle = |\psi_{k+1}\rangle, \qquad Q_B|\psi_{k+1}\rangle = -\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\mathbf{P}\mathcal{G}[\Psi_k^{n-1}] + \mathcal{O}(g_s^{k+2})\,. \tag{2.28}$$

Possible obstruction to solving these equations arise from the failure to satisfy (2.28). In [14] we showed that the solution to (2.28) exists iff

$$\mathcal{E}_{k+1}(\phi) \equiv \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\langle\phi|c_0^-\mathcal{G}|[\Psi_k{}^{n-1}]\rangle = \mathcal{O}(g_s^{k+2})\,, \tag{2.29}$$

for any BRST invariant zero momentum state $|\phi\rangle \in \widetilde{\mathcal{H}}_T$ of ghost number two and $L_0^+ = 0$.[4] Therefore $\mathcal{E}_{k+1}(\phi)$ represents an obstruction to extending the vacuum solution beyond order g_s^k. It was shown that the condition is trivially satisfied if $|\phi\rangle$ is BRST trivial. Hence the non-trivial constraints come from zero momentum non-trivial elements of the BRST cohomology — the vertex operators of zero momentum massless bosonic states. These obstructions correspond to existence of massless tadpoles in the theory. Therefore the absence of massless tadpoles to order g_s^{k+1} will correspond to (2.29).

While finding solutions to (2.28) we have the freedom of adding to $|\psi_{k+1}\rangle$ any state of the form

$$\sum_{\alpha} a_\alpha|\varphi_\alpha\rangle \tag{2.30}$$

where $\{|\varphi_\alpha\rangle\}$ is a basis of zero momentum, NS sector BRST invariant states in $\widehat{\mathcal{H}}_T$ and a_α's are arbitrary coefficients. Some of these a_α's could get fixed while trying to ensure (2.29) at higher order. Those that do not get fixed represent moduli and can be given arbitrary values.

[3] Throughout this paper we shall assume that the vacuum solution has expansion in powers of g_s. This includes the case of perturbative vacuum where the solution will have expansion in powers of g_s^2 — we simply will get $|\Psi_{2k+1}\rangle = |\Psi_{2k}\rangle$ for all integer k. There may also be cases where the vacuum solution has an expansion in powers of g_s^α for some α in the range $0 < \alpha < 1$. Our analysis can be extended to this case as well by replacing g_s by g_s^α everywhere.

[4] Since we are dealing with NS sector states, there is no distinction between $\widetilde{\mathcal{H}}_T$ and $\widehat{\mathcal{H}}_T$ and $\mathcal{G}$ in (2.27)–(2.29) can be replaced by identity operators.

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If $|\Psi_{\rm vac}\rangle$ denotes a vacuum solution of (2.25), then the gauge symmetries which preserve the solution correspond to global symmetries. Therefore they satisfy

$$Q_B|\Lambda_{\rm global}\rangle + \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{G}[\Psi_{\rm vac}^n\Lambda_{\rm global}] = 0\,. \tag{2.31}$$

Such global symmetries arising in the R-sector of heterotic string theory and RNS and NSR sectors of type II string theories correspond to global supersymmetries. In [15] we described a systematic procedure for solving these equations iteratively. If $|\Lambda_k\rangle$ denotes the solution to (2.31) to order g_s^k then we have

$$|\Lambda_k\rangle = -\sum_{n=0}^{\infty}\frac{1}{n!}\frac{b_0^+}{L_0^+}(1-\mathbf{P})\mathcal{G}[\Psi_{\rm vac}^n\Lambda_{k-1}] + |\lambda_k\rangle\,, \tag{2.32}$$

where $\mathbf{P}$ denotes the projection operator into $L_0^+=0$ states and $|\lambda_k\rangle$ is an $L_0^+=0$ state satisfying

$$Q_B|\lambda_k\rangle = -\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{P}\mathcal{G}[\Psi_{\rm vac}^n\Lambda_{k-1}] + \mathcal{O}(g_s^{k+1})\,. \tag{2.33}$$

The possible obstruction to solving (2.31) arises from (2.33). The latter equation can be solved only if

$$\mathcal{L}_k(\hat\phi) \equiv \langle\hat\phi|c_0^-\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{G}|[\Psi_{\rm vac}^n\Lambda_{k-1}]\rangle = \mathcal{O}(g_s^{k+1})\,, \tag{2.34}$$

for any BRST invariant state $|\hat\phi\rangle \in \widetilde{\mathcal{H}}_T$ of ghost number 3 and $L_0^+=0$. Therefore $\mathcal{L}_k(\hat\phi)$ represents an obstruction to finding global (super-)symmetry transformation parameter beyond order g_s^{k-1}. A non-vanishing $\mathcal{L}_k(\hat\phi)$ signals spontaneous breakdown of the global (super-)symmetry at order g_s^k, with the state conjugate to $|\hat\phi\rangle$ representing the candidate goldstone/goldstino state.

Since one of the main goals of the paper will be to explore the possibility of spontaneous breakdown of global supersymmetry, it will be useful to have a list of possible candidate states $|\mathcal{V}_{\rm G}^c\rangle$ for $\hat\phi$ in the fermionic sector. For this we shall restrict our discussion to the heterotic string theory, but generalization to type II string theories is straightforward. In this case $|\mathcal{V}_{\rm G}^c\rangle$ is an element of the BRST cohomology carrying ghost number 3, picture number $-3/2$ and zero momentum. The possible candidates are of the form

$$\mathcal{V}_{\rm G}^c \quad\equiv\quad -4(\partial c+\bar\partial\bar c)\bar c c e^{-3\phi/2}V^f\,,\quad -4(\partial c+\bar\partial\bar c)\bar c c\bar\partial^2\bar c\,\partial\xi e^{-5\phi/2}\,\hat\Sigma\,, \tag{2.35}$$

where V^f is a dimension $(1,5/8)$ operator and $\hat\Sigma$ is a dimension $(0,5/8)$ operator — both in the R-sector of the matter CFT — carrying space-time chirality consistent with GSO projection rules and satisfying

$$T_F(z)V^f(w) = \mathcal{O}\left((z-w)^{-1/2}\right),\quad T_F(z)\hat\Sigma(w) = \mathcal{O}\left((z-w)^{-1/2}\right)\,. \tag{2.36}$$

For simplicity we have dropped the spinor indices. $\hat\Sigma$ in fact represents the matter part of an operator in the zeroth order global supersymmetry transformation parameter: $\Lambda_0 =$

$\lambda_0 = ce^{-\phi/2}\hat{\Sigma}$. For SO(32) heterotic string theory compactified on Calabi-Yau 3-folds, possible choices for $\hat{\Sigma}$ are listed in (6.8). The -4 factors in the definition of these vertex operators will be useful later. The subscript 'G' and the superscript 'c' in $\mathcal{V}_{\rm G}^c$ stands for the fact that these states are conjugate to candidate goldstino states $\mathcal{V}_{\rm G}$:

$$\mathcal{V}_{\rm G} \qquad \propto \qquad \bar{c}ce^{-\phi/2}(V^f)^c, \quad c\eta e^{\phi/2}\hat{\Sigma}^c \tag{2.37}$$

where $(V^f)^c$ and $\hat{\Sigma}^c$ are conjugate operators of V^f and $\hat{\Sigma}$ in the matter sector, satisfying relations similar to (2.36). These are BRST invariant states of ghost number 2 and picture number $-1/2$. A non-vanishing $\mathcal{L}_k(\mathcal{V}_{\rm G}^c)$ would imply that the right hand side of (2.33) has a component along $\mathcal{V}_{\rm G}$, leading to a failure to solving this equation and consequently a breakdown of global supersymmetry.

For evaluation of (2.34) it is also useful to list the operators

$$\mathcal{G}\mathcal{V}_{\rm G}^c = \mathcal{X}_0\mathcal{V}_{\rm G}^c \qquad = \qquad \bar{c}c\eta e^{\phi/2}V^f, \quad \bar{c}c\bar{\partial}^2\bar{c}\, e^{-\phi/2}\,\hat{\Sigma}\,. \tag{2.38}$$

One can show that these are also non-trivial elements of the BRST cohomology.

Given a string field configuration $|\Psi_{\rm vac}\rangle$ satisfying (2.25), we define

$$\begin{aligned}
&\{A_1\cdots A_k\}'' \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\{\Psi_{\rm vac}^n A_1\cdots A_k\}\,, \qquad \text{for } k\geq 3\,,\\
&[A_1\cdots A_k]'' \equiv \sum_{n=0}^{\infty}\frac{1}{n!}[\Psi_{\rm vac}^n A_1\cdots A_k]\,, \qquad \text{for } k\geq 2\,,\\
&\{A_1\}''\equiv 0, \qquad [\,]''\equiv 0, \qquad \{A_1A_2\}''\equiv 0, \quad [A_1]''\equiv 0\,,\\
&\widehat{Q}_B|A\rangle \equiv Q_B|A\rangle + \sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{G}[\Psi_{\rm vac}^k A]\,.
\end{aligned} \tag{2.39}$$

$\widehat{Q}_B$ defined in (2.39) can be expressed as

$$\widehat{Q}_B = Q_B + \mathcal{G}\,K\,, \qquad K\,|A\rangle \equiv \sum_{k=0}^{\infty}\frac{1}{k!}[{\Psi_{\rm vac}}^k A]\,. \tag{2.40}$$

$\widehat{Q}_B$ and K act naturally on states in $\widehat{\mathcal{H}}_T$ defined in (2.18). Using the definition of K given in (2.40), the equations of motion (2.25) satisfied by $|\Psi_{\rm vac}\rangle$, and the identities (2.20)–(2.24) one can prove the following useful identity:

$$Q_BK + KQ_B + K\mathcal{G}K = 0\,. \tag{2.41}$$

From this and (2.17) it follows that

$$\widehat{Q}_B{}^2 = 0\,. \tag{2.42}$$

Using (2.20)–(2.25) one also finds the identities

$$\{A_1A_2\cdots A_{i-1}A_{i+1}A_iA_{i+2}\cdots A_N\}'' = (-1)^{\gamma_i\gamma_{i+1}}\{A_1A_2\cdots A_N\}''\,, \tag{2.43}$$

$$\sum_{i=1}^{N}(-1)^{\gamma_1+\cdots\gamma_{i-1}}\{A_1\cdots A_{i-1}(\widehat{Q}_B A_i)A_{i+1}\cdots A_N\}''$$
$$= -\frac{1}{2}\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\sum_{\substack{\{i_a;a=1,\cdots\ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}}\sigma(\{i_a\},\{j_b\})\{A_{i_1}\cdots A_{i_\ell}\mathcal{G}[A_{j_1}\cdots A_{j_k}]''\}'' \tag{2.44}$$

$$[A_1\cdots A_{i-1}A_{i+1}A_iA_{i+2}\cdots A_N]'' = (-1)^{\gamma_i\gamma_{i+1}}[A_1\cdots A_N]'', \tag{2.45}$$

and

$$\widehat{Q}_B\mathcal{G}[A_1\cdots A_N]'' + \sum_{i=1}^{N}(-1)^{\gamma_1+\cdots\gamma_{i-1}}\mathcal{G}[A_1\cdots A_{i-1}(\widehat{Q}_B A_i)A_{i+1}\cdots A_N]''$$
$$= -\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\sum_{\substack{\{i_a;a=1,\cdots\ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}}\sigma(\{i_a\},\{j_b\})\,\mathcal{G}[A_{i_1}\cdots A_{i_\ell}\mathcal{G}\,[A_{j_1}\cdots A_{j_k}]'']''\,. \tag{2.46}$$

For future use, we also define

$$\widetilde{Q}_B = Q_B + K\,\mathcal{G}\,, \tag{2.47}$$

where K has been defined in (2.40). $\widetilde{Q}_B$ acts naturally on states in $\widetilde{\mathcal{H}}_T$ defined in (2.18). It is easy to verify using (2.17), (2.40) and (2.47) that

$$\widehat{Q}_B\mathcal{G} = \mathcal{G}\widetilde{Q}_B\,. \tag{2.48}$$

Furthermore, using (2.40) and (2.41) one can prove the nilpotence of $\widetilde{Q}_B$ and the identities

$$\langle A|c_0^-\widehat{Q}_B|B\rangle = (-1)^{\gamma_A}\langle\widetilde{Q}_B A|c_0^-|B\rangle\,,\qquad \langle B|c_0^-\widetilde{Q}_B|A\rangle = (-1)^{\gamma_B}\langle\widehat{Q}_B B|c_0^-|A\rangle\,. \tag{2.49}$$

If we expand the string field as $|\Psi\rangle = |\Psi_{\rm vac}\rangle + |\Phi\rangle$ then the equations of motion for $|\Phi\rangle$ take the form

$$\widehat{Q}_B|\Phi\rangle + \sum_{n=2}^{\infty}\frac{1}{n!}\mathcal{G}[\Phi^n]'' = 0\,. \tag{2.50}$$

Therefore the linearized equations of motion for $|\Phi\rangle$ are

$$\widehat{Q}_B|\Phi_{\rm linear}\rangle = 0\,. \tag{2.51}$$

The spectrum of physical states around the vacuum solution $|\Psi_{\rm vac}\rangle$ is given by examining the momentum k carried by $|\Phi_{\rm linear}\rangle$ for which (2.51) has solutions. There are families of solutions to (2.51) which exist for all momenta, — these are pure gauge solutions of the form $\widehat{Q}_B|\Lambda\rangle$ for some $|\Lambda\rangle$. There are additional solutions which appear for definite values of k^2 — these represent the physical states and the values of k^2 at which these solutions appear give the physical mass^2 of the states.

In [14, 15] we described a systematic procedure for finding the solutions to (2.51) in a power series expansion in g_s. If $|\Phi_n\rangle$ denotes a solution to (2.51) to order g_s^n then we determine $|\Phi_n\rangle$ in the 'Siegel gauge'[5] using the recursion relation:

$$|\Phi_0\rangle = |\phi_n\rangle, \qquad |\Phi_{\ell+1}\rangle = -\frac{b_0^+}{L_0^+}(1-P)\mathcal{G}K|\Phi_\ell\rangle + |\phi_n\rangle + \mathcal{O}\left(g_s^{\ell+2}\right), \quad \text{for} \quad 0 \le \ell \le n-1\,, \tag{2.52}$$

where $|\phi_n\rangle$ satisfies

$$P|\phi_n\rangle = |\phi_n\rangle\,, \tag{2.53}$$

$$Q_B|\phi_n\rangle = -P\mathcal{G}K|\Phi_{n-1}\rangle + \mathcal{O}(g_s^{n+1})\,. \tag{2.54}$$

Here P is a projection operator defined as follows. Let us choose a basis of states in the CFT which are momentum eigenstates and also L_0^+ eigenstates. We define the mass level m via the relation $L_0^+ = (k^2+m^2)/2$ i.e. $m^2/2$ is the contribution to the L_0^+ eigenvalue from the internal part of the matter CFT and the oscillators of free matter and ghost fields. P will denote the projection operator to states of some fixed mass level m. In this case (2.54) at zeroth order, i.e the equation $Q_B|\phi_0\rangle = 0$, has non-trivial solution only when $k^2 = -m^2$ and hence m gives the tree level mass of the state. Therefore the value of k^2 at which (2.52)–(2.54) have perturbative solution describing a physical state is expected to be of order $-m^2 + \mathcal{O}(g_s)$ after inclusion of quantum corrections. At this momentum the states onto which we project by P all have $L_0^+ = (k^2+m^2)/2 \sim g_s$, while all other states will have $|L_0^+| \gtrsim 1$. The projection operator $(1-P)$ in (2.52) ensures that the $(L_0^+)^{-1}$ operator in (2.52) never gives any power of g_s in the denominator. As a result (2.52) leads to a well defined expansion of $|\Phi_n\rangle$ in powers of g_s, expressing it as a linear function of $|\phi_n\rangle$. After solving for $|\Phi_n\rangle$ this way we solve (2.53), (2.54) to determine $|\phi_n\rangle$.[6] Since for given momentum P projects onto a finite dimensional subspace of $\mathcal{H}_T$, (2.54) gives a finite set of linear equations. It will have a set of solutions which exist for all k. These are of the form $P\widehat{Q}_B|\Lambda\rangle$ for some ghost number 1 state $|\Lambda\rangle$ carrying momentum k, and are associated with pure gauge states. There is also another class of solutions which exist for specific values of $-k^2$. These describe physical states, with the value of $-k^2$ at which the solution exists giving the physical mass^2.

For future reference we note that (2.52)–(2.54) can be written as

$$|\Phi_{\rm linear}\rangle = \sum_{\ell=0}^{\infty}\left(-\frac{b_0^+}{L_0^+}(1-P)\mathcal{G}K\right)^{\ell}|\phi\rangle = \left(1+\frac{b_0^+}{L_0^+}(1-P)\mathcal{G}K\right)^{-1}|\phi\rangle\,, \tag{2.55}$$

$$P|\phi\rangle = |\phi\rangle\,, \tag{2.56}$$

$$Q_B|\phi\rangle = -P\mathcal{G}K|\Phi_{\rm linear}\rangle\,, \tag{2.57}$$

[5]Siegel gauge here refers to the gauge in which all states other than those projected by P are annihilated by b_0^+. For describing solutions, we shall follow this convention throughout this paper not only for the projection operator P but also for the projection operator $\mathbf{P}$ introduced earlier.

[6]It may seem somewhat strange that we first determine $|\Phi_{\ell+1}\rangle$ for all ℓ between 0 and $n-1$ iteratively in terms of $|\phi_n\rangle$ and determine $|\phi_n\rangle$ at the end at one step by solving a linear equation in the subspace projected by P. The reason for this is that for the physical states the allowed value of k^2 changes at each order. Since a small change in k is not described by small change in the vertex operator it is best not to compute $|\phi_n\rangle$ iteratively but compute it at one step at the very end.

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where $|\Phi_{\rm linear}\rangle$ denotes the solution to the linearized equations of motion to any order. It is understood that to calculate $|\Phi_{\rm linear}\rangle$ to any given order we have to expand the right hand side of (2.55) to that order, substitute this into the right hand side of (2.57) to get a linear equation for $|\phi\rangle$ in the subspace projected by P, and then solve this equation to determine $|\phi\rangle$ and the momentum carried by $|\phi\rangle$.

We shall conclude this section by introducing a common notation for describing states projected by $\mathbf{P}$ and P. $\mathbf{P}$ projects on to the zero momentum states with $L_0^+ = 0$. On the other hand for given momentum k at which some states become on-shell, P projects on to states which have $L_0^+ = \mathcal{O}(g_s)$. These are the states which would have $L_0^+ = 0$ at string tree level, but since the k-values shift due to mass renormalization L_0^+ also shifts by terms of order g_s. We shall collectively call all such states (including zero momentum states with $L_0^+ = 0$) as $L_0^+ \simeq 0$ states. In this notation both $\mathbf{P}$ and P project on to $L_0^+ \simeq 0$ states.

3 S-matrix from the action

In this section we shall discuss the construction of *tree level* amplitudes associated with the equations of motion (2.50); this is supposed to give the full quantum amplitudes in string theory. We shall also compare the amplitudes obtained this way to the usual amplitudes in string theory expressed as integrals of certain CFT correlation functions over the moduli spaces of Riemann surfaces.

3.1 Construction of off-shell amplitudes and S-matrix elements

Even though the equations of motion can in principle be used for computing the tree level S-matrix directly, we shall follow a simpler approach. Following [15] we shall first introduce an additional set of fields in terms of which one can write down an action from which the equations of motion can be derived, and then use this action to compute the S-matrix elements. These additional fields are described by a grassmann even state $|\widetilde{\Psi}\rangle \in \widetilde{\mathcal{H}}_T$ carrying ghost number 2, with $\widetilde{\mathcal{H}}_T$ defined as in (2.18). We now consider the action

$$S = g_s^{-2}\left[-\frac{1}{2}\langle\widetilde{\Psi}|c_0^- Q_B \mathcal{G}|\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}|c_0^- Q_B|\Psi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\{\Psi^n\}\right]. \tag{3.1}$$

It is easy to see that the action (3.1) is invariant under the infinitesimal gauge transformation

$$|\delta\Psi\rangle = Q_B|\Lambda\rangle + \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{G}[\Psi^n\Lambda]\,, \quad |\delta\widetilde{\Psi}\rangle = Q_B|\widetilde{\Lambda}\rangle + \sum_{n=0}^{\infty}\frac{1}{n!}[\Psi^n\Lambda]\,, \tag{3.2}$$

where $|\Lambda\rangle \in \widehat{\mathcal{H}}_T$, $|\widetilde{\Lambda}\rangle \in \widetilde{\mathcal{H}}_T$, and both carry ghost number 1. The equations of motion derived from (3.1) can be written as

$$Q_B(|\Psi\rangle - \mathcal{G}|\widetilde{\Psi}\rangle) = 0\,, \tag{3.3}$$

$$Q_B|\widetilde{\Psi}\rangle + \sum_{n=1}^{\infty}\frac{1}{(n-1)!}[\Psi^{n-1}] = 0\,. \tag{3.4}$$

Applying $\mathcal{G}$ on (3.4) and using (3.3) we recover the equation of motion (2.25) of $|\Psi\rangle$.

Given a vacuum solution $|\Psi_{\rm vac}\rangle$ in the NS sector satisfying (2.25), we can find a solution to (3.3) and (3.4) by setting $|\widetilde{\Psi}_{\rm vac}\rangle = |\Psi_{\rm vac}\rangle$ since $\mathcal{G}|\Psi_{\rm vac}\rangle = |\Psi_{\rm vac}\rangle$. Defining shifted fields

$$|\Phi\rangle = |\Psi\rangle - |\Psi_{\rm vac}\rangle, \qquad |\widetilde{\Phi}\rangle = |\widetilde{\Psi}\rangle - |\widetilde{\Psi}_{\rm vac}\rangle = |\widetilde{\Psi}\rangle - |\Psi_{\rm vac}\rangle \tag{3.5}$$

the action (3.1) and the gauge transformation laws (3.2) can be written as

$$S = g_s^{-2}\left[-\frac{1}{2}\langle\widetilde{\Phi}|c_0^- Q_B \mathcal{G}|\widetilde{\Phi}\rangle + \langle\widetilde{\Phi}|c_0^- Q_B|\Phi\rangle + \frac{1}{2}\langle\Phi|c_0^- K|\Phi\rangle + \sum_{n=3}^{\infty}\frac{1}{n!}\{\Phi^n\}''\right], \tag{3.6}$$

$$|\delta\Phi\rangle = \widehat{Q}_B|\Lambda\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{G}[\Phi^n\Lambda]'', \quad |\delta\widetilde{\Phi}\rangle = Q_B|\widetilde{\Lambda}\rangle + K|\Lambda\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}[\Phi^n\Lambda]'', \tag{3.7}$$

with K defined as in (2.40). The equations of motion derived from (3.6) are

$$Q_B(|\Phi\rangle - \mathcal{G}|\widetilde{\Phi}\rangle) = 0\,, \tag{3.8}$$

$$Q_B|\widetilde{\Phi}\rangle + K|\Phi\rangle + \sum_{n=3}^{\infty}\frac{1}{(n-1)!}[\Phi^{n-1}]'' = 0\,. \tag{3.9}$$

Applying $\mathcal{G}$ on (3.9) and using (2.40), (3.8) we recover the equation of motion (2.50) of $|\Phi\rangle$.

Even though this 1PI action gives the correct equations of motion, it has too many degrees of freedom. For example physical states in the R-sector will arise both in picture number $-1/2$ and picture number $-3/2$ sectors. We avoid this by imposing a constraint[7]

$$\mathcal{G}|\widetilde{\Psi}\rangle - |\Psi\rangle = 0 \quad \Rightarrow \quad \mathcal{G}|\widetilde{\Phi}\rangle - |\Phi\rangle = 0 \tag{3.10}$$

on the external states. Since this is consistent with the equations of motion and we are only doing a tree level computation with the 1PI effecting action, this is a consistent truncation. One might wonder if this imposes some additional restriction on the external states compared to the restrictions imposed by (2.51); we have shown in appendix B that as far as perturbative amplitudes are concerned, (3.10) does not give any additional constraint besides (2.51).

While determining the Feynman rules for computing tree amplitudes from the action (3.6), we shall use the weight factor of e^S in the path integral. Therefore n-point vertices in a Feynman diagram with external state $|A_1\rangle, \cdots |A_n\rangle$ are given directly by $g_s^{-2}\{A_1 \cdots A_n\}''$, while the propagator is negative of the inverse of the kinetic operator giving the quadratic part of S. If we had used e^{-S} or e^{iS} then the Feynman rules for the propagators and vertices will have additional phase. The effect of this will be to change each of the tree amplitudes computed from this action by an overall phase factor, without affecting the relative phases of different terms contributing to a given amplitude. This is related to the fact that tree level amplitudes computed from an action are really sensitive only to the equations of motion and not the normalization of the action.

In particular we shall see later (see last paragraph of section 3.4) that in our convention the kinetic operator of a field of mass m is $-(k^2 + m^2)$ up to a positive constant of

[7] This kind of doubling trick accompanied by a constraint has been attempted before in the context of Berkovits' open string field theory [55].

proportionality. Therefore S has the correct sign for being interpreted as the action for Lorentzian signature, and the correct weight factor in the path integral for Lorentzian signature is e^{iS}. The amplitudes computed with this rule can be obtained by multiplying the amplitudes computed here by a factor of i.

We gauge fix the theory in the Siegel gauge $b_0^+|\Phi\rangle = 0$, $b_0^+|\widetilde{\Phi}\rangle = 0$. In this gauge $Q_B = c_0^+ L_0^+$ and the kinetic term of the action (3.6) in the $(|\widetilde{\Phi}\rangle, |\Phi\rangle)$ space takes the form

$$g_s^{-2}\left[c_0^- c_0^+ L_0^+ \begin{pmatrix} -\mathcal{G} & 1 \\ 1 & 0 \end{pmatrix} + c_0^- \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}\right] . \tag{3.11}$$

Inverting this and multiplying by -1 we get the propagator

$$-\,g_s^2 \begin{pmatrix} \check{\Delta} & \bar{\Delta} \\ \widetilde{\Delta} & \Delta \end{pmatrix} \tag{3.12}$$

where

$$\check{\Delta} = \left[-\frac{b_0^+}{L_0^+}K\frac{b_0^+}{L_0^+} + \frac{b_0^+}{L_0^+}K\frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+} + \cdots\right] b_0^- \delta_{L_0^-} , \tag{3.13}$$

$$\bar{\Delta} = \left[\frac{b_0^+}{L_0^+} - \frac{b_0^+}{L_0^+}K\frac{b_0^+}{L_0^+}\mathcal{G} + \frac{b_0^+}{L_0^+}K\frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+}\mathcal{G} + \cdots\right] b_0^- \delta_{L_0^-} , \tag{3.14}$$

$$\widetilde{\Delta} = \left[\frac{b_0^+}{L_0^+} - \frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+} + \frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+} + \cdots\right] b_0^- \delta_{L_0^-} , \tag{3.15}$$

$$\Delta = \left[\frac{b_0^+}{L_0^+}\mathcal{G} - \frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+}\mathcal{G} + \frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+}\mathcal{G}K\frac{b_0^+}{L_0^+}\mathcal{G} + \cdots\right] b_0^- \delta_{L_0^-} . \tag{3.16}$$

The minus sign in (3.12) is a reflection of the fact that we use e^S as the weight factor in the path integral rather than e^{-S}. $\check{\Delta}$, $\bar{\Delta}$, $\widetilde{\Delta}$ and Δ act naturally on states in $\widehat{\mathcal{H}}_T$, $\widetilde{\mathcal{H}}_T$, $\widehat{\mathcal{H}}_T$ and $\widetilde{\mathcal{H}}_T$ to produce states in $\widetilde{\mathcal{H}}_T$, $\widetilde{\mathcal{H}}_T$, $\widehat{\mathcal{H}}_T$ and $\widehat{\mathcal{H}}_T$ respectively. One important property of Δ that will be useful later is the relation

$$\widehat{Q}_B \Delta c_0^- + \Delta c_0^- \widetilde{Q}_B = \delta_{L_0^-}\,\mathcal{G} , \tag{3.17}$$

acting on states in $\widetilde{\mathcal{H}}_T$. This can be derived using (2.47), (2.41) and other well-known (anti-)commutators involving Q_B.

Our goal will be to compute off-shell Green's functions with external propagators truncated, involving external Φ fields, using the propagator (3.16) and the interaction term given by the last term in (3.6). Since the interaction term as well the external states involve only the Φ fields, only the Δ term in the propagator is relevant for our computation. Therefore from now on in our computation we can forget about the $\widetilde{\Phi}$ fields altogether and work only with Φ fields with propagator $-g_s^2\Delta$ and interaction terms $g_s^{-2}\sum_{n=3}^{\infty}\{\Phi^n\}''/n!$. If we denote by $g_s^{-2}\Gamma^{(N)}(|A_1\rangle, \cdots |A_N\rangle)$ the truncated off-shell Green's function with external states $|A_1\rangle, \cdots |A_N\rangle \in \widehat{\mathcal{H}}_T$ with ghost number 2, then $g_s^{-2}\Gamma^{(N)}$ is computed by drawing all possible *tree* graphs with the states $|A_1\rangle, \cdots |A_N\rangle$ as external states, $g_s^{-2}\{\Phi^n\}''/n!$ for $n \geq 3$ as vertices and propagator $-g_s^2\Delta$ given in (3.16). The

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normalization factor g_s^{-2} in the definition of $\Gamma^{(N)}$ is by convention, and has been included so that the 1PI contribution to $\Gamma^{(N)}(|A_1\rangle, \cdots |A_N\rangle)$ is given by $\{A_1 \cdots A_N\}$ without any further normalization factor. S-matrix elements are obtained from $\Gamma^{(N)}$ by setting the external states on-shell and multiplying the result by $\prod_{i=1}^{N}(Z_i)^{-1/2}$ where Z_i is the wave-function renormalization factor associated with the i-th external state.

3.2 Comparison with the usual formulation

We shall use the description of the S-matrix elements given in section 3.1 for most of our analysis. Nevertheless it is useful to compare this definition of S-matrix to the usual description where at each loop order a given amplitude has a single term involving integration over the full moduli space. This is what we shall do in this subsection. A summary of the results of this subsection has been given in section 3.5; so readers interested in only the final results can skip this subsection.

If instead of working with the propagator Δ, the interaction $\sum_{n\geq 3}\{\Phi^n\}''/n!$ and the definition of on-shell states given in (2.51), we had used the unshifted action (3.1) to determine the propagator and interactions and used the definition of on-shell states given by $Q_B|\Psi\rangle = 0$, then the propagator of the $|\Psi\rangle$ field will be given by [15]

$$\Delta_0 \equiv b_0^+ b_0^- \mathcal{G}(L_0^+)^{-1}\delta_{L_0^-}\,, \tag{3.18}$$

and the interactions will be given by the 1PI amplitudes in the unshifted background. It then follows from standard results in string field theory (see e.g. [2, 13]) that the on-shell amplitudes computed this way would agree with the standard amplitudes of string theory. This would of course suffer the same divergences associated with mass and wave-function renormalization and massless tadpoles as discussed e.g. in [36, 37]. Our goal will be to bring the expression for the S-matrix given in section 3.1 as close as possible to the expression involving integration over the full moduli space of Riemann surfaces, and at the same time point out the crucial differences that removes the divergences in the latter expression associated with mass renormalization and massless tadpoles.

Our strategy will be to carry out a detailed comparison of the prescription of section 3.1 with the one where we use the unshifted action (3.1) and solutions to the linearized equations $Q_B|\Psi\rangle = 0$. One of the main differences between these two approaches is that term $\sum_{n=3}^{\infty}\{\Phi^n\}''/n!$ in the action does not contain any one point or two point vertices while $\sum_{n=1}^{\infty}\{\Psi^n\}/n!$ does contain such terms. As we shall discuss, at a crude level the absence of the $\{\Phi\}''$ term is compensated by inclusion of the shift by $|\Psi_{\rm vac}\rangle$ in the definition of the vertices $\{\Phi^n\}''/n!$ for $n \geq 3$, while the absence of the $\{\Phi^2\}''/2!$ term is compensated for by the use of the modified propagator (3.16) that includes loop corrections encoded in K and also in the fact that external states are taken to be annihilated by $\widehat{Q}_B$ instead of Q_B. Our goal will be to explore to what extent these compensations fail to be exact since that encodes the difference between the two approaches.

For clarity we shall divide our analysis into two parts.

1. First we shall suppose for the sake of argument that $\{A\} = 0$ (or equivalently $[\,] = 0$) so that there is no constant term in the equations of motion (2.25) and we can set

$|\Psi_{\rm vac}\rangle = 0$. In this case the one point function $\{\Psi\}$ vanishes even in the conventional approach, and the interaction terms $\{\Psi^n\}/n!$ and $\{\Phi^n\}''/n!$ give the same vertex for $n \geq 3$. Therefore the difference between the conventional approach and our approach lies in the two point vertex computed from $\{\Psi^2\}/2!$ which is present in the conventional approach but is absent in our approach. We shall examine how use of the modified propagator Δ instead of Δ_0 and modified external states compensate for it.

2. Next we shall relax the assumption that $\{A\} = 0$ so that $|\Psi_{\rm vac}\rangle$ no longer vanishes. Now the conventional approach will have also one point function that is absent in our approach. But the interaction vertices computed from the $\{\Phi^n\}''/n!$ term in our approach now differ from the interaction vertices computed from $\{\Psi^n\}/n!$ in the conventional approach. We shall examine to what extent these two effects compensate each other.

So let us begin with the first step, assuming that $\{A\} = 0$, $|\Psi_{\rm vac}\rangle = 0$. In this case $\langle B|c_0^- K|A\rangle = \langle B|c_0^-|[A]\rangle = \{BA\}$ represents the usual two point amplitude in the conventional approach based on (3.1). Therefore the two point function corresponds to the operator $c_0^- K$ and has to be inserted between a pair of internal propagators Δ_0. Summing over multiple insertions of this type would convert the internal propagators to

$$-\Delta_0 + \Delta_0(c_0^- K)\Delta_0 - \Delta_0(c_0^- K)\Delta_0(c_0^- K)\Delta_0 + \cdots\,. \tag{3.19}$$

This is precisely the propagator $-\Delta$ given in (3.16).

This shows that in our approach, the effect of using the propagator $-\Delta$ instead of $-\Delta_0$ precisely compensates for the missing two point vertices inserted on the internal lines in the conventional approach. However since the definition of $\Gamma^{(N)}$ involves truncating the full external propagator $-\Delta$, while the usual string amplitudes would correspond to first expressing Δ as in (3.19) and then truncating the left-most $-\Delta_0$, the prescription of section 3.1 would appear to be missing the contributions involving (multiple) insertion of the two point function $(c_0^- K)$ on the external legs. The result can still be expressed as an integral over the moduli space, but the integration runs over only a subspace of the full moduli space.

To understand how these missing contributions arise in our formalism, we have to recall that in the computation of the S-matrix element the external states are taken to be annihilated by $\widehat{Q}_B$ and not by Q_B. For states which undergo mass renormalization this shifts the on-shell value of the momentum. However this is not the only difference — there are other differences which affect even states which do not undergo mass renormalization. For this we recall that in (2.52)–(2.54) we described a systematic way of constructing solution to the $\widehat{Q}_B|\Phi_{\rm linear}\rangle = 0$ equation in a power series in the string coupling g_s. Let us examine how the modification of the external state as described there affects the computation of the S-matrix. As described in (2.55), the equation (2.52) for $|\Phi_{\rm linear}\rangle$ can be summarized as

$$|\Phi_{\rm linear}\rangle = \sum_{\ell=0}^{\infty}(-\Delta_0' c_0^- K)^\ell|\phi\rangle = (1 + \Delta_0' c_0^- K)^{-1}|\phi\rangle\,. \tag{3.20}$$

where

$$\Delta_0' = \frac{b_0^+ b_0^-}{L_0^+}\delta_{L_0^-}(1-P)\mathcal{G}\,, \tag{3.21}$$

denotes a modified propagator in which we have removed the contributions from a subset of states using the projection operator $(1-P)$. While expressing (2.55) as (3.20) we have used $\delta_{L_0^-}K = K$, $b_0^- K = 0$. (3.20) denotes repeated application of $-\Delta_0'$ and the two point vertex $c_0^- K$. If we replace Δ_0' by Δ_0, and $|\phi\rangle$ by a Q_B invariant state in (3.20), then (3.20) would exactly supply the missing contribution involving insertions of $c_0^- K$ on external legs that is needed to make our prescription agree with the usual amplitude computed in string theory. However since Δ_0' is not Δ_0 and $|\phi\rangle$ to any given order in g_s is obtained by solving eqs. (2.52)–(2.54) instead of $Q_B|\phi\rangle = 0$, our prescription is not exactly the same as that used for computing usual string amplitude. Due to this projector $(1-P)$ in Δ_0' our procedure is free from the infrared divergences associated with mass renormalization. The price we pay is that the external state that has to be used for computing the amplitude is $|\phi\rangle = P|\Phi_{\rm linear}\rangle$ obtained by solving (2.55)–(2.57). Even though this is a linear combination of the states at the same mass level as the state that we have at the string tree level, this is not in general a BRST invariant state.

Now let us consider the general case when $\{A\}$ and $[\,]$ are non-zero and hence $|\Psi_{\rm vac}\rangle$ is also non-zero. Since by Lorentz invariance $\{A\}$ vanishes when $|A\rangle \in \mathcal{H}_R$, we can take $|\Psi_{\rm vac}\rangle$ to be in the NS sector. Now the rules for computing the amplitude described in section 3.1 differ from the conventional prescription based on the action (3.1) expanded around $|\Psi\rangle = 0$ on two counts. First of all in the conventional approach we have to include the effect of one point function $\{\Psi\}$ in the computation of the amplitudes. This will require us to include tadpole graphs which are absent in our approach. Second, in our approach the vertex uses $\{\Phi^n\}''/n!$ whereas in the unshifted background the vertex uses $\{\Psi^n\}/n!$. Now using the vertex computed from $\{\Phi^n\}''/n!$ is equivalent to using the vertex computed from $\{\Psi^n\}/n!$ but inserting arbitrary number of $\Psi_{\rm vac}$ in the amplitude. To analyze the effect of these insertions we recall that in (2.27), (2.28) we described an algorithm for computing $|\Psi_{\rm vac}\rangle$ in a power series expansion in the string coupling. It is easy to convince oneself that if in (2.27) we did not have the $|\psi_{k+1}\rangle$ term and replaced $(1-\mathbf{P})$ by 1, then arbitrary number of insertions of $|\Psi_{\rm vac}\rangle$ in the amplitude in our approach will generate back the missing tadpole contributions that arise in the conventional approach. However the presence of the projector $1-\mathbf{P}$ in our approach has the effect that it removes the contribution of the $L_0^+ = 0$ states from the zero momentum propagator which will arise in the computation of the tadpole diagrams, thereby rendering the amplitude manifestly free from infrared divergences associated with propagators of massless states carrying zero momentum. However we have to now supplement this by inserting into the amplitude arbitrary number of insertions of $L_0^+ = 0$ states $|\psi\rangle$, whose order g_s^{k+1} expression $|\psi_{k+1}\rangle$ is obtained by solving (2.28). Since $|\psi\rangle$ has a power series expansion in g_s starting at order g_s or higher power of g_s, in any given order in g_s we only need a finite number of $|\psi\rangle$ insertions.

3.3 Orientation of the moduli space

Once we have fixed our conventions, the amplitudes are determined including the overall sign and hence comes with a prescription for how to choose the orientation of the moduli space over which we integrate. We shall now illustrate how this works for the genus zero four point function. Since the picture changing operators play no role in this analysis we shall carry out the analysis in bosonic string theory. Identical results will hold in superstring theory. (What we refer to as matter sector here will stand for matter plus the β, γ ghost system in superstring theory.)

Let us suppose that we have four on-shell vertex operators $\mathcal{V}_i$ of the form $\bar{c}cV_i$ where V_i are matter sector primaries of dimension (1,1). Now the 1PI part of the four point amplitude has the form

$$\frac{1}{2\pi i}\int dz\wedge d\bar{z}\left\langle \oint_z b(w)dw \oint_z d\bar{w}\bar{b}(\bar{w})\mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}_3(z_3)\mathcal{V}_4(z)\right\rangle \tag{3.22}$$

where $\oint_z$ denotes integration along a contour enclosing z with the normalization convention $\oint_z dw/(w-z) = 1$, $\oint_z d\bar{w}/(\bar{w}-\bar{z}) = 1$. The overall normalization in (3.22) is part of the definition of the normalization convention of $\{A_1\cdots A_N\}$ — the complete factor being $(2\pi i)^{-(3g-3+n)}$ for a g-loop n-point contribution. Now we can carry out the contour integrals yielding the result

$$\frac{1}{2\pi i}\int dz\wedge d\bar{z}\ \langle\mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}_3(z_3)V_4(z)\rangle\ . \tag{3.23}$$

Let us suppose that we do not a priori know what the orientation of the sphere labelled by z is. In that case we can write

$$dz\wedge d\bar{z} = -2\,i\,d(\mathrm{Re}z)\wedge d(\mathrm{Im}z) = -2\,i\,\varepsilon\,d^2z \tag{3.24}$$

where d^2z by definition is an integration measure whose integral over any subspace of the z-plane gives positive result. ε is a sign which we want to determine.

A direct determination of ε requires recalling some part of the analysis in [14, 15, 19]. There in the proof of various factorization identities of the integration measure we had used the plumbing fixture relation $zw = q$ with $q = \exp(-s+i\theta)$ and had taken $ds\wedge d\theta$ to define positive integration measure. This would correspond to $dq\wedge d\bar{q}$ describing i times a positive integration measure. Since z is related to q by an analytic transformation, we see that $dz\wedge d\bar{z}$ should also correspond to i times a positive integration measure and hence we must have $\varepsilon = -1$. This gives

$$dz\wedge d\bar{z} = 2\,i\,d^2z\,. \tag{3.25}$$

We shall now verify this by directly analyzing the amplitudes. Even though (3.23) is supposed to describe the 1PI part of the vertex, the contribution from the 1PR parts, obtained by gluing two three point vertices by a propagator, should have the same form with the integration over z covering different parts of the moduli space. Together they will cover the whole moduli space. Now suppose that there is a matter sector vertex operator V of dimension $(1+h, 1+h)$ for small h that appears in the operator product of $V_4(z)$ and $V_3(z)$

$$V_4(z)V_3(z_3) = A\,(z-z_3)^{-1+h}(\bar{z}-\bar{z}_3)^{-1+h}V(z_3)+\cdots \tag{3.26}$$

where A is some coefficient. We shall assume that V is normalized as

$$V(z)V(w) = (z-w)^{-2-2h}(\bar{z}-\bar{w})^{-2-2h}\,. \tag{3.27}$$

In this case the integral (3.23) will receive a contribution from the $z \simeq z_3$ region of the form

$$-\frac{1}{\pi}\,\varepsilon\,A \int d^2z(z-z_3)^{-1+h}(\bar{z}-\bar{z}_3)^{-1+h}\,\langle \mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}(z_3)\rangle \tag{3.28}$$

where

$$\mathcal{V}(z_3) = \bar{c}cV(z_3)\,. \tag{3.29}$$

This can be expressed as

$$-\frac{1}{\pi}\,\varepsilon\,A\,\langle \mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}(z_3)\rangle \int_0^1 2\pi r dr r^{-2+2h} \simeq -\varepsilon\,A\,h^{-1}\,\langle \mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}(z_3)\rangle\,. \tag{3.30}$$

Let us now compare this with what we expect from the 1PR amplitude in the effective field theory. The contribution comes from the diagram where a three point vertex containing $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}$ is connected by a propagator to a three point vertex connecting $\mathcal{V}_3$, $\mathcal{V}_4$ and $\mathcal{V}$. If we denote by $\mathcal{V}^c$ the state conjugate to $\mathcal{V}$:

$$|\mathcal{V}^c\rangle \equiv c_0\bar{c}_0\bar{c}_1c_1|V\rangle, \quad \langle\mathcal{V}^c|\mathcal{V}\rangle = 1\,, \tag{3.31}$$

then, using the fact that the zeroth order propagator is given by $-b_0^+b_0^-\delta_{L_0^-}(L_0^+)^{-1}$, we get the 1PR contribution to the four point function to be

$$-\langle \mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}(w_1)\rangle\langle \mathcal{V}(w_2)\mathcal{V}_3(z_3)\mathcal{V}_4(z_4)\rangle\langle\mathcal{V}^c|b_0^+b_0^-\delta_{L_0^-}(L_0^+)^{-1}|\mathcal{V}^c\rangle \tag{3.32}$$

where w_1 and w_2 are arbitrary points on which the result does not depend to leading order in h. Using the result $b_0^+b_0^- = 2\bar{b}_0b_0$, $L_0^+|\mathcal{V}^c\rangle = 2h|\mathcal{V}^c\rangle$ and eqs. (3.27), (3.31), (3.26), (2.9) we can express (3.32) as

$$A\,h^{-1}\,\langle \mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\mathcal{V}(z_3)\rangle \tag{3.33}$$

Comparing this with (3.30) we see that we must have $\varepsilon = -1$.

We could recover the more standard convention $dz\wedge d\bar{z} = -2id^2z$ by including an additional factor of $(-1)^{3g-3+n}$ in the definition of $\{A_1\cdots A_n\}$ since the latter involves integration over $3g-3+n$ dimensional complex manifold. This would replace the normalization factor $(2\pi i)^{-(3g-3+n)}$ in the definition of $\{A_1\cdots A_n\}$ by $(-2\pi i)^{-(3g-3+n)}$. We shall not attempt to do so in this paper.

3.4 Reality conditions on string fields

In computing tree level Feynman amplitudes from a field theory we do not need to know what the reality condition on the fields are — the path integral over various fields can be taken to run over arbitrary contours passing through the origin in the complexified field space. However when we discuss the choice of vacuum solutions encoded in the a_α's entering (2.30), then in order to see what classical backgrounds are physically possible we need to know the reality conditions on the a_α's. This is the issue we shall address now. For definiteness we shall restrict our analysis to the heterotic string theory, but the generalization to type II string theories is straightforward.

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We shall choose the basis $|\varphi_\alpha\rangle$ to be normalized as

$$\langle\varphi_\alpha|c_0^- c_0^+|\varphi_\beta\rangle = -\frac{1}{2}\delta_{\alpha\beta}\,. \tag{3.34}$$

The reason for choosing the $-$ sign will be apparent soon. Most zero momentum BRST invariant states satisfying this normalization are of the form

$$|\varphi_\alpha\rangle = \bar{c}ce^{-\phi}V_\alpha(0)|0\rangle\,, \tag{3.35}$$

where V_α is a dimension (1,1/2) superconformal primary in the matter sector satisfying

$$V_\alpha(z)V_\beta(w) = (z-w)^{-1}(\bar{z}-\bar{w})^{-2} + \cdots\,, \tag{3.36}$$

where $\cdots$ denotes less singular terms. Using (2.3), (2.9), (2.11) and the fact that $e^{-\phi}$ anti-commutes with V_α we can easily verify that (3.36) implies (3.34)

Let us for definiteness suppose that we are analyzing a four point function of the form given in (3.22), but in heterotic string theory instead of in the bosonic string theory. Let us take $\mathcal{V}_4$ to be the operator φ_α given in (3.35), and the other three vertex operators to be some fixed operators in the NS sector. In that case we need to insert two PCO's. We shall take the location of one of them to coincide with the location of the vertex $\mathcal{V}_4$. This converts $\mathcal{V}_4 = \varphi_\alpha$ to the form

$$\mathcal{X}_0|\varphi_\alpha\rangle = \bar{c}c\widetilde{V}_\alpha(0)|0\rangle - \frac{1}{4}\bar{c}\eta e^{\phi}V_\alpha(0)|0\rangle\,, \tag{3.37}$$

where

$$|\widetilde{V}_\alpha\rangle = -\oint dz T_F(z)|V_\alpha\rangle \equiv -G_{-1/2}|V_\alpha\rangle\,, \tag{3.38}$$

and $G_n \equiv \oint dz\, z^{n+1/2}T_F(z)$. It follows from (3.36) and $\{G_{1/2}, G_{-1/2}\} = (L_0)_{\rm matter}/2$ that

$$\widetilde{V}_\alpha(z)\widetilde{V}_\beta(w) = \frac{1}{4}(z-w)^{-2}(\bar{z}-\bar{w})^{-2} + \cdots\,. \tag{3.39}$$

Let us consider the effect of the first term on the right hand side of (3.37). Following the steps that led from (3.22) to (3.23) we see that the contour integrals of b and $\bar{b}$ remove the $\bar{c}c$ factor from the vertex operator and leaves us with only the operator $\widetilde{V}_\alpha$. It now follows from (3.23) and (3.25) that the effect of switching on a background of the form given in (2.30) is to insert $(1/\pi)\sum_\alpha a_\alpha \int d^2z\widetilde{V}_\alpha(z,\bar{z})$ into the CFT correlation functions. From the normalization condition (3.39) it follows that this is an allowed deformation of the CFT for real a_α. Therefore a field configuration of the form given in (2.30) should be declared real for real a_α.

As an aside we note from (3.34), (2.9) and the fact that the L_0^+ eigenvalues of the GSO even states in $\widehat{H}_T$ are of the form $(k^2+m^2)/2$ with positive constant m^2, that the kinetic term of the NS sector states in the action given in (3.1) comes with coefficient proportional to $-(k^2+m^2)$ multiplying the square of the real fields. Therefore e^S has damped kinetic term for large real values of the fields. This is the conventional definition of the euclidean path integral; so our action S is really the negative of the conventional euclidean action. Even though we shall not use the result, it may be useful to keep this in mind for future applications.

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3.5 Summary

Since the analysis has been somewhat technical, we shall now summarize the main results of this section. Our main aim will be to explain how the prescription for computing S-matrix elements that we arrive at differs from the usual Polyakov prescription for computing amplitudes.

1. The usual Polyakov prescription expresses an amplitude as an integral over the moduli space of Riemann surfaces with punctures, with the integrand given by certain correlation functions of vertex operators inserted at the punctures, ghost fields and picture changing operators inserted in an appropriate way as explained in [20]. For describing our modified prescription we first need to divide the integral over moduli space into separate sectors containing 1PI contributions and various one particle reducible (1PR) contributions. The latter contain Riemann surfaces obtained by gluing 1PI Riemann surfaces using plumbing fixture relation
$$zw = e^{-s+i\theta}, \quad 0 \le s < \infty, \quad 0 \le \theta < 2\pi \,, \tag{3.40}$$
where z and w are the local coordinates at the punctures that are glued. The integration over s and θ — which describe two of the moduli of the 1PR Riemann surface — yields the propagator
$$-\frac{1}{2\pi}\int_0^\infty ds \int_0^{2\pi} d\theta\, e^{-sL_0^+} e^{i\theta L_0^-}\, b_0^+ b_0^- \mathcal{G} = -\frac{b_0^+ b_0^-}{L_0^+}\delta_{L_0^-}\mathcal{G}\,. \tag{3.41}$$
The minus sign is convention dependent and has been explained below (3.16). This by itself of course does not change the result for the integral over the moduli space, but just involves organizing the integral as sum over different terms each of which in turn can be regarded as 1PI amplitudes connected by propagators. We shall now describe how the prescription we arrive at differs from the standard prescription.

2. In a 1PR contribution, any propagator that is forced to carry zero momentum due to momentum conservation (and hence is part of a 'tadpole' contribution) is replaced by the modified propagator
$$-\frac{b_0^+ b_0^-}{L_0^+}\delta_{L_0^-}(1-\mathbf{P})\mathcal{G}\,, \tag{3.42}$$
where $\mathbf{P}$ is the projector into the zero momentum $L_0^+ = 0$ states. This makes these contributions free from infrared divergences.

3. If we want results accurate up to order g_s^n, then in any amplitude we sum over arbitrary number k of insertions of the state $|\psi_n\rangle$ satisfying (2.28), weighted by a factor of $1/k!$, for $0 \le k \le n$.

4. The external momenta are set equal to their renormalized on-shell values for which eqs. (2.52)–(2.54) have solutions, and not their classical on-shell values obtained by demanding BRST invariance of the vertex operator.

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5. The external states that are inserted into the amplitude are not what we had at tree level, but are given by the states $|\phi_n\rangle$ obtained by solving (2.52)–(2.54) if we want results accurate up to order g_s^n.

6. In a 1PR contribution, if any propagator is forced to carry momentum equal to the momentum k_i carried by one of the external states (and hence is part of a mass / wave-function renormalization diagram on the external leg) we replace it by

$$-\frac{b_0^+ b_0^-}{L_0^+}\delta_{L_0^-}(1-P_i)\mathcal{G}\,, \tag{3.43}$$

where P_i is the projector into the states carrying momentum k_i and mass level equal to the tree level mass of that external state. These are the states carrying $L_0^+ \simeq 0$ in the sense described at the end of section 2. Therefore the insertion of $(1-P_i)$ removes states with $L_0^+ \simeq 0$ and makes the amplitude free from the IR divergences associated with mass renormalization.

7. Finally we multiply the result by $\prod_{i=1}^N (Z_i)^{-1/2}$ where Z_i is the wave-function renormalization factor associated with the i-th external state. These can be computed from the residues at the poles of the two point function.

Steps 2 and 3 above deal with massless tadpole contributions while the steps 4, 5, 6 and 7 deal with mass and wave-function renormalization issues. The prescription of steps 2 and 6 imply that whenever a propagator is forced to carry momentum that either vanishes or is equal to one of the external momenta, we remove the contribution from the $L_0^+ \simeq 0$ states from it, where $L_0^+ \simeq 0$ states have been defined at the end of section 2.

There are two more unrelated but useful results derived in this section. In section 3.3 we derived the sign of the integration measure over the moduli space that is compatible with the rest of our conventions. We found for example that if $(z,\bar{z})$ denote the complex coordinates denoting the location of a vertex operator on the Riemann surface then we must define the orientation of the moduli space such that

$$dz \wedge d\bar{z} = 2\,i\,d^2z\,, \tag{3.44}$$

where the integration measure d^2z is defined such that its integral over any subspace of the Riemann surface gives a positive result. Alternatively we can use the conventional orientation where $dz\wedge d\bar{z} = -2\,i\,d^2z$, but multiply every genus g, n-point 1PI amplitude by an additional factor of $(-1)^{3g-3+n}$.

The second result deals with the reality condition on the string fields. Let us suppose that in heterotic string theory we have a field configuration of the form

$$\lambda\,\bar{c}ce^{-\phi}V(0)|0\rangle\,, \tag{3.45}$$

where λ is a complex number and V is a dimension (1,1/2) superconformal primary in the matter sector satisfying

$$V(z)\,V(w) = (z-w)^{-1}(\bar{z}-\bar{w})^{-2} + \cdots\,. \tag{3.46}$$

Then this field configuration is real if λ is real.

4 Consequences of global and local (super-)symmetry

In this section we shall discuss consequences of local (super-)symmetry and also unbroken global (super-)symmetry. To simplify notation we shall drop the spinor index carried by the supersymmetry generator and the fermionic vertex operators. Our formulæ may be interpreted as relations involving particular components of the supersymmetry generator and fermionic vertex operators.

The results of this section may be summarized by saying that they are identical to those found in [36, 37] except for the following differences:

1. These Ward identities hold also for external states which undergo mass renormalization and for backgrounds which undergo non-trivial vacuum shift.

2. The amplitudes that enter the Ward identities are the modified amplitudes computed according to the prescription summarized in section 3.5. As a result these amplitudes are free from infrared divergences.

3. Using these Ward identities we can prove the equality of renormalized masses of bosons and fermions paired by supersymmetry. This is possible only due to the fact that the Ward identities hold for external states which undergo mass renormalization.

4.1 Bose-Fermi degeneracy for global supersymmetry

Let $|\Psi_{\rm vac}\rangle$ be a vacuum solution to the equations of motion (2.25), and let us suppose that we have a global supersymmetry transformation parameter $|\Lambda_{\rm global}\rangle$ that preserves this vacuum solution. Therefore $|\Lambda_{\rm global}\rangle$ satisfies (2.31) which can also be expressed as

$$\widehat{Q}_B|\Lambda_{\rm global}\rangle = 0\,. \tag{4.1}$$

Let $|\Phi_{\rm linear}\rangle$ be a solution to the linearized equations of motion around the background i.e. it satisfies $\widehat{Q}_B|\Phi_{\rm linear}\rangle = 0$. Then it follows from (2.39), (2.46) and (4.1) that

$$\widehat{Q}_B\mathcal{G}\left[\Lambda_{\rm global}\Phi_{\rm linear}\right]'' = 0\,. \tag{4.2}$$

Therefore $\mathcal{G}[\Lambda_{\rm global}\Phi_{\rm linear}]''$ also satisfies the linearized equations of motion. Since $|\Lambda_{\rm global}\rangle$ is fermionic, this provides a map between the bosonic and fermionic solutions to the linearized equations of motion. Since $|\Lambda_{\rm global}\rangle$ carries zero momentum, these solutions occur at the same values of momentum. Furthermore if the solution $|\Phi_{\rm linear}\rangle$ exists for all values of momenta so does the solution $\mathcal{G}[\Lambda_{\rm global}\Phi_{\rm linear}]''$ and if the solution $|\Phi_{\rm linear}\rangle$ exists for special values of k^2, the solution $\mathcal{G}[\Lambda_{\rm global}\Phi_{\rm linear}]''$ also exists for the same special values of k^2. Therefore this procedure pairs pure gauge solutions in the bosonic and fermionic sector and also physical solutions in the two sectors. In particular since the physical solutions occur at the same values of the k^2, it establishes the equality of the masses of bosons and fermions (even though each of them may get renormalized by perturbative corrections of string theory).[8]

[8]The only exception to this is the situation where $\mathcal{G}[\Lambda_{\rm global}\Phi_{\rm linear}]''$ vanishes. However typically in such situations one can identify another component of the supersymmetry transformation parameter which does the pairing.

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Note that in general $\mathcal{G}[\Lambda_{\rm global}\Phi_{\rm linear}]''$ is not in the Siegel gauge. Therefore if $|\widehat{\Phi}_{\rm linear}\rangle$ denotes the Siegel gauge physical state in the same sector obtained by solving (2.52)–(2.54), then we have

$$\mathcal{G}[\Lambda_{\rm global}\Phi_{\rm linear}]'' = \beta\,|\widehat{\Phi}_{\rm linear}\rangle + \widehat{Q}_B|\widehat{\Lambda}\rangle \tag{4.3}$$

for some $|\widehat{\Lambda}\rangle$. Here β is a normalization factor and we have assumed that there is a unique state with the required quantum numbers and physical mass. If there is a degeneracy even after quantum corrections then β will in general be a matrix.

For later use it will be useful to develop a procedure for computing the constant β. For this we recall from (2.55)–(2.57) that $|\widehat{\Phi}_{\rm linear}\rangle$ can be regarded as the result of recursively solving the equations

$$|\widehat{\Phi}_{\rm linear}\rangle = -\frac{b_0^+}{L_0^+}(1-P)\mathcal{G}K|\widehat{\Phi}_{\rm linear}\rangle + |\widehat{\phi}\rangle\,, \tag{4.4}$$

where $|\widehat{\phi}\rangle$ satisfies

$$P|\widehat{\phi}\rangle = |\widehat{\phi}\rangle\,, \quad Q_B|\widehat{\phi}\rangle = -P\mathcal{G}K|\widehat{\Phi}_{\rm linear}\rangle\,. \tag{4.5}$$

$|\widehat{\Phi}_{\rm linear}\rangle$ and $|\widehat{\phi}\rangle$ are states in $\widehat{\mathcal{H}}_T$ of ghost number 2. Let $|\widetilde{\Phi}_{\rm linear}\rangle$ and $|\widetilde{\phi}\rangle$ be conjugate states in $\widetilde{\mathcal{H}}_T$ of ghost number 3, satisfying

$$|\widetilde{\Phi}_{\rm linear}\rangle = -\frac{b_0^+}{L_0^+}(1-P)K\mathcal{G}|\widetilde{\Phi}_{\rm linear}\rangle + |\widetilde{\phi}\rangle\,, \tag{4.6}$$

$$P|\widetilde{\phi}\rangle = |\widetilde{\phi}\rangle\,, \quad Q_B|\widetilde{\phi}\rangle = -PK\mathcal{G}|\widetilde{\Phi}_{\rm linear}\rangle\,, \tag{4.7}$$

$$\langle\widetilde{\phi}|c_0^-|\widehat{\phi}\rangle = 1\,. \tag{4.8}$$

It follows from (4.4)–(4.8) that

$$\langle\widetilde{\Phi}_{\rm linear}|c_0^-|\widehat{\Phi}_{\rm linear}\rangle = \langle\widetilde{\phi}|c_0^-|\widehat{\phi}\rangle = 1\,, \tag{4.9}$$

and

$$\widetilde{Q}_B|\widetilde{\Phi}_{\rm linear}\rangle = 0\,, \tag{4.10}$$

where $\widetilde{Q}_B$ has been defined in (2.47). Taking the inner product of $\langle\widetilde{\Phi}_{\rm linear}|c_0^-$ with (4.3), and using (2.49), (4.9), (4.10) we get

$$\beta = \langle\widetilde{\Phi}_{\rm linear}|c_0^-\mathcal{G}[\Lambda_{\rm global}\widehat{\Phi}_{\rm linear}]''\rangle = \{(\mathcal{G}\widetilde{\Phi}_{\rm linear})\Lambda_{\rm global}\widehat{\Phi}_{\rm linear}\}''\,. \tag{4.11}$$

The right hand side of (4.11) is the 1PI three point function of external states $\mathcal{G}|\widetilde{\Phi}_{\rm linear}\rangle$, $|\Lambda_{\rm global}\rangle$ and $|\widehat{\Phi}_{\rm linear}\rangle$. Following the analysis of section 3.2 we can interpret this as the usual three point amplitude with external states $\mathcal{G}|\widetilde{\phi}\rangle$, $|\lambda\rangle$ and $|\widehat{\phi}\rangle$ — the projections of $\mathcal{G}|\widetilde{\Phi}_{\rm linear}\rangle$, $|\Lambda_{\rm global}\rangle$ and $|\widehat{\Phi}_{\rm linear}\rangle$ to the $L_0^+\simeq 0$ sector, and arbitrary number of insertions of $|\psi\rangle$ — the $L_0^+=0$ component of the vacuum solution, with the contribution from $L_0^+\simeq 0$ sectors removed from the propagators. As a result β is free from infrared divergences.

4.2 Ward identities for local (super-)symmetry

In this subsection we shall derive the Ward identities for S-matrix elements computed from the 1PI effective theory. As remarked at the end of section 3.1, the S-matrix elements can be computed from the truncated Green's functions $\Gamma^{(N)}$ by setting the external states on-shell and multiplying the result by appropriate wave-function renormalization factors for each external leg. We shall first show that the $\Gamma^{(N)}$'s satisfy the identities:

$$\sum_{i=1}^{N}(-1)^{\gamma_1+\cdots\gamma_{i-1}}\Gamma^{(N)}(|A_1\rangle,\cdots|A_{i-1}\rangle,\widehat{Q}_B|A_i\rangle,|A_{i+1}\rangle,\cdots|A_N\rangle)=0\,. \tag{4.12}$$

The proof of (4.12) goes as follows. In any Feynman diagram the external legs are divided into subsets, with the members of a given subset all part of the same 1PI vertex. We group the sum over i in the left hand side of (4.12) by these subsets. The result can now be organized using (2.44), with two kinds of contributions. In the first kind we have terms with $\widehat{Q}_B$ acting on the external states of the 1PI vertex which are not external states of $\Gamma^{(N)}$. These are the terms that would be present on the left hand side of (2.44) but are missing from the left hand side of (4.12). In the second kind we have contributions coming from the right hand side of (2.44). The first kind of terms, taken in pairs in which $\widehat{Q}_B$ acts on the states at the two ends of a given *internal propagator*, can be simplified using (3.17). It is easy to see that after summing over all graphs, these two kinds of terms cancel, with a pair of terms of the first kind, combined using (3.17), canceling a term of the second type.

We shall illustrate this with an example. Consider a contribution where two 1PI vertices are joined by a propagator, with external states $|A_1\rangle,\cdots|A_M\rangle$ attached to the first 1PI vertex and $|A_{M+1}\rangle,\cdots|A_N\rangle$ attached to the second 1PI vertex, and $\widehat{Q}_B$ acts in turn on each of these external states as on the left hand side of (4.12). In that case terms of the first kind have the form

$$\begin{aligned}&(-1)^{\gamma_1+\cdots\gamma_M}\{A_1\cdots A_M(\widehat{Q}_B\varphi_r)\}''\langle\varphi^r|c_0^-\Delta c_0^-|\varphi^s\rangle\{\varphi_s A_{M+1}\cdots A_N\}''\\ &+(-1)^{\gamma_1+\cdots\gamma_M+\gamma_{\varphi_s}}\{A_1\cdots A_M\varphi_r\}''\langle\varphi^r|c_0^-\Delta c_0^-|\varphi^s\rangle\{(\widehat{Q}_B\varphi_s)A_{M+1}\cdots A_N\}''\,.\end{aligned} \tag{4.13}$$

There are overall minus signs in the two terms from having to take these terms from the left to the right hand side of (2.44), but they cancel the minus sign in the propagator (3.12). Now we apply (2.19) in reverse on the last factor in each term, and use the results

$$\begin{aligned}&\widehat{Q}_B|\varphi_r\rangle\langle\varphi^r|c_0^-\Delta c_0^-|\varphi^s\rangle\langle\varphi_s|c_0^- = \widehat{Q}_B b_0^- c_0^-\Delta c_0^- b_0^- c_0^- = \widehat{Q}_B\Delta c_0^-\\ &|\varphi_r\rangle\langle\varphi^r|c_0^-\Delta c_0^-|\varphi^s\rangle\langle\widehat{Q}_B\varphi_s|c_0^- = b_0^- c_0^-\Delta c_0^-|\varphi^s\rangle\langle\varphi_s|c_0^-\widetilde{Q}_B(-1)^{\gamma_{\varphi_s}} = b_0^- c_0^-\Delta c_0^- b_0^- c_0^-\widetilde{Q}_B(-1)^{\gamma_{\varphi_s}}\\ &\qquad\qquad = (-1)^{\gamma_{\varphi_s}}\Delta c_0^-\widetilde{Q}_B\,.\end{aligned} \tag{4.14}$$

In arriving at (4.14) we have used (2.13), (2.49) and the fact that Δ defined in (3.16) is annihilated by b_0^- from the left as well as from the right. Substituting these into (4.13) we can bring it to the form

$$\begin{aligned}&(-1)^{\gamma_1+\cdots\gamma_M}\left\{A_1\cdots A_M\left((\widehat{Q}_B\Delta c_0^- + \Delta c_0^-\widetilde{Q}_B)[A_{M+1}\cdots A_N]''\right)\right\}''\\ &\quad = (-1)^{\gamma_1+\cdots\gamma_M}\{A_1\cdots A_M\mathcal{G}[A_{M+1}\cdots A_N]''\}''\end{aligned} \tag{4.15}$$

where in arriving at the second expression we have used (3.17) and the fact that

$$\delta_{L_0^-}[A_{M+1}\cdots A_N]'' = [A_{M+1}\cdots A_N]''\,. \tag{4.16}$$

We shall now combine this with contributions that come from a single 1PI vertex with external states $|A_1\rangle, \cdots |A_N\rangle$, with $\widehat{Q}_B$ acting on one of the states on the left hand side. After summing over the N terms in which $\widehat{Q}_B$ acts in turn on $|A_1\rangle, \cdots |A_N\rangle$, a direct application of (2.44) reduces this to a sum of terms of the second kind, one of which takes the form

$$-\,(-1)^{\gamma_1+\cdots\gamma_M}\{A_1\cdots A_M\mathcal{G}[A_{M+1}\cdots A_N]''\}''\,. \tag{4.17}$$

The overall minus sign comes from the minus sign on the right hand side of (2.44). The factor of $1/2$ on the right hand side of (2.44) has been cancelled by a factor of 2 that arises by combining pairs of terms — one where $A_{M+1}\cdots A_N$ is inside the square bracket and the other where $A_1\cdots A_M$ is inside the square bracket. We now see that (4.17) cancels (4.15). Similar cancellations occur for all terms, leading to (4.12).

Let us now suppose that we have a set of physical external states $|\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle$ satisfying

$$\widehat{Q}_B|\mathcal{A}_i\rangle = 0\,, \qquad \text{for} \qquad 1\le i\le N\,. \tag{4.18}$$

Let us also suppose that we have a local gauge transformation parameter $|\Lambda\rangle$ belonging either to the fermionic sector or to the bosonic sector. Then $\widehat{Q}_B|\Lambda\rangle$ represents a pure gauge state. It now follows from (4.12) with N replaced by $N+1$ and the states $|A_1\rangle, \cdots |A_{N+1}\rangle$ replaced by $|\Lambda\rangle, |\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle$ that

$$\Gamma^{(N+1)}(\widehat{Q}_B|\Lambda\rangle, |\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle) = 0\,. \tag{4.19}$$

Since S-matrix elements with external states $\widehat{Q}_B|\Lambda\rangle, |\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle$ are given by multiplying $\Gamma^{(N+1)}(\widehat{Q}_B|\Lambda\rangle, |\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle)$ by wave-function renormalization factors, vanishing of (4.19) will also imply the vanishing of this S-matrix element. This shows that pure gauge states of the form $\widehat{Q}_B|\Lambda\rangle$ decouple from the S-matrix of physical states. Note that since we have taken $|\mathcal{A}_i\rangle$'s to satisfy (4.18) which takes into account the effect of string loop corrections in the definition of $\widehat{Q}_B$, the decoupling of pure gauge states occurs even in the presence of external states that suffer mass renormalization.

The $\Gamma^{(N+1)}$ appearing in (4.19) corresponds to truncated Green's functions and hence, regarded as amplitudes with external states $|\mathcal{A}_i\rangle$ and $\widehat{Q}_B|\Lambda\rangle$, it will be given by integral over a subspace of the full moduli space of Riemann surfaces in which the contributions corresponding to diagrams 1PR in external legs are removed. However since $|\mathcal{A}_i\rangle$ are physical states annihilated by $\widehat{Q}_B$, we can use the arguments given below (3.21) to show that the missing 1PR parts associated with the external states $|\mathcal{A}_i\rangle$ are added back to the amplitude if we replace $|\mathcal{A}_i\rangle$'s by $P|\mathcal{A}_i\rangle$'s, P being the projection operator to $L_0^+\simeq 0$ states defined below (2.54) — except that the contribution due to the $L_0^+\simeq 0$ states are subtracted from the propagators on the external lines and hence the amplitude does not suffer from any divergence due to mass / wave-function renormalization. However this does not directly apply to the external line $\widehat{Q}_B|\Lambda\rangle$. We shall now show how this can also be rearranged so that we include also the contribution from the regions of the moduli space that are 1PR in the leg $\widehat{Q}_B|\Lambda\rangle$. For this we choose

$$|\Lambda\rangle = (1+\Delta_0 c_0^- K)^{-1}|\bar{\Lambda}\rangle = \sum_{k=0}^{\infty}(-\Delta_0 c_0^- K)^k|\bar{\Lambda}\rangle \tag{4.20}$$

for some state $|\bar{\Lambda}\rangle$. Now it follows from (2.41) and the expression for Δ_0 given in (3.18) that

$$[Q_B, \Delta_0 c_0^- K] = (1+\Delta_0 c_0^- K)\mathcal{G}K \tag{4.21}$$

acting on states in $\widehat{\mathcal{H}}$, and hence

$$[Q_B, (1+\Delta_0 c_0^- K)^{-1}] = -(1+\Delta_0 c_0^- K)^{-1}[Q_B, \Delta_0 c_0^- K](1+\Delta_0 c_0^- K)^{-1} = -\mathcal{G}K(1+\Delta_0 c_0^- K)^{-1}\,. \tag{4.22}$$

This gives

$$\widehat{Q}_B|\Lambda\rangle = (Q_B+\mathcal{G}K)(1+\Delta_0 c_0^- K)^{-1}|\bar{\Lambda}\rangle = (1+\Delta_0 c_0^- K)^{-1}Q_B|\bar{\Lambda}\rangle\,. \tag{4.23}$$

Therefore in (4.19) we can replace $\widehat{Q}_B|\Lambda\rangle$ by the right hand side of (4.23). Expanding this as $\sum_{k=0}^{\infty}(-\Delta_0 c_0^- K)^k Q_B|\bar{\Lambda}\rangle$ we see that we get back the missing 1PR contributions, but the external state $\widehat{Q}_B|\Lambda\rangle$ is replaced by $Q_B|\bar{\Lambda}\rangle$.

To summarize, we can interpret (4.19) as a statement of vanishing of the full off-shell string amplitude involving external states $Q_B|\bar{\Lambda}\rangle$ and the $L_0^+\simeq 0$ components of the $|\mathcal{A}_i\rangle$'s. However in computing this amplitude we have to subtract the contribution due to the $L_0^+\simeq 0$ states from all internal propagators which carry momentum equal to that of any of the external momenta carried by the states $|\mathcal{A}_i\rangle$. This removes the potential infrared divergences associated with mass renormalization. We also need to remove the contribution of $L_0^+ = 0$ states from internal propagators carrying zero momentum and sum over arbitrary number of insertions of $|\psi\rangle$ — the $L_0^+ = 0$ component of the vacuum solution. This removes the potential infrared divergences associated with tadpole graphs. This agrees with the general form of the Ward identity described in [36, 37], except for the subtraction of the infrared divergent contribution mentioned above and the use of $L_0^+\simeq 0$ components of $|\mathcal{A}_j\rangle$ instead of BRST invariant states as external states.

4.3 Ward identities for global (super-)symmetry

We shall now explore the consequences of global (super-)symmetry on the S-matrix. The existence of such a symmetry is signaled by a gauge transformation parameter $|\Lambda_{\rm global}\rangle$ satisfying

$$\widehat{Q}_B|\Lambda_{\rm global}\rangle = 0\,. \tag{4.24}$$

Typically $|\Lambda_{\rm global}\rangle$ carries zero momentum. Now if we used (4.19) with $|\Lambda\rangle$ replaced by $|\Lambda_{\rm global}\rangle$ then the resulting identity is trivial. To get something non-trivial we proceed somewhat differently. We first define a new object $\widetilde{\Gamma}^{(N)}(|A_1\rangle,\cdots|A_N\rangle)$ where the first argument $|A_1\rangle$ plays a somewhat different role compared to the other arguments. To define $\widetilde{\Gamma}^{(N)}$ we begin with the expression for the truncated Green's function $\Gamma^{(N)}$ as sum of Feynman diagrams built from 1PI vertices and propagators, and delete from this all terms

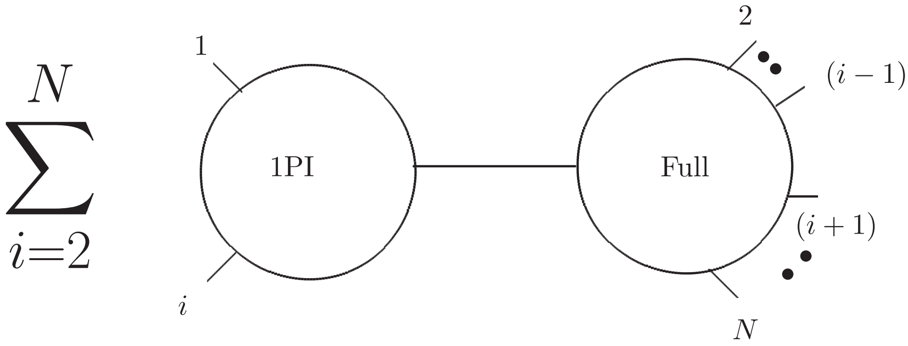

Figure 1. The contributions to be excluded from the definition of $\widetilde{\Gamma}^{(N)}$. Here the blob marked 1PI represents the 1PI vertex $\{\ \}''$, the blob marked Full represent the full truncated Green's function, the horizontal line connecting the two blobs represent the full propagator Δ and the short lines represent external states.

where by removing a single propagator we can separate the external states $|\widetilde{A}_1\rangle$ and one more $|A_i\rangle$ from the rest of the $|A_i\rangle$'s. This has been shown in figure 1. If we take the $|A_i\rangle$'s to be states carrying fixed momenta k_i then this means that we remove all terms where one of the internal propagators carry momentum $k_1 + k_i$ for any i between 2 and N. We can now derive an identity analogous to (4.12) for $\widetilde{\Gamma}^{(N)}$ using similar method, but now there are additional terms since the contributions of the first kind involving $\widehat{Q}_B \Delta c_0^- + \Delta c_0^- \widetilde{Q}_B$ with Δ representing a propagator carrying momentum $k_1 + k_i$ for $2 \leq i \leq N$ are absent. As a result some of the contributions of the second kind are not canceled and we get

$$\begin{aligned}&\sum_{i=1}^{N}(-1)^{\gamma_1+\cdots\gamma_{i-1}}\widetilde{\Gamma}^{(N)}(|A_1\rangle,\cdots|A_{i-1}\rangle,\widehat{Q}_B|A_i\rangle,|A_{i+1}\rangle,\cdots|A_N\rangle)\\&+\sum_{i=2}^{N}(-1)^{(\gamma_1+1)(\gamma_2+\cdots\gamma_{i-1})}\Gamma^{(N-1)}(|A_2\rangle,\cdots|A_{i-1}\rangle,\mathcal{G}[A_1A_i]'',|A_{i+1}\rangle,\cdots|A_N\rangle)=0\,.\end{aligned}\tag{4.25}$$

Note that the second term involves $\Gamma^{(N-1)}$ and not $\widetilde{\Gamma}^{(N-1)}$. The proof of this relation follows the same logic as the one used in arriving at (4.12).

Let us now suppose that we have a set of physical external states $|\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle$ satisfying (4.18) and a global (super-)symmetry transformation parameter $|\Lambda_{\rm global}\rangle$ satisfying (4.24). Then a direct application of (4.25) with N replaced by $N+1$, and the states $|A_1\rangle, \cdots |A_{N+1}\rangle$ taken as $|\Lambda_{\rm global}\rangle, |\mathcal{A}_1\rangle, \cdots |\mathcal{A}_N\rangle$ gives

$$\sum_{i=1}^{N}(-1)^{(\gamma_\Lambda+1)(\gamma_1+\cdots\gamma_{i-1})}\Gamma^{(N)}(|\mathcal{A}_1\rangle,\cdots|\mathcal{A}_{i-1}\rangle,\mathcal{G}[\Lambda_{\rm global}\mathcal{A}_i]'',|\mathcal{A}_{i+1}\rangle,\cdots|\mathcal{A}_N\rangle)=0\,,\tag{4.26}$$

where γ_Λ and γ_i now stand for the grassmannality of $|\Lambda_{\rm global}\rangle$ and $|\mathcal{A}_i\rangle$ respectively. Noting that according to (4.2), $\mathcal{G}[\Lambda_{\rm global}\mathcal{A}_i]''$ represents the on-shell state which is the transform of $|\mathcal{A}_i\rangle$ under the infinitesimal global (super-)symmetry generated by $|\Lambda_{\rm global}\rangle$, we recognize (4.26) as the Ward identities associated with the global (super-)symmetry generated by $|\Lambda_{\rm global}\rangle$.

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We could again examine to what extent the right hand side of (4.26) can be interpreted as usual string amplitudes involving integration over the full moduli space. For this we use (4.3) to replace $\mathcal{G}[\Lambda_{\rm global}\mathcal{A}_i]''$ by $\beta_{(i)}|\Phi^{(i)}\rangle + \widehat{Q}_B|\Lambda^{(i)}\rangle$ where $|\Phi^{(i)}\rangle$ is a solution to the linearized equations of motion in the Siegel gauge obtained by solving (2.52)–(2.54). The amplitude involving $\widehat{Q}_B|\Lambda^{(i)}\rangle$ vanishes by (4.19) and we can express (4.26) as

$$\sum_{i=1}^{N}(-1)^{(\gamma_\Lambda+1)(\gamma_1+\cdots\gamma_{i-1})}\beta_{(i)}\,\Gamma^{(N)}(|\mathcal{A}_1\rangle,\cdots|\mathcal{A}_{i-1}\rangle,|\widehat{\Phi}^{(i)}\rangle,|\mathcal{A}_{i+1}\rangle,\cdots|\mathcal{A}_N\rangle)=0\,. \tag{4.27}$$

Using the results summarized in section 3.5 we can represent the amplitude involving $|\widehat{\Phi}^{(i)}\rangle$ and the $|\mathcal{A}_j\rangle$'s as the usual string amplitude with external states given by the projections of $|\widehat{\Phi}^{(i)}\rangle$ and $|\mathcal{A}_j\rangle$'s to $L_0^+\simeq 0$ sector, arbitrary number of insertions of the $L_0^+=0$ component of the vacuum solution and appropriate subtractions from propagators. On the other hand using the result obtained in section 4.1 we can interpret $\beta_{(i)}$ as the usual three point amplitude of string theory with external states $\mathcal{G}|\widetilde{\phi}^{(i)}\rangle$, $|\lambda\rangle$ and $|\phi^{(i)}\rangle$ with arbitrary number of insertions of the $L_0^+=0$ component of the vacuum solution and appropriate subtractions from propagators. Here $|\lambda\rangle$ and $|\phi^{(i)}\rangle$ are the projections of $|\Lambda_{\rm global}\rangle$ and $|\Phi^{(i)}\rangle$ to $L_0^+\simeq 0$ states and $|\widetilde{\phi}^{(i)}\rangle$ is the ghost number 3 and picture number $-3/2$ state conjugate to $|\phi^{(i)}\rangle$ satisfying $\langle\widetilde{\phi}^{(i)}|c_0^-|\phi^{(i)}\rangle=0$ and (4.6), (4.7) with $|\widetilde{\phi}\rangle$ replaced by $|\widetilde{\phi}^{(i)}\rangle$. Again this form of the Ward identity agrees with those given in [36, 37] except for the subtraction schemes for the propagators and modification of external states.

5 Supersymmetry and massless tadpoles

In our analysis in the previous section, we have assumed the existence of a vacuum solution $|\Psi_{\rm vac}\rangle$ to all orders in g_s, and have derived various Ward identities associated with local and global (super-)symmetries. If instead the vacuum solution (and global (super-)symmetry transformation parameter) is assumed to exist to a certain order g_s^n, then the Ward identities will also be valid to that order. In this section we address a slightly different problem. We shall assume that we have a vacuum solution $|\Psi_{\rm vac}\rangle$ to a certain order in g_s and also unbroken supersymmetry to certain order in g_s. We shall then examine to what extent unbroken supersymmetry may help us extend the vacuum solution to higher order.

For simplicity we shall carry out our discussion in the context of heterotic string theory, but an identical analysis holds for NSNS sector tadpoles in type II string theory. Also we shall assume that the vacuum solution has a power series expansion in g_s containing both even and odd powers of g_s. This allows us to include the cases where under quantum corrections fields may get shifted by terms of order g_s [18]. If we are considering a vacuum described by a string field configuration containing only even powers of g_s then the natural expansion parameter is g_s^2 and in all subsequent formulæ we have to set the coefficients of all odd powers of g_s to zero. We shall give the result for this case explicitly in (5.16). As mentioned in footnote 3, there may also be cases where the vacuum solution has expansion in powers of g_s^α for some α in the range $0<\alpha<1$. In such cases we have to replace g_s by g_s^α in the following analysis.

We shall begin by assuming that we have a translationally invariant vacuum solution $|\Psi_p\rangle$ to the classical equations of motion to order g_s^p for some integer p, i.e.[9]

$$Q_B|\Psi_p\rangle + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\mathcal{G}[{\Psi_p}^{n-1}] = \mathcal{O}(g_s^{p+1})\,. \tag{5.1}$$

Our goal will be to see under what condition we can extend $|\Psi_{\rm vac}\rangle$ to the next order. If we denote by $|\mathcal{V}_{\rm S}\rangle$ a zero momentum BRST invariant NS sector Lorentz scalar state carrying ghost number 2 and picture number -1, then the obstruction to extending $|\Psi_{\rm vac}\rangle$ to the next order is encoded in the matrix element

$$\mathcal{E}_{p+1}(\mathcal{V}_{\rm S}) \equiv \langle \mathcal{V}_{\rm S}|c_0^- \left(Q_B|\Psi_p\rangle + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\mathcal{G}[{\Psi_p}^{n-1}] \right)\,. \tag{5.2}$$

By (5.1), $\mathcal{E}_{p+1}(\mathcal{V}_{\rm S})$ is already of order g_s^{p+1}. If we can show that $\mathcal{E}_{p+1}(\mathcal{V}_{\rm S})$ is of order g_s^{p+2} for every zero momentum BRST invariant state $|\mathcal{V}_{\rm S}\rangle$, then we can extend the solution to the next order [14]. This is known to hold as a consequence of (5.1) when $|\mathcal{V}_{\rm S}\rangle$ is a BRST trivial state [14], so we focus on the cases where $|\mathcal{V}_{\rm S}\rangle$ represents a non-trivial element of the BRST cohomology. Since Lorentz invariance is unbroken, it will be sufficient to consider only Lorentz scalar $|\mathcal{V}_{\rm S}\rangle$'s.

We shall consider string theories with unbroken supersymmetry at tree level. It has been shown in appendix C that in such theories, for every $\mathcal{V}_{\rm S}$ we can find a BRST invariant Ramond sector vertex operator $\mathcal{V}_{\rm F}$ of ghost number 2 and picture number $-1/2$, and another BRST invariant ghost number 1, picture number $-1/2$ state $|\Lambda_0\rangle$ representing a leading order global supersymmetry transformation, such that

$$|\mathcal{V}_{\rm S}\rangle = [\Lambda_0 \mathcal{V}_{\rm F}]_0 \tag{5.3}$$

up to addition of BRST trivial states. The subscript 0 on $[\cdots]$ indicates that we evaluate $[\cdots]$ only at genus 0. In the definition of $|\mathcal{V}_{\rm F}\rangle$ and $|\Lambda_0\rangle$ we shall not include multiplication by grassmann odd variable as would be required to promote them to fermionic string fields and supersymmetry transformation parameters respectively. Therefore $|\mathcal{V}_{\rm F}\rangle$ is grassmann odd and $|\Lambda_0\rangle$ is grassmann even.

Let us now suppose that Λ_0 can be extended to a global supersymmetry transformation parameter to order g_s^q for some $q \leq p$, i.e. there is a state $|\Lambda_q\rangle \in \mathcal{H}_T$ carrying zero momentum, ghost number 1 and picture number $-1/2$, satisfying

$$\widehat{Q}_B|\Lambda_q\rangle \equiv (Q_B + \mathcal{G}K_p)|\Lambda_q\rangle = \mathcal{O}(g_s^{q+1})\,, \quad |\Lambda_q\rangle = |\Lambda_0\rangle + \mathcal{O}(g_s)\,, \tag{5.4}$$

where K_p is the operator K defined in (2.40) with $|\Psi_{\rm vac}\rangle$ replaced by its p-th order solution $|\Psi_p\rangle$. Then we can use (5.3) to write

$$|\mathcal{V}_{\rm S}\rangle = [\Lambda_q \mathcal{V}_{\rm F}] + \mathcal{O}(g_s) = \sum_{m=0}^{\infty} \frac{1}{m!}[\Lambda_q \mathcal{V}_{\rm F} {\Psi_p}^m] + \mathcal{O}(g_s)\,, \tag{5.5}$$

[9] As usual we can restrict the sum to $n \leq p+1$ to get terms accurate to order g_s^p. Similar remark holds for subsequent sums.

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up to addition of BRST trivial terms. In the last step we have included a sum over m by exploiting the fact that since Ψ_p has its expansion beginning at order g_s, all but the $m=0$ term will be of order g_s. Using (5.2), (5.5) and (5.1) we get

$$\mathcal{E}_{p+1}(\mathcal{V}_\mathrm{S}) = \sum_{m=0}^{\infty} \frac{1}{m!} \langle [\Lambda_q \mathcal{V}_\mathrm{F} {\Psi_p}^m] | c_0^- \left(Q_B |\Psi_p\rangle + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \mathcal{G}[{\Psi_p}^{n-1}] \right) + \mathcal{O}(g_s^{p+2}) \,, \tag{5.6}$$

Using the fact that $|\mathcal{V}_\mathrm{F}\rangle$ is grassmann odd whereas $|\Lambda_q\rangle$ and $|\Psi_p\rangle$ are grassmann even we can express (5.6) as

$$\begin{aligned}
\mathcal{E}_{p+1}(\mathcal{V}_\mathrm{S}) = & -\sum_{m=0}^{\infty} \frac{1}{m!} \left\{ \Lambda_q \mathcal{V}_\mathrm{F} {\Psi_p}^m (Q_B \Psi_p) \right\} \\
& -\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m!(n-1)!} \left\{ \Lambda_q \mathcal{V}_\mathrm{F} {\Psi_p}^m \mathcal{G}[{\Psi_p}^{n-1}] \right\} + \mathcal{O}(g_s^{p+2}) \,.
\end{aligned} \tag{5.7}$$

Using (2.22), (2.24) and BRST invariance of $\mathcal{V}_\mathrm{F}$, we can rewrite this equation as

$$\begin{aligned}
\mathcal{E}_{p+1}(\mathcal{V}_\mathrm{S}) = & -\sum_{m=0}^{\infty} \frac{1}{m!} \left\{ (Q_B \Lambda_q) \mathcal{V}_\mathrm{F} {\Psi_p}^m \right\} - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left\{ \Lambda_q {\Psi_p}^m \mathcal{G}[\mathcal{V}_\mathrm{F} {\Psi_p}^n] \right\} \\
& + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left\{ \Lambda_q \mathcal{V}_\mathrm{F} {\Psi_p}^m \mathcal{G}[{\Psi_p}^n] \right\} \\
& - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m!(n-1)!} \left\{ \Lambda_q \mathcal{V}_\mathrm{F} {\Psi_p}^m \mathcal{G}[{\Psi_p}^{n-1}] \right\} + \mathcal{O}(g_s^{p+2}) \\
= & -\sum_{m=0}^{\infty} \frac{1}{m!} \left\{ (Q_B \Lambda_q) \mathcal{V}_\mathrm{F} {\Psi_p}^m \right\} - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left\{ \Lambda_q {\Psi_p}^m \mathcal{G}[\mathcal{V}_\mathrm{F} {\Psi_p}^n] \right\} \\
& + \mathcal{O}(g_s^{p+2}) \,.
\end{aligned} \tag{5.8}$$

The first three terms in the middle expression of (5.8) arise from manipulating the first term on the right hand side of (5.7) using (2.22) and the symmetry property (2.24). Using the definition of K_p that follows from (2.40) we can express (5.8) as

$$\mathcal{E}_{p+1}(\mathcal{V}_\mathrm{S}) = -\langle K_p \mathcal{V}_\mathrm{F} | c_0^- \widehat{Q}_B | \Lambda_q \rangle + \mathcal{O}(g_s^{p+2}) = \mathcal{O}(g_s^{q+2}, g_s^{p+2}) \,, \tag{5.9}$$

where $\mathcal{O}(g_s^{q+2}, g_s^{p+2})$ denotes that the error is given by the dominant term among g_s^{q+2} and g_s^{p+2}. The $\mathcal{O}(g_s^{q+2})$ correction comes from possible contributions of order g_s to $K_p|\mathcal{V}_\mathrm{F}\rangle$ and order g_s^{q+1} to $\widehat{Q}_B|\Lambda_q\rangle$.

Since $q \le p$, we have to consider two possibilities. For $q \le p-1$, the order g_s^{q+2} term dominates. Since this is of order g_s^{p+1} or larger, (5.9) does not allow us to extend the vacuum solution to the next order; this would require the order g_s^{p+1} contribution to $\mathcal{E}_{p+1}(\mathcal{V}_\mathrm{S})$ to vanish. On the other hand for $q=p$, this equation gives $\mathcal{E}_{p+1}(\mathcal{V}_\mathrm{S}) = \mathcal{O}(g_s^{p+2})$. This now allows us to extend $|\Psi_p\rangle$ to satisfy the classical equations of motion to next order, i.e. replace p by $p+1 = q+1$ in (5.1). Therefore our analysis shows that if supersymmetry is unbroken to order g_s^q then we can construct the vacuum solution to order g_s^{q+1}. This is in the spirit of the results of [36, 37]. Note however that in this analysis we have not made any assumption about $|\Psi_p\rangle$ being the perturbative vacuum. In particular by allowing $|\Psi_p\rangle$ to begin its expansion at order g_s instead of at order g_s^2, we have allowed for the possibility of non-trivial background $|\Psi_p\rangle$ — this will be illustrated in the next few sections.

Since for given q we can extend the classical solution to order g_s^p for $p=q+1$, we shall from now on set $p=q+1$. (5.9) can now be used to compute the order g_s^{q+2} contribution to $\mathcal{E}_{q+2}(\mathcal{V}_\mathrm{S})$ — referred henceforth as the tadpole contribution since in the conventional perturbation theory a non-vanishing $\mathcal{E}_{q+2}(\mathcal{V}_\mathrm{S})$ will show up as a tadpole divergence. In this case the order g_s^{q+2} contribution to the tadpole is given by the first term in the middle expression of (5.9). Now in $\mathcal{H}_T$ one can choose a basis in which the inner product $\langle A|c_0^-|B\rangle$ pairs physical states with physical states and pure gauge states with unphysical states where physical states are defined as those which are Q_B invariant but not Q_B trivial, unphysical states are not Q_B invariant and pure gauge states are Q_B trivial. This allows us to express c_0^- appearing in (5.9) as

$$c_0^- = c_0^-\,|\mathrm{phys}\rangle\langle\mathrm{phys}|c_0^- + c_0^-|\mathrm{unphys}\rangle\langle\mathrm{unphys}|Q_B c_0^- + c_0^- Q_B|\mathrm{unphys}\rangle\langle\mathrm{unphys}|c_0^-\,, \tag{5.10}$$

where we have used the fact that pure gauge states can be written as $Q_B|\mathrm{unphys}\rangle$. On the right hand side of (5.10) it is understood that a term like $|\mathrm{phys}\rangle\langle\mathrm{phys}|$ represents sum over all physical states in $\mathcal{H}_T$. Furthermore, as a consequence of (5.1), (5.4) with $p>q$ and BRST invariance of $|\mathcal{V}_\mathrm{F}\rangle$, we have the relations [14, 15]

$$\begin{aligned} Q_B\sum_{m=0}^{\infty}\frac{1}{m!}[\Lambda_q{\Psi_{q+1}}^m] &= \mathcal{O}(g_s^{q+2}), \qquad & Q_B\sum_{m=0}^{\infty}\frac{1}{m!}[\mathcal{V}_\mathrm{F}{\Psi_{q+1}}^m] &= \mathcal{O}(g_s^2),\\ Q_B\widehat{Q}_B|\Lambda_q\rangle &= \mathcal{O}(g_s^{q+2}), & Q_B K_{q+1}|\mathcal{V}_\mathrm{F}\rangle &= \mathcal{O}(g_s^2)\,, \end{aligned} \tag{5.11}$$

where the equations in the second line follow from those in the first line and the definitions of $\widehat{Q}_B$, K_p. Substituting (5.10) into (5.9) with $p=q+1$, and using (5.11) and (5.4) we see that the contribution to the right hand side of (5.9) from the last two terms on the right hand side of (5.10) is of order g_s^{q+3}. Therefore we can now express (5.9) for $p=q+1$ as

$$\begin{aligned} \mathcal{E}_{q+2}(\mathcal{V}_\mathrm{S}) &= -\sum_a \langle K_{q+1}\mathcal{V}_\mathrm{F}|c_0^-|\zeta_a\rangle\langle\zeta^a|c_0^-\widehat{Q}_B|\Lambda_q\rangle\\ &= -\sum_a\left(\sum_{m=0}^{\infty}\frac{1}{m!}\langle[\mathcal{V}_\mathrm{F}{\Psi_{q+1}}^m]|c_0^-|\zeta_a\rangle\right)\Bigg|_{g_s}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\langle\zeta^a|c_0^-\mathcal{G}|[\Lambda_q{\Psi_{q+1}}^n]\rangle\right)\Bigg|_{g_s^{q+1}}\\ &\quad+\mathcal{O}(g_s^{q+3})\,, \end{aligned} \tag{5.12}$$

where the sum over a runs over finite number of physical states $|\zeta_a\rangle\in\mathcal{H}_T$ carrying zero momentum and $|\zeta^a\rangle$ are conjugate physical states satisfying $\langle\zeta_a|c_0^-|\zeta^b\rangle = {\delta_a}^b$. In the last step of (5.12) we have used the fact that Q_B annihilates physical states. The subscripts g_s and g_s^{q+1} denote that we have to compute the correlation functions to order g_s and g_s^{q+1} respectively, — indeed by assumption these quantities receive possible contribution only at these orders and beyond. Ghost and picture number conservation shows that $|\zeta_a\rangle$ carries ghost number 2 and picture number $-1/2$ while its conjugate state $|\zeta^a\rangle$ carries ghost number 3 and picture number $-3/2$.

Physically the amplitude

$$\left(\sum_{m=0}^{\infty}\frac{1}{m!}\langle[\mathcal{V}_{\rm F}\Psi_{q+1}{}^m]|c_0^-|\zeta_a\rangle\right)\Bigg|_{g_s} = \sum_{m=0}^{\infty}\frac{1}{m!}\{\zeta_a\mathcal{V}_{\rm F}\Psi_{q+1}{}^m\}|_{g_s} \tag{5.13}$$

represents the 1PI two point function of $\mathcal{V}_{\rm F}$ and ζ_a in the background $|\Psi_{q+1}\rangle$ to order g_s. Since for two point function the 1PI amplitude is also the truncated Green's function, we can identify this as $g_s\Gamma^{(2,1)}(\zeta_a,\mathcal{V}_{\rm F}) = -g_s\Gamma^{(2,1)}(\mathcal{V}_{\rm F},\zeta_a)$ where $g_s^k\,\Gamma^{(n,k)}$ denotes the order g_s^k contribution to $\Gamma^{(n)}$. Similarly the amplitude

$$\left(\sum_{n=0}^{\infty}\frac{1}{n!}\langle\zeta^a|c_0^-\mathcal{G}|[\Lambda_q\Psi_{q+1}{}^n]\rangle\right)\Bigg|_{g_s^{q+1}} = \sum_{n=0}^{\infty}\frac{1}{n!}\{(\mathcal{G}\zeta^a)\,\Lambda_q\Psi_{q+1}{}^n\}|_{g_s^{q+1}} \tag{5.14}$$

can be identified as $g_s^{q+1}\Gamma^{(2,q+1)}(\mathcal{G}\zeta^a,\Lambda_q)$. Since ζ^a is a Ramond sector state, $\mathcal{G}\zeta^a = \mathcal{X}_0\zeta^a$. Therefore we can express (5.12) as

$$\mathcal{E}_{q+2}(\mathcal{V}_{\rm S}) = g_s^{q+2}\sum_a\Gamma^{(2,1)}(\mathcal{V}_{\rm F},\zeta_a)\,\Gamma^{(2,q+1)}(\mathcal{X}_0\zeta^a,\Lambda_q) + \mathcal{O}(g_s^{q+3})\,. \tag{5.15}$$

As remarked at the beginning of the section, if we are considering perturbative vacuum where the vacuum solution has expansion in powers of g_s^2, then the coefficients of all the odd powers of g_s vanish. In this case all factors of g_s in the above analysis can be replaced by g_s^2, and consequently Ψ_p, $\mathcal{E}_p$, Λ_q, K_p and $\Gamma^{(n,k)}$ factors will have to be replaced respectively by Ψ_{2p}, $\mathcal{E}_{2p}$, Λ_{2q}, K_{2p} and $\Gamma^{(n,2k)}$. (5.15) will now take the form

$$\mathcal{E}_{2q+4}(\mathcal{V}_{\rm S}) = g_s^{2q+4}\sum_a\Gamma^{(2,2)}(\mathcal{V}_{\rm F},\zeta_a)\,\Gamma^{(2,2q+2)}(\mathcal{X}_0\zeta^a,\Lambda_{2q}) + \mathcal{O}(g_s^{2q+6})\,. \tag{5.16}$$

In (5.15), $\Gamma^{(2,1)}(\mathcal{V}_{\rm F},\zeta_a)$ receives contribution only from the $m=1$ term in (5.13), with $\{\,\}$ replaced by its genus zero contribution and $|\Psi_{q+1}\rangle$ replaced by its order g_s contribution. Physically, this is the genus zero 3-point amplitude of ζ_a, $\mathcal{V}_{\rm F}$ and Ψ_1. Similarly, using the fact that Λ_q is constructed from Λ_0 by the iterative process described in (2.32), (2.33), one can follow the method of section 3.2 to conclude that $g_s^{q+1}\Gamma^{(2,q+1)}(\mathcal{X}_0\zeta^a,\Lambda_q)$ factor in (5.15) can be interpreted as the full two point function of $(\mathcal{X}_0\zeta^a)$ and λ_q — the projection of Λ_q to $L_0^+=0$ state — to order g_s^{q+1}, with the subtraction and insertion rules described in section 3.2.

If we had chosen to work at the perturbative vacuum where $|\Psi_{\rm vac}\rangle$ had its expansion in powers of g_s^2, then the factor $\Gamma^{(2,2)}(\mathcal{V}_{\rm F},\zeta_a)$ appearing in (5.16) will receive two contributions from the right hand side of (5.13) — one from genus zero with one insertion of $|\Psi_{2q+2}\rangle$ at order g_s^2 and a genus one contribution with no insertion of background $|\Psi_{2q+2}\rangle$. Following the analysis of section 3.2 we can interpret the sum of the two terms as the full genus one two point function of $\mathcal{V}_{\rm F}$ and ζ_a. Similarly, the factor $\Gamma^{(2,2q+2)}(\mathcal{X}_0\zeta^a,\Lambda_{2q})$ in (5.16) can be interpreted as the full two point function of $(\mathcal{X}_0\zeta^a)$ and λ_q — the projection of Λ_q to $L_0^+=0$ state — to order ${g_s}^{2q+2}$, with the subtraction and insertion rules described in section 3.2. With these interpretations, (5.16) agrees with the results of [36, 37] for the perturbative vacuum, except for the modified procedure for dealing with divergences associated with tadpoles and two point functions on external lines, as summarized at the beginning of section 3.

We also note that the states $|\zeta^a\rangle$, which represent elements of BRST cohomology with ghost number 3 and picture number $-3/2$, correspond to the candidate states $\mathcal{V}_{\rm G}^c$ listed in (2.35), and the conjugate states $|\zeta_a\rangle$ are the goldstino candidates listed in (2.37). Furthermore comparing (5.14) with (2.34) we see that $g_s^{q+1}\Gamma^{(2,q+1)}(\mathcal{X}_0\zeta^a, \Lambda_q)$ can be identified with $\mathcal{L}_{q+1}(\zeta^a)$ — the obstruction to finding global supersymmetry generator beyond order g_s^q. This is of course expected, since if we could extend the global supersymmetry generator to order g_s^{q+1} then $\mathcal{E}_{p+1}(\mathcal{V}_{\rm S})$ would vanish to order g_s^{q+2}.

6 SO(32) heterotic string theory on Calabi-Yau three folds

So far our analysis has been very general without referring to any specific background. In the rest of the paper we shall apply this general analysis to a specific class of backgrounds — SO(32) heterotic string theory on Calabi-Yau manifolds, with an unbroken U(1) gauge group at the tree level. Keeping this in mind in this section we shall review some basic facts about this theory.

6.1 Low energy effective field theory description

It is known from the analysis of [47–49] that one loop quantum corrections in this theory will generate a Fayet-Iliopoulos D-term. We shall not review these arguments in detail, but note that one of the effects of the D-term is to generate a one loop effective potential for a complex scalar field χ of the following form[10]

$$C\, g_s^{-2}\, (\chi^*\chi - K\, g_s^2)^2\,, \tag{6.1}$$

where g_s is the string coupling and C and K are positive numerical constants which can be computed in any given theory. At tree level the potential is $C g_s^{-2}(\chi^*\chi)^2$ and the vacuum is at $\chi = 0$. However this perturbative vacuum becomes unstable at one loop since the field χ becomes tachyonic and supersymmetry is broken. On the other hand it is clear from (6.1) that there is a stable supersymmetric vacuum at $|\chi| = K^{1/2} g_s$. We shall for definiteness take the expectation value of χ to be real, i.e. only the real part $\chi_R \equiv (\chi + \chi^*)/\sqrt{2}$ of χ gets a vacuum expectation value.

In order that the minimum of (6.1) describes a supersymmetric extremum we need to assume that there is no F-term potential for the field χ and also that there is no other D-term potential for this field. It has been shown in [48] that left-right symmetric compactification of SO(32) heterotic string theory on a Calabi-Yau manifold always contains a field χ satisfying these criteria. The world-sheet properties of the vertex operator of χ will be described in section 6.2.

[10] In general there can be more than one chiral multiplet charged under the U(1), in which case the $\chi^*\chi$ term in (6.1) is replaced by $\sum_i q_i\chi_i^*\chi_i$ where q_i are constants proportional to the U(1) charge carried by χ_i. We shall assume that even if such multiple scalar fields are present, only one of them — which we shall denote by χ — acquires a vacuum expectation value. It is easy to generalize the analysis to cases where multiple fields of this type acquire vacuum expectation values.

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6.2 World-sheet superconformal field theory

In this subsection we shall review some of the properties of the matter sector of the world-sheet superconformal field theory (SCFT) describing SO(32) heterotic string theory on a Calabi-Yau 3-fold. We shall consider backgrounds with 'spin connection embedded in gauge connection' preserving (2,2) world-sheet supersymmetry, but in principle our analysis can be generalized to compactification preserving (0,2) world-sheet supersymmetry as well. We shall only quote the relevant results; more details may be found in [48] and the references given there.

The SCFT consists of three parts. One part contains four free scalars describing the non-compact target space coordinates X^μ for $0 \le \mu \le 3$ and their right-moving superpartners ψ^μ. The second part contains 26 free left-moving fermions λ^a transforming in the vector representation of the unbroken SO(26) gauge group. Finally the third part involves an interacting theory of compact target space coordinates, their right-moving superpartners and 6 left-moving fermions transforming in the vector representation of the SO(6) subgroup of the original SO(32) gauge group. Together they describe a (2,2) superconformal field theory with central charge 9.

We shall work in the $\alpha' = 1$ unit in which X^μ and its fermionic partner ψ^μ have the following operator product expansion:

$$\begin{aligned} \partial X^\mu(z)\partial X^\nu(w) &= -\frac{\eta^{\mu\nu}}{2(z-w)^2} + \cdots, \quad \psi^\mu(z)\psi^\nu(w) = -\frac{\eta^{\mu\nu}}{2(z-w)} + \cdots, \\ \bar\partial X^\mu(\bar z)\bar\partial X^\nu(\bar w) &= -\frac{\eta^{\mu\nu}}{2(\bar z-\bar w)^2} + \cdots, \end{aligned} \tag{6.2}$$

where $\cdots$ denote less singular terms whose knowledge will not not be needed for our analysis. The matter energy momentum tensor $T(z)$ and its superpartner $T_F(z)$ have the following form

$$\begin{aligned} T(z) &= -\partial X^\mu\partial X^\nu\eta_{\mu\nu} + \psi_\mu\partial\psi^\mu + T_{\rm int}, \quad T_F(z) = -\psi_\mu\partial X^\mu + (T_F)_{\rm int}\,, \\ \bar T(\bar z) &= -\bar\partial X^\mu\bar\partial X^\nu\eta_{\mu\nu} + \bar T_{\rm int} + \bar T_\lambda\,, \end{aligned} \tag{6.3}$$

where the subscript ${}_{\rm int}$ denotes contributions from the (2,2) SCFT and $\bar T_\lambda$ denotes the energy momentum tensor of the 26 free left moving fermions λ^a. Following standard procedure of bosonization of the free fermions ψ^μ we can introduce spin fields $\Sigma^{(4)}_\alpha$, $\Sigma^{(4)}_{\dot\alpha}$ of dimension $(0, 1/4)$ with α, $\dot\alpha$ each taking two values. Since they carry dimension 1/4, we cannot assign definite grassmannality to these fields. Later we shall construct fields with definite grassmannality using these spin fields, and write down their operator product expansion with each other and with other fields.

Let us now turn to the interacting SCFT with (2,2) world-sheet supersymmetry. As a consequence of the (2,2) supersymmetry there is a right-moving R-symmetry current $J(z)$. We shall normalize it as

$$J(z)J(w) = \frac{3}{(z-w)^2} + \cdots\,. \tag{6.4}$$

With the help of this current we can construct conjugate pair of internal spin fields $\Sigma^{(6)}$ and $\Sigma^{(6)c}$ of dimensions (0,3/8) each, carrying J charges $\pm 3/2$ [56, 57]. Also as a consequence of

(2,2) supersymmetry $(T_F)_{\rm int}$ can be expressed as $T_F^+ + T_F^-$ with operator product expansion

$$T_F^+(z)T_F^-(w) = \frac{3}{4(z-w)^3} + \frac{1}{4(z-w)^2}J(w) + \frac{1}{4(z-w)}T(w) + \frac{1}{8(z-w)}\partial J(w) + \cdots \,,$$
$$T_F^+(z)T_F^+(w) = \text{non-singular}, \quad T_F^-(z)T_F^-(w) = \text{non-singular}\,. \tag{6.5}$$

The J-charges carried by $T_F^\pm$, $\Sigma^{(6)}$ and $\Sigma^{(6)c}$ are as follows:

$$T_F^+ : 1, \quad T_F^- : -1, \quad \Sigma^{(6)} : \frac{3}{2}, \quad \Sigma^{(6)c} : -\frac{3}{2}\,. \tag{6.6}$$

This means for example that $J(z)T_F^\pm(w) = \pm(z-w)^{-1}T_F^\pm(w) + \cdots$ etc.

Finally since the (2,2) supersymmetric theory is left-right symmetric it also has a left-moving U(1) current $\bar J$ that is responsible for an unbroken U(1) gauge symmetry of the string theory at tree level. This has operator product expansion

$$\bar J(z)\bar J(w) = \frac{3}{(\bar z - \bar w)^2} + \cdots\,. \tag{6.7}$$

The left-moving images of the other operators will not play any special role in our analysis and so we shall not discuss them.

A useful set of composite operators in this theory are the dimension (0,5/8) spin fields Σ_α, $\Sigma^c_{\dot\alpha}$, obtained by combining the spin fields coming from the compact directions and the non-compact directions:

$$\Sigma_\alpha = \Sigma^{(6)}\,\Sigma^{(4)}_\alpha, \quad \Sigma^c_{\dot\alpha} = \Sigma^{(6)c}\,\Sigma^{(4)}_{\dot\alpha}\,. \tag{6.8}$$

At the leading order, the global supersymmetry transformation parameter Λ_0 in the $-1/2$ picture can be constructed in terms of these fields. They are given by

$$c\,e^{-\phi/2}\Sigma_\alpha\,, \quad c\,e^{-\phi/2}\Sigma^c_{\dot\alpha}\,. \tag{6.9}$$

For describing states in the $-3/2$ picture it will also be useful to introduce spin fields of 'wrong chirality' as follows:

$$\widetilde\Sigma^c_\alpha = \Sigma^{(6)c}\,\Sigma^{(4)}_\alpha, \quad \widetilde\Sigma_{\dot\alpha} = \Sigma^{(6)}\,\Sigma^{(4)}_{\dot\alpha}\,. \tag{6.10}$$

Since the operators $\Sigma^{(6)}$, $\Sigma^{(6)c}$, $\Sigma^{(4)}_\alpha$, $\Sigma^{(4)}_\alpha$, $e^{-\phi/2}$ etc. have fractional values of $2\times$ dimensions, it is hard to assign them definite grassmann parities and hence their correlation functions will have phase ambiguities. For this reason we shall now choose spin fields of definite grassmann parity by combining the matter and ghost sector spin fields. They are taken to be

$$e^{-(2n+1)\phi/2}\Sigma_\alpha, \quad e^{-(2n+1)\phi/2}\Sigma^c_{\dot\alpha}, \quad e^{-(2n-1)\phi/2}\widetilde\Sigma^c_\alpha, \quad e^{-(2n-1)\phi/2}\widetilde\Sigma_{\dot\alpha}, \tag{6.11}$$

and declared to be GSO even and grassmann odd for n even and GSO odd and grassmann even for n odd. This is consistent with the fact that the Ramond sector vertex operators in the $-1/2$ picture (e.g. $\bar c c e^{-\phi/2}\Sigma_\alpha \bar J$) are taken to be grassmann odd and $e^{\pm\phi}$ are also grassmann odd. The operator products of these operators with $e^{m\phi}$ for integer m are determined from (2.3) and the fact that $e^{m\phi}$ has grassmann parity $(-1)^m$. Their operator products with ψ^μ and $(T_F)_{\rm int}$ and with each other are given as follows:

$$\begin{aligned}
\psi^\mu(z)\, e^{-\phi/2}\Sigma_\alpha(w) &= \frac{i}{2}(z-w)^{-1/2}(\gamma^\mu)_\alpha{}^{\dot\beta} e^{-\phi/2}\widetilde\Sigma_{\dot\beta}(w)+\cdots\,, \\
\psi^\mu(z)\, e^{-\phi/2}\Sigma^c_{\dot\alpha}(w) &= \frac{i}{2}(z-w)^{-1/2}(\gamma^\mu)_{\dot\alpha}{}^{\beta} e^{-\phi/2}\widetilde\Sigma^c_{\beta}(w)+\cdots\,, \\
e^{-\phi/2}\Sigma_\alpha(z)\; e^{-3\phi/2}\widetilde\Sigma^c_\beta(w) &= \varepsilon_{\alpha\beta}(z-w)^{-2}e^{-2\phi}(w)+\cdots\,, \\
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)\; e^{-3\phi/2}\widetilde\Sigma_{\dot\beta}(w) &= \varepsilon_{\dot\alpha\dot\beta}(z-w)^{-2}e^{-2\phi}(w)+\cdots\,, \\
(T_F)_{\rm int}(z)\, e^{-\phi/2}\Sigma_\alpha(w) &\sim (z-w)^{-1/2}, \qquad (T_F)_{\rm int}(z)\, e^{-\phi/2}\Sigma^c_{\dot\alpha}(w)\sim (z-w)^{-1/2}, \\
(T_F)_{\rm int}(z)\, e^{-3\phi/2}\widetilde\Sigma^c_\alpha(w) &\sim (z-w)^{-1/2}, \quad (T_F)_{\rm int}(z)\, e^{-3\phi/2}\widetilde\Sigma_{\dot\alpha}(w)\sim (z-w)^{-1/2}\,,
\end{aligned} \tag{6.12}$$

where $\varepsilon=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and γ^μ are four dimensional γ-matrices, normalized as

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbf{1}\,. \tag{6.13}$$

The operator product of a pair of spin fields in which both contain $\Sigma^{(6)}$ or both contain $\Sigma^{(6)c}$ are less singular and will not be needed for our analysis.

It follows from the first two equations of (6.12) and the second equation of (6.2) that we also have

$$\begin{aligned}
\psi^\mu(z)\, e^{-\phi/2}\widetilde\Sigma^c_\alpha(w) &= \frac{i}{2}(z-w)^{-1/2}(\gamma^\mu)_\alpha{}^{\dot\beta} e^{-\phi/2}\Sigma^c_{\dot\beta}(w)+\cdots\,, \\
\psi^\mu(z)\, e^{-\phi/2}\widetilde\Sigma_{\dot\alpha}(w) &= \frac{i}{2}(z-w)^{-1/2}(\gamma^\mu)_{\dot\alpha}{}^{\beta} e^{-\phi/2}\Sigma_{\beta}(w)+\cdots\,.
\end{aligned} \tag{6.14}$$

From these we can derive all other operator products, e.g. we have the following useful relations involving GSO even operators:

$$\begin{aligned}
e^{-\phi}\psi^\mu(z)\, e^{-\phi/2}\Sigma_\alpha(w) &= \frac{i}{2}(z-w)^{-1}(\gamma^\mu)_\alpha{}^{\dot\beta} e^{-3\phi/2}\widetilde\Sigma_{\dot\beta}(w)+\cdots\,, \\
e^{-\phi}\psi^\mu(z)\, e^{-\phi/2}\Sigma^c_{\dot\alpha}(w) &= \frac{i}{2}(z-w)^{-1}(\gamma^\mu)_{\dot\alpha}{}^{\beta} e^{-3\phi/2}\widetilde\Sigma^c_{\beta}(w)+\cdots\,, \\
e^{\phi}\psi^\mu(z)\, e^{-3\phi/2}\widetilde\Sigma^c_\alpha(w) &= -\frac{i}{2}(z-w)(\gamma^\mu)_\alpha{}^{\dot\beta} e^{-\phi/2}\Sigma^c_{\dot\beta}(w)+\cdots\,, \\
e^{\phi}\psi^\mu(z)\, e^{-3\phi/2}\widetilde\Sigma_{\dot\alpha}(w) &= -\frac{i}{2}(z-w)(\gamma^\mu)_{\dot\alpha}{}^{\beta} e^{-\phi/2}\Sigma_{\beta}(w)+\cdots\,,
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
e^{-\phi/2}\Sigma_\alpha(z)e^{-\phi/2}\Sigma^c_{\dot\beta}(w) &= i(z-w)^{-1}\gamma^\mu_{\alpha\dot\beta}e^{-\phi(w)}\psi^\mu(w)+\cdots\,, \\
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)e^{-\phi/2}\Sigma_{\beta}(w) &= i(z-w)^{-1}\gamma^\mu_{\dot\alpha\beta}e^{-\phi(w)}\psi^\mu(w)+\cdots\,, \\
\gamma^\mu_{\alpha\dot\beta}\equiv(\gamma^\mu)_\alpha{}^{\dot\gamma}\varepsilon_{\dot\gamma\dot\beta}, \quad \gamma^\mu_{\dot\alpha\beta} &\equiv (\gamma^\mu)_{\dot\alpha}{}^{\gamma}\varepsilon_{\gamma\beta}, \quad (\gamma^\mu)_{\alpha\dot\beta}=(\gamma^\mu)_{\dot\beta\alpha}\,,
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
e^{-\phi/2}\Sigma_\alpha(z)e^{\phi/2}\widetilde\Sigma^c_\beta(w) &= -\varepsilon_{\alpha\beta}(z-w)^{-1}+\cdots \\
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)e^{\phi/2}\widetilde\Sigma_{\dot\beta}(w) &= -\varepsilon_{\dot\alpha\dot\beta}(z-w)^{-1}+\cdots\,.
\end{aligned} \tag{6.17}$$

etc.

For our analysis we shall also need the first subleading terms in the operator product expansion in (6.17). This must be of the form of a constant multiplying a dimension (0,1) current. Since J is a dimension (0,1) current, the next term in each operator product expansion could contain J multiplied by a constant. The constant can be determined by analyzing the sphere three point function

$$\langle e^{-2\phi} J(y) e^{-\phi/2} \Sigma_\alpha(z) e^{\phi/2} \widetilde{\Sigma}^c_\beta(w) \rangle \tag{6.18}$$

and its counterpart with dotted indices. Using (6.6), (6.8), (6.10) and (6.17) we can express this correlator as

$$\begin{aligned} &\frac{3}{2}\left(\frac{1}{y-z} - \frac{1}{y-w}\right) \langle e^{-2\phi}(y) e^{-\phi/2} \Sigma_\alpha(z) e^{\phi/2} \widetilde{\Sigma}^c_\beta(w) \rangle \\ &\quad = \frac{3}{2} \frac{(z-w)}{(y-z)(y-w)} (-\varepsilon_{\alpha\beta})(y-z)^{-1}(y-w)(z-w)^{-1} \\ &\quad = -\frac{3}{2}\varepsilon_{\alpha\beta}(y-z)^{-2}, \end{aligned} \tag{6.19}$$

where in going from the first to the second line we have evaluated the correlator in the first line using the general form of the three point function and the operator product expansion (6.17). On the other hand if the operator product $e^{-\phi/2}\Sigma_\alpha(z) e^{\phi/2}\widetilde{\Sigma}^c_\beta(w)$ had contained a term of the form $c_{\alpha\beta}\, J(w)$ then in the $z \to w$ limit (6.18) would have behaved as

$$c_{\alpha\beta} \langle e^{-2\phi} J(y) J(w) \rangle = 3\, c_{\alpha\beta}(y-w)^{-2}, \tag{6.20}$$

using (6.4). Comparing this with the $z \to w$ limit of (6.19) we get $c_{\alpha\beta} = -\varepsilon_{\alpha\beta}/2$. A similar analysis for the correlator involving dotted indices gives the coefficient $\varepsilon_{\dot\alpha\dot\beta}/2$ of J — the additional minus sign arising from the fact that $\Sigma^c_{\dot\alpha}$ and Σ_α carry opposite J charges.

What other operators could the operator products (6.17) contain on the right hand side? First of all they could contain the ghost current $\partial\phi$. They could also contain bilinears of free fermions $\psi^\mu\psi^\nu$. Therefore we can write

$$\begin{aligned} e^{-\phi/2}\Sigma_\alpha(z) e^{\phi/2}\widetilde{\Sigma}^c_\beta(w) &= -\varepsilon_{\alpha\beta}\left\{(z-w)^{-1} + \frac{1}{2}J(w)\right\} + \mathcal{O}(\partial\phi) + \mathcal{O}(\psi^\mu\psi^\nu) + \mathcal{O}(z-w) \\ e^{-\phi/2}\Sigma^c_{\dot\alpha}(z) e^{\phi/2}\widetilde{\Sigma}_{\dot\beta}(w) &= -\varepsilon_{\dot\alpha\dot\beta}\left\{(z-w)^{-1} - \frac{1}{2}J(w)\right\} + \mathcal{O}(\partial\phi) + \mathcal{O}(\psi^\mu\psi^\nu) + \mathcal{O}(z-w). \end{aligned} \tag{6.21}$$

This summarizes the relevant information on the holomorphic and anti-holomorphic operators in the (2,2) SCFT. However for our analysis we shall also need to review the properties of certain non-(anti)-holomorphic operators in this theory — namely those which describe the vertex operators associated with target space fields belonging to a special class of chiral and anti-chiral multiplets. We describe the relevant properties below:

1. Let σ be a complex scalar field in the target space, describing the scalar component of such a massless chiral superfield. The -1 picture vertex operators of the field σ and its complex conjugate take the form

$$\mathcal{V}_\sigma = \bar{c}\, c\, e^{-\phi} V_\sigma, \quad \mathcal{V}_{\sigma^*} = \bar{c}\, c\, e^{-\phi} V_{\sigma^*}, \tag{6.22}$$

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where V_σ, V_{σ^*} are matter sector operators of dimension $(1,1/2)$ in the (2,2) SCFT associated with the compact directions. We normalize V_σ, V_{σ^*} so that

$$V_\sigma(z)V_{\sigma^*}(w) = (z-w)^{-1}(\bar z-\bar w)^{-2}, \quad e^{-\phi}V_\sigma(z)e^{-\phi}V_{\sigma^*}(w) = -(z-w)^{-2}(\bar z-\bar w)^{-2}e^{-2\phi(w)}, \tag{6.23}$$

in accordance with (3.34), (3.36).

2. The J-charges carried by V_σ and V_{σ^*} are

$$V_\sigma\ :\ 1, \qquad V_{\sigma^*}\ :\ -1\,. \tag{6.24}$$

3. The operator product of $T_F^\pm$ with V_σ, V_{σ^*} take the form

$$\begin{aligned} &T_F^-(z)V_\sigma(w) = -(w-z)^{-1}\widetilde V_\sigma(w)+\cdots, && T_F^+(z)V_{\sigma^*}(w) = -(w-z)^{-1}\widetilde V_{\sigma^*}(w)+\cdots,\\ &T_F^-(z)V_{\sigma^*}(w) = \text{non-singular}, && T_F^+(z)V_\sigma(w) = \text{non-singular}, \end{aligned} \tag{6.25}$$

where $\widetilde V_\sigma$ and $\widetilde V_{\sigma^*}$ are matter sector vertex operators of dimension (1,1). It follows from (6.6), (6.24) that the J-charges carried by $\widetilde V_\sigma$ and $\widetilde V_{\sigma^*}$ are

$$\widetilde V_\sigma\ :\ 0, \qquad \widetilde V_{\sigma^*}\ :\ 0\,. \tag{6.26}$$

$\widetilde V_\sigma$ and $\widetilde V_{\sigma^*}$ will be useful for constructing zero picture vertex operators.

4. Using the two dimensional superconformal algebra one can show that these vertex operators satisfy the identities

$$\begin{aligned} T_F^+(w)\widetilde V_\sigma(z) &= -\frac14(w-z)^{-2}\,V_\sigma(z) - \frac14(w-z)^{-1}\partial_z V_\sigma(z) + \text{non-singular terms}\\ T_F^-(w)\widetilde V_{\sigma^*}(z) &= -\frac14(w-z)^{-2}\,V_{\sigma^*}(z) - \frac14(w-z)^{-1}\partial_z V_{\sigma^*}(z) + \text{non-singular terms}\\ T_F^-(w)\,\widetilde V_\sigma(z) &= \text{non-singular terms}, \quad T_F^+(w)\,\widetilde V_{\sigma^*}(z) = \text{non-singular terms}\,. \end{aligned} \tag{6.27}$$

5. Given a chiral multiplet field σ there are also Ramond sector vertex operators V^f_σ, $V^f_{\sigma^*}$ of dimension $(1,3/8)$ in the internal CFT from which we can build the full vertex operator of the space-time fermions. In the spirit of (6.8), (6.10) we combine them with the spin fields $\Sigma^{(4)}_\alpha$, $\Sigma^{(4)}_{\dot\alpha}$ to define

$$V^f_{\sigma,\alpha} \equiv V^f_\sigma \Sigma^{(4)}_\alpha, \qquad V^f_{\sigma,\dot\alpha} \equiv V^f_\sigma \Sigma^{(4)}_{\dot\alpha}, \qquad V^f_{\sigma^*,\alpha} \equiv V^f_{\sigma^*} \Sigma^{(4)}_\alpha, \qquad V^f_{\sigma^*,\dot\alpha} \equiv V^f_{\sigma^*} \Sigma^{(4)}_{\dot\alpha}\,. \tag{6.28}$$

The vertex operators of the fermionic partners of σ^* and σ in the $-1/2$ picture are given by, respectively,

$$\bar c c e^{-\phi/2} V^f_{\sigma^*,\dot\alpha}, \quad \bar c c e^{-\phi/2} V^f_{\sigma,\alpha}\,. \tag{6.29}$$

The basic operator products involving these operators are:

$$\begin{aligned} V_{\sigma^*}(w)\,e^{-\phi/2}\widetilde\Sigma_{\dot\alpha}(z) &= (w-z)^{-1/2}e^{-\phi/2}V^f_{\sigma^*,\dot\alpha}(z)+\cdots,\\ V_\sigma(w)\,e^{-\phi/2}\widetilde\Sigma^c_\alpha(z) &= (w-z)^{-1/2}e^{-\phi/2}V^f_{\sigma,\alpha}(w)+\cdots, \end{aligned}$$

$$
\begin{aligned}
V_{\sigma^*}(w)\, e^{-\phi/2}\Sigma_\alpha(z) &= -(w-z)^{-1/2} e^{-\phi/2} V^f_{\sigma^*,\alpha}(z) + \cdots\,, \\
V_{\sigma}(w)\, e^{-\phi/2}\Sigma^c_{\dot\alpha}(z) &= -(w-z)^{-1/2} e^{-\phi/2} V^f_{\sigma,\dot\alpha}(w) + \cdots\,.
\end{aligned} \tag{6.30}
$$

The relative minus signs between the first two terms and the last two terms in (6.30) has been included to ensure that the operator product of ψ^μ with $e^{-\phi/2}V^f_{\sigma,\alpha}$ etc. follow the same pattern as the operator product of ψ^μ with $e^{-\phi/2}\Sigma_\alpha$ etc. given in (6.12), (6.14):

$$
\begin{aligned}
\psi^\mu(w)\, e^{-\phi/2} V^f_{\sigma,\alpha}(z) &= \frac{i}{2}(w-z)^{-1/2}\, (\gamma^\mu)_\alpha{}^{\dot\beta}\, e^{-\phi/2} V^f_{\sigma,\dot\beta}(z) \\
\psi^\mu(w)\, e^{-\phi/2} V^f_{\sigma,\dot\alpha}(z) &= \frac{i}{2}(w-z)^{-1/2}\, (\gamma^\mu)_{\dot\alpha}{}^{\beta}\, e^{-\phi/2} V^f_{\sigma,\beta}(z) \\
\psi^\mu(w)\, e^{-\phi/2} V^f_{\sigma^*,\dot\alpha}(z) &= \frac{i}{2}(w-z)^{-1/2}\, (\gamma^\mu)_{\dot\alpha}{}^{\beta}\, e^{-\phi/2} V^f_{\sigma^*,\beta}(z) \\
\psi^\mu(w)\, e^{-\phi/2} V^f_{\sigma^*,\alpha}(z) &= \frac{i}{2}(w-z)^{-1/2}\, (\gamma^\mu)_\alpha{}^{\dot\beta}\, e^{-\phi/2} V^f_{\sigma^*,\dot\beta}(z)\,.
\end{aligned} \tag{6.31}
$$

In arriving at (6.31) we have used the fact that V_σ and V_{σ^*} anti-commute with ψ^μ. We also have

$$
\begin{aligned}
V_{\sigma^*}(w)\, e^{-\phi/2}\Sigma^c_{\dot\alpha}(z) &= \mathcal{O}((w-z)^{1/2})\,, \\
V_{\sigma}(w)\, e^{-\phi/2}\Sigma_{\alpha}(z) &= \mathcal{O}((w-z)^{1/2})\,, \\
V_{\sigma^*}(w)\, e^{-\phi/2}\widetilde\Sigma^c_{\alpha}(z) &= \mathcal{O}((w-z)^{1/2})\,, \\
V_{\sigma}(w)\, e^{-\phi/2}\widetilde\Sigma_{\dot\alpha}(z) &= \mathcal{O}((w-z)^{1/2})\,.
\end{aligned} \tag{6.32}
$$

Using (6.23) and (6.30) we get

$$
\begin{aligned}
V_{\sigma}(w)\, e^{-\phi/2} V^f_{\sigma^*,\dot\alpha}(z) &= (w-z)^{-1/2}(\bar w-\bar z)^{-2} e^{-\phi/2}\widetilde\Sigma_{\dot\alpha}(z) + \cdots \\
V_{\sigma^*}(w)\, e^{-\phi/2} V^f_{\sigma,\alpha}(z) &= (w-z)^{-1/2}(\bar w-\bar z)^{-2} e^{-\phi/2}\widetilde\Sigma^c_{\alpha}(z) + \cdots\,, \\
V_{\sigma}(w)\, e^{-\phi/2} V^f_{\sigma^*,\alpha}(z) &= -(w-z)^{-1/2}(\bar w-\bar z)^{-2} e^{-\phi/2}\Sigma^c_{\dot\alpha}(z) + \cdots \\
V_{\sigma^*}(w)\, e^{-\phi/2} V^f_{\sigma,\dot\alpha}(z) &= -(w-z)^{-1/2}(\bar w-\bar z)^{-2} e^{-\phi/2}\Sigma_{\alpha}(z) + \cdots\,.
\end{aligned} \tag{6.33}
$$

From (6.12), (6.30), (6.33) and standard manipulations in conformal field theory we can derive other required operator product expansions. Some useful relations involving GSO even operators are given below:

$$
\begin{aligned}
e^{-\phi/2}\Sigma_\alpha(z)\; e^{-\phi}V_{\sigma^*}(w) &= (z-w)^{-1} e^{-3\phi/2} V^f_{\sigma^*,\alpha}(w) + \cdots\,, \\
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)\; e^{-\phi}V_{\sigma}(w) &= (z-w)^{-1} e^{-3\phi/2} V^f_{\sigma,\dot\alpha}(w) + \cdots\,, \\
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)\; e^{-\phi}V_{\sigma^*}(w) &= \text{non-singular terms}\,, \\
e^{-\phi/2}\Sigma_{\alpha}(z)\; e^{-\phi}V_{\sigma}(w) &= \text{non-singular terms}\,, \\
e^{-3\phi/2}V^f_{\sigma^*,\beta}(z)\; e^{-\phi/2}V^f_{\sigma,\alpha}(w) &= (z-w)^{-2}(\bar z-\bar w)^{-2}\varepsilon_{\beta\alpha} e^{-2\phi(w)} + \cdots\,, \\
e^{-3\phi/2}V^f_{\sigma,\dot\beta}(z)\; e^{-\phi/2}V^f_{\sigma^*,\dot\alpha}(w) &= (z-w)^{-2}(\bar z-\bar w)^{-2}\varepsilon_{\dot\beta\dot\alpha} e^{-2\phi(w)} + \cdots\,, \\
e^{-\phi/2}\Sigma_\alpha(z)\; e^{-\phi/2}V^f_{\sigma,\beta}(w) &= (z-w)^{-1}\varepsilon_{\alpha\beta} e^{-\phi}V_\sigma(w) + \cdots\,,
\end{aligned}
$$

$$
\begin{aligned}
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)\; e^{-\phi/2}V^f_{\sigma^*,\dot\beta}(w) &= (z-w)^{-1}\varepsilon_{\dot\alpha\dot\beta}e^{-\phi}V_{\sigma^*}(w)+\cdots\,,\\
e^{-\phi/2}\Sigma_{\alpha}(z)\; e^{-\phi/2}V^f_{\sigma^*,\dot\beta}(w) &= \text{non-singular terms}\,,\\
e^{-\phi/2}\Sigma^c_{\dot\alpha}(z)\; e^{-\phi/2}V^f_{\sigma,\beta}(w) &= \text{non-singular terms}\,,\\
e^{\phi}\psi^\mu(z)\, e^{-3\phi/2}V^f_{\sigma^*,\alpha}(w) &= -\frac{i}{2}(z-w)\,(\gamma^\mu)_\alpha{}^{\dot\beta}e^{-\phi/2}V^f_{\sigma^*,\dot\beta}(w)+\cdots\,,\\
e^{\phi}\psi^\mu(z)\, e^{-3\phi/2}V^f_{\sigma,\dot\alpha}(w) &= -\frac{i}{2}(z-w)(\gamma^\mu)_{\dot\alpha}{}^{\beta}e^{-\phi/2}V^f_{\sigma,\beta}(w)+\cdots\,,\\
e^{-\phi}\psi^\mu(z)\; e^{-\phi/2}V^f_{\sigma^*,\dot\alpha}(w) &= \frac{i}{2}(z-w)^{-1}(\gamma^\mu)_{\dot\alpha}{}^{\beta}e^{-3\phi/2}V^f_{\sigma^*,\beta}(w)+\cdots\,,\\
e^{-\phi}\psi^\mu(z)\; e^{-\phi/2}V^f_{\sigma,\alpha}(w) &= \frac{i}{2}(z-w)^{-1}(\gamma^\mu)_{\alpha}{}^{\dot\beta}e^{-3\phi/2}V^f_{\sigma,\dot\beta}(w)+\cdots\\
e^{\phi}\psi^\mu(z)\; e^{-5\phi/2}V^f_{\sigma^*,\dot\alpha}(w) &= \frac{i}{2}(z-w)^{2}(\gamma^\mu)_{\dot\alpha}{}^{\beta}e^{-3\phi/2}V^f_{\sigma^*,\beta}(w)+\cdots\,,\\
e^{\phi}\psi^\mu(z)\; e^{-5\phi/2}V^f_{\sigma,\alpha}(w) &= \frac{i}{2}(z-w)^{2}(\gamma^\mu)_{\alpha}{}^{\dot\beta}e^{-3\phi/2}V^f_{\sigma,\dot\beta}(w)+\cdots\,.
\end{aligned}
\tag{6.34}
$$

6. It follows from (6.6), (6.24) and (6.30) that the J-charges carried by these new internal Ramond sector operators are

$$
V^f_\sigma : -\frac{1}{2}, \quad V^f_{\sigma^*} : \frac{1}{2}\,. \tag{6.35}
$$

7. Finally let us turn to the $\bar J$ charge carried by the relevant fields. Here there is no universal result for all chiral and anti-chiral multiplets since different fields may transform in different representations of the gauge group. However the theory contains special chiral multiplet fields which are singlets under the SO(26) gauge group but carry ± 2 unit of charge under the U(1) gauge group. The number of such fields depends on the Hodge numbers of the Calabi-Yau manifold. In order to simplify our analysis we shall assume that there is a unique chiral multiplet field χ of this type carrying $\bar J$ charge 2 — this would arise e.g. for a Calabi-Yau manifold with $h_{1,1}=1$, $h_{1,2}=0$. This field is the one that appears in (6.1) and is important because it will condense to restore supersymmetry which otherwise is broken by a Fayet-Iliopoulos term generated at one loop [48]. The $\bar J$ charges carried by the associated vertex operators are

$$
V_\chi\,:\,2, \qquad V_{\chi^*}\,:\,-2, \qquad \widetilde V_\chi\,:\,2, \qquad \widetilde V_{\chi^*}\,:\,-2\,. \tag{6.36}
$$

7 Supersymmetry restoration

We shall now apply the general analysis carried out earlier in this paper to SO(32) heterotic string theory compactified on a Calabi-Yau manifold. As reviewed in section 6.1, typically one loop quantum corrections will generate a Fayet-Iliopoulos D-term [47–49]. This causes supersymmetry to be spontaneously broken at the original perturbative vacuum. However using effective field theory it is easy to see that there should be a nearby vacuum where supersymmetry is restored. Our goal will be to demonstrate how this result can be derived from superstring perturbation theory without invoking low energy effective field theory.

This will also tell us how to compute perturbative superstring amplitudes around the shifted vacuum going beyond what is possible in low energy effective field theory.

7.1 Construction of the vacuum solutions in superstring perturbation theory

If we work in the subspace of the field space in which $\chi = (\chi_R + i\chi_I)/\sqrt{2}$ is real, then at the level of order g_s^3 contribution to the equations of motion there are three solutions in the low energy effective field theory described by the potential (6.1). These correspond to $\chi_R = 0, \pm g_s\sqrt{2\,K}$. The existence of these three solutions would show up in the analysis of section 2 through the existence of three possible solutions for $|\psi_1\rangle$ in (2.27), (2.28) when we study the contribution to the equation of motion to order g_s^3. Now from the point of view of low energy effective field theory it is clear that the solution corresponding to $\chi_R = 0$ should break supersymmetry while the solutions corresponding to $\chi_R = \pm g_s\sqrt{2K}$ should restore supersymmetry. Our goal will be to prove, from the point of view of superstring perturbation theory, that this is indeed what happens.

Let

$$\mathcal{V}_{\chi_R} = \bar{c}ce^{-\phi}V_{\chi_R} \tag{7.1}$$

denote the vertex operator of χ_R in the -1 picture where V_{χ_R} is a dimension (1,1/2) operator in the matter SCFT associated with compact directions. We shall normalize V_χ as in (6.23). It then follows from the analysis of section 3.4 that the coefficient of $\mathcal{V}_{\chi_R}$ in the expansion of the string field is real. We now proceed to solve the equations of motion following (2.27), (2.28), beginning with the ansatz

$$|\Psi_1\rangle = |\psi_1\rangle = \beta\, g_s|\mathcal{V}_{\chi_R}\rangle\,, \tag{7.2}$$

where β is a real constant. At this order β remains undetermined since $Q_B|\Psi_1\rangle = 0$ for all β. At the next order we get

$$\begin{aligned} |\Psi_2\rangle &= -\frac{b_0^+}{L_0^+}(1-\mathbf{P})[\,]_1 - \frac{1}{2}\beta^2\, g_s^2\,\frac{b_0^+}{L_0^+}(1-\mathbf{P})[\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}]_0 + |\psi_2\rangle && (7.3)\\ |\psi_2\rangle &= \beta\, g_s\,|\mathcal{V}_{\chi_R}\rangle + |\tilde{\psi}_2\rangle\,, && (7.4)\end{aligned}$$

where $|\tilde{\psi}_2\rangle$ is a contribution of order g_s^2, determined from the equation

$$Q_B|\tilde{\psi}_2\rangle = -\mathbf{P}\left([\,]_1 + \frac{1}{2}\beta^2\, g_s^2[\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}]_0\right)\,. \tag{7.5}$$

In (7.3), (7.5), $[\cdots]_g$ denotes that $[\cdots]$ has to be computed to g-loop order, and we have made use of the fact that $[\,]_0 = 0$, $[A]_0 = 0$ for all $|A\rangle \in \widehat{\mathcal{H}}_T$. We have shown in appendix D that both terms on the right hand side of (7.5) vanish and hence we can take

$$|\tilde{\psi}_2\rangle = 0\,. \tag{7.6}$$

According to (2.29), in order to extend the solution to order g_s^3 we need to ensure that for any BRST invariant physical state $|p\rangle$ of ghost number 2, $L_0^+ = 0$ and picture number -1, we have

$$\langle p|c_0^-|\left([\,]_1 + [\Psi_1]_1 + \frac{1}{2}[\Psi_2{}^2]_0 + \frac{1}{6}[\Psi_1{}^3]_0\right)\rangle = \mathcal{O}(g_s^4)\,. \tag{7.7}$$

In writing down this equation we have used the fact that $[\Psi]$ receives contribution from 1-loop and higher and hence to get $[\Psi]$ accurate up to order g_s^3 it is enough the keep terms up to order g_s in $|\Psi\rangle$. On the other hand $[\Psi^2]$, $[\Psi^3]$ etc. receive contribution from tree level, with the expansion for Ψ beginning at order g_s; therefore in $[\Psi^2]$ is is enough to keep terms in $|\Psi\rangle$ accurate to order g_s^2 and in $[\Psi^3]$ it is enough to keep terms in $|\Psi\rangle$ accurate up to order g_s. Substituting (7.2)–(7.4) and (7.6) into (7.7) we get

$$\Big\langle p\Big|c_0^-\Big|\Big([\,]_1+\beta\, g_s\,[\mathcal{V}_{\chi R}]_1+\frac{1}{2}\beta^2 g_s^2\,[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0-\beta\, g_s\,[\mathcal{V}_{\chi R}b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\,]_1]_0$$
$$-\frac{1}{2}\beta^3 g_s^3[\mathcal{V}_{\chi R}b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0]_0+\frac{1}{6}\beta^3\, g_s^3\,[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\Big)$$
$$=\mathcal{O}(g_s^4)\,. \tag{7.8}$$

As already mentioned below (7.5), the analysis of appendix D shows that the first and the third terms on the left hand side of (7.8) vanish. To analyze the rest of the terms we recall that $\mathcal{V}_{\chi R}$ can be expressed as $(\mathcal{V}_\chi+\mathcal{V}_{\chi^*})/\sqrt{2}$ where $\mathcal{V}_\chi$ and $\mathcal{V}_{\chi^*}$ carry $\bar{J}$ charges 2 and -2 respectively (see (6.36)). Since the second and fourth terms each has a single $\mathcal{V}_{\chi R}$, in order to get a non-zero result $|p\rangle$ must carry $\bar{J}$ charge 2 or -2. Since the fifth and the sixth terms each has three $\mathcal{V}_{\chi R}$'s, to get a non-zero result $|p\rangle$ must have $\bar{J}$ charge ± 2 or ± 6. Now, the $\bar{L}_0$ eigenvalue of a state is bounded from below by $-1+\bar{j}^2/6$, $\bar{j}$ being the $\bar{J}$ charge of the state. Therefore, there are no states carrying $\bar{j}=\pm 6$ and $L_0^+=0$. By our assumption stated in the last paragraph of section 6, the only zero momentum states that can have $\bar{j}=\pm 2$ and $L_0^+=0$ are $\mathcal{V}_\chi$ and $\mathcal{V}_{\chi^*}$. Furthermore there is a Z_2 conjugation symmetry under which χ_R is even and the imaginary part χ_I of χ is odd. This fixes $|p\rangle$ uniquely to be $|\mathcal{V}_{\chi R}\rangle$ for getting non-zero contribution to various terms on the left hand side of (7.8). Therefore (7.8) can be written as[11]

$$-\frac{1}{2}\beta^3\, g_s^3\,\big\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\,\big(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\big)\big\}_0+\frac{1}{6}\beta^3\, g_s^3\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_0$$
$$+\beta\, g_s\,\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_1-\beta\, g_s\,\big\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\,\big(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\,]_1\big)\big\}_0=\mathcal{O}(g_s^4)\,, \tag{7.9}$$

where, as for $[\cdots]$, $\{\cdots\}_g$ denotes that we have to compute the contribution to $\{\cdots\}$ up to genus g.

Eq. (7.9) can be expressed in a more convenient form by defining

$$\begin{aligned}G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)&=\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_0\\&\quad-3\,\big\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\,\big(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\big)\big\}_0\,,\\ g_s^2\,G^{(2,2)}(0,\chi_R;0,\chi_R)&=\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_1-\big\{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\,\big(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\,]_1\big)\big\}_0\,.\end{aligned} \tag{7.10}$$

We recognize $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$ as the tree level on-shell four point function of four external zero momentum χ_R state in the $\beta=0$ vacuum, with the first term giving

[11] If the theory contains multiple fields carrying $\bar{J}$ charges ± 2, then there will be more candidates for $|p\rangle$ leading to more equations. But there will also be more parameters labelling the solution since the leading order solution can be taken to be an arbitrary linear combination of these states. Therefore we shall have multiple equations involving multiple variables. This is analogous to the situation in effective field theory stated in footnote 10.

the 1PI part of the amplitude and the second term giving the 1PR part, with the sum over s, t and u-channel graphs giving the factor of 3. Similarly $g_s^2 G^{(2,2)}(0,\chi_R;0,\chi_R)$ gives the one loop on-shell two point function of two external zero momentum χ_R states in the $\beta=0$ vacuum, with the first term giving the 1PI contribution and the second term giving the 1PR contribution where a tree level 3-point vertex is attached by a propagator to a one loop one point vertex.[12] Both $G^{(4,0)}$ and $G^{(2,2)}$ have been normalized so that they do not carry any factor of g_s. (7.9) can now be expressed as

$$\frac{1}{6}\beta^3\, g_s^3 G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R) + g_s^3\,\beta\, G^{(2,2)}(0,\chi_R;0,\chi_R) = 0\,. \tag{7.11}$$

(7.11) has a trivial solution $\beta=0$ that describes the original perturbative vacuum, and non-trivial solutions at

$$\beta^2 = -6\, G^{(2,2)}(0,\chi_R;0,\chi_R)/G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)\,. \tag{7.12}$$

It is possible to show by explicit calculation that these solutions correspond to real values of β. This in turn shows the existence of three solutions to the equations of motion to order g_s^3, in agreement with what we get from the low energy effective field theory.

In the next subsection we shall show that while supersymmetry is broken at order g_s^2 in the $\beta=0$ vacuum, in the other two vacua at $\beta\sim 1$, supersymmetry is restored at least to order g_s^2.

7.2 Supersymmetry of the vacuum solution

We shall now examine whether or not the vacuum solutions of section 7.1 possess global supersymmetry to order g_s^2. This corresponds to existence of solution to (2.31) to order g_s^2 for some Ramond sector state $|\Lambda_{\rm global}\rangle$. Using (2.34) for $k=2$, and using the fact that the expansion of $|\Psi_{\rm vac}\rangle$ begins at order g_s and that of $|\Lambda_{\rm global}\rangle$ begins at order g_s^0, we see that the existence of unbroken global supersymmetry $|\Lambda_{\rm global}\rangle$ to order g_s^2 will require us to show that

$$\mathcal{L}_1(\mathcal{V}^c_{\rm G}) \equiv \langle \mathcal{V}^c_{\rm G}|c_0^-\mathcal{X}_0|[\Psi_1\Lambda_0]_0\rangle = \mathcal{O}(g_s^2)\,, \tag{7.13}$$

and

$$\mathcal{L}_2(\mathcal{V}^c_{\rm G}) \equiv \left\langle \mathcal{V}^c_{\rm G}\middle|c_0^-\mathcal{X}_0\middle|\left([\Psi_2\Lambda_1]_0 + [\Lambda_0]_1 + \frac{1}{2}[\Psi_1\Psi_1\Lambda_0]_0\right)\right\rangle = \mathcal{O}(g_s^3)\,, \tag{7.14}$$

for any BRST invariant state $|\mathcal{V}^c_{\rm G}\rangle$ of ghost number 3, picture number $-3/2$ and $L_0^+=0$.

$|\Lambda_0\rangle$ is given by the zeroth order global supersymmetry transformation parameter

$$|\Lambda_0\rangle = |\lambda_0\rangle = |ce^{-\phi/2}\Sigma\rangle\,, \tag{7.15}$$

where Σ is a matter sector spin field — one of the operators Σ_α or $\Sigma^c_{\dot\alpha}$ given in (6.8). The existence of $|\Lambda_1\rangle$ requires (7.13), which can be interpreted as $\{(\mathcal{X}_0\mathcal{V}^c_{\rm G})\Psi_1\Lambda_0\}_0 = \mathcal{O}(g_s^2)$. $\mathcal{V}^c_{\rm G}$

[12]Since $\mathbf{P}[\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}]_0$ and $\mathbf{P}[\;]_1$ vanish according to the analysis of appendix D, we can drop the $1-\mathbf{P}$ factors in (7.10). Therefore $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$ and $G^{(2,2)}(0,\chi_R;0,\chi_R)$ can be interpreted as the usual on-shell amplitudes of string theory in the perturbative vacuum without any need for the subtraction, or insertion of external zero momentum state $|\psi_2\rangle$ of the kind discussed in section 3.

is a physical state of ghost number 3 and picture number $-3/2$. The general form of these states is given in (2.35) and the result of the action of $\mathcal{X}_0$ on these states is given in (2.38). Combining this with the form of $\mathcal{V}_{\chi R}$ and λ_0 given respectively in (7.1) and (7.15) we have

$$\mathcal{V}_{\chi R} : \bar{c}ce^{-\phi}V_{\chi R}, \quad \lambda_0 : ce^{-\phi/2}\Sigma, \quad \mathcal{X}_0\mathcal{V}_{\rm G}^c : \bar{c}c\eta e^{\phi/2}V^f, \quad \bar{c}c\bar{\partial}^2\bar{c}\, e^{-\phi/2}\,\hat{\Sigma}\,, \tag{7.16}$$

where both Σ and $\hat{\Sigma}$ represent some operator from the list $(\Sigma_\alpha, \Sigma^c_{\dot{\alpha}})$ given in (6.8). The contribution to $\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\mathcal{V}_{\chi R}\lambda_0\}_0$ from the first candidate for $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ vanishes by ϕ charge conservation (and also by $\xi - \eta$ charge conservation). The contribution from the second candidate for $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ involves the following matter part of the correlation function:

$$\langle V_{\chi R}\,\Sigma\,\hat{\Sigma}\rangle\,. \tag{7.17}$$

Since $V_{\chi R} = (V_\chi + V_{\chi^*})/\sqrt{2}$ where V_χ and V_{χ^*} carry $\bar{J}$ charges 2 and -2 respectively and Σ, $\hat{\Sigma}$ are neutral under $\bar{J}$, such a three point function vanishes by $\bar{J}$ charge conservation. Therefore (7.13) holds.

$|\Lambda_1\rangle$ can now be computed using (2.32), (2.33) and (7.2):

$$|\Lambda_1\rangle = |\lambda_1\rangle - \frac{b_0^+}{L_0^+}(1-\mathbf{P})\mathcal{X}_0[\Psi_1\Lambda_0]_0 = |\lambda_0\rangle - \beta\, g_s\, \frac{b_0^+}{L_0^+}(1-\mathbf{P})\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0\,, \tag{7.18}$$

$$|\lambda_1\rangle = |\lambda_0\rangle + |\tilde{\lambda}_1\rangle, \qquad Q_B|\tilde{\lambda}_1\rangle = -\mathbf{P}\mathcal{X}_0[\Psi_1\Lambda_0]_0 = -\beta g_s \mathbf{P}\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0\,. \tag{7.19}$$

We shall now argue that the right hand side of the second equation in (7.19) vanishes. For this we have to show that the inner product of $c_0^-\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0$ with all states in $\mathcal{H}_T$ with $L_0^+ = 0$ vanishes. Due to ghost and picture number conservation we can restrict to states carrying ghost number 3 and picture number $-3/2$. The analysis described in the paragraph containing (7.16), (7.17) already shows that the inner product $\langle\mathcal{V}_{\rm G}^c|c_0^-|\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0\rangle$ vanishes for all candidate $\mathcal{V}_{\rm G}^c$. There is only one more class of states carrying ghost number 3, picture number $-3/2$ and $L_0^+ = 0$ — they are the unphysical states $\bar{c}c\bar{\partial}^2\bar{c}e^{-3\phi/2}\hat{\Sigma}$ with $\hat{\Sigma}$ given by one of the operators from the list $(\Sigma_\alpha, \Sigma^c_{\dot{\alpha}})$. However $\bar{J}$ charge conservation prevents this from having a non-zero inner product with $\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0$. Therefore we conclude that the right hand side of the second equation in (7.19) vanishes, and hence we can take

$$|\tilde{\lambda}_1\rangle = 0\,. \tag{7.20}$$

Substituting (7.2)–(7.4), (7.6) and (7.18)–(7.20) into (7.14) we get

$$\begin{aligned}\mathcal{L}_2(\mathcal{V}_{\rm G}^c) = \;& \beta\, g_s\, \{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\mathcal{V}_{\chi R}\lambda_0\}_0 + \{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\}_1 \\ & - \left\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\;]_1\right)\right\}_0 \\ & - \frac{1}{2}\beta^2 g_s^2\left\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\right)\right\}_0 \\ & + \frac{1}{2}\beta^2 g_s^2\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_0 \\ & - \beta^2 g_s^2\left\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\mathcal{V}_{\chi R}\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0\right)\right\}_0 \\ & + \mathcal{O}(g_s^3)\,.\end{aligned} \tag{7.21}$$

The first term is the same as the one given in (7.13) and has been shown to be zero already. Therefore the condition for unbroken supersymmetry to order g_s^2 can now be expressed as

$$
\begin{aligned}
&\{(\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c)\lambda_0\}_1 - \left\{(\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c)\lambda_0\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\,]_1\right)\right\}_0 \\
&-\frac{1}{2}\beta^2 g_s^2\left\{(\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c)\lambda_0\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\right)\right\}_0 + \frac{1}{2}\beta^2 g_s^2\{(\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c)\lambda_0\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_0 \\
&-\beta^2 g_s^2\left\{(\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c)\mathcal{V}_{\chi R}\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0\right)\right\}_0 \\
&\quad = \mathcal{O}(g_s^3)\,,
\end{aligned}
\tag{7.22}
$$

for all $\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c$ of the form given in (7.16). We have shown in appendix E that for all but one $\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c$ listed in (7.16), each term on the left hand side of (7.22) vanishes identically to the required order. The particular $\mathcal{V}_{\mathrm{G}}^c$ for which the terms do not vanish is of the form

$$
\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c = \bar{c}\,c\,\eta\,e^{\phi/2}\,\widetilde{\Sigma}\,\bar{J}\,, \tag{7.23}
$$

where $\widetilde{\Sigma}$ is a matter sector spin field of 'wrong chirality' — one of the operators in the list (6.10) — satisfying

$$
\widetilde{\Sigma}(z)\Sigma(w) \propto (z-w)^{-5/8} + \cdots\,. \tag{7.24}
$$

For example if Σ is chosen to be Σ_α then we have to take $\widetilde{\Sigma}$ to be $\varepsilon^{\alpha\beta}\widetilde{\Sigma}_\beta$. The $\mathcal{V}_{\mathrm{G}}^c$ corresponding to (7.23) is given by $-4(\partial c+\bar{\partial}\bar{c})\bar{c}ce^{-3\phi/2}\widetilde{\Sigma}\,\bar{J}$. This is turn is conjugate to $\bar{c}ce^{-\phi/2}\Sigma\,\bar{J}$ — the zero momentum gaugino vertex operator in the $-1/2$ picture associated with the U(1) gauge group — up to a proportionality constant. This is related to the fact that in the present situation this gaugino acts as the goldstino in the situation when supersymmetry is broken.

Let us now analyze the fate of global supersymmetry in the three vacua obtained in section 7.1. From (7.22) we see that for $\beta=0$ only the first two terms survive. The sum of these terms give the one loop two point function of $\mathcal{X}_0\mathcal{V}_{\mathrm{G}}^c$ and λ_0 in the perturbative vacuum. We shall see explicitly in section 9 (see (9.24)) that this two point function is non-zero. Therefore at the $\beta=0$ vacuum supersymmetry is broken at order g_s^2.

Since the left hand side of eq. (7.22) is quadratic in β, there are two other values of $\beta\sim 1$, related by a change of sign, where the the left hand side of (7.22) vanishes to order g_s^2. In fact since the sum of the first two terms in (7.22) is the genus 1 two point function, it is of order g_s^2. Therefore we can factor out an overall factor of g_s^2 from the left hand side of (7.22) and get a g_s independent quadratic equation for β whose solutions are at $\beta\sim 1$. Therefore at these values of β supersymmetry is restored to order g_s^2. How are these related to the value of β obtained in section 7.1? One can of course try to do a direct computation and compare the results. However we can avoid doing this by using the result of section 5 that unbroken supersymmetry to order g_s^2 implies vanishing tadpole to order g_s^3. Therefore once β has been adjusted to make the left hand side of (7.22) vanish to order g_s^2, it also makes the left hand side of (7.9) vanish to order g_s^3. In other words the non-zero solutions for β obtained from (7.9) and (7.22) must be the same.

This establishes that while supersymmetry is broken at order g_s^2 at the perturbative vacuum $\beta=0$ it is restored at least to order g_s^2 at the shifted vacuum where $\beta\sim 1$.

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8 Bose-Fermi degeneracy at the shifted vacuum

In this section we shall explicitly compute the renormalized mass^2 of χ_R and its fermionic partner to order g_s^2 at the shifted vacuum and show that they are equal.

8.1 Scalar mass^2 to order g_s^2

In this subsection we shall compute the renormalized mass^2 of the scalar field χ_R to order g_s^2 at the shifted vacuum. For this we shall follow the procedure described in (2.52)–(2.54). To order g_s^2, the relevant iterative equations are

$$
\begin{aligned}
|\Phi_0\rangle &= |\phi_2\rangle \\
|\Phi_1\rangle &= -\frac{b_0^+}{L_0^+}(1-P)K|\Phi_0\rangle + |\phi_2\rangle \\
|\Phi_2\rangle &= -\frac{b_0^+}{L_0^+}(1-P)K|\Phi_1\rangle + |\phi_2\rangle \\
Q_B|\phi_2\rangle &= -PK|\Phi_1\rangle = PK\frac{b_0^+}{L_0^+}(1-P)K|\phi_2\rangle - PK|\phi_2\rangle\,,
\end{aligned} \tag{8.1}
$$

where P now denotes the projection operator onto mass level zero states. Using the definition of K, and keeping terms to order g_s^2, we can express the last equation as

$$
\begin{aligned}
Q_B|\phi_2\rangle &= P[\Psi_1 b_0^+ (L_0^+)^{-1}(1-P)[\Psi_1\phi_2]_0]_0 - P[\phi_2]_1 - P[\Psi_2\phi_2]_0 - \frac{1}{2}P[{\Psi_1}^2\phi_2]_0 \\
&= \beta^2 g_s^2 P[\mathcal{V}_{\chi R} b_0^+ (L_0^+)^{-1}(1-P)[\mathcal{V}_{\chi R}\phi_2]_0]_0 - P[\phi_2]_1 \\
&\quad -\beta g_s P[\mathcal{V}_{\chi R}\phi_2]_0 + P[(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\ ]_1)\phi_2]_0 \\
&\quad +\frac{1}{2}\beta^2 g_s^2 P[(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0)\phi_2]_0 - \frac{1}{2}\beta^2 g_s^2 P[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\phi_2]_0\,,
\end{aligned} \tag{8.2}
$$

where in the last step we have used the expression for the approximations $|\Psi_1\rangle$ and $|\Psi_2\rangle$ to the vacuum solution given in (7.2)–(7.6).

So far our discussion has been for general state, but for computing the mass renormalization of χ_R we shall consider the following ansatz for ϕ_2:[13]

$$
\phi_2 = \mathcal{V}_{\chi R}(k), \quad \mathcal{V}_{\chi R}(k) \equiv \mathcal{V}_{\chi R} e^{ik\cdot X} \tag{8.3}
$$

with $k^2 \simeq 0$. The non-trivial part of (8.2) comes from the inner product of this equation with arbitrary state of momentum $-k$ and $L_0^+ \simeq 0$. Using various charge conservation one can show that the only contribution comes from the inner product with the state $\langle \mathcal{V}_{\chi R}(-k)|c_0^-$. Using the normalization

$$
\langle \mathcal{V}_{\chi R}(-k)|c_0\bar{c}_0|\mathcal{V}_{\chi R}(k)\rangle = -1\,, \tag{8.4}
$$

[13] In general we should have allowed ϕ_2 to be an arbitrary linear combination of all mass level zero states, but in the present situation symmetry consideration prevents the mixing of $\mathcal{V}_{\chi R}$ with other states at mass level zero.

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which in turn follows from (2.9) and the normalization of $\mathcal{V}_\chi$ given in (6.23), we can express the inner product of (8.2) with $\langle \mathcal{V}_{\chi_R}(-k)|c_0^-$ as

$$\begin{aligned}
-\frac{k^2}{4} &= \beta^2 g_s^2 \{\mathcal{V}_{\chi_R}(-k)\mathcal{V}_{\chi_R} b_0^+ (L_0^+)^{-1}(1-P)[\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}(k)]_0\}_0 - \{\mathcal{V}_{\chi_R}(-k)\mathcal{V}_{\chi_R}(k)\}_1 \\
&\quad -\beta g_s \{\mathcal{V}_{\chi_R}(-k)\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}(k)\}_0 + \{\mathcal{V}_{\chi_R}(-k)(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\ ]_1)\mathcal{V}_{\chi_R}(k)\}_0 \\
&\quad +\frac{1}{2}\beta^2 g_s^2 \{\mathcal{V}_{\chi_R}(-k)(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}]_0)\mathcal{V}_{\chi_R}(k)\}_0 \\
&\quad -\frac{1}{2}\beta^2 g_s^2 \{\mathcal{V}_{\chi_R}(-k)\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}(k)\}_0 \,.
\end{aligned} \tag{8.5}$$

The $\{\mathcal{V}_{\chi_R}(-k)\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}(k)\}_0$ term vanishes due to the by now familiar $\bar{J}$ charge conservation rule. Now the renormalized mass m_B of χ_R is obtained by demanding that the above equation has a solution at $k^2 = -m_B^2$. Since we expect m_B to be of order g_s, and hence k^2 to be of order g_s^2, and since each term on the right hand side is already of order g_s^2, we can set $k = 0$ while evaluating the right hand side. In this case the projection operator P reduces to $\mathbf{P}$ and $\mathcal{V}_{\chi_R}(\pm k)$ reduces to $\mathcal{V}_{\chi_R}$. Setting $k^2 = -m_B^2$ on the left hand side of (8.5) we get

$$\begin{aligned}
\frac{m_B^2}{4} &= \frac{3}{2}\beta^2 g_s^2 \{\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R} b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}]_0\}_0 - \frac{1}{2}\beta^2 g_s^2 \{\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}\}_0 \\
&\quad -\{\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}\}_1 + \{\mathcal{V}_{\chi_R}\mathcal{V}_{\chi_R}(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\ ]_1)\}_0 \,.
\end{aligned} \tag{8.6}$$

Using (7.10) this can be written as

$$\frac{m_B^2}{4} = -\frac{1}{2}\beta^2 g_s^2 G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R) - g_s^2\, G^{(2,2)}(0,\chi_R;0,\chi_R)\,. \tag{8.7}$$

β has been determined in (7.12). Substituting (7.12) into (8.7), we get

$$m_B^2 = 8\, g_s^2\, G^{(2,2)}(0,\chi_R;0,\chi_R) = -\frac{4}{3}\beta^2 g_s^2 G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)\,. \tag{8.8}$$

Rest of this subsection will be devoted to the computation of $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$.

Before we proceed, we need to introduce some notations.

1. As in section 7 we shall denote by $\mathcal{V}_\chi$, $\mathcal{V}_{\chi^*}$ the unintegrated vertex operators for zero momentum χ, χ^* in -1 picture:

$$\mathcal{V}_\chi = \bar{c}\, c\, e^{-\phi} V_\chi, \quad \mathcal{V}_{\chi^*} = \bar{c}\, c\, e^{-\phi} V_{\chi^*}, \tag{8.9}$$

where V_χ, V_{χ^*} are matter sector operators of dimension $(1, 1/2)$. V_χ, V_{χ^*} are normalized as in (6.23). $\mathcal{V}_{\chi_R}$ is obtained from these via the relation

$$\mathcal{V}_{\chi_R} = \frac{1}{\sqrt{2}}(\mathcal{V}_\chi + \mathcal{V}_{\chi^*})\,. \tag{8.10}$$

From (8.9) we can calculate the zero picture unintegrated vertex operators:[14]

$$\widetilde{\mathcal{V}}_\chi(z) \equiv \lim_{w\to z} \mathcal{X}(w)\mathcal{V}_\chi(z) = \bar{c}c\widetilde{V}_\chi(z) - \frac{1}{4}\bar{c}\,\eta\, e^{\phi}\, V_\chi(z)$$

[14]Even though for brevity we use the label z to label the argument of a vertex operator, it is to be understood that it depends both on z and $\bar{z}$.

$$\widetilde{\mathcal{V}}_{\chi^*}(z) \equiv \lim_{w\to z} \mathcal{X}(w)\mathcal{V}_{\chi^*}(z) = \bar{c}c\widetilde{V}_{\chi^*}(z) - \frac{1}{4}\bar{c}\,\eta\, e^{\phi}\, V_{\chi^*}(z), \tag{8.11}$$

where $\widetilde{V}$ has been defined in (6.25).

2. Using (2.4)–(2.6) and (6.27) for $\sigma = \chi$ it follows that

$$\begin{aligned}
&\{Q_B, \mathcal{V}_\chi(z)\} = 0, \quad \{Q_B, \widetilde{\mathcal{V}}_\chi(z)\} = 0, \\
&\{Q_B, \widetilde{V}_\chi(z)\} = \partial_z \left(c\widetilde{V}_\chi(z) - \frac{1}{4}\eta\, e^{\phi}\, V_\chi(z)\right) + \partial_{\bar{z}}(\bar{c}\widetilde{V}_\chi(z))\,.
\end{aligned} \tag{8.12}$$

There are also similar identities with χ replaced by χ^*.

We now return to the computation of $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$. For this we need to choose the locations of the two PCO's. If we represent the amplitude as the integral of a four point correlation function of the χ_R vertex operators in the complex plane, then the PCO's are located at two points in this plane. The final result should be independent of their locations as long as they are chosen in a gluing compatible manner and are compatible with the permutation symmetries of the external vertices. Gluing compatibility requires that as we approach a degeneration limit where two of the vertex operators come close (which is conformally equivalent to other two vertex operators coming close) one of the PCO's should be close to the two vertex operators which are coming close while the other PCO should be at a finite distance away from them or at infinity (which is conformally equivalent to its being close to the other two vertex operators). This can be achieved by taking one of the PCO's to coincide with one of the χ_R vertex operators for all values of the moduli, converting this to a zero picture vertex operator and letting the other PCO be a function of the moduli such that near any degeneration the other PCO remains (conformally) away from the zero picture vertex operator. In general we have to make the prescription symmetric under the permutation of the four punctures by taking averages, but in this case that is not necessary since the vertex operators at the four punctures are identical.

Once we have fixed the choice of the PCO locations, the computation of $G^{(4,0)}$ involves computing the appropriate world-sheet correlator and integrating the result over the moduli space of four punctured sphere. For this we can keep three of the punctures at fixed locations and integrate over the location of the fourth puncture. The final result is independent of which of the puncture locations we choose to integrate over, and we exploit this freedom to integrate over the location of the zero picture vertex operator.

We shall begin by writing down the general expression for the four point amplitude where the two PCO's are inserted at arbitrary points u and v and then specialize to the case where one of the PCO's approach the location of one of the vertices. We insert three vertex operators at fixed positions z_1, z_2, z_3 and the fourth one at a variable position z and let the PCO locations u and v depend on z and $\bar{z}$. The general expression, following the rules described in [19], is

$$\begin{aligned}
&G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R) \\
&= \frac{1}{2\pi i}\int dz\wedge d\bar{z}\Big\langle \mathcal{V}_{\chi_R}(z_1)\mathcal{V}_{\chi_R}(z_2)\mathcal{V}_{\chi_R}(z_3)
\end{aligned} \tag{8.13}$$

$$\Big(\mathcal{X}(u)\mathcal{X}(v)b_z\bar{b}_{\bar{z}}+(\mathcal{X}(v)\partial\xi(u)\partial_z u+\mathcal{X}(u)\partial\xi(v)\partial_z v)\bar{b}_{\bar{z}}$$
$$-(\mathcal{X}(v)\partial\xi(u)\partial_{\bar{z}}u+\mathcal{X}(u)\partial\xi(v)\partial_{\bar{z}}v)b_z+\partial\xi(u)\partial\xi(v)(\partial_z u\partial_{\bar{z}}v-\partial_z v\partial_{\bar{z}}u)\Big)\mathcal{V}_{\chi_R}(z)\Big\rangle,$$

where

$$b_z\equiv\oint_z b(w)dw,\qquad \bar{b}_{\bar{z}}=\oint_z \bar{b}(\bar{w})d\bar{w}\,. \tag{8.14}$$

Here $\oint_z$ denotes an integration contour encircling z, normalized so that $\oint_z dw(w-z)^{-1}=1$, $\oint_z d\bar{w}(\bar{w}-\bar{z})^{-1}=1$. The contours must be chosen *so as to keep u and v outside the contours.* Using (8.10) we can now replace each of the $\mathcal{V}_{\chi_R}$ factor in terms of $\mathcal{V}_\chi$ and $\mathcal{V}_{\chi^*}$. The resulting correlator will have 16 terms, but only six of them, containing equal number of χ's and χ^*'s, will be non-zero. A typical term is given by

$$\mathcal{A}^{(4,0)}\equiv\frac{1}{4}\frac{1}{2\pi i}\int dz\wedge d\bar{z}\Big\langle\mathcal{V}_{\chi^*}(z_1)\mathcal{V}_\chi(z_2)\mathcal{V}_{\chi^*}(z_3) \tag{8.15}$$
$$\Big(\mathcal{X}(u)\mathcal{X}(v)b_z\bar{b}_{\bar{z}}+(\mathcal{X}(v)\partial\xi(u)\partial_z u+\mathcal{X}(u)\partial\xi(v)\partial_z v)\bar{b}_{\bar{z}}$$
$$-(\mathcal{X}(v)\partial\xi(u)\partial_{\bar{z}}u+\mathcal{X}(u)\partial\xi(v)\partial_{\bar{z}}v)b_z+\partial\xi(u)\partial\xi(v)(\partial_z u\partial_{\bar{z}}v-\partial_z v\partial_{\bar{z}}u)\Big)\mathcal{V}_\chi(z)\Big\rangle.$$

The other terms are related to this by different assignments of χ and χ^* to different punctures. They can be generated from (8.15) by first summing over cyclic permutation of z_1,z_2,z_3, and then summing over the exchange of all the χ's with χ^*'s in each of these terms. Now the correlator has a symmetry under the exchange of all the χ's with χ^*'s; hence summing over this exchange produces a factor of 2. Therefore if we denote by $\bar{\mathcal{A}}^{(4,0)}$ the result of averaging $\mathcal{A}^{(4,0)}$ over the cyclic permutation of z_1,z_2,z_3, then we can write

$$G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)=6\,\bar{\mathcal{A}}^{(4,0)}\,. \tag{8.16}$$

We now turn to the evaluation of $\mathcal{A}^{(4.0)}$. Using the result (3.25) we can replace $dz\wedge d\bar{z}$ by $2id^2z$. Now we take the limit $v\to z$. In this limit $\partial_{\bar{z}}v=0$, $\partial_z v=1$. In the process of taking the limit we have to pass v through the integration contours involved in the definitions of b_z and $\bar{b}_{\bar{z}}$. Using the relations $\oint_v dwb(w)\mathcal{X}(v)=\partial\xi(v)$, $\oint_v d\bar{w}\bar{b}(\bar{w})\mathcal{X}(v)=0$ one can show that all the terms involving $\partial\xi(v)$ cancel. Therefore we are left with

$$\mathcal{A}^{(4,0)}$$
$$=\frac{1}{4}\int\frac{d^2z}{\pi}\Big\langle\mathcal{V}_{\chi^*}(z_1)\mathcal{V}_\chi(z_2)\mathcal{V}_{\chi^*}(z_3)$$
$$\Big(\mathcal{X}(u)\oint_z b(w)dw\oint_z\bar{b}(\bar{w})d\bar{w}+\partial\xi(u)\,\partial_z u\oint_z\bar{b}(\bar{w})d\bar{w}-\partial\xi(u)\,\partial_{\bar{z}}u\oint_z b(w)dw\Big)\widetilde{\mathcal{V}}_\chi(z)\Big\rangle$$
$$=\frac{1}{4}\int\frac{d^2z}{\pi}\Big\langle\mathcal{V}_{\chi^*}(z_1)\mathcal{V}_\chi(z_2)\mathcal{V}_{\chi^*}(z_3)$$
$$\Big(\mathcal{X}(u)\widetilde{V}_\chi(z)+\partial\xi(u)\,\partial_z u\Big(c\widetilde{V}_\chi(z)-\frac{1}{4}\eta\,e^\phi\,V_\chi(z)\Big)+\partial\xi(u)\,\partial_{\bar{z}}u\,\bar{c}\widetilde{V}_\chi(z)\Big)\Big\rangle. \tag{8.17}$$

For reasons that will become clear soon, we now introduce an auxiliary quantity $\widetilde{\mathcal{A}}^{(4,0)}$ by taking the $u \to z_2$ limit of (8.17). In this limit $\partial_z u$ and $\partial_{\bar{z}} u$ vanish, and we get

$$\widetilde{\mathcal{A}}^{(4,0)} = \frac{1}{4} \int \frac{d^2 z}{\pi} \Big\langle \mathcal{V}_{\chi^*}(z_1) \widetilde{\mathcal{V}}_{\chi}(z_2) \mathcal{V}_{\chi^*}(z_3) \widetilde{V}_{\chi}(z) \Big\rangle . \tag{8.18}$$

Using (8.9) and (8.11) we now see that in (8.18), ϕ charge conservation forces us to pick the $\bar{c}c\widetilde{V}_{\chi}(z_2)$ term from the zero picture vertex operators $\widetilde{\mathcal{V}}_{\chi}(z_2)$. Therefore the matter part of the correlation function now involves two factors of $\widetilde{V}_{\chi}$ and two factors of V_{χ^*}. Eqs. (6.24), (6.26) then show that the total $J(z)$ charge carried by all the vertex operators in the correlation function add up to -2 and hence the result vanishes by J-charge conservation. Therefore $\widetilde{\mathcal{A}}^{(4,0)}$ vanishes identically and we can write

$$\mathcal{A}^{(4,0)} = \mathcal{A}^{(4,0)} - \widetilde{\mathcal{A}}^{(4,0)} . \tag{8.19}$$

Our strategy will be to express the right hand side of (8.19) as a total derivative in the moduli space. This can then be expressed as sum of boundary terms which are easier to evaluate.

We use the relations

$$\widetilde{\mathcal{V}}_{\chi}(z_2) = \mathcal{X}(z_2) \mathcal{V}_{\chi}(z_2) \tag{8.20}$$

and

$$\mathcal{X}(u) - \mathcal{X}(z_2) = \{Q_B, \xi(u) - \xi(z_2)\} \tag{8.21}$$

to write

$$\mathcal{X}(u)\mathcal{V}_{\chi}(z_2) - \widetilde{\mathcal{V}}_{\chi}(z_2) = \{Q_B, \xi(u) - \xi(z_2)\}\, \mathcal{V}_{\chi}(z_2) . \tag{8.22}$$

Using this we get

$$\begin{aligned} &\mathcal{A}^{(4,0)} - \widetilde{\mathcal{A}}^{(4,0)} \\ &= \frac{1}{4} \int \frac{d^2 z}{\pi} \Big\langle \mathcal{V}_{\chi^*}(z_1) \mathcal{V}_{\chi}(z_2) \mathcal{V}_{\chi^*}(z_3) \\ &\bigg(\{Q_B, \xi(u) - \xi(z_2)\} \widetilde{V}_{\chi}(z) + \partial\xi(u)\, \partial_z u \left(c\widetilde{V}_{\chi}(z) - \frac{1}{4} \eta\, e^{\phi}\, V_{\chi}(z) \right) + \partial\xi(u)\, \partial_{\bar{z}} u\, \bar{c} \widetilde{V}_{\chi}(z) \bigg) \Big\rangle . \end{aligned} \tag{8.23}$$

We can now deform the BRST contour and use the relations (8.12) to arrive at

$$\begin{aligned} &\mathcal{A}^{(4,0)} - \widetilde{\mathcal{A}}^{(4,0)} \\ &= \frac{1}{4} \int \frac{d^2 z}{\pi}\, \partial_z \Big\langle (\xi(u) - \xi(z_2))\, \mathcal{V}_{\chi^*}(z_1) \mathcal{V}_{\chi}(z_2) \mathcal{V}_{\chi^*}(z_3) \left(c\widetilde{V}_{\chi}(z) - \frac{1}{4} \eta\, e^{\phi}\, V_{\chi}(z) \right) \Big\rangle \\ &\quad + \frac{1}{4} \int \frac{d^2 z}{\pi}\, \partial_{\bar{z}} \Big\langle (\xi(u) - \xi(z_2))\, \mathcal{V}_{\chi^*}(z_1) \mathcal{V}_{\chi}(z_2) \mathcal{V}_{\chi^*}(z_3) \left(\bar{c}(\bar{z}) \widetilde{V}_{\chi}(z) \right) \Big\rangle . \end{aligned} \tag{8.24}$$

Now on the sphere a non-zero correlation function requires the number of c insertions minus the number of b insertions to be 3, and similarly the number of $\bar{c}$ insertions minus the number of $\bar{b}$ insertions to be 3. Since each of $\mathcal{V}_{\chi^*}(z_1)$, $\mathcal{V}_{\chi}(z_2)$ and $\mathcal{V}_{\chi^*}(z_3)$ contains a factor of $\bar{c}c$, we see that the only non-vanishing term in (8.24) is the term involving η. Using (8.19) we can now express (8.24) as

$$\mathcal{A}^{(4,0)} = -\frac{1}{16}\int \frac{d^2z}{\pi}\,\partial_z\Big\langle (\xi(u)-\xi(z_2))\,\mathcal{V}_{\chi^*}(z_1)\mathcal{V}_\chi(z_2)\mathcal{V}_{\chi^*}(z_3)\eta\, e^{\phi}\,V_\chi(z)\Big\rangle\,. \tag{8.25}$$

Since this is the integral of a total derivative, the result can be expressed as boundary contributions. There are three relevant boundaries corresponding to z coming close to z_1, z_2 and z_3. In order to get a non-zero contribution from the boundary $z \to z_i$, the term inside the total derivative must be of the form

$$(z-z_i)^{-\alpha}(\bar{z}-\bar{z}_i)^{-\alpha-1} \tag{8.26}$$

for $\alpha \geq 0$. We shall now examine the contribution near each of these boundaries.

1. First let us examine the contribution from the boundary near $z = z_2$. This is controlled by the operator product of $\eta e^{\phi} V_\chi(z)$ and $(\xi(u)-\xi(z_2))\mathcal{V}_\chi(z_2) = (\xi(u)-\xi(z_2))\bar{c}ce^{-\phi}V_\chi(z_2)$. Possible negative powers of $(\bar{z}-\bar{z}_2)$ can come from the operator product of $V_\chi(z)$ and $V_\chi(z_2)$. Now since V_χ carries $\bar{J}$ charge of 2, any operator appearing in the product $V_\chi(z)V_\chi(z_2)$ must have $\bar{J}$ charge 4. Standard CFT results and (6.4) now tells us that the left-handed conformal weight $\bar{h}$ of such an operator has a lower bound of $4^2/6 = 8/3$. Therefore the lowest power of $(\bar{z}-\bar{z}_2)$ that we can get in the operator product of $V_\chi(z)$ and $V_\chi(z_2)$ is $(\bar{z}-\bar{z}_2)^{2/3}$. Comparison with (8.26) now shows that it is not possible to get a non-vanishing boundary contribution from z near z_2.

2. Next we turn to the boundary contribution from z near z_1. In this case the matter part of the operator product involves the combination $V_\chi(z)V_{\chi^*}(z_1)$ carrying total $\bar{J}$ charge zero, and hence we may get sufficiently negative power of $(\bar{z}-\bar{z}_1)$ so as to get a non-zero boundary contribution. To evaluate this we have to carefully study the full operator product expansion. Now as discussed earlier, in this limit we have to keep u away from z (and hence also z_1). The relevant operator product that could produce a singular term of the form (8.26) as $z \to z_1$ is

$$\eta\, e^{\phi}\, V_\chi(z)\,\mathcal{V}_{\chi^*}(z_1) = -\eta(z)\,\bar{c}(z_1)\,c(z_1)\,e^{\phi}(z)\,e^{-\phi(z_1)}\,V_\chi(z)V_{\chi^*}(z_1)\,. \tag{8.27}$$

Since $e^{\phi(z)}e^{-\phi(z_1)} = (z-z_1)+\mathcal{O}\left((z-z_1)^2\right)$, in order to get a term of the form (8.26), we must pick those terms in the operator product $V_\chi(z)V_{\chi^*}(z_1)$ whose holomorphic part has at least a singularity of order $(z-z_1)^{-1}$. Using the fact that V_χ and V_{χ^*} are operators of conformal weight (1,1/2), and the normalization condition (6.23), we see that the relevant terms in the operator product $V_\chi(z)V_{\chi^*}(z_1)$ are of the form

$$V_\chi(z)V_{\chi^*}(z_1) = (z-z_1)^{-1}(\bar{z}-\bar{z}_1)^{-2} + (z-z_1)^{-1}(\bar{z}-\bar{z}_1)^{-1}\,\bar{\mathcal{J}}(\bar{z}_1) + \text{less singular terms} \tag{8.28}$$

where $\bar{\mathcal{J}}$ is some dimension (1,0) left-handed current. Assuming that $\bar{J}$ is the only left-handed U(1) current in the matter CFT associated with the compact directions, we see that $\bar{\mathcal{J}}$ must be proportional to $\bar{J}$. In order to find the constant of proportionality we use (6.36) to write

$$\begin{aligned}
&\left\langle \bar{J}(\bar{w})V_\chi(z)V_{\chi^*}(z_1)\right\rangle = 2\left(\frac{1}{\bar{w}-\bar{z}}-\frac{1}{\bar{w}-\bar{z}_1}\right)\langle V_\chi(z)V_{\chi^*}(z_1)\rangle \\
&= 2\left(\frac{1}{\bar{w}-\bar{z}}-\frac{1}{\bar{w}-\bar{z}_1}\right)(z-z_1)^{-1}(\bar{z}-\bar{z}_1)^{-2} \\
&= 2(\bar{w}-\bar{z})^{-1}(\bar{w}-\bar{z}_1)^{-1}(z-z_1)^{-1}(\bar{z}-\bar{z}_1)^{-1}\,. \qquad (8.29)
\end{aligned}$$

Taking $z\to z_1$ limit on both sides and using (8.28) we get

$$(z-z_1)^{-1}(\bar{z}-\bar{z}_1)^{-1}\langle \bar{J}(\bar{w})\bar{\mathcal{J}}(\bar{z}_1)\rangle = 2(\bar{w}-\bar{z}_1)^{-2}(z-z_1)^{-1}(\bar{z}-\bar{z}_1)^{-1}\,. \qquad (8.30)$$

Comparing with (6.7) we now get

$$\bar{\mathcal{J}} = \frac{2}{3}\bar{J}\,. \qquad (8.31)$$

Using (8.27), (8.28) and (8.31) we see that the relevant part of the operator product expansion involved in the computation of the boundary contribution to (8.25) from $z=z_1$ is given by

$$\eta\, e^{\phi}\, V_\chi(z)\,\mathcal{V}_{\chi^*}(z_1) = -\eta(z_1)\,\bar{c}(z_1)\,c(z_1)\,\left[(\bar{z}-\bar{z}_1)^{-2}+\frac{2}{3}(\bar{z}-\bar{z}_1)^{-1}\,\bar{J}(\bar{z}_1)\right]. \qquad (8.32)$$

Comparing with (8.26) we see that only the second term inside the square bracket contributes to the boundary term from the $z=z_1$ end. Furthermore one can easily show [19] that the boundary contribution from the $z=z_1$ end of $\int d^2z\partial_z(1/\bar{z})$ is given by $-\pi$. Using this the net contribution to the right hand side of (8.25) from the $z=z_1$ boundary is given by

$$-\frac{1}{16}\frac{1}{\pi}(-\pi)(-1)\frac{2}{3}\left\langle(\xi(u)-\xi(z_2))\eta(z_1)\,\bar{c}(z_1)\,c(z_1)\bar{J}(\bar{z}_1)\mathcal{V}_\chi(z_2)\mathcal{V}_{\chi^*}(z_3)\right\rangle\,. \qquad (8.33)$$

This correlation function can be easily evaluated using (2.9) and the known operator product expansion between various fields, including (8.29). The net result is

$$\frac{1}{12}\,\frac{z_2-u}{u-z_1}\,\frac{z_1-z_3}{z_2-z_3}\,. \qquad (8.34)$$

3. The analysis of the boundary contribution in the $z\to z_3$ limit can be obtained by exchanging z_1 and z_3 in the above results. This gives

$$\frac{1}{12}\,\frac{z_2-u}{u-z_3}\,\frac{z_3-z_1}{z_2-z_1}\,. \qquad (8.35)$$

Adding (8.34), (8.35) and averaging over cyclic permutations of z_1, z_2 and z_3, we get a net contribution

$$\bar{\mathcal{A}}^{(4,0)} = -\frac{1}{12}. \qquad (8.36)$$

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Substituting (8.36) into (8.16) we get

$$G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R) = -\frac{1}{2}\,. \tag{8.37}$$

Note that the final result is independent of the location u of the PCO and also of z_1, z_2 and z_3, as is expected from the general arguments of [14, 19].

Using (8.8) we now get

$$m_B^2 = \frac{2}{3}\,\beta^2 g_s^2\,, \tag{8.38}$$

i.e.

$$m_B = \sqrt{\frac{2}{3}}\,\beta\, g_s\,. \tag{8.39}$$

In order to fully determine m_B we need to determine β via (7.12). This in turn requires determination of $G^{(2,2)}(0,\chi_R;0,\chi_R)$. This was done in [48, 49] and has been partially reviewed in appendix F. Using the result for $G^{(2,0)}(0,\chi_R;0,\chi_R)$ given in appendix F and the result for $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$ given in (8.37) we can determine the actual value of β and of m_B and verify that they are real. However for testing the equality of the masses of fermions and bosons we shall not need this result, since the fermion mass will also be determined in terms of β.

8.2 Fermion mass to order g_s

We shall now compute the mass of the fermion that is the superpartner of χ_R to order g_s and compare this with the scalar mass m_B given in (8.39) to test supersymmetry restoration at the shifted vacuum. For this we have to solve the linearized equation of motion (2.51) in the fermionic sector and determine the on-shell value of k^2. Equating this with $-m_F^2$ we can determine the mass of the fermionic partner of χ_R.

Since for fermions the quantity that enters the linearized equation of motion is the mass and not mass^2, it is enough to compute correction to first order in g_s for determining the mass to order g_s. Therefore the iterative equations (2.52)–(2.54) take the form

$$|\Phi_0\rangle = |\phi_1\rangle, \quad P|\phi_1\rangle = |\phi_1\rangle\,, \tag{8.40}$$

$$|\Phi_1\rangle = -\frac{b_0^+}{L_0^+}(1-P)\mathcal{X}_0 K|\Phi_0\rangle + |\phi_1\rangle\,, \tag{8.41}$$

$$Q_B|\phi_1\rangle = -P\mathcal{X}_0 K|\Phi_0\rangle = -P\mathcal{X}_0[\Psi_1\phi_1] = -\beta g_s P\mathcal{X}_0[\mathcal{V}_R\phi_1]\,. \tag{8.42}$$

We shall look for solution to (8.42) using the ansatz

$$|\phi_1\rangle = |Y_\alpha\rangle f^\alpha + |Z_{\dot\alpha}\rangle g^{\dot\alpha} + |\tilde\phi_1\rangle\,, \tag{8.43}$$

where

$$|Y_\alpha\rangle \equiv \bar c c e^{-\phi/2} V^f_{\chi,\alpha} e^{ik.X}(0)|0\rangle, \quad |Z_{\dot\alpha}\rangle \equiv \frac{1}{\sqrt{3}}\bar c c e^{-\phi/2}\Sigma^c_{\dot\alpha}\bar J e^{ik\cdot X}(0)|0\rangle\,, \tag{8.44}$$

f^α and $g^{\dot\alpha}$ are grassmann odd variables, and $|\tilde\phi_1\rangle$ is an order g_s correction. We shall for now proceed by ignoring the effect of $|\tilde\phi_1\rangle$, but will return to discuss its role at the end of this section. $|Y_\alpha\rangle$ and $|Z_{\dot\alpha}\rangle$ represent respectively the fermionic partners of χ and the U(1) gauge fields at the zeroth order. Introducing the states

$$|\widetilde{A}_{\dot\beta}\rangle = (\partial c + \bar\partial\bar c)\bar c c\partial\xi e^{-5\phi/2} V^f_{\chi^*,\dot\beta} e^{-ik\cdot X}(0)|0\rangle, \quad |\widetilde{B}_\beta\rangle = \frac{1}{\sqrt{3}}(\partial c + \bar\partial\bar c)\bar c c\partial\xi e^{-5\phi/2}\Sigma_\beta \bar J e^{-ik\cdot X}(0)|0\rangle\,, \tag{8.45}$$

the two linearly independent equations derived from (8.42) can be taken to be

$$\langle\widetilde{A}_{\dot\beta}|c_0^-\left(Q_B|\phi_1\rangle + \beta g_s \mathcal{X}_0[\mathcal{V}_R\phi_1]\right) = 0, \quad \langle\widetilde{B}_\beta|c_0^-\left(Q_B|\phi_1\rangle + \beta g_s\mathcal{X}_0[\mathcal{V}_R\phi_1]\right)\rangle = 0\,. \tag{8.46}$$

Note that we have dropped the projection operator P since $|\widetilde{A}_\beta\rangle$ and $|\widetilde{B}_{\dot\beta}\rangle$ are P invariant states. Using (6.12), (6.34), (2.9), (2.4)–(2.6)) and $\bar J$ charge conservation, we find

$$\begin{aligned} \langle\widetilde{A}_{\dot\beta}|c_0^- Q_B|Y_\alpha\rangle &= -\frac{1}{4}k_\mu\gamma^\mu_{\dot\beta\alpha}, & \langle\widetilde{B}_\beta|c_0^- Q_B|Z_{\dot\alpha}\rangle &= -\frac{1}{4}k_\mu\gamma^\mu_{\beta\dot\alpha}, \\ \langle\widetilde{A}_{\dot\beta}|c_0^- Q_B|Z_{\dot\alpha}\rangle &= 0, & \langle\widetilde{B}_\beta|c_0^- Q_B|Y_\alpha\rangle &= 0, \end{aligned} \tag{8.47}$$

$$\langle\widetilde{A}_{\dot\beta}|c_0^-\mathcal{X}_0|[\mathcal{V}_{\chi R}Y_\alpha]\rangle = 0, \qquad \langle\widetilde{B}_\beta|c_0^-\mathcal{X}_0|[\mathcal{V}_{\chi R}Z_{\dot\alpha}]\rangle = 0\,. \tag{8.48}$$

$\gamma^\mu_{\beta\dot\alpha}$ and $\gamma^\mu_{\dot\beta\alpha}$ have been defined in (6.16). Furthermore, using Lorentz invariance we can write

$$\langle\widetilde{A}_{\dot\beta}|c_0^-\mathcal{X}_0|[\mathcal{V}_{\chi R}Z_{\dot\alpha}]\rangle = -C\varepsilon_{\dot\beta\dot\alpha}, \quad \langle\widetilde{B}_\beta|c_0^-\mathcal{X}_0|[\mathcal{V}_{\chi R}Y_\alpha]\rangle = D\,\varepsilon_{\beta\alpha}\,, \tag{8.49}$$

where C and D are two constants to be determined. This allows us to express (8.46) as

$$-\frac{1}{4}k_\mu(\gamma^\mu)_{\dot\beta\alpha}f^\alpha - \beta\, g_s\, C\,\varepsilon_{\dot\beta\dot\alpha}g^{\dot\alpha} = 0, \quad \beta\, g_s\, D\,\varepsilon_{\beta\alpha}f^\alpha - \frac{1}{4}k_\mu(\gamma^\mu)_{\beta\dot\alpha}g^{\dot\alpha} = 0\,, \tag{8.50}$$

ignoring the contribution from $|\tilde\phi_1\rangle$. Multiplying the first equation by $k_\nu(\gamma^\nu)_\beta{}^{\dot\beta}$ and using the second equation to eliminate $g^{\dot\alpha}$ we get

$$\left\{\frac{k^2}{16} + \beta^2\, g_s^2\, CD\right\}\varepsilon_{\beta\alpha}f^\alpha = 0\,. \tag{8.51}$$

Demanding that f^α is non-zero (so that we have a non-trivial solution to the linearized equations) and comparing (8.51) with the on-shell condition $k^2 + m_F^2 = 0$, we get the fermion mass m_F:

$$m_F = 4\,\beta\, g_s\sqrt{CD}\,. \tag{8.52}$$

We shall now compute C and D. Using (2.7), (2.16), (6.2), (6.3), (6.12) and (8.45) we get

$$\begin{aligned} \mathcal{X}_0|\widetilde{A}_{\dot\beta}\rangle &= \frac{1}{4}\,|A_{\dot\beta}\rangle, & |A_{\dot\beta}\rangle &\equiv \bar c c e^{-\phi/2}V^f_{\chi^*,\dot\beta}e^{-ik\cdot X}(0)|0\rangle, \\ \mathcal{X}_0|\widetilde{B}_\beta\rangle &= \frac{1}{4}\,|B_\beta\rangle, & |B_\beta\rangle &\equiv \frac{1}{\sqrt{3}}\bar c c e^{-\phi/2}\Sigma_\beta\bar J e^{-ik\cdot X}(0)|0\rangle\,. \end{aligned} \tag{8.53}$$

Comparison with (8.49) yields

$$\frac{1}{4}\,\{A_{\dot\beta}\mathcal{V}_{\chi R}Z_{\dot\alpha}\} = -C\,\varepsilon_{\dot\beta\dot\alpha}, \quad \frac{1}{4}\,\{B_\beta\mathcal{V}_{\chi R}Y_\alpha\} = D\,\varepsilon_{\beta\alpha}\,. \tag{8.54}$$

Now using the fact that $\mathcal{V}_{\chi R} = (\mathcal{V}_\chi + \mathcal{V}_{\chi^*})/\sqrt{2}$ and using $\bar{J}$ charge conservation, we get

$$\{A_{\dot\beta}\mathcal{V}_{\chi R}Z_{\dot\alpha}\} = \frac{1}{\sqrt{2}}\{A_{\dot\beta}\mathcal{V}_\chi Z_{\dot\alpha}\}\,, \quad \{B_\beta\mathcal{V}_{\chi R}Y_\alpha\} = \frac{1}{\sqrt{2}}\{B_\beta\mathcal{V}_{\chi^*}Y_\alpha\} = -\frac{1}{\sqrt{2}}\{Y_\alpha\mathcal{V}_{\chi^*}B_\beta\} \tag{8.55}$$

where in the last step we have used the fact that the $\{\cdots\}$ product is anti-symmetric under the exchange of two Ramond sector states of ghost number 2. Using (8.55), (2.9) and the various operator product expansions described in section 6.2 we now get

$$\begin{aligned}
\{A_{\dot\beta}\mathcal{V}_{\chi R}Z_{\dot\alpha}\} &= \frac{1}{\sqrt{6}}\left\langle\left(\bar{c}ce^{-\phi/2}V^f_{\chi^*,\dot\beta}e^{-ik\cdot X}\right)(z_1)\left(\bar{c}ce^{-\phi}V_\chi\right)(z_2)\left(\bar{c}ce^{-\phi/2}\Sigma^c_{\dot\alpha}\bar{J}e^{ik\cdot X}\right)(z_3)\right\rangle \\
&= -\sqrt{\frac{2}{3}}\varepsilon_{\dot\beta\dot\alpha}\,, \\
\{B_\beta\mathcal{V}_{\chi R}Y_\alpha\} &= -\frac{1}{\sqrt{6}}\left\langle\left(\bar{c}ce^{-\phi/2}V^f_{\chi,\alpha}e^{ik\cdot X}\right)(z_1)\left(\bar{c}ce^{-\phi}V_{\chi^*}\right)(z_2)\left(\bar{c}ce^{-\phi/2}\Sigma_\beta\bar{J}e^{-ik\cdot X}\right)(z_3)\right\rangle \\
&= -\sqrt{\frac{2}{3}}\varepsilon_{\alpha\beta}\,.
\end{aligned} \tag{8.56}$$

Comparison with (8.54) and (8.56) gives

$$C = \frac{1}{4}\sqrt{\frac{2}{3}}, \quad D = \frac{1}{4}\sqrt{\frac{2}{3}}, \tag{8.57}$$

and hence

$$m_F = \sqrt{\frac{2}{3}}\,\beta\,g_s\,. \tag{8.58}$$

This matches m_B given in (8.39) confirming the prediction of unbroken supersymmetry at the shifted vacuum.

Let us now discuss the role of $|\tilde\phi_1\rangle$ appearing in (8.43). As mentioned there, $|\tilde\phi_1\rangle$ is an order g_s contribution. It could contain a linear combination of unphysical, physical and pure gauge states at the same mass level — where the classification of the states into these three categories refer to the property they would have at $k^2 = 0$. The pure gauge contribution can be removed by a gauge transformation while the physical state contribution will have the effect of renormalizing the constants f^α and $g^{\dot\alpha}$ appearing in (8.43). Therefore we focus on unphysical state contribution to $|\tilde\phi_1\rangle$. This can be e.g. of the form $(\bar\partial\bar{c} + \partial c)ce^{-\phi/2}\Sigma^c_{\dot\alpha}e^{ik\cdot X}(0)|0\rangle$. Now using the fact that $|\tilde\phi_1\rangle$ is of order g_s, and that $Q_B|\widetilde{A}_{\dot\beta}\rangle \sim g_s$, $Q_B|\widetilde{B}_\beta\rangle \sim g_s$ (counting k^μ to be of order g_s) one can show that the contribution of $|\tilde\phi_1\rangle$ to the two equations given in (8.46) is of order g_s^2. Since the rest of the terms in these equations, given in (8.50), are of order g_s we see that the effect of $|\tilde\phi_1\rangle$ to these equations is subleading, and hence does not affect the computation of the fermion mass to the leading order. However $|\tilde\phi_1\rangle$ plays a crucial role when we take the inner product of (8.42) with an unphysical state $\langle s|c_0^-$. In this case the left hand side gets a contribution of order g_s from the $\langle s|c_0^- Q_B|\tilde\phi_1\rangle$ term. $|\tilde\phi_1\rangle$ now has to be adjusted to cancel the order g_s contribution from the rest of the terms.

9 Two loop dilaton tadpole in the perturbative vacuum

We shall now use the result of section 5 to compute the two loop dilaton tadpole in SO(32) heterotic string theory compactified on a Calabi-Yau 3-fold in the perturbative vacuum corresponding to $\beta = 0$. More precisely we shall compute $\mathcal{E}_4(\mathcal{V}_{\rm S})$ in this theory for various zero momentum, ghost number two states in the BRST cohomology and show that the result does not vanish. This will then show that the $\beta = 0$ vacuum becomes inconsistent at two loops.

Now in the $\beta = 0$ vacuum the natural expansion parameter is g_s^2 since the classical solution is of order g_s^2. Therefore, the relevant equation is (5.16). Since in (5.16) the error term is of order g_s^{2q+6}, we see that it is enough to take $q = 0$ in order to compute $\mathcal{E}_4(\mathcal{V}_{\rm S})$. Also as discussed in the last paragraph of section 5, $|\zeta_a\rangle$ must be one of the candidate goldstino states $|\mathcal{V}_{\rm G}\rangle$ appearing in (2.37), and $|\zeta^a\rangle$ must be one of the conjugate states $|\mathcal{V}_{\rm G}^c\rangle$ appearing in (2.35). Furthermore we can also use the arguments in appendix E to argue that the choice of $\mathcal{V}_{\rm G}^c$ in this case is given uniquely by

$$\mathcal{X}_0\zeta^a = \mathcal{X}_0\mathcal{V}_{\rm G}^c = \frac{1}{\sqrt{3}}\bar{c}\,c\,\eta\,e^{\phi/2}\,\widetilde{\Sigma}\,\bar{J}\,, \tag{9.1}$$

where $\widetilde{\Sigma}$ is one of the operators that appear in (6.10) and the $1/\sqrt{3}$ factor has been included for convenience. This gives

$$\begin{aligned} \zeta^a &= \mathcal{V}_{\rm G}^c = -\frac{4}{\sqrt{3}}(\partial c + \bar{\partial}\bar{c})\bar{c}\,c\,e^{-3\phi/2}\,\widetilde{\Sigma}\,\bar{J} \\ \zeta_a &= \mathcal{V}_{\rm G} = -\frac{1}{4\sqrt{3}}\bar{c}\,c\,e^{-\phi/2}\,\Sigma^c\bar{J}\,, \end{aligned} \tag{9.2}$$

where Σ^c is a matter sector spin field from the list (6.8) that is conjugate to $\widetilde{\Sigma}$, i.e. satisfies

$$e^{-\phi/2}\Sigma^c(z)e^{-3\phi/2}\widetilde{\Sigma}(w) = (z-w)^{-2}e^{-2\phi}(w)\,. \tag{9.3}$$

eq. (5.16) with $q = 0$ now gives

$$\mathcal{E}_4(\mathcal{V}_{\rm S}) = g_s^4\,\Gamma_P^{(2,2)}(\mathcal{V}_{\rm F},\mathcal{V}_{\rm G})\,\Gamma_P^{(2,2)}(\mathcal{X}_0\mathcal{V}_{\rm G}^c,\Lambda_0)\,, \tag{9.4}$$

where the subscript P of Γ denotes that we are referring to the amplitude in the perturbative vacuum ($\beta = 0$). Since there is no two point function on the sphere, and since for torus amplitudes there is no distinction between truncated Green's function and full Green's function, $g_s^2\,\Gamma_P^{(2,2)}$ denotes the full torus two point function in the perturbative vacuum. In the next two subsections we shall evaluate the two factors on the right hand side of (9.4).

9.1 Goldstino coupling to the supersymmetry generator

In this subsection we shall evaluate $\Gamma_P^{(2,2)}(\mathcal{X}_0\mathcal{V}_{\rm G}^c,\Lambda_0)$. We now put in the explicit Lorentz spinor indices and choose

$$\Lambda_0 = ce^{-\phi/2}\Sigma_\alpha, \quad \mathcal{X}_0\mathcal{V}_{\rm G}^c = \frac{1}{\sqrt{3}}\bar{c}\,c\,\eta\,e^{\phi/2}\,\widetilde{\Sigma}_\beta^c\,\bar{J}\,. \tag{9.5}$$

We shall follow the convention of section 7 of [19] for our computation. We denote by u the coordinate on the torus with the identification $u \equiv u+1 \equiv u+\tau$, place the vertex operator $\mathcal{X}_0\mathcal{V}_{\rm G}^{\rm c}$ at the origin $u=0$ and the vertex operator Λ_0 at a point $u=y$ that will eventually be integrated over. Since both vertex operators are dimension zero primaries, the choice of local coordinate system does not affect the amplitude. Nevertheless it will be useful to fix some local coordinate system for the choice of Beltrami differentials. We take the local coordinates around the punctures to be $w_1 = u$ and $w_2 = u-y$. We also require the choice of the location of a PCO consistent with the factorization property, namely that when the two vertex operators approach each other, the PCO should be away from both vertex operators. In the u coordinate system we take this to be at a point v that in general depends on $y, \bar y, \tau, \bar\tau$. Then following the procedure described in section 7 of [19] we can express the amplitude as[15]

$$\Gamma_P^{(2,2)}(\mathcal{X}_0\mathcal{V}_{\rm G}^{\rm c},\Lambda_0) = -\frac{1}{4\pi^2}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\Big\langle\{\mathcal{X}(v)b_\tau\bar b_{\bar\tau}b_y\bar b_{\bar y}+\cdots\}\mathcal{X}_0\mathcal{V}_{\rm G}^{\rm c}(0)\,\Lambda_0(y)\Big\rangle \quad (9.6)$$

where

$$b_y = \oint_y dw b(w), \quad \bar b_{\bar y} = \oint_y d\bar w \bar b(\bar w), \quad b_\tau = \frac{1}{2\pi i}\int_a dw b(w), \quad \bar b_{\bar\tau} = -\frac{1}{2\pi i}\int_a d\bar w\bar b(\bar w)\,. \quad (9.7)$$

Here $\oint_y$ denotes a contour around y with the normalization $\oint_y dw(w-y)^{-1}=1$, $\oint_y d\bar w(\bar w - \bar y)^{-1}=1$, and $\int_a$ denotes a contour around the a-cycle of the torus, connecting u to $u+1$, with the normalization $\int_a dw = 1$, $\int_a d\bar w = 1$. The overall factor of $-1/4\pi^2$ arises from the $(2\pi i)^{-3g+3-n}$ normalization factor in the g-loop, n-point amplitude. The $\cdots$ inside the curly bracket is a sum of four terms, in each one of the factors b_τ, $\bar b_{\bar\tau}$, b_y and $\bar b_{\bar y}$ is replaced by $\partial\xi(v)\partial_\tau v$, $\partial\xi(v)\partial_{\bar\tau}v$, $\partial\xi(v)\partial_y v$ and $\partial\xi(v)\partial_{\bar y}v$ respectively and the $\mathcal{X}(v)$ factor is dropped.

Now from the expression for Λ_0 given in (9.5) we see that $\bar b(\bar w)$ has no pole at $\bar w = \bar y$. Hence the contribution from the $\bar b_{\bar y}$ term in (9.6) vanishes, and the only non-vanishing contribution comes from the term in which $\bar b_{\bar y}$ is replaced by $\partial\xi(v)\partial_{\bar y}v$ and the $\mathcal{X}(v)$ factor is dropped:

$$\Gamma_P^{(2,2)}(\mathcal{X}_0\mathcal{V}_{\rm G}^{\rm c},\Lambda_0) = -\frac{1}{4\pi^2}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\,\Big\langle b_\tau\,\bar b_{\bar\tau}\,b_y\,\partial\xi(v)\,\partial_{\bar y}\,v\,\mathcal{X}_0\mathcal{V}_{\rm G}^{\rm c}(0)\,\Lambda_0(y)\Big\rangle \quad (9.8)$$

Naively one might expect that the term proportional to $\partial_{\bar y}v$ can also be made to vanish by keeping v fixed at a position away from 0, since this will be consistent with the factorization relation which requires that in the $y\to 0$ limit the PCO location v must be away from 0. This would make the whole amplitude (9.8) vanish. However it is not in general possible to keep v fixed at a $y, \bar y$ independent position due to the existence of spurious poles, and as a result the amplitude is not identically zero. We shall proceed by taking v to be independent of y, $\bar y$, τ and $\bar\tau$ for most of the range of these variables except for a small tubular neighborhood around the spurious pole(s). Inside the tubular neighborhood we take the PCO to be located at another constant value w in the u plane. In that case the non-zero

[15]The $(y,\bar y)$ integral will run over only half of the torus due to the involution symmetry $u\to -u$. We shall include a factor of 1/2 in the definition of the correlation function $\langle\cdots\rangle$ and allow the integral over $(y,\bar y)$ to run over the full torus.

contribution to the amplitude comes only from the boundary of this tubular neighborhood at which the PCO location jumps discontinuously from v to w. The result is [19, 20]

$$\Gamma_P^{(2,2)}(\mathcal{X}_0\mathcal{V}_{\rm G}^c,\Lambda_0)=\frac{1}{4\pi^2}\int_S d\tau\wedge d\bar\tau\wedge dy\left\langle b_\tau\,\bar b_{\bar\tau}\,(\xi(v)-\xi(w))\,\mathcal{X}_0\mathcal{V}_{\rm G}^c(0)\,b_y\,\Lambda_0(y)\right\rangle \tag{9.9}$$

where S denotes the boundary of the tubular neighborhood enclosing the spurious pole. For fixed $\tau,\bar\tau$, S corresponds to an anti-clockwise contour enclosing the spurious pole. Note that we have moved the b_y factor next to Λ_0 so that we can use it to remove the $c(y)$ factor from Λ_0. Since $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ is grassmann even, this generates only one minus sign from having to pass b_y through $(\xi(v)-\xi(w))$. In going from (9.8) to (9.9), two other minus signs have cancelled. One of them comes from the fact that since we are carrying out integration over $\bar y$ first, we have to rearrange $dy\wedge d\bar y$ as $-d\bar y\wedge dy$. The second minus sign comes from the fact that the measure is defined so that $d(\mathrm{Im}\,y)\wedge d(\mathrm{Re}\,y)$ gives positive volume. As this is opposite of what is used conventionally, we would have an extra minus sign (otherwise the y integral would run along a clockwise contour). We now see from (9.9) that if near the location y_s of the spurious pole we have

$$F(y)\equiv\sqrt{3}\left\langle b_\tau\,\bar b_{\bar\tau}\,(\xi(v)-\xi(w))\,\mathcal{X}_0\mathcal{V}_{\rm G}^c(0)\,b_y\,\Lambda_0(y)\right\rangle=\frac{A_s(\tau,\bar\tau)}{y-y_s}+\text{non-singular terms}\,, \tag{9.10}$$

then we have

$$\Gamma_P^{(2,2)}(\mathcal{X}_0\mathcal{V}_{\rm G}^c,\Lambda_0)=-\frac{1}{2\sqrt{3}\pi i}\int_{\mathcal{F}}d\tau\wedge d\bar\tau A_s(\tau,\bar\tau)\,, \tag{9.11}$$

where $\mathcal{F}$ denotes the fundamental domain of the moduli space of one punctured genus one Riemann surface. If there are multiple spurious poles then we have to sum over the residues at all the poles. The multiplicative factor of $\sqrt{3}$ has been included in (9.10) to get rid of the $1/\sqrt{3}$ factor in $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ given in (9.5).

To compute A_s we need to evaluate the correlation function, locate the position of the spurious poles and find the residues at the poles. To locate the spurious poles we use the general expression for the correlation functions of ξ's, η's and $e^{q\phi}$'s in the 'large Hilbert space' [54, 58, 59]. On genus 1 Riemann surface it takes the simple form

$$\begin{aligned}&\left\langle\prod_{i=1}^{n+1}\xi(x_i)\prod_{j=1}^{n}\eta(y_j)\prod_{k=1}^{m}e^{q_k\phi(z_k)}\right\rangle_\delta'\\&=\frac{\prod_{j=1}^n\vartheta_\delta(-y_j+\sum_{i=1}^{n+1}x_i-\sum_{i=1}^n y_i+\sum_k q_k z_k)}{\prod_{j=1}^{n+1}\vartheta_\delta(-x_j+\sum_{i=1}^{n+1}x_i-\sum_{i=1}^n y_i+\sum_k q_k z_k)}\frac{\prod_{i<i'}E(x_i,x_{i'})\prod_{j<j'}E(y_j,y_{j'})}{\prod_{i,j}E(x_i,y_j)\prod_{k<\ell}E(z_k,z_\ell)^{q_kq_\ell}}\,,\\&\quad\times\delta_{\sum_{k=1}^m q_k,0}\,,\end{aligned} \tag{9.12}$$

where the prime indicates that we are referring to the correlation function in the large Hilbert space. δ labels spin structure, ϑ denotes the Jacobi theta functions and $E(x,y)$ denotes the prime form which on genus one surface takes the form

$$E(x,y)=\vartheta_1(x-y)/\vartheta_1'(0)\,. \tag{9.13}$$

Note that if q_i's are not integers the correlation function suffers from the usual phase ambiguity; these will be fixed later. Since (9.12) gives the correlation function in the 'large Hilbert space', we have one more ξ compared to η. To compute a correlation function in the small Hilbert space where there are equal number of ξ's and η's, and ξ's always appear with a derivative or a difference operator acting on it (e.g. in (9.10) we have $\xi(v) - \xi(w)$), we can simply insert a factor of $\xi(p)$ for some arbitrary point p and interpret this as a correlation function in the large Hilbert space. The general structure of the correlators guarantees that the result is independent of the choice of the point p.

Note that (9.12) has poles when $\vartheta_\delta(-x_j + \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n} y_i + \sum_k q_k\, z_k)$ vanishes. As this happens when no operators are coincident in general, they are referred to as spurious poles. In the following we shall generalize the notion a bit to include any pole that depends on the position of the PCO, including those which occur when a PCO collides with another PCO or a vertex operator.

We shall now use (9.12) to compute (9.10). For this we insert a factor of $\xi(w)$ into the correlator in (9.10) to interpret this as a correlation function in the large Hilbert space. Since the $\xi(w)\xi(w)$ term vanishes, we get, after using (9.5) and using the b_y factor to remove the c factor of Λ_0,

$$\begin{aligned} F(y) &= \left\langle b_\tau\, \bar b_{\bar\tau} \xi(w)\xi(v)\bar c\, c\, \eta\, e^{\phi/2}\, \widetilde\Sigma^c_\beta\, \bar J(0)\, e^{-\phi/2}\Sigma_\alpha(y)\right\rangle' \\ &= -\left\langle b_\tau\, \bar b_{\bar\tau} \xi(w)\xi(v)\bar c(0)\, c(0)\, \eta(0)\, e^{-\phi/2}\Sigma_\alpha(y)\; e^{\phi/2}\, \widetilde\Sigma^c_\beta\, \bar J(0)\right\rangle' . \end{aligned} \tag{9.14}$$

We can now use (9.12) to evaluate this correlation function, and get

$$F(y) = -\frac{1}{2}\varepsilon \sum_\delta \frac{\vartheta_\delta(w+v-\frac{y}{2})}{\vartheta_\delta(v-\frac{y}{2})\vartheta_\delta(w-\frac{y}{2})}\, \frac{E(w,v)}{E(v,0)E(w,0)}\, (E(y,0))^{1/4} \left\langle b_\tau\, \bar b_{\bar\tau}\, \bar c(0)\, c(0) \Sigma_\alpha(y) \widetilde\Sigma^c_\beta(0)\, \bar J(0)\right\rangle_\delta , \tag{9.15}$$

where 1/2 is the usual factor accompanying sum over spin structures and ε is a phase to be determined. Indeed, as it stands the right hand side of (9.15) is ill defined since there is a fractional power of $E(y,0)$ and the correlator contains the product $\Sigma_\alpha(y)\widetilde\Sigma^c_\beta(0)$ which has fractional power of y in the operator product expansion. This ambiguity is resolved by noting that in the $v \to 0$ limit the correlation function appearing in (9.14) must reduce to v^{-1} times the correlation function with the $\xi(v)$ and $\eta(0)$ factors dropped, while in (9.15) the product of the ϑ_δ's and E's reduce to $(E(y,0))^{1/4}/\,(v\,\vartheta_\delta(-y/2))$ in this limit. Therefore an unambiguous way of writing (9.15) is

$$\begin{aligned} F(y) = -\frac{1}{2} \sum_\delta \frac{\vartheta_\delta(w+v-\frac{y}{2})}{\vartheta_\delta(v-\frac{y}{2})\vartheta_\delta(w-\frac{y}{2})}\, \frac{E(w,v)}{E(v,0)E(w,0)}\, \vartheta_\delta(-y/2) \\ \times \left\langle b_\tau\, \bar b_{\bar\tau}\, \xi(w)\, \bar c(0)\, c(0) e^{-\phi/2}\Sigma_\alpha(y) e^{\phi/2}\widetilde\Sigma^c_\beta(0)\, \bar J(0)\right\rangle'_\delta . \end{aligned} \tag{9.16}$$

The spurious poles can be identified as the value of y at which the $\vartheta_\delta(v-\frac{y}{2})$ factor in the denominator vanishes.[16] Since we have spin fields in the correlator, we can relate the contribution from different spin structures by shifting y by the periods of the torus. This allows

[16] The other pole at the zero of $\vartheta_\delta(w-\frac{y}{2})$ appears after we have shifted the location of the PCO from v

us to focus on only one spin structure — which we shall take to be the periodic-periodic (PP) spin structure — at the cost of extending the range of y to over a parallelogram of sides 2 and 2τ and picking up the contribution from all the poles in this range. This gives

$$F(y) = -\frac{1}{2}\frac{\vartheta_1(w+v-\frac{y}{2})}{\vartheta_1(v-\frac{y}{2})\vartheta_1(w-\frac{y}{2})}\frac{E(w,v)}{E(v,0)E(w,0)}\vartheta_1(-y/2)$$
$$\left\langle b_\tau\,\bar{b}_{\bar\tau}\,\bar{c}(0)\,c(0)e^{-\phi/2}\Sigma_\alpha(y)e^{\phi/2}\widetilde{\Sigma}^c_\beta(0)\,\bar{J}(0)\right\rangle_{PP}, \tag{9.17}$$

with the subscript PP denoting periodic-periodic sector (odd spin structure). Note that we have dropped the $\xi(w)$ factor and returned to the correlator evaluated in the small Hilbert space. Within a parallelogram of sides 2 and 2τ, the spurious pole is at $y = 2v$. The residue at this pole is given by

$$A_s = \frac{1}{\vartheta_1'(0)}\frac{\vartheta_1(w)}{\vartheta_1(w-v)}\frac{E(w,v)}{E(v,0)E(w,0)}\vartheta_1(-v)\left\langle b_\tau\,\bar{b}_{\bar\tau}\,\bar{c}(0)\,c(0)e^{-\phi/2}\Sigma_\alpha(2v)e^{\phi/2}\widetilde{\Sigma}^c_\beta(0)\,\bar{J}(0)\right\rangle_{PP}. \tag{9.18}$$

Using (9.13) this can be rewritten as

$$A_s = -\left\langle b_\tau\,\bar{b}_{\bar\tau}\,\bar{c}(0)\,c(0)e^{-\phi/2}\Sigma_\alpha(2v)e^{\phi/2}\widetilde{\Sigma}^c_\beta(0)\,\bar{J}(0)\right\rangle_{PP}. \tag{9.19}$$

In order to calculate this, we note the following:

1. The correlator is a doubly periodic function of v with periods 1 and τ.
2. The operators product singularities of the correlation function occur at $2v = 0$ mod 1 or τ. This gives $v = 0, 1/2, \tau/2$ and $(1+\tau)/2$.
3. According to (9.12) the ϕ correlator has a spurious pole due to the $\vartheta_1(v)$ factor in the denominator. This occurs at $v = 0$.
4. Since all the poles occur at positions where two vertex operators coincide in some spin structure, we can determine the residues at the singularities using (6.21). For example for $v \to 0$, the leading term in the expansion is

$$\varepsilon_{\alpha\beta}\,(2v)^{-1}\left\langle b_\tau\,\bar{b}_{\bar\tau}\,\bar{c}(0)\,c(0)\bar{J}(0)\right\rangle_{PP}. \tag{9.20}$$

The leading singularities of (9.19) near $1/2$, $\tau/2$ and $(1+\tau)/2$ can be found by noting that translating v by $1/2$, $\tau/2$ and $(1+\tau)/2$ in (9.19) we can access the correlator in the other spin structures. Therefore the behavior near one of these poles will be given by

$$\varepsilon_{\alpha\beta}\,(2v-s_\delta)^{-1}\left\langle b_\tau\,\bar{b}_{\bar\tau}\,\bar{c}(0)\,c(0)\bar{J}(0)\right\rangle_{\delta}, \tag{9.21}$$

for some even spin structure δ. s_δ takes values 1, τ and $1+\tau$ for different spin structures.

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to w inside the tubular neighborhood around the spurious pole and plays no role since by construction it is outside the tubular neighborhood of the original spurious pole.

5. Since in the PP sector there are zero modes of the free fermions ψ^μ as well as of the bosonic ghost fields β, γ, correlation function of the form (9.20) is somewhat ill defined. For this reason it is simplest to analyze the correlator (9.19) for non-zero v first and then take the limit. Using (9.12) we see that the correlator appearing in (9.19) in the ghost sector goes as $E(2v,0)^{1/4}/\vartheta_1(v) \sim v^{-3/4}$ for small v. The operator product expansion of the matter sector spin fields goes as $v^{-5/4}$. However the spin field correlator in the free fermion sector in spin structure δ gives an explicit factor of $\vartheta_\delta(v)^2$ in the numerator [60], which for PP sector translates to $\vartheta_1(v)^2$. Since this goes as v^2 for small v, we see that the correlator does not have any singularity for $v \to 0$.

6. In the case of even spin structure, there are no zero modes and the correlation function (9.21) is unambiguous. In fact this vanishes due to the vanishing of the one point function of $\bar{J}$ in the matter sector (see the discussion at the end of appendix D).

Therefore the net result is that the correlator (9.19) has no poles in the v plane. Since this is a doubly periodic function of v it must be a constant. This in turn shows that we can evaluate it by calculating it at any point in the v plane. We shall choose to evaluate it at one of the points $1/2$, $\tau/2$ or $(1+\tau)/2$ where there is no ambiguity associated with zero modes and we can use operator product expansion. The constant term is determined by the first subleading correction in the expansion (6.21). The $\partial\phi$ term does not contribute due to the vanishing of one point function of $\bar{J}$ in the matter sector, and the $\psi^\mu\psi^\nu$ terms does not contribute due to Lorentz invariance and anti-symmetry under $\mu \leftrightarrow \nu$. Therefore the result is

$$A_s = \frac{1}{2}\varepsilon_{\alpha\beta} \left\langle b_\tau\, \bar{b}_{\bar\tau}\, \bar{c}(0)\, c(0) J(0) \bar{J}(0) \right\rangle_{\delta_e}, \tag{9.22}$$

where δ_e is any of the even spin structures. Since the result does not depend on the spin structure we can in fact take the average over the three even spin structures. Furthermore the right hand side of (9.22) vanishes if we replace δ_e by the odd spin structure PP due to zero modes of ψ^μ, and hence we can include in the sum the PP spin structure as well to write[17]

$$\begin{aligned} A_s &= \frac{1}{3}\frac{1}{2}\varepsilon_{\alpha\beta} \sum_\delta \left\langle b_\tau\, \bar{b}_{\bar\tau}\, \bar{c}(0)\, c(0) J(0) \bar{J}(0) \right\rangle_\delta \\ &= \frac{1}{3}\,\varepsilon_{\alpha\beta} \left\langle b_\tau\, \bar{b}_{\bar\tau}\, \bar{c}(0)\, c(0) J(0) \bar{J}(0) \right\rangle, \end{aligned} \tag{9.23}$$

where we have reverted to the earlier notation that correlator without a subscript denotes implicit sum over spin structures accompanied by a factor of $1/2$. Substituting this into (9.11) we get

$$\Gamma_P^{(2,2)}(\mathcal{X}_0 \mathcal{V}_{\rm G}^c, \Lambda_0) = -\varepsilon_{\alpha\beta}\, \frac{1}{6\sqrt{3}\pi i} \int_{\mathcal{F}} d\tau \wedge d\bar\tau \left\langle b_\tau\, \bar{b}_{\bar\tau}\, \bar{c}(0)\, c(0) J(0) \bar{J}(0) \right\rangle = -\varepsilon_{\alpha\beta}\, \Xi\,, \tag{9.24}$$

[17] Again the result in the PP sector is somewhat ambiguous due to existence of both fermionic and bosonic zero modes, but we can define it by inserting an additional pair of operators $e^{-\phi/2}\Sigma_\gamma(x_1) e^{\phi/2}\widetilde{\Sigma}^c_\delta(x_2)$ at two arbitrary points x_1 and x_2 and picking up the coefficient of the $\varepsilon_{\gamma\delta}(x_1 - x_2)^{-1}$ term as $x_1 \to x_2$. This can be shown to vanish following logic similar to the one showing the absence of pole of (9.19) at $v = 0$. In any case, the relevant quantity is really (9.22), and the equality of (9.23) and (9.22) can be taken as the definition of what appears on the right hand side of (9.23).

where

$$\Xi = \frac{1}{3\pi\sqrt{3}} \int_{\mathcal{F}} d^2\tau \Big\langle b_\tau \, \bar{b}_{\bar{\tau}} \, \bar{c}(0) \, c(0) J(0) \bar{J}(0) \Big\rangle . \tag{9.25}$$

Note that in (9.25) we have replaced $d\tau \wedge d\bar{\tau}$ by $2id^2\tau$ in the spirit of (3.25). Up to normalization, the same factor appears in the expression for the one loop renormalized mass^2 of the scalar χ_R in the shifted vacuum given in (F.10).

Ξ has been calculated in [48, 49] with non-zero result — the actual value depends on the massless field content of the theory. We shall not try to repeat the analysis here, but express all further results in terms of this quantity.

9.2 Goldstino-Dilatino coupling

Next we turn to the computation of $\Gamma_P^{(2,2)}(\mathcal{V}_{\rm F}, \mathcal{V}_{\rm G})$. Associated with the choice of $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ given in (9.5), we have from (9.2)

$$\begin{aligned} \zeta^a &= \mathcal{V}_{\rm G}^c = -\frac{4}{\sqrt{3}} (\partial c + \bar{\partial}\bar{c}) \bar{c} \, c \, e^{-3\phi/2} \, \widetilde{\Sigma}^c_\beta \, \bar{J} \, , \\ \zeta_a &= \mathcal{V}_{\rm G} = \frac{1}{4\sqrt{3}} \bar{c} \, c \, e^{-\phi/2} \, \Sigma^\beta \bar{J} \, , \end{aligned} \tag{9.26}$$

where $\Sigma^\beta \equiv \varepsilon^{\beta\gamma}\Sigma_\gamma$ and we have used $\varepsilon^{\beta\gamma}\varepsilon_{\gamma\alpha} = -\delta^\beta{}_\alpha$.

There are two possible candidates for $\mathcal{V}_{\rm S}$ whose tadpoles may be generated at this order:

$$\mathcal{V}_{\rm S} \propto \bar{c}ce^{-\phi}\psi_\mu \bar{\partial} X^\mu, \quad c\eta \, , \tag{9.27}$$

representing the zero momentum vertex operators for the trace of the graviton and the dilaton. Their fermionic partners have been constructed in appendix C, yielding the results

$$\mathcal{V}_{\rm F} \propto \bar{c}ce^{-\phi/2} (\gamma_\mu)_\gamma{}^{\dot{\alpha}} \Sigma^c_{\dot{\alpha}} \bar{\partial} X^\mu, \quad c\eta e^{\phi/2} \widetilde{\Sigma}^c_\gamma \, , \tag{9.28}$$

and similar operators related to the above by the exchange of dotted and undotted indices.

First consider the case where $\mathcal{V}_{\rm F} = c\eta e^{\phi/2} \widetilde{\Sigma}^c_\gamma$. In this case the computation of the two point function proceeds in a way very similar to that of the previous subsection. We take the vertex operator $\zeta_a = \mathcal{V}_{\rm G}$ to be at fixed position 0 and the vertex operator $c\eta e^{\phi/2}\widetilde{\Sigma}^c_\gamma$ at y. Then the analog of (9.6) will be

$$\Gamma_P^{(2,2)}(\mathcal{V}_{\rm G}, \mathcal{V}_{\rm F}) = -\frac{1}{4\pi^2} \int d\tau \wedge d\bar{\tau} \wedge dy \wedge d\bar{y} \Big\langle \{ \mathcal{X}(v) b_\tau \bar{b}_{\bar{\tau}} b_y \bar{b}_{\bar{y}} + \cdots \} \mathcal{V}_{\rm G}(0) \, \mathcal{V}_{\rm F}(y) \Big\rangle \tag{9.29}$$

where $b_\tau, \bar{b}_{\bar{\tau}}, b_y, \bar{b}_{\bar{y}}$ have been defined in (9.7) and as before the $\cdots$ inside the curly bracket represent sum of four terms, in each one of the factors b_τ, $\bar{b}_{\bar{\tau}}$, b_y and $\bar{b}_{\bar{y}}$ is replaced by $\partial\xi(v)\partial_\tau v$, $\partial\xi(v)\partial_{\bar{\tau}} v$, $\partial\xi(v)\partial_y v$ and $\partial\xi(v)\partial_{\bar{y}} v$ respectively and the $\mathcal{X}(v)$ factor is dropped. We now notice that since $\mathcal{V}_{\rm F}(y) = c\eta e^{\phi/2}\widetilde{\Sigma}^c_\gamma(y)$ is annihilated by $\bar{b}_{\bar{y}}$, we must pick the term where $\bar{b}_{\bar{y}}$ is replaced by $\partial\xi(v)\partial_{\bar{y}} v$ and the $\mathcal{X}(v)$ factor is dropped. This gives the analog of (9.8):

$$\Gamma_P^{(2,2)}(\mathcal{V}_{\rm G}, \mathcal{V}_{\rm F}) = -\frac{1}{4\pi^2} \int d\tau \wedge d\bar{\tau} \wedge dy \wedge d\bar{y} \Big\langle b_\tau \bar{b}_{\bar{\tau}} \, b_y \partial\xi(v)\partial_{\bar{y}} v \mathcal{V}_{\rm G}(0) \, \mathcal{V}_{\rm F}(y) \Big\rangle . \tag{9.30}$$

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Following the same logic as in the previous section we see that if we take v to be constant on most of the torus outside a small disk containing the spurious pole and take it to be another constant w inside the disk so that there is no spurious pole inside the disk, then the integrand vanishes both inside and outside the disk and picks up non-zero contribution only from the boundary of the disk where the PCO location jumps. This leads to the analog of (9.9)

$$\Gamma_P^{(2,2)}(\mathcal{V}_\mathrm{G},\mathcal{V}_\mathrm{F}) = -\frac{1}{4\pi^2}\int_S d\tau\wedge d\bar\tau\wedge dy\left\langle b_\tau\,\bar b_{\bar\tau}\,(\xi(v)-\xi(w))\,\mathcal{V}_\mathrm{G}(0)\,b_y\,\mathcal{V}_\mathrm{F}(y)\right\rangle \tag{9.31}$$

where S denotes the boundary of the tubular neighborhood in the τ, y space enclosing the spurious pole. From this we can reach the analog of (9.10), (9.11), i.e. if near the location of the spurious pole y_s we have

$$\widetilde F(y) \equiv 4\sqrt{3}\left\langle b_\tau\,\bar b_{\bar\tau}\,(\xi(v)-\xi(w))\,\mathcal{V}_\mathrm{G}(0)\,b_y\,\mathcal{V}_\mathrm{F}(y)\right\rangle = \frac{\widetilde A_s(\tau,\bar\tau)}{y-y_s} + \text{non-singular terms}\,, \tag{9.32}$$

then we have

$$\Gamma_P^{(2,2)}(\mathcal{V}_\mathrm{G},\mathcal{V}_\mathrm{F}) = \frac{1}{8\sqrt{3}\pi i}\int_\mathcal{F} d\tau\wedge d\bar\tau\widetilde A_s(\tau,\bar\tau)\,. \tag{9.33}$$

Now following the same logic that led to (9.16) we can manipulate the expression for $\widetilde F(y)$ given in (9.32) to the form

$$\begin{aligned}\widetilde F(y) &= \frac{1}{2}\sum_\delta \frac{\vartheta_\delta(w+v-\frac{3}{2}y)}{\vartheta_\delta(v-\frac{1}{2}y)\vartheta_\delta(w-\frac{1}{2}y)}\frac{E(w,v)}{E(w,y)E(v,y)}\vartheta_\delta(y/2)\\ &\quad\times\left\langle b_\tau\,\bar b_{\bar\tau}\,\xi(w)\,e^{\phi/2}\,\widetilde\Sigma^c_\gamma(y)\;\bar c\,c\,e^{-\phi/2}\,\Sigma^\beta\bar J(0)\right\rangle'_\delta\,.\end{aligned} \tag{9.34}$$

The spurious poles now arise from two sets of points — at the zeroes of $\vartheta_\delta(v-\frac{1}{2}y)$ and at the zero of $E(v,y)$. First let us analyze the contribution from the first set of poles. For this we restrict the sum over δ to be in the PP sector only at the cost of extending the range of y to a parallelogram of sides 2 and 2τ. The spurious pole is at $y=2v$. Evaluating the residue at the pole we get the analog of (9.19):

$$\widetilde A_s^{(1)} = \left\langle b_\tau\,\bar b_{\bar\tau}\,e^{\phi/2}\,\widetilde\Sigma^c_\gamma(2v)\;\bar c\,c\,e^{-\phi/2}\,\Sigma^\beta\bar J(0)\right\rangle_{PP}\,. \tag{9.35}$$

Next we consider the contribution from the pole at $y=v$ arising from the zero of $E(v,y)$. The residue at this pole gives

$$\widetilde A_s^{(2)} = -\frac{1}{2}\sum_\delta\left\langle b_\tau\,\bar b_{\bar\tau}\,\xi(w)\,e^{\phi/2}\,\widetilde\Sigma^c_\gamma(v)\;\bar c\,c\,e^{-\phi/2}\,\Sigma^\beta\bar J(0)\right\rangle'_\delta\,. \tag{9.36}$$

We can evaluate the correlator in any spin structure and then get the result for the other spin structures by shifting v by 1, τ or $1+\tau$. Now in the PP sector the correlator is similar to the one that appeared in (9.19). Following the same logic given below (9.19) one can show that (9.36) is independent of v. Therefore (9.36) is also independent of δ and we can express the result as the contribution from the PP sector multiplied by a factor of 4. This gives

$$\widetilde A_s^{(2)} = -2\left\langle b_\tau\,\bar b_{\bar\tau}\,e^{\phi/2}\,\widetilde\Sigma^c_\gamma(v)\;\bar c\,c\,e^{-\phi/2}\,\Sigma^\beta\bar J(0)\right\rangle_{PP}\,. \tag{9.37}$$

Adding this to (9.35) and using the fact that both expressions are independent of v, we get

$$\widetilde{A}_s = -\left\langle b_\tau\, \bar{b}_{\bar\tau}\, e^{\phi/2}\, \widetilde{\Sigma}^c_\gamma(2v)\ \bar{c}\, c\, e^{-\phi/2}\, \Sigma^\beta \bar{J}(0)\right\rangle_{PP} . \tag{9.38}$$

Following the same logic that led from (9.19) to (9.23) we get

$$\widetilde{A}_s = \frac{1}{3}\delta^\beta{}_\gamma \left\langle b_\tau\, \bar{b}_{\bar\tau}\bar{c}(0)c(0)J(0)\bar{J}(0)\right\rangle . \tag{9.39}$$

Substituting this into (9.33) and using $d\tau\wedge d\bar\tau = 2i d^2\tau$, we get

$$\Gamma^{(2,2)}_P(\mathcal{V}_\mathrm{G},\mathcal{V}_\mathrm{F}) = \frac{1}{12\sqrt{3}\pi}\delta^\beta{}_\gamma \int_{\mathcal{F}} d^2\tau \left\langle b_\tau\, \bar{b}_{\bar\tau}\bar{c}(0)c(0)J(0)\bar{J}(0)\right\rangle = \frac{1}{4}\delta^\beta{}_\gamma\Xi\,, \tag{9.40}$$

where Ξ has been defined in (9.25).

Let us now turn to the second candidate for $\mathcal{V}_\mathrm{F}$:

$$\mathcal{V}_\mathrm{F} = \bar{c}ce^{-\phi/2}(\gamma_\mu)_\gamma{}^{\dot\alpha}\Sigma^c_{\dot\alpha}\bar\partial X^\mu\,. \tag{9.41}$$

Eq. (9.29) has identical form

$$\Gamma^{(2,2)}_P(\mathcal{V}_\mathrm{G},\mathcal{V}_\mathrm{F}) = -\frac{1}{4\pi^2}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\left\langle \{\mathcal{X}(v)b_\tau\bar b_{\bar\tau}b_y\bar b_{\bar y}+\cdots\}\mathcal{V}_\mathrm{G}(0)\,\mathcal{V}_\mathrm{F}(y)\right\rangle. \tag{9.42}$$

However since now both $\mathcal{V}_\mathrm{G}$ and $\mathcal{V}_\mathrm{F}$ carry factors of $e^{-\phi/2}$, ϕ charge conservation tells us that the contribution from the $\cdots$ terms inside the curly bracket must vanish. Furthermore we must pick the $e^\phi T_F(v)$ term inside the PCO $\mathcal{X}(v)$. Using the b_y, $\bar b_{\bar y}$ factors to remove the $\bar c c$ factor from $\mathcal{V}_\mathrm{F}(y)$ we can express (9.42) as

$$\begin{aligned}\Gamma^{(2,2)}_P(\mathcal{V}_\mathrm{G},\mathcal{V}_\mathrm{F}) = &-\frac{1}{16\sqrt{3}\,\pi^2}\,(\gamma_\mu)_\gamma{}^{\dot\alpha}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\\ &\left\langle e^\phi T_F(v)\, b_\tau\bar b_{\bar\tau}\ \bar c c e^{-\phi/2}\Sigma^\beta\bar J(0)\ e^{-\phi/2}\Sigma^c_{\dot\alpha}\bar\partial X^\mu(y)\right\rangle.\end{aligned} \tag{9.43}$$

Furthermore in order to get a non-zero correlation function involving the X^μ fields we must pick the contribution $-\psi_\nu\partial X^\nu$ from T_F. Using the fact that the two point function $\partial X^\nu(v)\bar\partial X^\mu(y)$ on the torus gives a factor of $-\pi\eta^{\mu\nu}/(2\tau_2)$, we can express this amplitude as

$$\begin{aligned}\Gamma^{(2,2)}_P(\mathcal{V}_\mathrm{G},\mathcal{V}_\mathrm{F}) = &-\frac{1}{32\sqrt{3}\,\pi}\,(\gamma_\mu)_\gamma{}^{\dot\alpha}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\,(\tau_2)^{-1}\\ &\left\langle e^\phi\psi^\mu(v)\, b_\tau\,\bar b_{\bar\tau}\,\bar c\, c\, e^{-\phi/2}\,\Sigma^\beta\bar J(0)\ e^{-\phi/2}\Sigma^c_{\dot\alpha}(y)\right\rangle.\end{aligned} \tag{9.44}$$

We now notice that the operator $e^\phi\psi^\mu(v)$ has no singularity near either of the operators inserted at 0 or y. In each spin structure δ it can have at most one spurious pole at the zero of $\vartheta_\delta(v-y/2)$. Since it is a doubly periodic function of v in each spin structure and since the only doubly periodic function with ≤ 1 poles is a constant, we conclude that the correlation function is v independent. Therefore we can evaluate it at any value of v which we shall choose to be $v = y$. Using the operator product expansion (6.12), $e^{\phi}\psi^{\mu}(v)e^{-\phi/2}\Sigma^c_{\dot\alpha}(y)$ in the $v \to y$ limit reduces to $(i/2)(\gamma^{\mu})_{\dot\alpha}{}^{\delta} e^{\phi/2}\widetilde{\Sigma}^c_{\delta}(y)$. Therefore the correlator becomes

$$\begin{aligned}&\Gamma^{(2,2)}_P(\mathcal{V}_{\rm G}, \mathcal{V}_{\rm F}) \\ &= -i\,\frac{1}{16\sqrt{3}\,\pi}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\,(\tau_2)^{-1}\Big\langle b_\tau\,\bar b_{\bar\tau}\,\bar c\,c\,e^{-\phi/2}\,\Sigma^{\beta}\,\bar J(0)\;e^{\phi/2}\widetilde{\Sigma}^c_{\gamma}(y)\Big\rangle\,.\end{aligned} \tag{9.45}$$

We can get the correlator in any spin structure by starting with the one in the PP sector and then translating y by 1, τ and $1+\tau$. We now note that in the PP sector the correlator is identical in form to the one that appears in (9.38). Hence it is independent of y and is given by (9.39) with a sign flip due to different ordering of the two operators inside the correlator. Multiplying this by a factor of 4 due to the four spin structures, and including a factor of 1/2 by which we need to multiply the sum over spin structures, we get

$$\Gamma^{(2,2)}_P(\mathcal{V}_{\rm G}, \mathcal{V}_{\rm F}) = -i\,\delta^{\beta}{}_{\gamma}\frac{1}{24\sqrt{3}\,\pi}\int d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y\,(\tau_2)^{-1}\Big\langle b_\tau\,\bar b_{\bar\tau}\,\bar c\,c\,J\,\bar J(0)\Big\rangle\,. \tag{9.46}$$

Expressing $d\tau\wedge d\bar\tau\wedge dy\wedge d\bar y$ as $-4d^2\tau d^2y$ and noting that the y integral gives a factor of τ_2 we finally arrive at

$$\Gamma^{(2,2)}_P(\mathcal{V}_{\rm G}, \mathcal{V}_{\rm F}) = i\,\delta^{\beta}{}_{\gamma}\frac{1}{6\sqrt{3}\,\pi}\int_{\mathcal{F}} d^2\tau\,\Big\langle b_\tau\,\bar b_{\bar\tau}\,\bar c\,c\,J\,\bar J(0)\Big\rangle = \frac{1}{2}i\,\delta^{\beta}{}_{\gamma}\Xi\,, \tag{9.47}$$

where Ξ has been defined in (9.25).

9.3 Tadpoles

Finally let us assemble all the pieces to compute the two loop tadpoles. Our first task will be to compute $\mathcal{V}_{\rm S} = [\Lambda_0\mathcal{V}_{\rm F}]_0$ with precise normalization factors for the candidate $\mathcal{V}_{\rm F}$'s given in (9.28). This can be easily done using the fact that up to BRST exact terms, $[\Lambda_0\mathcal{V}_{\rm F}]_0$ can be found by computing the operator product of Λ_0 and $\mathcal{V}_{\rm F}$ and then applying the b_0^- operator on the resulting state. This gives

$$\begin{aligned}\Big[\Lambda_0\,\Big(\bar c c e^{-\phi/2}(\gamma_\mu)_{\gamma}{}^{\dot\alpha}\Sigma^c_{\dot\alpha}\bar\partial X^{\mu}\Big)\Big]_0 &= i\,(\gamma_\mu\gamma^{\nu})_{\gamma}{}^{\beta}\,\varepsilon_{\beta\alpha}\,\bar c c e^{-\phi}\psi_\nu\bar\partial X^{\mu}\,,\\ \Big[\Lambda_0\,\Big(c\eta e^{\phi/2}\widetilde{\Sigma}^c_{\gamma}\Big)\Big]_0 &= \varepsilon_{\gamma\alpha}c\eta\,,\end{aligned} \tag{9.48}$$

for $\Lambda_0 = ce^{-\phi/2}\Sigma_\alpha$. Taking the product of (9.24) and (9.40) and a minus sign to account for the fact that in (9.40) the arguments of $\Gamma^{(2,2)}_P$ are exchanged compared to those in (9.4), we get

$$\varepsilon_{\gamma\alpha}\,\mathcal{E}_4(c\eta) = \frac{1}{4}\,g_s^4\,\varepsilon_{\alpha\gamma}\,\Xi^2 = -\frac{1}{4}\,g_s^4\,\varepsilon_{\gamma\alpha}\,\Xi^2\,. \tag{9.49}$$

Similarly taking the product of (9.24) and (9.47) and a minus sign, we get

$$i\,(\gamma_\mu\gamma^{\nu})_{\gamma}{}^{\beta}\,\varepsilon_{\beta\alpha}\,\mathcal{E}_4\Big(\bar c c e^{-\phi}\psi_\nu\bar\partial X^{\mu}\Big) = \frac{i}{2}\,\varepsilon_{\alpha\gamma}\,g_s^4\,\Xi^2 = -\frac{i}{2}\,\varepsilon_{\gamma\alpha}\,g_s^4\,\Xi^2\,. \tag{9.50}$$

We now use the fact that due to Lorentz invariance of the background, we must have

$$\mathcal{E}_4\Big(\bar c c e^{-\phi}\psi_\nu\bar\partial X^{\mu}\Big) = \frac{1}{4}\delta_{\nu}{}^{\mu}\,\mathcal{E}_4\Big(\bar c c e^{-\phi}\psi_\rho\bar\partial X^{\rho}\Big)\,. \tag{9.51}$$

Substituting this into (9.50) we get

$$i\,\varepsilon_{\gamma\alpha}\,\mathcal{E}_4\left(\bar{c}ce^{-\phi}\psi_\rho\bar{\partial}X^\rho\right) = -\frac{i}{2}\,\varepsilon_{\gamma\alpha}\,g_s^4\,\Xi^2\,. \tag{9.52}$$

This gives

$$\mathcal{E}_4(c\eta) = -\frac{1}{4}\,g_s^4\,\Xi^2\,,\qquad \mathcal{E}_4\left(\bar{c}ce^{-\phi}\psi_\rho\bar{\partial}X^\rho\right) = -\frac{1}{2}\,g_s^4\,\Xi^2,\qquad \mathcal{E}_4\left(\bar{c}ce^{-\phi}\psi_\mu\bar{\partial}X^\nu\right) = -\frac{1}{8}\,\delta_\mu{}^\nu\,g_s^4\,\Xi^2\,. \tag{9.53}$$

In appendix G we shall verify that these results and the result for the one loop renormalized mass^2 of the scalar given in (F.10) are all consistent with the predictions of low energy effective supersymmetric field theory.

10 Two loop dilaton tadpole in the shifted vacuum

In this short section we shall compute the dilaton tadpole at the shifted vacuum to order g_s^4 and show that it vanishes.

The relevant formula in this case is (5.15). Since we have already demonstrated that the shifted vacuum has unbroken global supersymmetry to order g_s^2, the result of section 5 shows that all tadpoles vanish to order g_s^3. The dilaton tadpole in the shifted vacuum to order g_s^4 can be computed using (5.15) for $q=2$. Denoting by $\Gamma_S^{(n,k)}$ the $\Gamma^{(n,k)}$'s in the shifted vacuum ($\beta\neq 0$), we get

$$\mathcal{E}_4(\mathcal{V}_\mathrm{S}) = g_s^4\sum_a \Gamma_S^{(2,1)}(\mathcal{V}_\mathrm{F},\zeta_a)\,\Gamma_S^{(2,3)}(\mathcal{X}_0\zeta^a,\Lambda_2) + \mathcal{O}(g_s^5)\,. \tag{10.1}$$

As discussed below (5.16), $\Gamma_S^{(2,1)}(\mathcal{V}_\mathrm{F},\zeta_a)$ is given by

$$\{\mathcal{V}_\mathrm{F}\zeta_a\Psi_1\}_0 = \beta g_s\{\mathcal{V}_\mathrm{F}\zeta_a\mathcal{V}_{\chi R}\}_0 \tag{10.2}$$

where, as in section 7, the subscript 0 on { } reflects genus zero contribution, and we have used the result $\Psi_1=\beta g_s\mathcal{V}_{\chi R}$. Now for the dilaton tadpole the relevant $\mathcal{V}_\mathrm{F}$'s have been given in (9.28):

$$\mathcal{V}_\mathrm{F}\propto \bar{c}ce^{-\phi/2}(\gamma_\mu)_\beta{}^{\dot\alpha}\Sigma^c_{\dot\alpha}\bar{\partial}X^\mu,\quad c\eta e^{\phi/2}\widetilde{\Sigma}^c_\beta\,, \tag{10.3}$$

and other operators related to the above by the exchange of dotted and undotted indices. On the other hand we have $\mathcal{V}_{\chi R}=\bar{c}ce^{-\phi}V_{\chi R}$. Finally the candidate goldstino states $|\zeta_a\rangle$ have been listed in (2.37), and the only candidates that could possibly contribute satisfying $\bar{J}$ charge conservation are

$$\mathcal{V}_\mathrm{G}\propto \bar{c}ce^{-\phi/2}V^f_{\chi,\alpha},\quad \bar{c}ce^{-\phi/2}V^f_{\chi^*,\dot\alpha}\,. \tag{10.4}$$

It is now easy to see that the sphere 3-point amplitude appearing on the right hand side of (10.2) vanishes for all the candidates. For the first candidate for $\mathcal{V}_\mathrm{F}$ given in (10.3) the amplitude vanishes due to the vanishing of one point function of $\bar{\partial}X^\mu$ on the sphere. For the second candidate for $\mathcal{V}_\mathrm{F}$ the amplitude vanishes by ϕ charge conservation. This shows the vanishing of the dilaton tadpole in the shifted vacuum to order ${g_s}^4$.

Note that this analysis leaves open the possibility that there may be other massless fields whose tadpoles do not vanish to this order. Additional work will be necessary to rule out the existence of such tadpoles.

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A Glossary of symbols

In this appendix we shall give a glossary of symbols that have been used extensively in the paper. This can be used as a quick reference while reading the paper. For superconformal ghosts and total Virasoro generators we use standard notation b, c, β, γ, ξ, η, ϕ, L_n, $\bar{L}_n$ that will not be listed below. Unless stated otherwise, the description below refers to the heterotic string theory, but can be easily generalized for type II string theories.

The following symbols refer to the general part of our analysis.

1. Q_B denotes BRST charge.
2. $b_0^\pm$, $c_0^\pm$ and $L_0^\pm$ denote respectively $(b_0 \pm \bar{b}_0)$, $(c_0 \pm \bar{c}_0)/2$ and $(L_0 \pm \bar{L}_0)$.
3. T_m, $\bar{T}_m$ denote energy momentum tensors in the matter sector.
4. T_F denotes the world-sheet superpartner of T_m.
5. $\mathcal{X}$ denotes the picture changing operator. For type II string theory we also have $\bar{\mathcal{X}}$ representing picture changing operator in the left-moving sector.
6. $\mathcal{X}_0$ is the zero mode of the picture changing operator.
7. For the heterotic string $\mathcal{G}$ is identity in the NS sector and $\mathcal{X}_0$ in the R sector. For type II strings $\mathcal{G}$ is identity in the NSNS sector, $\mathcal{X}_0$ in the NSR sector, $\bar{\mathcal{X}}_0$ in the RNS sector and $\mathcal{X}_0\bar{\mathcal{X}}_0$ in the RR sector.
8. $\mathcal{H}_T$ is the subspace of states in the Hilbert space of the matter and ghost CFT annihilated by b_0^- and L_0^-.
9. $\widehat{\mathcal{H}}_T$ is the subspace of $\mathcal{H}_T$ in which NS sector states carry picture number -1 and R sector states carry picture number $-1/2$.
10. $\widetilde{\mathcal{H}}_T$ is the subspace of $\mathcal{H}_T$ in which NS sector states carry picture number -1 and R sector states carry picture number $-3/2$.
11. $\{A_1 \cdots A_N\}$ is the 1PI amplitude with external states $|A_1\rangle, \cdots |A_N\rangle \in \widehat{\mathcal{H}}_T$, normalized so that tree level amplitudes do not carry any factor of g_s.
12. $[A_1 \cdots A_N]$ is a state in $\widetilde{\mathcal{H}}_T$ such that $\langle A_0|c_0^-|[A_1 \cdots A_N]\rangle$ is $\{A_0 A_1 \cdots A_N\}$ for all states $\langle A_0|$.
13. $|\Psi\rangle$ denotes the string field which is an arbitrary state in $\widehat{\mathcal{H}}_T$ of ghost number 2.

14. $|\Psi_{\rm vac}\rangle$ denotes a Lorentz invariant classical solution to the equations of motion carrying zero momentum, representing a candidate vacuum state.

15. $|\Psi_k\rangle$ is the approximation to $|\Psi_{\rm vac}\rangle$ to order g_s^k.

16. $|\psi_k\rangle$ is the projection of $|\Psi_k\rangle$ to $L_0^+=0$ sector.

17. $\mathbf{P}$ is the projection operator to $L_0^+=0$ states.

18. $|\tilde\psi_k\rangle = |\psi_k\rangle - |\psi_{k-1}\rangle$ is the order g_s^k contribution to $|\Psi_{\rm vac}\rangle$, projected to the $L_0^+=0$ sector.

19. $|\Phi\rangle$ denotes the shifted field $|\Psi\rangle - |\Psi_{\rm vac}\rangle$.

20. $\widehat{Q}_B = Q_B + \mathcal{G}K$ is the kinetic operator around the shifted vacuum where K is the operator defined in (2.40).

21. $\widetilde{Q}_B$ is the operator $Q_B + K\mathcal{G}$.

22. $\{A_1\cdots A_N\}''$ for $N\geq 3$ is the 1PI amplitude with external states $|A_1\rangle,\cdots|A_N\rangle$ around the shifted vacuum.

23. $[A_1\cdots A_N]''$ for $N\geq 2$ is a state in $\widetilde{\mathcal{H}}_T$ such that $\langle A_0|c_0^-|[A_1\cdots A_N]''\rangle$ is $\{A_0A_1\cdots A_N\}''$ for all states $\langle A_0|$.

24. $|\Lambda\rangle$ is the local gauge transformation parameter, represented by an arbitrary state in $\widehat{\mathcal{H}}_T$ of ghost number 1.

25. $|\Lambda_{\rm global}\rangle$ is a gauge transformation parameter satisfying $\widehat{Q}_B|\Lambda_{\rm global}\rangle = 0$ and represents a global symmetry transformation.

26. $|\Lambda_k\rangle$ is a gauge transformation parameter satisfying $\widehat{Q}_B|\Lambda_k\rangle = \mathcal{O}(g_s^{k+1})$. It represents a global symmetry transformation parameter to order g_s^k.

27. $|\lambda_k\rangle$ is the projection of $|\Lambda_k\rangle$ to the $L_0^+=0$ sector.

28. $|\Phi_{\rm linear}\rangle$ is a solution to the linearized equations of motion $\widehat{Q}_B|\Phi_{\rm linear}\rangle = 0$ around the shifted vacuum.

29. $|\Phi_\ell\rangle$ is a solution to the linearized equation around the shifted vacuum to order g_s^ℓ.

30. $L_0^+\simeq 0$ states are those whose L_0^+ value for on-shell momentum is of order g_s or less.

31. P denotes projection operator to $L_0^+\simeq 0$ states.

32. $|\phi_\ell\rangle$ is the projection of $|\Phi_\ell\rangle$ to $L_0^+\simeq 0$ sector.

33. $\Gamma^{(n)}(k_1,a_1;\cdots k_n,a_n)$ denotes the n-point Green's function with external propagators truncated, normalized so that the 1PI contribution to this agrees with $\{A_1,\cdots A_n\}$. $k_1,\cdots k_n$ label the momenta carried by the external states and $a_1,\cdots a_n$ denote other quantum numbers.

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34. $g_s^\ell \Gamma^{(n,\ell)}(k_1, a_1; \cdots k_n, a_n)$ denotes the order g_s^ℓ contribution to $\Gamma^{(n)}(k_1, a_1; \cdots k_n, a_n)$

35. $|\mathcal{V}_{\rm S}\rangle$ represents an arbitrary zero momentum Lorentz scalar state of ghost number 2, picture number -1 satisfying $Q_B|\mathcal{V}_{\rm S}\rangle = 0$.

36. $|\mathcal{V}_{\rm F}\rangle$ is the superpartner of $|\mathcal{V}_{\rm S}\rangle$ at the leading order in g_s defined through (5.3).

37. $|\mathcal{V}_{\rm G}\rangle$ is a zero momentum, ghost number 2, picture number $-1/2$ element of the BRST cohomology representing a candidate for the goldstino state.

38. $|\mathcal{V}_{\rm G}^c\rangle$ is a zero momentum, ghost number 3, picture number $-3/2$ element of the BRST cohomology representing dual of the goldstino state.

39. $\{|\zeta^a\rangle\}$ describe the collection of all candidate states for $\mathcal{V}_{\rm G}^c$ given in (2.35).

40. $\{|\zeta_a\rangle\}$ describe the collection of all candidate states for $\mathcal{V}_{\rm G}$ listed in (2.37).

The following symbols are used specifically in the analysis of SO(32) heterotic string theory on Calabi-Yau 3-folds.

1. χ denotes the chiral multiplet complex scalar whose real part $\chi_R = (\chi + \chi^*)/\sqrt{2}$ condenses at one loop to produce the supersymmetric vacuum.

2. σ represents a generic massless complex scalar field belonging to a chiral multiplet.

3. J is the holomorphic U(1) current associated with the R-symmetry of the (2,2) supersymmetry algebra associated with the compact directions.

4. $\bar{J}$ is the anti-holomorphic U(1) current associated with the R-symmetry of the (2,2) supersymmetry algebra associated with the compact directions. This is also the U(1) current whose associated gauge symmetry is anomalous.

5. $\mathcal{V}_\sigma = \bar{c}ce^{-\phi}V_\sigma$ is the zero momentum vertex operator for σ in -1 picture. Here V_σ is a matter sector vertex operator of dimension (1,1/2).

6. $\widetilde{V}_\sigma(w) = -\oint_w dz T_F(z) V_\sigma(w)$ is used for the construction of the vertex operator for σ in the 0 picture. Here T_F denotes the matter part of the super-stress tensor.

7. $\mathcal{V}_\chi$ is the zero momentum vertex operator for χ in -1 picture, obtained by taking $\sigma = \chi$ in the above definition.

8. $\mathcal{V}_{\chi_R} = (\mathcal{V}_\chi + \mathcal{V}_{\chi^*})/\sqrt{2}$ is the vertex operator for the real part of χ.

9. $\Sigma_\alpha^{(4)}$, $\Sigma_{\dot\alpha}^{(4)}$ are spin fields of dimension $(0, 1/4)$ associated with free fermions.

10. $\Sigma^{(6)}$, $\Sigma^{(6)c}$ are spin fields of dimension $(0, 3/8)$ associated with the compact directions.

11. Σ_α, $\Sigma_{\dot\alpha}^c$, $\widetilde{\Sigma}_{\dot\alpha}$ and $\widetilde{\Sigma}_\alpha^c$ are various products of $\Sigma^{(6)}$ and $\Sigma^{(4)}$ given in (6.8) and (6.10).

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12. $\bar{c}ce^{-\phi/2}V^f_{\sigma,\alpha}$ is the zero momentum vertex operator for the fermionic partner of σ in $-1/2$ picture. Schematically $V^f_{\sigma,\alpha}$ can be expressed as $V^f_\sigma \Sigma^{(4)}_\alpha$ with V^f_σ denoting a matter sector vertex operator of dimension (1, 3/8) in the CFT associated with the compact directions. Similar definitions apply to $V^f_{\sigma,\dot\alpha}$, $V^f_{\sigma^*,\alpha}$ and $V^f_{\sigma^*,\dot\alpha}$, the details of which can be found in (6.28).

13. $\bar{c}ce^{-\phi/2}V^f_{\chi,\alpha}$ is the zero momentum vertex operator for the fermionic partner of χ in $-1/2$ picture.

14. $G^{(n,2g)}(0,\chi_R;\cdots 0,\chi_R)$ denotes the full n-point, genus g amplitude of n external zero momentum χ_R states in the perturbative vacuum.

15. $\mathcal{A}^{(4,0)}$ is a particular contribution to $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$, defined in (8.15).

16. $\widetilde{\mathcal{A}}^{(4,0)}$ is $\mathcal{A}^{(4,0)}$ computed with wrong PCO arrangement that makes it vanish.

17. $\Gamma^{(n,2k)}_P$ denotes $\Gamma^{(n,2k)}$ computed in the perturbative vacuum in which supersymmetry is broken at one loop order.

18. $\Gamma^{(n,2k)}_S$ denotes $\Gamma^{(n,2k)}$ computed in the shifted vacuum in which supersymmetry is restored at one loop order.

B Pairing of physical states between different picture numbers

Eq. (2.51) gives the linearized equation of motion around the shifted vacuum

$$\widehat{Q}_B|\Phi_{\rm linear}\rangle \equiv Q_B|\Phi_{\rm linear}\rangle + \mathcal{G}\,K|\Phi_{\rm linear}\rangle = 0\,. \tag{B.1}$$

The solutions to this equation give the possible external states in the S-matrix element. On the other hand (3.10) imposes an additional constraint

$$\mathcal{G}|\widetilde{\Phi}_{\rm linear}\rangle - |\Phi_{\rm linear}\rangle = 0\,. \tag{B.2}$$

The goal of this appendix will be to show that given a perturbative solution to (B.1) we can always find a $|\widetilde{\Phi}_{\rm linear}\rangle$ satisfying (B.2), possibly after adding a pure gauge term to $|\Phi_{\rm linear}\rangle$. Hence (B.2) does not impose any additional constraint on possible choices of external states. Since in the NS sector $\mathcal{G}$ is the identity operator, the result holds trivially there by choosing $|\widetilde{\Phi}_{\rm linear}\rangle = |\Phi_{\rm linear}\rangle$. Therefore we need to focus on the Ramond sector.

For this analysis we shall restrict ourselves to the case when $|\Phi_{\rm linear}\rangle$ carries non-zero momentum. Let us now define $|\widetilde{\Phi}_\ell\rangle$ for $0\le\ell\le n$ to be the solution to the following recursion relations analogous to (2.52)–(2.54):

$$|\widetilde{\Phi}_0\rangle = |\widetilde{\phi}_n\rangle, \qquad |\widetilde{\Phi}_{\ell+1}\rangle = -\frac{b_0^+}{L_0^+}(1-P)K\mathcal{G}|\widetilde{\Phi}_\ell\rangle + |\widetilde{\phi}_n\rangle + \mathcal{O}\left(g_s^{\ell+2}\right)\,, \quad \text{for} \quad 0\le\ell\le n-1\,, \tag{B.3}$$

where $|\widetilde{\phi}_n\rangle$ satisfies

$$P|\widetilde{\phi}_n\rangle = |\widetilde{\phi}_n\rangle\,, \tag{B.4}$$

$$Q_B|\widetilde{\phi}_n\rangle = -PK\mathcal{G}|\widetilde{\Phi}_{n-1}\rangle + \mathcal{O}(g_s^{n+1})\,. \tag{B.5}$$

Given a solution to (B.3)–(B.5), we can find a solution to (2.52)–(2.54) by setting

$$|\phi_n\rangle = \mathcal{G}|\widetilde{\phi}_n\rangle, \qquad |\Phi_\ell\rangle = \mathcal{G}|\widetilde{\Phi}_\ell\rangle\,, \quad \text{for} \quad 0 \le \ell \le n\,. \tag{B.6}$$

Non-trivial solutions to (B.3)–(B.5) can be found by starting with a physical state at order g_s^0, and then systematically correcting the state as well as the momenta carried by the state by solving the recursion relations. On the other hand from the result of [61] it follows that to order g_s^0, there is a one to one correspondence between the physical states in picture numbers $-1/2$ and $-3/2$ at non-zero momentum. Therefore given a physical state at picture number $-1/2$, we can first construct a physical state at picture number $-3/2$, then correct it by solving the recursion relations (B.3)–(B.5), and then find the corrected physical state in picture number $-1/2$ using (B.6). This shows that in perturbation theory we do not lose any physical state by imposing the additional condition (B.2).

C Bose-Fermi pairing at zero momentum

For non-zero momentum supersymmetry pairs NS and R-sector vertex operators. However at zero momentum some of this pairing may break down since the BRST cohomology at zero momentum is not always given by the analytic continuation from non-zero momentum [61]. On the other hand, for the analysis of section 5 we need to assume the existence of such pairing. In this appendix we shall prove the results which were used in the analysis of section 5. More specifically we shall prove that at zero momentum, for every BRST invariant, Lorentz scalar state $|\mathcal{V}_{\rm S}\rangle \in \widehat{\mathcal{H}}_T$ of ghost number 2, we can find a BRST invariant Ramond sector state $|\mathcal{V}_{\rm F}\rangle \in \widehat{\mathcal{H}}_T$ of ghost number 2, and another BRST invariant Ramond sector state $|\Lambda_0\rangle \in \widehat{\mathcal{H}}_T$ of ghost number 1 — representing a global supersymmetry transformation parameter at zeroth order — such that

$$|\mathcal{V}_{\rm S}\rangle = [\Lambda_0 \mathcal{V}_{\rm F}]_0\,, \tag{C.1}$$

up to addition of BRST trivial states. Here Λ_0 is the leading contribution to a global supersymmetry generator, and the subscript 0 on $[\cdots]$ denotes that we only include genus zero contribution in computing $[\cdots]$. There may be more than one $|\mathcal{V}_{\rm F}\rangle$ satisfying the desired relation, but for us the existence of one such state will be enough.

To prove (C.1) we take a brute force approach and write down the general form of possible $|\mathcal{V}_{\rm S}\rangle$. Up to addition of BRST exact terms, there are two kinds of BRST invariant scalar vertex operators at zero momentum (see e.g. [19]):

$$\mathcal{V}_{\rm S} : \bar{c} c e^{-\phi} W, \quad c\eta\,, \tag{C.2}$$

where W denotes a dimension (1,1/2) superconformal primary in the NS sector of the matter CFT. The first set of vertex operators are familiar, the second one gives the zero momentum dilaton vertex operator in the -1 picture. Their conjugate BRST invariant states $(\mathcal{V}_{\rm S})^c$ in the ghost number 3 sector are, up to normalization and addition of pure gauge states

$$(\mathcal{V}_{\rm S})^c : (\partial c + \bar{\partial}\bar{c})\bar{c} c e^{-\phi} W^c, \quad (\partial c + \bar{\partial}\bar{c})\bar{c}\,\bar{\partial}^2\bar{c}\, c\partial\xi e^{-2\phi}\,, \tag{C.3}$$

where W^c denotes the operator in the matter sector conjugate to W.

On the other hand the leading contribution to the supersymmetry generators Λ_0 have the form

$$\Lambda_0 : ce^{-\phi/2}\Sigma\,, \tag{C.4}$$

where Σ is a dimension $(0, 5/8)$ operator in the R sector of the matter CFT. For example for four dimensional theories preserving $\mathcal{N} = 1$ supersymmetry, Σ is one of the operators $\Sigma^c_{\dot\alpha}$ or Σ_α introduced in (6.8). Let $\widetilde{\Sigma}$ be the dimension (0,5/8) operator 'conjugate' to Σ satisfying

$$\Sigma(z)\widetilde{\Sigma}(w) = (z-w)^{-5/4} + \cdots\,. \tag{C.5}$$

For four dimensional theories preserving $\mathcal{N} = 1$ supersymmetry, $\widetilde{\Sigma}$ is one of the operators $\widetilde{\Sigma}_{\dot\alpha}$ or $\widetilde{\Sigma}^c_\alpha$ introduced in (6.8).

Now space-time supersymmetry at the leading order ensures existence of matter primaries V^f of dimension (1,5/8) in the Ramond sector such that for some conjugate pair Σ, $\widetilde{\Sigma}$,

$$\widetilde{\Sigma}(z)W(w) \sim (z-w)^{-1/2}V^f(w) + \cdots\,, \qquad \Sigma(z)V^f(w) \sim (z-w)^{-3/4}W(w) + \cdots \tag{C.6}$$

where $\cdots$ denotes less singular terms. The first equation in (C.6) can be taken to be the defining equation for V^f; the second equation then follows as a consequence.

We now consider the Ramond sector vertex operators

$$\mathcal{V}_\mathrm{F} : \bar{c}ce^{-\phi/2}V^f, \quad c\eta e^{\phi/2}\widetilde{\Sigma}\,. \tag{C.7}$$

One can easily show that these represent not-trivial elements of the BRST cohomology. Indeed, these are precisely the candidate goldstino states listed in (2.37). Furthermore the three point function of Λ_0, $\mathcal{V}_\mathrm{F}$ and $(\mathcal{V}_\mathrm{S})^c$ on a sphere can be shown to be non-zero, establishing that $\langle(\mathcal{V}_\mathrm{S})^c|c_0^-[\Lambda_0\mathcal{V}_\mathrm{F}]_0\rangle \neq 0$. Therefore $[\Lambda_0\mathcal{V}_\mathrm{F}]_0$ for $\mathcal{V}_\mathrm{F}$ given in (C.7) indeed produces the states $\mathcal{V}_\mathrm{S}$ given in (C.2) up to addition of BRST exact states. This shows that for every zero momentum scalar vertex operators $\mathcal{V}_\mathrm{S}$ in the -1 picture we can find a zero momentum fermionic vertex operator $\mathcal{V}_\mathrm{F}$ in the $-1/2$ picture satisfying (C.1).

D Vanishing of $|\tilde{\psi}_2\rangle$

For SO(32) heterotic string theory on a Calabi-Yau manifold, the order g_s^2 contribution to the vacuum solution, projected to the L_0^+ sector, is given by $|\tilde{\psi}_2\rangle$ satisfying (7.5):

$$Q_B|\tilde{\psi}_2\rangle = -\mathbf{P}\left(\frac{1}{2}\beta^2\, g_s^2[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0 + [\;]_1\right)\,. \tag{D.1}$$

In this appendix we shall show that each term on the right hand side of (D.1) vanishes. This in turn would imply that $|\tilde{\psi}_2\rangle$ can be taken to be zero.

We begin with the first term. In order to show that it vanishes, we need to show that for an arbitrary state $|A\rangle \in \mathcal{H}_T$ with $L_0^+ = 0$, ghost number 2 and picture number -1,

$$\langle A|c_0^-[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\rangle = \{\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}A\}_0 \tag{D.2}$$

vanishes. Using $\mathcal{V}_{\chi R} = \bar{c}ce^{-\phi}V_{\chi R}$, expressing $V_{\chi R}$ as $(V_\chi + V_{\chi^*})/\sqrt{2}$ and using the result (6.36) that V_χ and V_{χ^*} carries $\bar{J}$ charges 2 and -2 respectively, we see that A must carry $\bar{J}$ charge 0 or ± 4. It is easy to see that there are no states in $\mathcal{H}_T$ with $L_0^+ = 0$ and $\bar{J} = \pm 4$ since the $\bar{L}_0$ eigenvalue is bounded from below by $-1 + \bar{j}^2/6$ where $\bar{j}$ is the $\bar{J}$ eigenvalue. Therefore A must carry $\bar{J}$ charge 0. This in turn allows us to express (D.2) as

$$\{\mathcal{V}_\chi \mathcal{V}_{\chi^*} A\}_0 \,. \tag{D.3}$$

We also know from Lorentz invariance that A must be a Lorentz scalar for (D.3) to be non-zero. Finally if A is a pure gauge then the result vanishes trivially to order g_s^2 since $\mathcal{V}_\chi$ and $\mathcal{V}_{\chi^*}$ are BRST invariant operators. Therefore we need to focus on $\bar{J}$ neutral, scalar vertex operators A of ghost number 2 and picture number -1, representing physical or unphysical states. For this we use the classification of possible choices of A given in [19] (section 5.5)

1. A can be either a chiral or an anti-chiral operator of the form $\mathcal{V}_\sigma = \bar{c}ce^{-\phi}V_\sigma$ or $\mathcal{V}_{\sigma^*} = \bar{c}ce^{-\phi}V_{\sigma^*}$ as given in (6.22). In either case the ϕ charge conservation tells us that in the single PCO that has to be inserted in the definition of $\{\mathcal{V}_\chi \mathcal{V}_{\chi^*} A\}$ we must pick the $e^\phi T_F$ term. Furthermore Lorentz invariance guarantees that the possible non-vanishing contributions come from the internal part $T_F^{(int)} = T_F^+ + T_F^-$ of T_F. Now consider the insertion involving $e^\phi T_F^+(w)$. It follows from the operator product expansion of the $e^{q\phi}$'s given in (2.3) and the operator product of $T_F^\pm$ given in (6.25) that as a function of w, there is a zero at the location of $\mathcal{V}_\chi$ and no poles at the locations of $\mathcal{V}_{\chi^*}$ and A. Furthermore since $e^\phi T_F^+(w)$ is a dimension zero primary, as $w \to \infty$ the function goes to a constant. Such a function, having no poles and one zero in the complex plane, vanishes identically. Similar argument can be given for $e^\phi T_F^-$, with the roles of $\mathcal{V}_\chi$ and $\mathcal{V}_{\chi^*}$ interchanged.

2. A can be the zero momentum dilaton vertex operators $\bar{c}ce^{-\phi}\psi_\mu\bar{\partial}X^\mu$. In this case the correlation function on the sphere vanishes due to the presence of the single $\bar{\partial}X^\mu$ factor. (The PCO can supply factors of ∂X^μ but no factor of $\bar{\partial}X^\mu$.)

3. A can be $c\eta$ — the second operator listed in (C.2). In this case by ϕ charge conservation we must pick the $c\partial\xi$ term from the PCO. However the correlator now contains 4 factors of c and two factors of $\bar{c}$, and hence vanishes by separate ghost charge conservation in the holomorphic and anti-holomorphic sectors.

4. This essentially exhausts all physical state candidates for $|A\rangle$. So we now turn to the cases when $|A\rangle$ is unphysical. One class of candidates listed in [19] (section 5.5) have the form

$$A = (\partial c + \bar{\partial}\bar{c})ce^{-\phi}V \,, \tag{D.4}$$

for some GSO odd matter sector vertex operator V of dimension $(0, 1/2)$. In the SO(32) heterotic string on a Calabi-Yau 3-fold there are no such operators that are Lorentz scalars.

5. Among the other class of unphysical operators listed in section 5.5 of [19] there is only one candidate that is neutral under $\bar{J}$:

$$A = (\partial c + \bar{\partial}\bar{c})c\partial\xi e^{-2\phi}\bar{c}\bar{J}\,. \tag{D.5}$$

Note that this operator is not BRST invariant but is still a dimension zero matter primary. Hence $\{\mathcal{V}_\chi\mathcal{V}_{\chi^*}A\}$ can be computed without specifying the choice of local coordinates at the punctures. Let $(z_3, \bar{z}_3)$ denote the argument of A, $(z_1, \bar{z}_1)$ denote the argument of $\mathcal{V}_\chi$, $(z_2, \bar{z}_2)$ denote the argument of $\mathcal{V}_{\chi^*}$ and w be the argument of the PCO. Then the amplitude is given by

$$\langle \mathcal{X}(w)A(z_3)\mathcal{V}_\chi(z_1)\mathcal{V}_{\chi^*}(z_2)\rangle\,. \tag{D.6}$$

Since A, $\mathcal{V}_\chi$, $\mathcal{V}_{\chi^*}$ are proportional to $e^{-2\phi}$, $e^{-\phi}$ and $e^{-\phi}$ respectively, ϕ charge conservation tells us that we must pick the term proportional to $e^{2\phi}$ from the PCO and hence there is no matter operator coming from the PCO. This allows us to evaluate the matter part of the correlator easily. First using (6.36) and a standard argument in CFT, we can remove the $\bar{J}$ from the correlation function at the cost of multiplying the result by a factor of

$$2\left(\frac{1}{\bar{z}_3 - \bar{z}_1} - \frac{1}{\bar{z}_3 - \bar{z}_2}\right)\,. \tag{D.7}$$

The rest of the correlation function in the matter sector can be easily evaluated as it involves a two point function on a sphere of operators of dimension (1,1/2). This gives a factor of $(z_1 - z_2)^{-1}(\bar{z}_1 - \bar{z}_2)^{-2}$. The net result, after combining with (D.7), is

$$2(z_1 - z_2)^{-1}(\bar{z}_1 - \bar{z}_2)^{-1}(\bar{z}_1 - \bar{z}_3)^{-1}(\bar{z}_2 - \bar{z}_3)^{-1}\,. \tag{D.8}$$

The important point to note is that this is symmetric under the exchange of $(z_1, \bar{z}_1)$ with $(z_2, \bar{z}_2)$. On the other hand the rest of the correlator involving the ghost insertions is of the form

$$\langle \mathcal{X}(w)\,(\partial c + \bar{\partial}\bar{c})c\partial\xi e^{-2\phi}\bar{c}(z_3)\,\bar{c}ce^{-\phi}(z_1)\,\bar{c}ce^{-\phi}(z_2)\rangle\,. \tag{D.9}$$

Note now that this is anti-symmetric under the exchange of z_1 and z_2. Combining this with the matter sector result we see that the net result is anti-symmetric under the exchange of $(z_1, \bar{z}_1)$ and $(z_2, \bar{z}_2)$. However the vertex is supposed to be symmetrized under the exchange of the external states, and hence under the exchange of z_1, z_2, z_3 keeping the external states fixed. This makes the contribution vanish. Indeed, if we had started with the amplitude (D.2) so that we have the three point amplitude of A, $\mathcal{V}_{\chi R}$ and $\mathcal{V}_{\chi R}$ placed at z_3, z_1 and z_2, and then expressed $\mathcal{V}_{\chi R}$ as $(\mathcal{V}_\chi + \mathcal{V}_{\chi^*})/\sqrt{2}$, we would have arrived at the average of two terms — the amplitude (D.6) and another one related to it by the exchange of z_1 and z_2. This vanishes due to the anti-symmetry of (D.6) under $z_1 \leftrightarrow z_2$ exchange.

Let us now turn to the second term on the right hand side of (D.1). We need to analyze, for an arbitrary $L_0^+ = 0$ state $|A\rangle$, the quantity

$$\langle A|c_0^-|[\ ]_1\rangle = \{A\}_1 \,. \tag{D.10}$$

This is one point function of A on the torus. For physical states described by chiral scalar vertex operators of the form $\mathcal{V}_\sigma$ or $\mathcal{V}_{\sigma^*}$, or the dilaton vertex operators $\bar{c}ce^{-\phi}\psi_\mu\bar{\partial}X^\mu$ or $c\eta$, this vanishes due to the result of section 5 and the existence of global supersymmetry at tree level string theory. The result also vanishes trivially for pure gauge states due to the identity (2.23). The only remaining candidate for A is the operator described in (D.5). We shall now argue that the one point function of (D.5) on the torus also vanishes. Since A given in (D.5) carries picture number -1, we have one PCO insertion on the torus and the ϕ charge conservation tells us that we need to pick up the term involving $e^{2\phi}$ from the PCO. Therefore the PCO insertion involves purely ghost operators. Let us consider the contribution from a given spin structure. Since for a given spin structure the correlation function in the matter and ghost sectors factorize, the one point function of A given in (D.5) can be regarded as the product of the one point function of $\bar{J}$ in the matter sector and an appropriate correlation function in the ghost sector. Due to translation invariance on the torus the one point function of $\bar{J}$ in the matter sector is independent of its location u. Now under $u \to -u$ transformation the spin structure remains unchanged but $\bar{J}$ changes sign. This shows that the one point function of $\bar{J}$ in the matter sector must vanish separately for each spin structure.

This establishes the vanishing of each term on the right hand side of (D.1), and in turn shows that $|\tilde{\psi}_2\rangle$ can be taken to vanish.

E Uniqueness of the goldstino candidate

For SO(32) heterotic string theory on Calabi-Yau manifolds, we derived in (7.22) the condition for unbroken supersymmetry to order g_s^2:

$$\begin{aligned}
&\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\}_1 - \left\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\ ]_1\right)\right\}_0 \\
&-\frac{1}{2}\beta^2 g_s^2\left\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})[\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}]_0\right)\right\}_0 + \frac{1}{2}\beta^2 g_s^2\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\lambda_0\mathcal{V}_{\chi R}\mathcal{V}_{\chi R}\}_0 \\
&-\beta^2 g_s^2\left\{(\mathcal{X}_0\mathcal{V}_{\rm G}^c)\mathcal{V}_{\chi R}\left(b_0^+(L_0^+)^{-1}(1-\mathbf{P})\mathcal{X}_0[\mathcal{V}_{\chi R}\lambda_0]_0\right)\right\}_0 \\
&\quad = \mathcal{O}(g_s^3)\,.
\end{aligned} \tag{E.1}$$

The condition must hold for each candidate state $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ listed in (2.38). In this appendix we shall show that there is only one $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ given in (7.23) for which the different terms in (E.1) are not zero. This is important for establishing the existence of global supersymmetry to this order, since otherwise by adjusting a single constant β we may not be able to make the left hand side of (E.1) vanish to order g_s^2 for different choices of $\mathcal{V}_{\rm G}^c$.

For our analysis it will be useful to note that since $\mathcal{V}_{\chi R}$ carries picture number -1 and λ_0 and $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ carry picture number $-1/2$, each of the terms in the left hand side of (E.1) requires insertion of one PCO. First let us consider the second candidate for

$\mathcal{X}_0\mathcal{V}_{\rm G}^c$ listed in (2.38), given by $\bar c c\bar\partial^2\bar c\, e^{-\phi/2}\,\hat\Sigma$. Now in each of the correlation functions appearing in (E.1), the only vertex operators carrying four dimensional Lorentz indices are $\lambda_0 = ce^{-\phi/2}\Sigma$ and $\mathcal{X}_0\mathcal{V}_{\rm G}^c$. Since the Lorentz invariant tensors are $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$, Σ appearing in λ_0 and $\hat\Sigma$ appearing in $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ must either both carry dotted indices or both carry undotted indices. Since both Σ and $\hat\Sigma$ are to be chosen from the list (6.8) we see that they must either both carry $\Sigma^{(6)}$ or both carry $\Sigma^{(6)c}$. Expressing $\mathcal{V}_{\chi R}$ as $(\mathcal{V}_\chi + \mathcal{V}_{\chi^*})/\sqrt{2}$ and using the U(1) charge assignments under the left and right handed U(1) currents J and $\bar J$ given in (6.6), (6.24) and (6.36) it is easy to check that none of the terms in (E.1) can satisfy both charge conservations even after including the effect of a possible factor of T_F coming from one PCO that needs to be inserted into the correlation functions. Hence the left hand side of (E.1) vanishes identically for this choice of $\mathcal{X}_0\mathcal{V}_{\rm G}^c$.

Next we turn to the first candidate for $\mathcal{X}_0\mathcal{V}_{\rm G}^c$ in (2.38) given by $\bar c c\eta e^{\phi/2}V^f$ where V^f is a dimension (1,5/8) Ramond vertex operator in the matter sector. Let us for definiteness assume that Σ appearing in λ_0 carries an undotted index, i.e. has the form $ce^{-\phi/2}\Sigma_\alpha$; an identical analysis can be carried out in the other case. In this case by Lorentz invariance, V^f must also carry an undotted index. We shall now consider several possibilities:

1. First let us assume that V^f has the form $V^f_{\sigma^*,\alpha}$ where σ^* denotes some anti-chiral superfield and $V^f_{\sigma^*,\alpha}$ has been introduced in (6.28). Note that the 'wrong' assignment, $V^f_{\sigma^*,\alpha}$ (instead of $V^f_{\sigma^*,\dot\alpha}$ or $V^f_{\sigma,\alpha}$ as in (6.29)) is necessary due to GSO projection rules since V^f is accompanied by the operator $\eta e^{\phi/2}$, which has opposite GSO parity compared to $e^{-\phi/2}$. Since according to (6.6), (6.35) $\Sigma^{(6)}$ carries J-charge 3/2 and $V^f_{\sigma^*}$ carries J-charge 1/2, the total J-charge carried by λ_0 and $\mathcal{V}_{\rm G}^c$ is $3/2+1/2=2$. Since the ϕ charges in each correlator in (E.1) add up to the correct value (0 on the torus and -2 on the sphere), we need to pick the $c\partial\xi$ term from the PCO which has no J-charge. Finally among the two $\mathcal{V}_{\chi R}$ insertions in the various terms in (E.1), the $\bar J$ charge conservation tells us that one of them must be $\mathcal{V}_\chi$ and the other one $\mathcal{V}_{\chi^*}$ once we express $\mathcal{V}_{\chi R}$ as $(\mathcal{V}_\chi+\mathcal{V}_{\chi^*})/\sqrt{2}$.[18] Therefore their J-charges also cancel according to (6.24). This tells us that we have a net J-charge 2 carried by all the operators in the correlation function. Hence such a contribution must vanish.

2. Next we consider the case where V^f has the form $\widetilde\Sigma^c_\alpha \bar J^a$ where $\bar J^a$ is some anti-holomorphic current in the matter SCFT associated with the compact directions. Again although this appears to have the 'wrong GSO parity' ($\widetilde\Sigma^c_\alpha$ instead of Σ_α), this is what is needed to compensate for the fact that $\eta e^{\phi/2}$ has opposite GSO charge compared to $e^{-\phi/2}$. Now we see that the total J charge of λ_0 and $\mathcal{V}_{\rm G}^c$ add up to $3/2-3/2=0$ and hence this matrix element can be non-zero. Indeed, for the choice $\bar J^a=\bar J$ this precisely gives the operator (7.23) that gives the non-vanishing contribution to (E.1). However as long as there are no other U(1) gauge groups in the theory, all other $\bar J^a$'s must be conserved currents associated with non-abelian groups.

[18] Had we picked two $\mathcal{V}_\chi$'s or two $\mathcal{V}_{\chi^*}$'s the total $\bar J$ charge carried by them would be ± 4. To compensate for this $V^f_{\sigma^*,\alpha}$ would have to carry $\bar J$ charge ∓ 4 which will give a lower bound of 8/3 to its $\bar L_0$ eigenvalue.

The matrix elements of such operators will vanish by the associated non-abelian global symmetry of the world-sheet theory.

3. Finally consider the case where $V^f = \bar{\partial}X^\mu(\gamma_\mu)_\alpha{}^{\dot{\beta}}\widetilde{\Sigma}_{\dot{\beta}}$. In this case since ϕ charge conservation requires us to pick the $c\partial\xi$ term from the PCO, there is no other ∂X^μ or $\bar{\partial}X^\mu$ insertions in the correlation function. As a result the correlator involving $\bar{\partial}X^\mu$ vanishes.

This shows that the operator given in (7.23) is the only operator that could contribute to any of the terms on the left hand side of (E.1).

F Calculation of $G^{(2,2)}(0,\chi_R;0,\chi_R)$

In order to compute the shift β of the scalar field using (7.12), we need two quantities: the tree level four point function $G^{(4,0)}(0,\chi_R;0,\chi_R;0,\chi_R;0,\chi_R)$ and the one loop two point function $G^{(2,2)}(0,\chi_R;0,\chi_R)$. The former was calculated in section 8.1 (eq. (8.37)) while the latter was calculated in [19, 36, 37, 48, 49]. In this short appendix we shall review the result for the latter in the normalization convention of this paper.

We shall use the result of [19] whose conventions we are using here. There are however some additional normalization factors we have to account for. First, using $\chi_R = (\chi + \chi^*)/\sqrt{2}$ we can write

$$G^{(2,2)}(0,\chi_R;0,\chi_R) = G^{(2,2)}(0,\chi;0,\chi^*)\,. \tag{F.1}$$

This is the amplitude given in eq. (7.13) of [19] with V_1 identified as V_χ and V_2 identified as V_{χ^*}. There are however two additional normalization factors to be accounted for:

1. The normalization factor of $(2\pi i)^{-3g+3-n}$ gives a factor of $-1/4\pi^2$ for $g=1$, $n=2$. This was not explicitly included in (7.13) of [19].

2. The $d^2\tau d^2y$ in eq. (7.13) of [19] should actually be $d\tau\wedge d\bar{\tau}\wedge dy\wedge d\bar{y}$. This translates to $-4d^2\tau d^2y$.

Together these give an additional factor of $1/\pi^2$. With this normalization factor included the final result (7.34) of [19] translates to

$$\frac{q}{4\pi}\int_{\mathcal{F}} d^2\tau\Big\langle b_\tau\bar{b}_{\bar{\tau}}\bar{c}(0)c(0)V_D(0)\Big\rangle \tag{F.2}$$

where q denotes the $\bar{J}$ charge carried by χ and V_D is a dimension (1,1) operator that appears in the operator product of V_χ and V_{χ^*} (eq. (7.30) of [19]):

$$V_\chi(0)V_{\chi^*}(y) = \frac{q}{\bar{y}}V_D(0)\,. \tag{F.3}$$

According to (6.36) we have $q=2$, and so

$$G^{(2,2)}(0,\chi_R;0,\chi_R) = \frac{1}{2\pi}\int_{\mathcal{F}} d^2\tau\Big\langle b_\tau\bar{b}_{\bar{\tau}}\bar{c}(0)c(0)V_D(0)\Big\rangle\,. \tag{F.4}$$

The relevant operator V_D is in fact proportional to $J\bar{J}$. To find the constant of proportionality, let $V_D = \lambda\, J\bar{J}$. Consider now the three point function on the sphere:

$$\langle J\bar{J}(w)V_\chi(z)V_{\chi^*}(y)\rangle_{\rm sphere}\,. \tag{F.5}$$

Using the results (6.24), (6.36) that V_χ and V_{χ^*} carry $(J,\bar{J})$ chargers $(1,2)$ and $(-1,-2)$ respectively, we can express this correlator as

$$\begin{aligned}
&2\left(\frac{1}{w-z}-\frac{1}{w-y}\right)\left(\frac{1}{\bar{w}-\bar{z}}-\frac{1}{\bar{w}-\bar{y}}\right)\langle V_\chi(z)V_{\chi^*}(y)\rangle_{\rm sphere}\\
&\quad=2|z-y|^2|w-z|^{-2}|w-y|^{-2}(z-y)^{-1}(\bar{z}-\bar{y})^{-2}\\
&\quad=2|w-z|^{-2}|w-y|^{-2}(\bar{z}-\bar{y})^{-1}\,,
\end{aligned}\tag{F.6}$$

where we have used the normalization (6.23) of V_χ, V_{χ^*}. On the other hand using (F.3) we see that in the $y\to z$ limit (F.5) has the form

$$q(\bar{y}-\bar{z})^{-1}\langle J(w)\bar{J}(\bar{w})V_D(z)\rangle = q\,\lambda(\bar{y}-\bar{z})^{-1}\langle J(w)\bar{J}(\bar{w})J(z)\bar{J}(\bar{z})\rangle = 9\,q\,\lambda(\bar{y}-\bar{z})^{-1}|w-z|^{-4}\,. \tag{F.7}$$

Using $q=2$ and comparing this with the $y\to z$ limit of (F.6) we get

$$\lambda = -\frac{1}{9}\,. \tag{F.8}$$

Therefore we can rewrite (F.4) as

$$G^{(2,2)}(0,\chi_R;0,\chi_R) = -\frac{1}{18\pi}\int_{\mathcal{F}} d^2\tau\Big\langle b_\tau\bar{b}_{\bar{\tau}}\bar{c}(0)c(0)J(0)\bar{J}(0)\Big\rangle = -\frac{1}{2\sqrt{3}}\Xi\,, \tag{F.9}$$

where Ξ has been defined in (9.25).

Using (8.8) we now get

$$m_B^2 = 8g_s^2 G^{(2,2)}(0,\chi_R;0,\chi_R) = -\frac{4}{\sqrt{3}}\,g_s^2\,\Xi\,. \tag{F.10}$$

G Consistency with low energy effective field theory

In this appendix we shall verify that in SO(32) heterotic string theory on Calabi-Yau manifolds, the results for the tadpoles at the perturbative vacuum given in (9.53) and the renormalized scalar mass^2 given in (F.10) are all consistent with the prediction of low energy supersymmetric field theory. This will be done in several steps.

Our first step will be to identify the metric field in the string field $|\Psi\rangle$. For this consider the configuration

$$\Psi = \tilde{h}_{\mu\nu}\bar{c}ce^{-\phi}\psi^\mu\bar{\partial}X^\nu\,, \tag{G.1}$$

where $\tilde{h}_{\mu\nu}$ are constants. Then following the analysis of section 3.3 and section 3.4 one can show that to linear order, shifting the background by (G.1) will correspond to the insertion of

$$-\frac{1}{2\pi}\int d^2z\,\tilde{h}_{\mu\nu}\,\partial X^\mu\bar{\partial}X^\nu \tag{G.2}$$

into the world-sheet correlation function. Here the $-1/2$ factor arises from the picture changing operation given in (3.38) and the $1/\pi$ is the same factor that appears by combining (3.23) and (3.25). On the other hand the world-sheet action involving the flat space-time coordinates in the $\alpha' = 1$ unit has the form

$$
\begin{aligned}
S_w &= \frac{1}{4\pi} \int d^2\,\xi\,\partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} = \frac{1}{\pi} \int d^2 z\,\partial X^\mu \bar\partial X^\nu \eta_{\mu\nu}\,, \\
z &\equiv \xi^1 + i\xi^2, \quad \bar z \equiv \xi^1 - i\xi^2, \quad d^2 z \equiv d\xi^1 d\xi^2, \quad \partial \equiv \frac{\partial}{\partial z}, \quad \bar\partial \equiv \frac{\partial}{\partial \bar z}\,.
\end{aligned} \tag{G.3}
$$

We now interpret the insertion of (G.2) as the addition of a term to $-S_w$ that appears in the exponent in the world-sheet path integral. This amounts to replacing $\eta_{\mu\nu}$ in (G.3) by $\eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} = \tilde h_{\mu\nu}/2$. Therefore to linear order in $h_{\mu\nu}$, the space-time metric $g_{\mu\nu}$ can be identified as $\eta_{\mu\nu} + h^S_{\mu\nu}$ and the background 2-form field can be identified as $h^A_{\mu\nu}$ where the superscripts S and A stand for symmetric and anti-symmetric parts respectively. For our analysis we shall be interested in the symmetric part of h only.

Our next task will be to identify the dilaton field. For this we consider a more general string field configuration of the form

$$
\begin{aligned}
\Psi = \Big[& 2h_{\mu\nu}\bar c c e^{-\phi}\psi^\mu \bar\partial X^\nu + \phi_1 \bar c \bar\partial^2 \bar c c \partial\xi e^{-2\phi} + \phi_2 c\eta + a_\nu(\partial c + \bar\partial \bar c)\bar c c \partial \xi e^{-2\phi}\bar\partial X^\nu \\
& + b_\mu(\partial c + \bar\partial\bar c) c e^{-\phi}\psi^\mu \Big] e^{ik\cdot X}\,,
\end{aligned} \tag{G.4}
$$

where $h_{\mu\nu}$, ϕ_1, ϕ_2, a_ν and b_μ are constants. By the analysis of the previous paragraph $\eta_{\mu\nu} + h_{\mu\nu}$ represents the metric $g_{\mu\nu}$ to linear order in $h_{\mu\nu}$. Now the linearized equations of motion $Q_B|\Psi\rangle = 0$ give

$$
\begin{aligned}
\frac{k^2}{2} h_{\mu\nu} - ik_\nu b_\mu - \frac{i}{2} k_\mu a_\nu &= 0 \\
\frac{k^2}{4}\phi_2 + \frac{i}{4} k^\mu b_\mu &= 0 \\
\frac{k^2}{4}\phi_1 - \frac{i}{4} k^\nu a_\nu &= 0 \\
ik_\nu \phi_2 + \frac{i}{2} k^\mu h_{\mu\nu} + \frac{1}{2} a_\nu &= 0 \\
b_\mu + \frac{i}{2} k^\rho h_{\mu\rho} - \frac{i}{2} k_\mu \phi_1 &= 0\,.
\end{aligned} \tag{G.5}
$$

Furthermore linearized gauge transformation $\delta|\Psi\rangle = Q_B|\Lambda\rangle$ leads to the following gauge symmetries

$$
\begin{aligned}
\delta h_{\mu\nu} &= i\epsilon_\mu k_\nu + ik_\mu \xi_\nu \\
\delta\phi_1 &= ik\cdot\xi + \zeta \\
\delta\phi_2 &= -\frac{i}{2} k\cdot\epsilon + \frac{1}{2}\zeta \\
\delta b_\mu &= \frac{k^2}{2}\epsilon_\mu + \frac{i}{2} k_\mu \zeta \\
\delta a_\nu &= k^2 \xi_\nu - ik_\nu \zeta\,,
\end{aligned} \tag{G.6}
$$

where ϵ_μ, ξ_ν and ζ are gauge transformation parameters. We shall from now on take $h_{\mu\nu}$ to be symmetric and set ϕ_1 to 0 using the ζ gauge transformations. The gauge symmetries of the resulting theory are generated by ξ_μ, with the other parameters fixed by the constraints $\epsilon_\mu = \xi_\mu$, $\zeta = -ik\cdot\xi$. Eliminating a_ν and b_μ using the last two equations in (G.5) and combining other equations we get

$$\begin{aligned} \frac{k^2}{2}h_{\mu\nu} - \frac{1}{2}k_\nu k^\rho h_{\mu\rho} - \frac{1}{2}k_\mu k^\rho h_{\rho\nu} - k_\mu k_\nu \phi_2 &= 0\,, \\ k^2\phi_2 + \frac{1}{2}k^\rho k^\sigma h_{\rho\sigma} &= 0\,. \end{aligned} \tag{G.7}$$

We now compare this with the linearized equations of motion derived from the low energy effective action involving the metric and the dilaton. Up to a constant of proportionality this action takes the form

$$\int d^4x\,\sqrt{-\det g}\,e^{-2\Phi}(R + 4D_\mu\Phi D^\mu\Phi) \tag{G.8}$$

leading to the following linearized equations of motion around $g_{\mu\nu} = \eta_{\mu\nu}$, $e^{2\Phi} = g_s^2$:

$$\begin{aligned} \frac{k^2}{2}h_{\mu\nu} - \frac{1}{2}k_\nu k^\rho h_{\mu\rho} - \frac{1}{2}k_\mu k^\rho h_{\rho\nu} + \frac{1}{2}k_\mu k_\nu h^\rho{}_\rho - 2k_\mu k_\nu\widetilde{\Phi} &= 0\,, \\ k^2 h^\rho{}_\rho - k^\rho k^\sigma h_{\rho\sigma} - 4k^2\widetilde{\Phi} &= 0\,. \end{aligned} \tag{G.9}$$

Here $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and $\widetilde{\Phi} = \Phi - \ln g_s$. Comparing (G.7) and (G.9) we arrive at the identification

$$\phi_2 = 2\widetilde{\Phi} - \frac{1}{2}h^\rho{}_\rho\,. \tag{G.10}$$

We shall now consider a zero momentum string field configuration describing change in the background metric and dilaton fields by $h_{\mu\nu}$ and $\widetilde{\Phi}$ respectively. Since at zero momentum a_μ and b_μ vanish by the last two equations of motion in (G.5) and we have chosen to work in the $\phi_1 = 0$ gauge, we see from (G.4) and (G.10) that this corresponds to the state

$$\Psi_0 = 2h_{\mu\nu}\bar{c}ce^{-\phi}\psi^\mu\bar{\partial}X^\nu + \left(2\widetilde{\Phi} - \frac{1}{2}h^\rho{}_\rho\right)c\eta\,. \tag{G.11}$$

Using (9.53) we get

$$\mathcal{E}_4(\Psi_0) = 2\frac{1}{4}h^\rho{}_\rho\left(-\frac{1}{2}g_s^4\,\Xi^2\right) + \left(2\widetilde{\Phi} - \frac{1}{2}h^\rho{}_\rho\right)\left(-\frac{1}{4}g_s^4\,\Xi^2\right) = -\frac{1}{4}g_s^4\,\Xi^2\left(2\widetilde{\Phi} + \frac{1}{2}h^\rho{}_\rho\right). \tag{G.12}$$

We shall now compare this with the term in the action linear in fields at the perturbative vacuum of the low energy effective supersymmetric field theory. The relevant fields for our analysis will be a U(1) gauge field A_μ, its associated field strength $F_{\mu\nu}$ and a scalar field χ carrying charge e under this gauge field. We shall also keep the metric $g_{\mu\nu}$ and Φ as background fields. The action takes the form

$$-\frac{1}{4}\int d^4x\,\sqrt{-\det g}\,e^{-2\Phi}\left[(\partial_\mu\chi^* - ieA_\mu\chi^*)(\partial^\mu\chi + ieA^\mu\chi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(e\chi^*\chi - c\,e^{2\Phi})^2\right]. \tag{G.13}$$

The overall factor of $-1/4$ ensures that the scalar kinetic terms in the action are given by $-k^2/4$ as in the string field theory action. c is a constant giving the Fayet-Iliopoulos term. Since it is generated at one loop, we have a factor of $e^{2\Phi}$ multiplying it. Now if we expand the action around the perturbative vacuum $\chi = 0$, $A_\mu = 0$, $g_{\mu\nu} = \eta_{\mu\nu}$, $e^{2\Phi} = g_s^2$ in powers of $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and $\widetilde{\Phi} = \Phi - \ln g_s$, we get, to linear order

$$-\frac{1}{8}\, g_s^2 \left(2\widetilde{\Phi} + \frac{1}{2} h^\rho{}_\rho\right) c^2 \,. \tag{G.14}$$

We have used the convention stated below (2.8) that the volume of space-time has been normalized to 1. This has to be compared with $g_s^{-2}\mathcal{E}_4(\Psi_0)$ given in (G.12), with the g_s^{-2} factor representing the overall normalization factor of g_s^{-2} multiplying the action (3.1) which was not reflected in the definition (2.25) of the equation of motion $\mathcal{E}$. This gives

$$c^2 = 2\,\Xi^2 \,. \tag{G.15}$$

Note that at this stage we already have a non-trivial consistency check between the 1PI effective string field theory result and low energy effective field theory result since both (G.12) and (G.14) have the combination $\left(2\widetilde{\Phi} + \frac{1}{2} h^\rho{}_\rho\right)$.

By expanding the action (G.13) around the shifted vacuum $e\chi^*\chi = ce^{2\Phi}$ we see that the tadpoles of $h^\rho{}_\rho$ and Φ vanish there, in agreement with the result of section 10. Furthermore by expanding the action around this vacuum to quadratic order in χ we learn that the mass of χ at the shifted vacuum is given by

$${m_B}^2 = 2c\, g_s^2\, e \,. \tag{G.16}$$

This has to be compared with (F.10), but for this we need to determine the value of e. To do this, we note from (G.13) that in momentum space, the three point coupling of an external χ and a χ^* carrying momenta $\pm k$ and a zero momentum gauge field of polarization ϵ^μ is given by

$$-\frac{1}{2} e\, k^\mu \epsilon_\mu \,. \tag{G.17}$$

On the other hand, this can be computed in string theory from the 3-point function of a pair of χ, χ^* vertex operators and the zero momentum gauge field vertex operator

$$i\sqrt{\frac{2}{3}}\, \bar{c}ce^{-\phi}\psi^\mu \bar{J} \,, \tag{G.18}$$

where the $i\sqrt{\frac{2}{3}}$ factor has been included to ensure that the vertex operator satisfies the normalization condition (3.36). Therefore the required 3-point function is given by

$$A_3 = i\sqrt{\frac{2}{3}}\, \epsilon_\mu \left\langle \bar{c}ce^{-\phi}V_\chi e^{ik\cdot X}(z_1)\, \bar{c}ce^{-\phi}V_\chi^* e^{-ik\cdot X}(z_2)\, \bar{c}ce^{-\phi}\psi^\mu \bar{J}(z_3)\, \mathcal{X}(w) \right\rangle \,. \tag{G.19}$$

The PCO location w can be chosen arbitrarily. We shall take the limit $w \to z_3$. This gives

$$A_3 = -i\sqrt{\frac{1}{6}}\, \epsilon_\mu \left\langle \bar{c}ce^{-\phi}V_\chi e^{ik\cdot X}(z_1)\, \bar{c}ce^{-\phi}V_\chi^* e^{-ik\cdot X}(z_2)\, \bar{c}c\partial X^\mu \bar{J}(z_3) \right\rangle \,. \tag{G.20}$$

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We can now use (6.2), (6.36) to remove the $\partial X^\mu(z_3)$ and $\bar J(\bar z_3)$ factors at the cost of picking up the factors

$$-\frac{ik^\mu}{2}\left(\frac{1}{z_3-z_1}-\frac{1}{z_3-z_2}\right)\times 2\left(\frac{1}{\bar z_3-\bar z_1}-\frac{1}{\bar z_3-\bar z_2}\right) = -ik^\mu|z_3-z_1|^{-2}|z_3-z_2|^{-2}|z_1-z_2|^2\,. \tag{G.21}$$

This gives

$$\begin{aligned} A_3 &= -\sqrt{\frac{1}{6}}\,\epsilon_\mu k^\mu \Big\langle \bar c c e^{-\phi} V_\chi e^{ik\cdot X}(z_1)\,\bar c c e^{-\phi} V_\chi^* e^{-ik\cdot X}(z_2)\,\bar c c(z_3)\Big\rangle |z_3-z_1|^{-2}|z_3-z_2|^{-2}|z_1-z_2|^2 \\ &= -\sqrt{\frac{1}{6}}\,\epsilon_\mu k^\mu\,, \end{aligned} \tag{G.22}$$

where in the last step we have used (6.23) and the normalization condition (2.9) to evaluate the correlation function. Comparing this with (G.17) we get

$$e = \sqrt{\frac{2}{3}}\,. \tag{G.23}$$

(G.16) and (G.15) now gives

$$(m_B^2)^2 = 4c^2\, g_s^4\, e^2 = 4\times 2\, g_s^4\, \Xi^2 \times \frac{2}{3} = \frac{16}{3} g_s^4\, \Xi^2\,. \tag{G.24}$$

This agrees with the result given in (F.10).

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