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The Roman (k, k)-domatic number of a graph*

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Abstract

Let k be a positive integer. A *Roman k-dominating function* on a graph G is a labelling $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex with label 0 has at least k neighbors with label 2. A set $\{f_1, f_2, \dots, f_d\}$ of distinct Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \le 2k$ for each $v \in V(G)$, is called a *Roman (k, k)-dominating family* (of functions) on G. The maximum number of functions in a Roman (k, k)-dominating family on G is the *Roman (k, k)-domatic number* of G, denoted by $d_R^k(G)$. Note that the Roman $(1, 1)$-domatic number $d_R^1(G)$ is the usual Roman domatic number $d_R(G)$. In this paper we initiate the study of the Roman (k, k)-domatic number in graphs and we present sharp bounds for $d_R^k(G)$. In addition, we determine the Roman (k, k)-domatic number of some graphs. Some of our results extend those given by Sheikholeslami and Volkmann in 2010 for the Roman domatic number.

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domination number, Roman (k, k)-domatic number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. Consult [4, 15] for the notation and terminology which are not defined here.

Let k be a positive integer. A subset S of vertices of G is a *k-dominating set* if $|N_G(v) \cap S| \geq k$ for every $v \in V(G) - S$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. A *k-domatic partition* is a partition of V into k-dominating sets, and the *k-domatic number* $d_k(G)$ is the largest number of sets in a k-domatic partition. The k-domatic number was introduced by Zelinka [16]. Further results on the k-domatic number can be found in the paper [5] by Kämmerling and Volkmann. For a good survey on the domatic numbers in graphs we refer the reader to [1]. Recently more domatic parameters are studied (see for instance [10, 11, 12]).

Let $k \geq 1$ be an integer. Following Kämmerling and Volkmann [6], a *Roman k-dominating function* (briefly RkDF) on a graph G is a labelling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has at least k neighbors with label 2. The *weight* of a Roman k-dominating function is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The minimum weight of a Roman k-dominating function on a graph G is called the *Roman k-domination number*, denoted by $\gamma_{kR}(G)$. Note that the Roman 1-domination number $\gamma_{1R}(G)$ is the usual Roman domination number $\gamma_R(G)$. A *$\gamma_{kR}(G)$-function* is a Roman k-dominating function of G with weight $\gamma_{kR}(G)$. A Roman k-dominating function $f : V \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a k-dominating set when f is an RkDF, and since placing weight 2 at the vertices of a k-dominating set yields an RkDF, in [6], it was observed that

$$\gamma_k(G) \leq \gamma_{kR}(G) \leq 2\gamma_k(G). \tag{1.1}$$

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 2k$ for each $v \in V(G)$ is called a *Roman (k, k)-dominating family* (of functions) on G. The maximum number of functions in a

Roman (k,k)-dominating family (briefly R(k,k)D family) on G is the *Roman* (k,k)-*domatic number* of G, denoted by $d_R^k(G)$. The Roman (k,k)-domatic number is well-defined and

$$d_R^k(G) \geq 1 \tag{1.2}$$

for all graphs G since the set consisting of any RkDF forms an R(k,k)D family on G and if $k \geq 2$, then

$$d_R^k(G) \geq 2 \tag{1.3}$$

since the functions $f_i : V(G) \to \{0,1,2\}$ defined by $f_i(v) = i$ for each $v \in V(G)$ and $i = 1,2$ forms an R(k,k)D family on G of order 2. In the special case when $k = 1$, $d_R^1(G)$ is the Roman domatic number $d_R(G)$ investigated in [8] and has been studied in [9].

The definition of the Roman dominating function was given implicitly by Stewart [14] and ReVelle and Rosing [7]. Cockayne et al. [3] as well as Chambers et al. [2] have given a lot of results on Roman domination.

Our purpose in this paper is to initiate the study of the Roman (k,k)-domatic number in graphs. We first study basic properties and bounds for the Roman (k,k)-domatic number of a graph. In addition, we determine the Roman (k,k)-domatic number of some classes of graphs.

The next known results are useful for our investigations.

Proposition A (Kämmerling, Volkmann [6] 2009)**.** *Let $k \geq 1$ be an integer, and let G be a graph of order n. If $n \leq 2k$, then $\gamma_{kR}(G) = n$. If $n \geq 2k+1$, then $\gamma_{kR}(G) \geq 2k$.*

Proposition B (Kämmerling, Volkmann [6] 2009)**.** *Let G be a graph of order n. Then $\gamma_{kR}(G) < n$ if and only if G contains a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $\deg_H(v) \geq k$ for each $v \in X$.*

Proposition C (Kämmerling, Volkmann [6] 2009)**.** *If G is a graph of order n and maximum degree $\Delta \geq k$, then*

$$\gamma_{kR}(G) \geq \left\lceil \frac{2n}{\frac{\Delta}{k} + 1} \right\rceil .$$

Proposition D (Sheikholeslami, Volkmann [8] 2010)**.** *If G is a graph, then*

$$d_R(G) = 1$$

if and only if G is empty.

Proposition E (Sheikholeslami, Volkmann [8] 2010)**.** *If G is a graph of order $n \geq 2$, then $d_R(G) = n$ if and only if G is the complete graph on n vertices.*

Proposition F (Sheikholeslami, Volkmann [8] 2010)**.** *Let K_n be the complete graph of order $n \geq 1$. Then $d_R(K_n) = n$.*

Proposition G (Sheikholeslami, Volkmann [13]). *Let $K_{p,q}$ be the complete bipartite graph of order $p+q$ such that $q \geq p \geq 1$. Then $\gamma_{kR}(K_{p,q}) = p+q$ when $p < k$ or $q = p = k$, $\gamma_{kR}(K_{p,q}) = k+p$ when $p+q \geq 2k+1$ and $k \leq p \leq 3k$ and $\gamma_{kR}(K_{p,q}) = 4k$ when $p \geq 3k$.*

We start with the following observations and properties. The first observation is an immediate consequence of (1.3) and Proposition D.

Observation 1.1. *If G is a graph, then $d_R^k(G) = 1$ if and only if $k = 1$ and G is empty.*

Observation 1.2. *If G is a graph and $k \geq 2$ is an integer, then $d_R^k(G) = 2$ if and only if G is trivial.*

Proof. If G is trivial, then obviously $d_R^k(G) = 2$. Now let G be nontrivial and let $v \in V(G)$. Define $f, g, h : V(G) \to \{0,1,2\}$ by

$$f(v) = 1 \text{ and } f(x) = 2 \text{ if } x \in V(G) - \{v\},$$

$$g(v) = 2 \text{ and } g(x) = 1 \text{ if } x \in V(G) - \{v\},$$

and

$$h(x) = 1 \text{ if } x \in V(G).$$

It is clear that $\{f, g, h\}$ is an R(k,k)D family of G and hence $d_R^k(G) \geq 3$. This completes the proof. □

Observation 1.3. *If G is a graph and $k \geq \Delta(G)+1$ is an integer, then $d_R^k(G) \leq 2k-1$.*

Proof. If $d_R^k(G) = 1$, then the statement is trivial. Let $d_R^k(G) \geq 2$. Since $k \geq \Delta(G)+1$, we have $\gamma_{kR}(G) = n$. Let $\{f_1, f_2, \ldots, f_d\}$ be an R(k,k)D family on G such that $d = d_R^k(G)$. Since $f_1, f_2, \ldots, f_d$ are distinct, we may assume $f_i(v) = 2$ for some i and some $v \in V(G)$. It follows from $\sum_{j=1}^{d} f_j(v) \leq 2k$ that $\sum_{j \neq i} f_j(v) \leq 2k-2$. Thus $d-1 \leq 2k-2$ as desired. □

Observation 1.4. *If $k \geq 2$ is an integer, and G is a graph of order $n \geq 2k-2$, then $d_R^k(G) \geq 2k-1$.*

Proof. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, then define $f_j : V(G) \to \{0,1,2\}$ by $f_j(v_j) = 2$ and $f_j(x) = 1$ for $x \in V(G) - \{v_j\}$ and $1 \leq j \leq 2k-2$ and $f_{2k-1} : V(G) \to \{0,1,2\}$ by $f_{2k-1}(x) = 1$ for each $x \in V(G)$. Then $f_1, f_2, \ldots, f_{2k-1}$ are distinct with $\sum_{i=1}^{2k-1} f_i(x) = 2k$ for each $x \in \{v_1, v_2, \ldots, v_{2k-2}\}$ and $\sum_{i=1}^{2k-1} f_i(x) = 2k-1$ otherwise. Therefore $\{f_1, f_2, \ldots, f_{2k-1}\}$ is an R(k,k)D family on G, and thus $d_R^k(G) \geq 2k-1$. □

The last two observations lead to the next result immediately.

Corollary 1.5. *Let $k \geq 2$ be an integer. If G is a graph of order $n \geq 2k-2$ and $k \geq \Delta(G)+1$, then $d_R^k(G) = 2k-1$.*

Observation 1.6. *If $k \geq 3$ is an integer, and G is a graph of order $n \geq 2k-4$, then $d_R^k(G) \geq 2k-2$.*

Proof. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, then define $f_j : V(G) \to \{0,1,2\}$ by $f_j(v_j) = 2$ and $f_j(x) = 1$ for $x \in V(G) - \{v_j\}$ and $1 \leq j \leq 2k-4$, $f_{2k-3} : V(G) \to \{0,1,2\}$ by $f_{2k-3}(x) = 1$ for each $x \in V(G)$ and $f_{2k-2} : V(G) \to \{0,1,2\}$ by $f_{2k-2}(x) = 2$ for each $x \in V(G)$. Then $f_1, f_2, \ldots, f_{2k-2}$ are distinct with $\sum_{i=1}^{2k-2} f_i(x) = 2k$ for each $x \in V(G)$. Therefore $\{f_1, f_2, \ldots, f_{2k-2}\}$ is an R(k,k)D family on G, and thus $d_R^k(G) \geq 2k-2$. □

Observation 1.7. *Let $k \geq 2$ be an integer. If G is a graph of order $n \leq 2k-3$ and $k \geq \Delta(G)+1$, then $d_R^k(G) \leq 2k-2$.*

Proof. If $n = 1$, then $d_R^k(G) = 2 \leq 2k-2$. Assume now that $n \geq 2$. Let $\{f_1, f_2, \ldots, f_d\}$ be an R(k,k)D family on G such that $d = d_R^k(G)$. Since $k \geq \Delta(G)+1$, we observe that $f_i(x) \geq 1$ for each $1 \leq i \leq d$ and each $x \in V(G)$. Suppose to the contrary that $d \geq 2k-1$. Since $f_1, f_2, \ldots, f_d$ are distinct, there exists a vertex $u \in V(G)$ such that $f_s(u) = f_t(u) = 2$ for two indices $s, t \in \{1, 2, \ldots, d\}$ with $s \neq t$. However, this leads to

$$\sum_{i=1}^{d} f_i(u) \geq \sum_{i=1}^{2k-1} f_i(u) \geq 4 + 2k - 3 = 2k+1,$$

a contradiction. Therefore $d_R^k(G) \leq 2k-2$, and the proof is complete. □

Theorem 1.8. *Let $k \geq 1$ be an integer, and let G be a graph of order n. If $k \geq 3 \cdot 2^{n-2}$, then $d_R^k(G) = 2^n$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be the set of all pairwise distinct functions from $V(G)$ into the set $\{1,2\}$. Then f_i is a Roman k-dominating function on G for $1 \leq i \leq d$, and it is well-known that $d = 2^n$. The hypothesis $k \geq 3 \cdot 2^{n-2}$ leads to

$$\sum_{i=1}^{d} f_i(v) = \sum_{i=1}^{2^n} f_i(v) = 2^{n-1} + 2^n = 3 \cdot 2^{n-1} \leq 2k$$

for each vertex $v \in V(G)$. Therefore $\{f_1, f_2, \ldots, f_d\}$ is an R(k,k)D family on G and thus $d_R^k(G) \geq 2^n$.

Now let $f : V(G) \longrightarrow \{0,1,2\}$ be a Roman k-dominating function on G. Since $k \geq 3 \cdot 2^{n-2} > n > \Delta(G)$, it is impossible that $f(x) = 0$ for any vertex $x \in V(G)$. Hence the number of Roman k-dominating functions on G is at most 2^n and so $d_R^k(G) \leq 2^n$. This yields the desired identity. □

Observation 1.9. *If $k \geq 1$ is an integer, then $\gamma_{kR}(K_n) = \min\{n, 2k\}$.*

Proof. If $n \leq 2k$, then Proposition A implies that $\gamma_{kR}(K_n) = n$.

Assume now that $n \geq 2k+1$. It follows from Proposition A that $\gamma_{kR}(K_n) \geq 2k$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, and define $f : V(K_n) \to \{0,1,2\}$ by $f(v_1) = f(v_2) = \ldots = f(v_k) = 2$ and $f(v_j) = 0$ for $k+1 \leq j \leq n$. Then f is an RkDF on K_n of weight $2k$ and thus $\gamma_{kR}(K_n) \leq 2k$, and the proof is complete. □

2. Properties of the Roman (k, k)-domatic number

In this section we present basic properties of $d_R^k(G)$ and sharp bounds on the Roman (k,k)-domatic number of a graph.

Theorem 2.1. *Let G be a graph of order n with Roman k-domination number $\gamma_{kR}(G)$ and Roman (k,k)-domatic number $d_R^k(G)$. Then*

$$\gamma_{kR}(G) \cdot d_R^k(G) \leq 2kn.$$

Moreover, if $\gamma_{kR}(G) \cdot d_R^k(G) = 2kn$, then for each $R(k,k)D$ family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_R^k(G)$, each function f_i is a $\gamma_{kR}(G)$-function and $\sum_{i=1}^d f_i(v) = 2k$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an R(k,k)D family on G such that $d = d_R^k(G)$ and let $v \in V$. Then

$$\begin{aligned} d \cdot \gamma_{kR}(G) &= \sum_{i=1}^{d} \gamma_{kR}(G) \\ &\leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ &= \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \\ &\leq \sum_{v \in V} 2k \\ &= 2kn. \end{aligned}$$

If $\gamma_{kR}(G) \cdot d_R^k(G) = 2kn$, then the two inequalities occurring in the proof become equalities. Hence for the R(k,k)D family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{kR}(G)$, thus each function f_i is a $\gamma_{kR}(G)$-function, and $\sum_{i=1}^d f_i(v) = 2k$ for all $v \in V$. □

Theorem 2.2. *Let G be a graph of order $n \geq 2$ and $k \geq 1$ be an integer. Then $\gamma_{kR}(G) = n$ and $d_R^k(G) = 2k$ if and only if G does not contain a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $\deg_H(v) \geq k$ for each $v \in X$ and G has $2k$ or $2k-1$ connected bipartite subgraphs $H_i = (X_i, Y_i)$ with $|X_i| = |Y_i|$, $\deg_{H_i}(v) \geq k$ for each $v \in X_i$ and $|\{i \mid u \in Y_i\}| = |\{i \mid u \in X_i\}| = k$ for each $u \in V(G)$.*

Proof. Let $\gamma_{kR}(G) = n$ and $d_R^k(G) = 2k$. It follows from Proposition B that G does not contain a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $\deg_H(v) \geq k$ for each $v \in X$. Let $\{f_1, \ldots, f_{2k}\}$ be a Roman (k,k)-dominating family on G. By Theorem 2.1, $\gamma_{kR}(G) = \omega(f_i) = n$ for each i. First suppose for each i, there exists a vertex x such that $f_i(x) \neq 1$. Assume that H_i is a subgraph of G with vertex set $V_0^{f_i} \cup V_2^{f_i}$ and edge set $E(V_0^{f_i}, V_2^{f_i})$. Since $\omega(f_i) = n$ and f_i is a Roman k-dominating function, $|V_2^{f_i}| = |V_0^{f_i}|$ and $\deg_{H_i}(v) \geq k$ for each $v \in V_0^{f_i}$. By Theorem 2.1, $\sum_{i=1}^{2k} f_i(v) = 2k$ for each $v \in V(G)$ which implies that $|\{i \mid v \in V_2^{f_i}\}| = |\{i \mid v \in V_0^{f_i}\}| = k$ for each $v \in V(G)$. Now suppose $f_i(x) = 1$ for each $x \in V(G)$ and some i, say $i = 2k$. Define the bipartite subgraphs H_i for $1 \leq i \leq 2k-1$ as above.

Conversely, assume that G does not contain a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $\deg_H(v) \geq k$ for each $v \in X$ and G has $2k$ or $2k-1$ connected bipartite subgraphs $H_i = (X_i, Y_i)$ with $|X_i| = |Y_i|$ and $\deg_{H_i}(v) \geq k$ for each $v \in X_i$. Then by Proposition B, $\gamma_{kR}(G) = n$. If G has $2k$ connected bipartite subgraphs H_i, then the mappings $f_i : V(G) \to \{0,1,2\}$ defined by

$$f_i(u) = 2 \text{ if } u \in Y_i,\ f_i(v) = 0 \text{ if } v \in X_i, \text{and } f_i(x) = 1 \text{ for each } x \in V - (X_i \cup Y_i)$$

are Roman k-dominating functions on G and $\{f_i \mid 1 \leq i \leq 2k\}$ is a Roman (k,k)-dominating family on G. If G has $2k-1$ connected bipartite subgraphs H_i, then the mappings $f_i, g : V(G) \to \{0,1,2\}$ defined by $g(x) = 1$ for each $x \in V(G)$ and

$$f_i(u) = 2 \text{ if } u \in Y_i,\ f_i(v) = 0 \text{ if } v \in X_i, \text{and } f_i(x) = 1 \text{ for each } x \in V - (X_i \cup Y_i)$$

are Roman k-dominating functions on G and $\{g, f_i \mid 1 \leq i \leq 2k-1\}$ is a Roman (k,k)-dominating family on G.

Thus $d_R^k(G) \geq 2k$. It follows from Theorem 2.1 that $d_R^k(G) = 2k$, and the proof is complete. □

The next corollary is an immediate consequence of Proposition C, Observation 1.3 and Theorem 2.1.

Corollary 2.3. *For every graph G of order n, $d_R^k(G) \leq \max\{\Delta, k-1\} + k$.*

Let $A_1 \cup A_2 \cup \ldots \cup A_d$ be a k-domatic partition of $V(G)$ into k-dominating sets such that $d = d_k(G)$. Then the set of functions $\{f_1, f_2, \ldots, f_d\}$ with $f_i(v) = 2$ if $v \in A_i$ and $f_i(v) = 0$ otherwise for $1 \leq i \leq d$ is an R(k,k)D family on G. This shows that $d_k(G) \leq d_R^k(G)$ for every graph G. Since $\gamma_{kR}(G) \geq \min\{n, \gamma_k(G)+k\}$ (cf. [6]), for each graph G of order $n \geq 2$, Theorem 2.1 implies that $d_R^k(G) \leq \frac{2kn}{\min\{n,\gamma_k(G)+k\}}$. Combining these two observations, we obtain the following result.

Corollary 2.4. *For any graph G of order n,*

$$d_k(G) \leq d_R^k(G) \leq \frac{2kn}{\min\{n, \gamma_k(G)+k\}}.$$

Theorem 2.5. *Let K_n be the complete graph of order n and k a positive integer. Then $d_R^k(K_n) = n$ if $n \geq 2k$, $d_R^k(K_n) \leq 2k-1$ if $n \leq 2k-1$ and $d_R^k(K_n) = 2k-1$ if $k \geq 2$ and $2k-2 \leq n \leq 2k-1$.*

Proof. By Proposition F, we may assume that $k \geq 2$. Assume that $V(K_n) = \{x_1, x_2, ..., x_n\}$. First let $n \geq 2k$. Since Observation 1.9 implies that $\gamma_{kR}(K_n) = 2k$, it follows from Theorem 2.1 that $d_R^k(K_n) \leq n$. For $1 \leq i \leq n$, define now $f_i : V(K_n) \to \{0, 1, 2\}$ by

$$f_i(x_i) = f_i(x_{i+1}) = \ldots = f_i(x_{i+k-1}) = 2 \text{ and } f_i(x) = 0 \text{ otherwise},$$

where the indices are taken modulo n. It is easy to see that $\{f_1, f_2, \ldots, f_n\}$ is an R$(k,k)D$ family on G and hence $d_R^k(K_n) \geq n$. Thus $d_R^k(K_n) = n$.

Now let $n \leq 2k-1$. Then Observation 1.9 yields $\gamma_{kR}(K_n) = n$, and it follows from Theorem 2.1 that $d_R^k(K_n) \leq 2k$. Suppose to the contrary that $d_R^k(K_n) = 2k$. Then by Theorem 2.1, each Roman k-dominating function f_i in any R(k,k)D family $\{f_1, f_2, , \ldots, f_{2k}\}$ on G is a $\gamma_{kR}(G)$-function. This implies that $f_i(x) = 1$ for each $x \in V(K_n)$. Hence $f_1 \equiv f_2 \equiv \cdots \equiv f_{2k}$ which is a contradiction. Thus $d_R^k(K_n) \leq 2k-1$.

In the special case $k \geq 2$ and $2k-2 \leq n \leq 2k-1$, Observation 1.4 shows that $d_R^k(K_n) \geq 2k-1$ and so $d_R^k(K_n) = 2k-1$. □

In view of Proposition G and Theorem 2.1 we obtain the next upper bounds for the Roman (k,k)-domatic number of complete bipartite graphs.

Corollary 2.6. *Let $K_{p,q}$ be the complete bipartite graph of order $p+q$ such that $q \geq p \geq 1$, and let k be a positive integer. Then $d_R^k(K_{p,q}) \leq 2k$ if $p < k$ or $q = p = k$, $d_R^k(K_{p,q}) \leq \frac{2k(p+q)}{k+p}$ if $p+q \geq 2k+1$ and $k \leq p \leq 3k$ and $d_R^k(K_{p,q}) \leq \frac{p+q}{2}$ if $p \geq 3k$.*

For some special cases of complete bipartite graphs, we can prove more.

Corollary 2.7. *Let $K_{p,p}$ be the complete bipartite graph of order $2p$, and let k be a positive integer. If $p \geq 3k$, then $d_R^k(K_{p,p}) = p$. If $p < k$, then $d_R^k(K_{p,p}) \leq 2k-1$. In particular, if $p = k-1$, then $d_R^k(K_{p,p}) = 2k-1$, and if $p = k-2$, then $d_R^k(K_{p,p}) = 2k-2$.*

Proof. Assume first that $p \geq 3k$. Let $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_p\}$ be the partite sets of the complete bipartite graph $K_{p,p}$. For $1 \leq i \leq p$, define $f_i : V(K_{p,p}) \to \{0, 1, 2\}$ by

$$f_i(u_i) = f_i(u_{i+1}) = \ldots = f_i(u_{i+k-1}) = f_i(v_i) = f_i(v_{i+1}) = \ldots = f_i(v_{i+k-1}) = 2$$

and $f_i(x) = 0$ otherwise, where the indices are taken modulo p. It is a simple matter to verify that $\{f_1, f_2, \ldots, f_p\}$ is an R$(k,k)D$ family on $K_{p,p}$ and hence $d_R^k(K_{p,p}) \geq p$. Using Corollary 2.6 for $p = q \geq 3k$, we obtain $d_R^k(K_{p,p}) = p$.

Assume next that $p < k$. Since $k > p = \Delta(K_{p,p})$, it follows from Observation 1.3 that $d_R^k(K_{p,p}) \leq 2k-1$.

Assume now that $p = k-1$. Then $k \geq 2$ and $n(K_{p,p}) = 2k-2$, and we deduce from Observation 1.4 that $d_R^k(K_{p,p}) \geq 2k-1$ and so $d_R^k(K_{p,p}) = 2k-1$.

Finally, assume that $p = k-2$. Then $k \geq 3$ and $n(K_{p,p}) = 2k-4$. It follows from Observation 1.6 that $d_R^k(K_{p,p}) \geq 2k-2$ and from Observation 1.7 that $d_R^k(K_{p,p}) \leq 2k-2$ and thus $d_R^k(K_{p,p}) = 2k-2$. □

Theorem 2.8. *If G is a graph of order $n \geq 2$, then*

$$\gamma_{kR}(G) + d_R^k(G) \leq n + 2k \tag{2.1}$$

with equality if and only if $\gamma_{kR}(G) = n$ and $d_R^k(G) = 2k$ or $\gamma_{kR}(G) = 2k$ and $d_R^k(G) = n$.

Proof. If $d_R^k(G) \leq 2k-1$, then obviously $\gamma_{kR}(G) + d_R^k(G) \leq n + 2k - 1$. Let now $d_R^k(G) \geq 2k$. If $\gamma_{kR}(G) \geq 2k$, Theorem 2.1 implies that $d_R^k(G) \leq n$. According to Theorem 2.1, we obtain

$$\gamma_{kR}(G) + d_R^k(G) \leq \frac{2kn}{d_R^k(G)} + d_R^k(G). \tag{2.2}$$

Using the fact that the function $g(x) = x + (2kn)/x$ is decreasing for $2k \leq x \leq \sqrt{2kn}$ and increasing for $\sqrt{2kn} \leq x \leq n$, this inequality leads to the desired bound immediately.

Now let $\gamma_{kR}(G) \leq 2k-1$. Since $\min\{n, \gamma_k(G) + k\} \leq \gamma_{kR}(G)$, we deduce that $\gamma_{kR}(G) = n$. According to Theorem 2.1, we obtain $d_R^k(G) \leq 2k$ and hence $d_R^k(G) = 2k$. Thus

$$\gamma_{kR}(G) + d_R^k(G) = n + 2k.$$

If $\gamma_{kR}(G) = n$ and $d_R^k(G) = 2k$ or $\gamma_{kR}(G) = 2k$ and $d_R^k(G) = n$, then obviously $\gamma_{kR}(G) + d_R^k(G) = n + 2k$.

Conversely, let equality hold in (2.1). It follows from (2.2) that

$$n + 2k = \gamma_{kR}(G) + d_R^k(G) \leq \frac{2kn}{d_R^k(G)} + d_R^k(G) \leq n + 2k,$$

which implies that $\gamma_{kR}(G) = \frac{2kn}{d_R^k(G)}$ and $d_R^k(G) = 2k$ or $d_R^k(G) = n$. This completes the proof. □

The special case $k = 1$ of the next result can be found in [8].

Theorem 2.9. *For every graph G and positive integer k,*

$$d_R^k(G) \leq \delta(G) + 2k.$$

Moreover, the upper bound is sharp.

Proof. If $d_R^k(G) \leq 2k$, the result is immediate. Let now $d_R^k(G) \geq 2k+1$ and let $\{f_1, f_2, \ldots, f_d\}$ be an R(k,k)D family on G such that $d = d_R^k(G)$. Assume that v is a vertex of minimum degree $\delta(G)$. Let ℓ be the number of sums $\sum_{u \in N[v]} f_i(u) = 1$ and let m be the number of those sums in which $\sum_{u \in N[v]} f_i(u) = 2$. Obviously, $l + 2m \leq 2k$.

We may assume, without loss of generality, that the equality $\sum_{u \in N[v]} f_i(u) = 1$ holds for $i = 1, \ldots, \ell$, if any, and the equality $\sum_{u \in N[v]} f_i(u) = 2$ holds for $i = \ell+1, \ldots, \ell+m$ when $m \geq 1$. In this case $f_i(v) = 1$ and $f_i(u) = 0$ for each

$u \in N(v)$ and $i = 1, \ldots, \ell$ and $f_i(v) = 2$ and $f_i(u) = 0$ for each $u \in N(v)$ and $i = \ell+1, \ldots, \ell+m$. Thus $f_i(v) = 0$ for $\ell+m+1 \leq i \leq d$, and thus $\sum_{u\in N[v]} f_i(u) \geq 2k$ for $\ell+m+1 \leq i \leq d$. Altogether we obtain

$$\begin{aligned} 2k(d-(\ell+m))+\ell+2m &\leq \sum_{i=1}^{d}\sum_{u\in N[v]} f_i(u) \\ &= \sum_{u\in N[v]}\sum_{i=1}^{d} f_i(u) \\ &\leq \sum_{u\in N[v]} 2k \\ &= 2k(\delta(G)+1). \end{aligned}$$

If $m = 0$, then the above inequality chain leads to

$$d \leq \delta(G)+1+\ell-\ell/(2k).$$

Since the function $g(x) = x + x/(2k)$ is increasing for $0 \leq x \leq 2k$, we deduce the desired bound as follows

$$d \leq \delta(G)+1+\ell-\ell/(2k) \leq \delta(G)+1+2k-(2k)/(2k) = \delta(G)+2k.$$

Now let $m \geq 1$. Then we obtain

$$d \leq \delta(G)+(\ell+m)+\frac{2k-\ell-2m}{2k}.$$

Since the last fraction in the sum is a rational number in $[0,1]$ and since $m \geq 1$, we deduce that

$$d \leq \delta(G)+(\ell+m)+\frac{2k-\ell-2m}{2k} \leq \delta(G)+(\ell+m)+1 \leq \delta(G)+\ell+2m \leq \delta(G)+2k$$

as desired.

To prove the sharpness of this inequality, let G_i be a copy of $K_{k^3+(2k+1)k}$ with vertex set $V(G_i) = \{v_1^i, v_2^i, \ldots, v_{k^3+(2k+1)k}^i\}$ for $1 \leq i \leq k$ and let the graph G be obtained from $\cup_{i=1}^k G_i$ by adding a new vertex v and joining v to each $v_1^i, \ldots, v_k^i$. Define the Roman k-dominating functions f_i^s, h_l for $1 \leq i \leq k, 0 \leq s \leq k-1$ and $1 \leq l \leq 2k$ as follows:

$$f_i^s(v_1^i) = \cdots = f_i^s(v_k^i) = 2,\ f_i^s(v^j_{(i-1)k^2+(s+1)k+1}) = \cdots = f_i^s(v^j_{(i-1)k^2+(s+1)k+k}) = 2$$
$$\text{if } j \in \{1,2,\ldots,k\}-\{i\} \text{ and } f_i^s(x) = 0 \ \text{ otherwise}$$

and for $1 \leq l \leq 2k$,

$$h_l(v) = 1, h_l(v^i_{k^3+lk+1}) = \ldots = h_l(v^i_{k^3+lk+k}) = 2\ (1 \leq i \leq k),$$

and $h_l(x) = 0$ otherwise.

It is easy to see that f_i^s and g_l are Roman k-dominating function on G for each $1 \le i \le k, 0 \le s \le k-1, 1 \le l \le 2k$ and $\{f_i^s, g_l \mid 1 \le i \le k, 0 \le s \le k-1 \text{ and } 1 \le l \le 2k\}$ is a Roman (k,k)-dominating family on G. Since $\delta(G) = k^2$, we have $d_R^k(G) = \delta(G) + 2k$. □

For regular graphs the following improvement of Theorem 2.9 is valid.

Theorem 2.10. *Let k be a positive integer. If G is a $\delta(G)$-regular graph, then*

$$d_R^k(G) \le \max\{2k-1, \delta(G)+k\} \le \delta(G) + 2k - 1.$$

Proof. If $k > \Delta(G) = \delta(G)$ then by Observation 1.7, $d_R^k(G) \le 2k-1$ and the desired bound is proved. If $k \le \Delta(G)$, then it follows from Corollary 2.3 that

$$d_R^k(G) \le \delta(G) + k,$$

and the proof is complete. □

As an application of Theorems 2.9 and 2.10, we will prove the following Nordhaus-Gaddum type result.

Theorem 2.11. *Let $k \ge 1$ be an integer. If G is a graph of order n, then*

$$d_R^k(G) + d_R^k(\overline{G}) \le n + 4k - 2, \tag{2.3}$$

with equality only for graphs with $\Delta(G) - \delta(G) = 1$.

Proof. It follows from Theorem 2.9 that

$$d_R^k(G) + d_R^k(\overline{G}) \le (\delta(G) + 2k) + (\delta(\overline{G}) + 2k) = (\delta(G) + 2k) + (n - \Delta(G) - 1 + 2k).$$

If G is not regular, then $\Delta(G) - \delta(G) \ge 1$, and hence this inequality implies the desired bound $d_R^k(G) + d_R^k(\overline{G}) \le n + 4k - 2$. If G is $\delta(G)$-regular, then we deduce from Theorem 2.10 that

$$d_R^k(G) + d_R^k(\overline{G}) \le (\delta(G) + 2k - 1) + (\delta(\overline{G}) + 2k - 1) = n + 4k - 3,$$

and the proof of the Nordhaus-Gaddum bound (2.3) is complete. Furthermore, the proof shows that we have equality in (2.3) only when $\Delta(G) - \delta(G) = 1$. □

Corollary 2.12 ([8]). *For every graph G of order n,*

$$d_R(G) + d_R(\overline{G}) \le n + 2,$$

with equality only for graphs with $\Delta(G) = \delta(G) + 1$.

For regular graphs we prove the following Nordhaus-Gaddum inequality.

Theorem 2.13. *Let $k \geq 1$ be an integer. If G is a δ-regular graph of order n, then*

$$d_R^k(G) + d_R^k(\overline{G}) \leq \max\{4k-2, n+2k-1, n+3k-2-\delta, 3k+\delta-1\}. \qquad (2.4)$$

Proof. Let $\delta(G) = \delta$ and $\delta(\overline{G}) = \overline{\delta}$. We distinguish four cases.

If $k \geq \delta + 1$ and $k \geq \overline{\delta} + 1$, then it follows from Observation 1.7 that

$$d_R^k(G) + d_R^k(\overline{G}) \leq (2k-1) + (2k-1) = 4k-2.$$

If $k \leq \delta$ and $k \leq \overline{\delta}$, then Corollary 2.3 implies that

$$d_R^k(G) + d_R^k(\overline{G}) \leq (\delta + k) + (\overline{\delta} + k) = \delta + 2k + n - 1 - \delta = n + 2k - 1.$$

If $k \geq \delta + 1$ and $k \leq \overline{\delta}$, then we deduce from Observation 1.7 and Corollary 2.3 that

$$d_R^k(G) + d_R^k(\overline{G}) \leq (2k-1) + (\overline{\delta} + k) = 3k - 1 + n - 1 - \delta = n + 3k - 2 - \delta.$$

If $k \leq \delta$ and $k \geq \overline{\delta} + 1$, then Observation 1.7 and Corollary 2.3 lead to

$$d_R^k(G) + d_R^k(\overline{G}) \leq (\delta + k) + (2k-1) = 3k + \delta - 1.$$

This completes the proof. □

If G is a δ-regular graph of order $n \geq 2$, then Theorem 2.13 leads to the following improvement of Theorem 2.11 for $k \geq 2$.

Corollary 2.14. *Let $k \geq 2$ be an integer. If G is a δ-regular graph of order $n \geq 2$, then*

$$d_R^k(G) + d_R^k(\overline{G}) \leq n + 4k - 4.$$

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