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Some inequalities for q -polygamma function and ζ_q -Riemann zeta functions

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Abstract

In this paper, we present some inequalities for q -polygamma functions and ζ_q -Riemann Zeta functions, using a q -analogue of Holder type inequality.

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1. Introduction and preliminaries

In this section, we provide a summary of notations and definitions used in this paper. For details, one may refer to [3, 5].

For $n = 1, 2, \dots$ we denote by $\psi_n(x) = \psi^{(n)}(x)$ the polygamma functions as the n -th derivative of the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$, where $\Gamma(x)$ denotes the usual gamma function.

Throughout this paper we will fix $q \in (0, 1)$. Let a be a complex number. The q -shifted factorials are defined by

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots,$$

$$(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n = \prod_{k \geq 0} (1 - aq^k).$$

Jackson [4] defined the q -gamma function as

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, \dots \quad (1.1)$$

It satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1, \quad (1.2)$$

where for x complex $[x]_q = \frac{1-q^x}{1-q}$.

The q -gamma function has the following integral representation (see [2])

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t = \int_0^{\frac{\infty}{1-q}} t^{x-1} E_q^{-qt} d_q t, \quad x > 0.$$

where $E_q^x = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]_q!} = (1 + (1-q)x)_q^\infty$, which is the q -analogue of the classical exponential function.

The q -analogue of the ψ function is defined as the logarithmic derivative of the q -gamma function

$$\psi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}, \quad x > 0. \quad (1.3)$$

The q -Jackson integral from 0 to a is defined by (see [4, 5])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n. \quad (1.4)$$

For $a = \infty$ the q -Jackson integral is defined by (see [4, 5])

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n \quad (1.5)$$

provided that sums in (1.4) and (1.5) converge absolutely.

In [2] the q -Riemman zeta function is defined as follows (see Section 2.3 for the definitions)

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}. \quad (1.6)$$

In relation to (1.3) and (1.6), K. Brahim [1], using a q -analogue of the generalized Schwarz inequality, proved the following Theorems.

Theorem 1.1. For $n = 1, 2, \dots$,

$$\psi_{q,n}(x) \psi_{q,m}(x) \geq \psi_{q, \frac{m+n}{2}}^2(x),$$

where $\psi_{q,n} = \psi_q^{(n)}$ is n -th derivative of ψ_q and $\frac{m+n}{2}$ is an integer.

Theorem 1.2. For all $s > 1$,

$$[s + 1]_q \frac{\zeta_q(s)}{\zeta_q(s + 1)} \geq q[s]_q \frac{\zeta_q(s + 1)}{\zeta_q(s + 2)}.$$

The aim of this paper is to present some inequalities for q -polygamma functions and q -zeta functions by using a q -analogue of Holder type inequality, similar to those in [1].

2. Main results

2.1. A lemma

In order to prove our main results, we need the following lemma.

Lemma 2.1. Let $a \in \mathbf{R}_+ \cup \{\infty\}$, let f and g be two nonnegative functions and let $p, t > 1$ such that $p^{-1} + t^{-1} = 1$. The following inequality holds

$$\int_0^a f(x)g(x)d_qx \leq \left(\int_0^a f^p(x)d_qx \right)^{\frac{1}{p}} \left(\int_0^a g^t(x)d_qx \right)^{\frac{1}{t}}.$$

Proof. Let $a > 0$. By (1.4) we have that

$$\int_0^a f(x)g(x)d_qx = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)g(aq^n)q^n. \quad (2.1)$$

By the use of the Holder's inequality for infinite sums, we obtain

$$\left(\sum_{n=0}^{\infty} f(aq^n)g(aq^n)q^n \right) \leq \left(\sum_{n=0}^{\infty} f^p(aq^n)q^n \right)^{\frac{1}{p}} \cdot \left(\sum_{n=0}^{\infty} g^t(aq^n)q^n \right)^{\frac{1}{t}}. \quad (2.2)$$

Hence

$$\begin{aligned} & (1 - q)a \left(\sum_{n=0}^{\infty} f(aq^n)g(aq^n)q^n \right) \\ & \leq ((1 - q)a)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} f^p(aq^n)q^n \right)^{\frac{1}{p}} \cdot ((1 - q)a)^{\frac{1}{t}} \left(\sum_{n=0}^{\infty} g^t(aq^n)q^n \right)^{\frac{1}{t}}. \end{aligned} \quad (2.3)$$

The result then follows from (2.1), (2.2) and (2.3). \square

2.2. The q -polygamma function

From (1.1) one can derive the following series representation for the function $\psi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}$:

$$\psi_q(x) = -\log(1 - q) + \log q \sum_{n \geq 1} \frac{q^{nx}}{1 - q^n}, \quad x > 0, \quad (2.4)$$

which implies that

$$\psi_q(x) = -\log(1-q) + \frac{\log q}{1-q} \int_0^q \frac{t^{x-1}}{1-t} d_q t. \quad (2.5)$$

Theorem 2.2. For $n = 2, 4, 6 \dots$ set $\psi_{q,n}(x) = \psi_q^{(n)}(x)$ the n -th derivative of the function ψ_q . Then for $p, t > 1$ such that $\frac{1}{p} + \frac{1}{t} = 1$ the following inequality holds

$$\psi_{q,n}\left(\frac{x}{p} + \frac{y}{t}\right) \leq \psi_{q,n}(x)^{\frac{1}{p}} \cdot \psi_{q,n}(y)^{\frac{1}{t}}. \quad (2.6)$$

Proof. From (2.5) we deduce that

$$\psi_{q,n}(x) = \frac{\log q}{1-q} \int_0^q \frac{(\log u)^n u^{x-1}}{1-u} d_q u, \quad (2.7)$$

hence

$$\psi_{q,n}\left(\frac{x}{p} + \frac{y}{t}\right) = \frac{\log q}{1-q} \int_0^q \frac{(\log u)^n u^{\frac{x}{p} + \frac{y}{t} - 1}}{1-u} d_q u.$$

By Lemma 2.1 with $a = q$ we have

$$\begin{aligned} \psi_{q,n}\left(\frac{x}{p} + \frac{y}{t}\right) &= \frac{\log q}{1-q} \int_0^q \left[\frac{(\log u)^n}{1-u}\right]^{\frac{1}{p}} u^{\frac{x-1}{p}} \left[\frac{(\log u)^n}{1-u}\right]^{\frac{1}{t}} u^{\frac{y-1}{t}} d_q u \\ &\leq \left(\frac{\log q}{1-q} \int_0^q \frac{(\log u)^n u^{x-1}}{1-u} d_q u\right)^{\frac{1}{p}} \left(\frac{\log q}{1-q} \int_0^q \frac{(\log u)^n u^{y-1}}{1-u} d_q u\right)^{\frac{1}{t}} \\ &= (\psi_{q,n}(x))^{\frac{1}{p}} (\psi_{q,n}(y))^{\frac{1}{t}} \end{aligned}$$

where $f(u) = \left(\frac{(\log u)^n}{1-u}\right)^p u^{\frac{x-1}{p}}$ and $g(u) = \left(\frac{(\log u)^n}{1-u}\right)^t u^{\frac{y-1}{t}}$. □

For $p = t = 2$ in (2.6) one has the following result.

Corollary 2.3. We have

$$\psi_{q,n}\left(\frac{x+y}{2}\right) \leq \sqrt{\psi_{q,n}(x) \cdot \psi_{q,n}(y)}.$$

2.3. q -zeta function

For $x > 0$ we set $\alpha(x) = \frac{\log x}{\log q} - E\left(\frac{\log x}{\log q}\right)$ and $\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}}$, where $E\left(\frac{\log x}{\log q}\right)$ is the integer part of $\frac{\log x}{\log q}$.

In [2] the q -zeta function is defined as follows

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}.$$

There ([2]) it is proved that ζ_q is a q -analogue of the classical Riemann Zeta function, and for all $s \in \mathbf{C}$ such that $\Re(s) > 1$, and for all $u > 0$ one has

$$\zeta_q(s) = \frac{1}{\tilde{\Gamma}_q(s)} \int_0^\infty u^{s-1} Z_q(u) d_q u,$$

where $Z_q(t) = \sum_{n=1}^\infty e_q^{-\{n\}_q t}$, $\tilde{\Gamma}_q(t) = \frac{\Gamma_q(t)}{K_q(t)}$, and

$$K_q(t) = \frac{(1-q)^{-s}}{1+(1-q)^{-1}} \cdot \frac{(-1-q)_\infty (-1-q)^{-1}; q)_\infty}{(-1-q)q^s; q)_\infty (-1-q)^{-1}q^{1-s}; q)_\infty}.$$

Theorem 2.4. For $\frac{1}{p} + \frac{1}{t} = 1$ and $\frac{x}{p} + \frac{y}{t} > 1$,

$$\frac{\tilde{\Gamma}_q\left(\frac{x}{p} + \frac{y}{t}\right)}{\tilde{\Gamma}_q^{\frac{1}{p}}(x) \cdot \tilde{\Gamma}_q^{\frac{1}{t}}(y)} \leq \frac{\zeta_q^{\frac{1}{p}}(x) \cdot \zeta_q^{\frac{1}{t}}(y)}{\zeta_q\left(\frac{x}{p} + \frac{y}{t}\right)}.$$

Proof. From Lemma 2.1 we have that

$$\begin{aligned} \int_0^\infty u^{\frac{x}{p} + \frac{y}{t} - 1} Z_q(u) d_q u &= \int_0^\infty u^{\frac{x-1}{p}} \cdot (Z_q(u))^{\frac{1}{p}} u^{\frac{y-1}{t}} \cdot (Z_q(u))^{\frac{1}{t}} d_q u \\ &\leq \left(\int_0^\infty u^{x-1} \cdot (Z_q(u)) d_q u \right)^{\frac{1}{p}} \cdot \left(\int_0^\infty u^{y-1} \cdot (Z_q(u)) d_q u \right)^{\frac{1}{t}}. \end{aligned}$$

For $f(u) = u^{\frac{x-1}{p}} \cdot (Z_q(u))^{\frac{1}{p}}$ and $g(u) = u^{\frac{y-1}{t}} \cdot (Z_q(u))^{\frac{1}{t}}$ we obtain that

$$\tilde{\Gamma}_q\left(\frac{x}{p} + \frac{y}{t}\right) \cdot \zeta_q\left(\frac{x}{p} + \frac{y}{t}\right) \leq \tilde{\Gamma}_q^{\frac{1}{p}}(x) \cdot \tilde{\Gamma}_q^{\frac{1}{t}}(y) \cdot \zeta_q^{\frac{1}{p}}(x) \cdot \zeta_q^{\frac{1}{t}}(y),$$

which completes the proof. □

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