

University of Bath



PHD

Machine Learning and Forward Looking Information in Option Prices

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Award date:
2018

Awarding institution:
University of Bath

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Machine Learning and Forward Looking Information in Option Prices

A thesis submitted for the degree of Doctor of Philosophy

University of Bath

submitted by

Qi Hu

September, 2018

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With great respect and appreciation,

*I would like to dedicate this thesis to my family, who encourage and support me
all the time.*

*I also would like to dedicate it to Professor David P. Newton who made this
work possible.*

Acknowledgements

Undertaking this PhD has been a long and life-changing experience for me. I could not have completed this without the support and encouragement from many people who I deeply appreciate in various ways.

First and foremost, I would like to thank my supervisor Professor David P. Newton. Without his encouragement and support, I probably would have given up along my PhD journey. His continuous emotional support, invaluable suggestions and friendship help me through the tough times. Also, I want to thank Professor Michael Tretyakov and Professor Ania Zalewska for their excellent guidance and supervision.

Further, I want to express my respect to Professor Stephen Ross, who proposed The Recovery Theorem. He will be remembered and missed by me. The implementation of The Recovery Theorem (or Ross Recovery) was a good start in my research. It has helped me to understand option pricing theory and asset pricing model fundamentally.

In addition, I am indebted to many of my colleagues for their friendship and support. I would like to thank: Soap (Haozhe Su), Tony (Hui Tian) and Harry (Yulin Wu). They are wonderful friends and the name order is random. Moreover, special thanks go to staff in the University of Bath and University of Nottingham. I am grateful for their help. I thank the University of Bath, particularly Professor Bernie Morley, for a tuition fee scholarship.

Last but not the least, I would like to thank my family, who always support

Acknowledgements

and love me. My PhD journey would be difficult without your love and support. I would like to express my deep appreciation to my husband Liang Liu, for his patience and continuous support.

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Abstract

The use of forward-looking information from option prices attracted a lot of attention after the 2008 financial crisis, which highlighting the difficulty of using historical data to predict extreme events. Although a considerable number of papers investigate extraction of forward-information from cross-sectional option prices, Figlewski (2008) argues that it is still an open question and none of the techniques is clearly superior.

This thesis focuses on getting information from option prices and investigates two broad topics: applying machine learning in extracting state price density and recovering natural probability from option prices. The estimation of state price density (often described as risk-neutral density in the option pricing literature) is of considerable importance since it contains valuable information about investors' expectations and risk preferences. However, this is a non-trivial task due to data limitation and complex arbitrage-free constraints. In this thesis, I develop a more efficient linear programming support vector machine (L_1 -SVM) estimator for state price density which incorporates no-arbitrage restrictions and bid-ask spread. This method does not depend on a particular approximation function and framework and is, therefore, universally applicable. In a parallel empirical study, I apply the method to options on the S&P 500, showing it to be comparatively accurate and smooth. In addition, since the existing literature has no consensus about what information is recovered from The Recovery Theorem, I empirically examine this recovery problem in a continuous diffusion setting. Us-

Abstract

ing the market data of S&P 500 index option and synthetic data generated by Ornstein–Uhlenbeck (OU) process, I show that the recovered probability is not the real-world probability. Finally, to further explain why The Recovery Theorem fails and show the existence of associated martingale component, I demonstrate a example bivariate recovery.

Notation and Abbreviations

| Items | Description |
|----------------------------|--|
| (Ω, \mathcal{F}, P) | Probability space |
| S | Underlying asset price, S_t is the price at time t , S_T is the price at maturity date |
| r | Interest rate, in option pricing theory, it can be assumed as constant, deterministic function or zero |
| T | Option maturity date |
| K | Option strike price |
| F^τ | Forward Price |
| $Z(S_T)$ | Option payoff |
| k | Forward moneyness $\frac{k}{F^\tau}$ |
| τ | Time to maturity of Option, it usually is calculated as $\frac{T-t}{365}$ |
| σ | Volatility of underlying |
| $C(k, \tau)$ | Call option price with strike k and time to maturity τ |
| $P(k, \tau)$ | Put option price with strike k and time to maturity τ |
| ITM | In the money, which refers strike price is less than underlying price |
| ATM | At the money, which refers strike price is equal than underlying price |

Continued on next page

| Items | Description |
|-----------------------|--|
| OTM | Out the money, which refers strike price is great than underlying price |
| SDF | Stochastic discount factor |
| EMH | Efficient market hypothesis |
| ARCH | Autoregressive conditional heteroskedasticity |
| GARCH | Generalized autoregressive conditional heteroskedasticity |
| AI | Artificial intelligence |
| AIC | Akaike Information Criterion |
| SVM | Support vector machine |
| QP | Quadratic Programming |
| LP | Linear Programming |
| \mathbb{P} | Real-world probability measure, probability density under \mathbb{P} is $f(S_T)$ |
| \mathbb{Q} | Risk neutral measure, the risk neutral density is $p^*(S_T)$, risk neutral distribution is $F(S_T)$ |
| \mathbb{Q}^T | Forward measure |
| $E_t(\cdot)$ | Expectation under real-world probability |
| $K(x, x)$ | Kernel function |
| \mathbf{k} | Vector of forward moneyness |
| $\boldsymbol{\tau}$ | Vector of time to maturity |
| α_i | Coefficient of L_1 -SVM approximation |
| C, λ | Trade-off parameters balance the estimated error and no arbitrage conditions |
| a, ε, ξ | Bound parameters of L_1 -SVM approximation |

Continued on next page

Notation and Abbreviations

| Items | Description |
|--------------|---------------------------------------|
| $r_k(x)$ | Derivatives vector of kernel function |
| $Y_k(Z_k)$ | Vector of target value y |
| \otimes | Kernel product of two matrix |
| S | State Price matrix |
| P | Real-world transition matrix |
| Q | Transition matrix |

Chapter 1

Introduction

In this chapter, I outline the research background, motivations, and main contributions and findings. Section 1.1 briefly introduces the history of options. Section 1.2 summaries research gaps and motivation. An overview of the main findings of this thesis is presented in Section 1.3. Section 1.4 provides the thesis structure.

1.1 Overview and Background

We live in a world full of contradiction and paradox, a fact of which perhaps the most fundamental illustration is this: that the existence of a problem of knowledge depends on the future being different than the past, while the possibility of the solution of the problem depends on the future being like the past.

—Knight (1921)¹

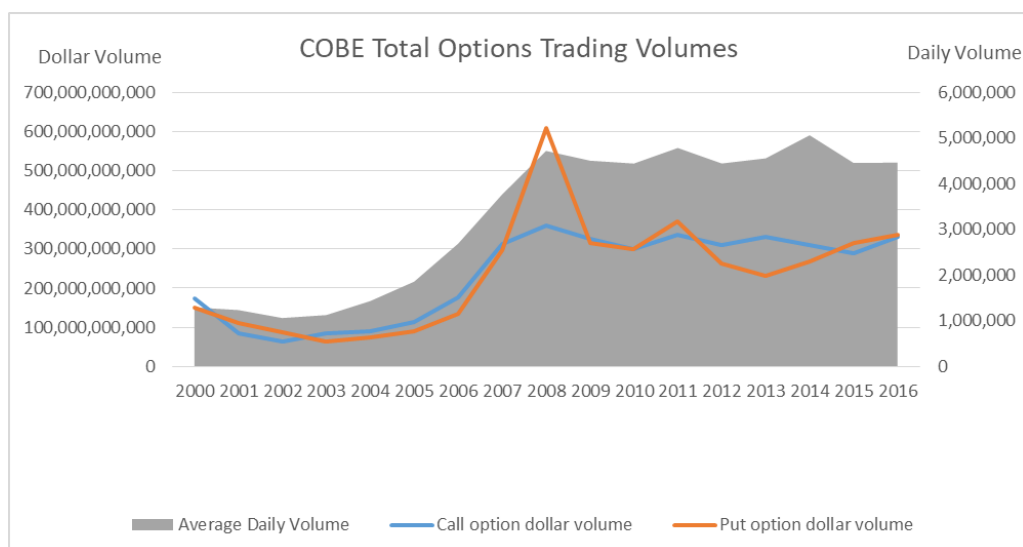
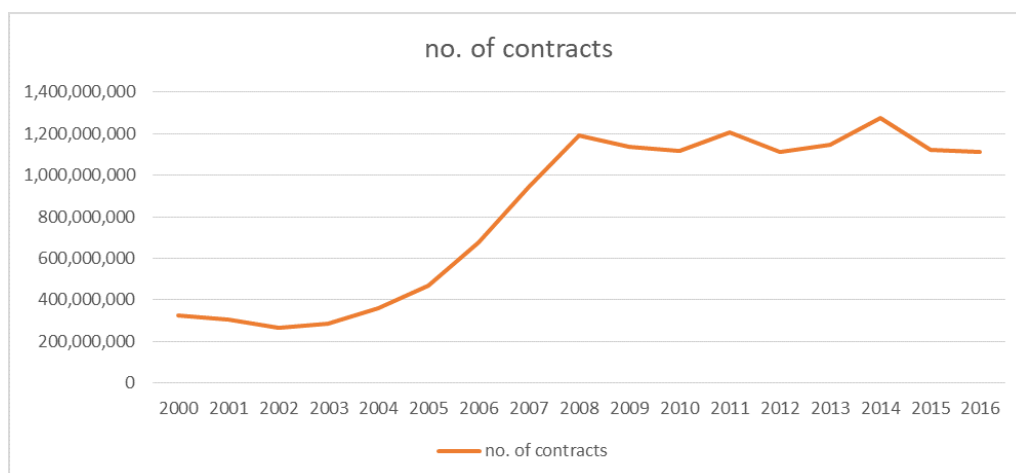
An option is a financial contract that gives a buyer (or a seller) a right to buy (or sell) a certain amount of something which I name the underlying at specific price on a specified date². The earliest example of using options can be traced

¹Page 313

²The key insight of option contract is: it gives the right but not the an obligation to exercise contract. This intuition is easy to understand as instead of an obligation, people always have a choice (option) in real life

1.1. Overview and Background

Figure 1.1.1: Number of Contracts and Trading Volume on CBOE



Note: The top panel plots the number of contracts traded on CBOE from 2000 to 2016. The bottom panel reports the average daily trading volumes and dollar volume of call and put options on CBOE in the same period. The orange line represents the dollar volume of call option while the blue line plots the dollar volume of put options. The gray areas plot shows the average daily volume changes. The data is available at <http://www.cboe.com/data/historical-options-data/annual-market-statistics>

1.1. Overview and Background

back to ancient Greece³, when Thales of Miletus made a successful speculation by designing an olive presses option contract. According to Aristotle⁴, Thales could predict weather conditions. Using this information, Thales successfully forecasted a rich harvest of olives in a particular year and paid a low deposit (option premium) to pre-book all olive presses in advance. This deposit gave him the right to use the olive-presses if he so choose. When the season came and the demand of olive-presses surged, he rented the olive presses to farmers at a higher price (exercising his options). This is a classic example of using call options to make a profit. However, not all people seeing similar opportunities got lucky. Thompson (2007) remarks that tulip-mania in 1637 was associated with the use of options. On 24th February 1637, Dutch florists announced a new trading rule which transformed the tulip-bulb forward contract into an option contract. Under the new rule, investors did not need to pay in the forward or spot market to buy tulip bulbs. They could pay only 3 percent of the forward contract price to reserve a right (without obligation) to buy tulip bulbs in the future. If the market soared, they profited like Thales of Miletus. If the market slipped, they would have lost just 3 percent. This announcement resulted in a massive rise in tulip options prices and many people invested their lifes' savings in tulip options. In the next few months, the strike price of tulip option traded as high as 10 times the spot price. Then the bubble burst and many went bankrupt as the tulip fever suddenly ended. This tulip disaster set a bad image of options over decades⁵. For a century, options trading was illegal in Great Britain. On the other hand, options trading in the US began and develop rapidly afterward. In 1872, an American

³Some researchers believe that the story (the marriage between Jacob and Laban's daughter) in The Bible is the first record option transaction

⁴Please refer to Aristotle (1999) for more detail. The original record is in Aristotle, Politics 1.1259a, Book I, Chapter 11, sections 5-10

⁵The other famous derivative disaster is Baring Bank in 1995, Nick Lesson traded aggressively in Nikkei 225 and SIMEX futures and options and lost 827 million pounds.

1.1. Overview and Background

businessman, Russel Sage, created call and put options. In 1973, a remarkable year in option history, the first official traded options market - Chicago Board Options Exchange (CBOE) was opened and Fischer Black, Myron Scholes and Robert Merton(Black and Scholes (1973); Merton (1973)) published their option valuation model. Since then, the option market has grown explosively. Exchanges were opened all over the world and products such index option, spread option, VIX options and Long-term Equity Anticipation Securities (LEAPS) began to be traded.

Nowadays, with introduction of electronic trading platforms and increased computation power, the trading of options has become more popular and easier. According to the Bank for International Settlements(BIS)⁶ and the Futures Industry Association(FIA)⁷, traded option contracts account for 10% of the global OTC (over-the-counter; i.e, not traded on an exchange but directly) derivative market and 40% of exchange traded contracts in 2017. Furthermore, both the number of tradable option contracts and trading volumes have soared since 2005. As shown in Figure 1.1.1, the number of options contracts has climbed from less than 400 million to more than 1200 million from 2000 to 2016. In particular, the number of contracts increased dramatically from 2005 to 2006. This significant change corresponds with the introduction of weekly, short-term options and VIX options in 2005 and 2006⁸. In addition, the daily trading volume and dollar volume maintained an historically high level during the past 16 years with more than 4 million options traded daily on the CBOE over 8 years.

⁶see <https://www.bis.org/statistics/derstats.htm> for more detail

⁷see <https://fia.org/categories/exchange-volume> for more detail

⁸see <http://www.cboe.com/aboutcboe/history> for more detail

1.2 Motivations and Research Gaps

This thesis focuses on extracting information from options market data: more precisely, the risk neutral density(RND), underlying natural probability and investors' risk preferences. As the earlier section has described, both Thales of Miletus and Dutch saw a similar money-making opportunity. However, one profited but the others' hopes ended badly. The main difference between them is underlying movement and investor's risk preference. Obviously, if any investors have true foresight about either item, they could make handsome profits (or avoid unnecessary speculation). However, this is impossible in real markets. How to forecast future underlying movement has been a fundamental concern of both scholars and practitioners for many years. There are numerous business, equity, financial analysts in Wall Street writing market outlook reports and thousands of researchers using sophisticated financial forecasting models every day. As Granger (2005) states "more likely the results are fragile, once you try to use them, they go away".

From time series econometrics and rational expectations perspective, scholars and practitioners always try to draw on insights from historical data (such as using autoregressive conditional heteroskedasticity (ARCH) and generalized autoregressive conditional heteroskedasticity (GARCH) model). However, this approach is directly challenged by prominent economists like Knight (1921)⁹ and the famous efficient market hypothesis (EMH) by Fama (1970). Fama (1970) argues that all available information in the market has already been reflected in the current asset price. Future asset price movements are only influenced by new information. Knight's and Fama's comments gained renewed interest after 2008 when conventional risk management techniques have been questioned for

⁹see quote before this chapter

their inability to manage tail risk and black swan events. Historical time series data do not offer many guidelines in predicting extreme events. Recently researchers have shifted their attention to inferring underlying probability densities from options markets. As an option is a forward-looking instrument, the benefits of focussing on cross-sectional option data seem obvious in principle. First, an option contains information on expectations about underlying future movement. Second, unlike traditional time series econometrics, inferring probabilities from option prices requires no prior guesses about the underlying distribution.

Starting with the remarkable Black-Scholes-Merton model (Black and Scholes (1973); Merton (1973)), a large body of option pricing literature has developed to calculate the fair value of an option price. Wilmott (2006) classifies the option pricing methods into four catalogues: Lattice(tree) method (Cox et al. (1979)), Monte Carlo method (Boyle (1977)), Finite Difference method (Brennan and Schwartz (1977)) and Numerical Integration (Andricopoulos et al. (2003, 2007); Chen et al. (2014))¹⁰. All of these are based on the risk neutral measure \mathbb{Q} . Following this theoretical framework, a number of studies have concentrated on developing techniques to extract the risk neutral probability density (RND) from option prices (see Figlewski (2018) and Chapter 5 for a brief review), but none shows superior performance. This is not a surprise since estimating the risk neutral density poses five challenges (see Chapter 5 for more detail).

- The estimation of RND is based on continuous strike price while the strike price in the real market is discrete
- The estimation of RND ignores the information contained in the bid-ask spread

¹⁰Because this thesis focus on the inverse problem of option pricing(a.k.a estimate underlying probability density from option price), I recommend reader refer Chapter2 and textbooks Wilmott (2006) and Hull and Basu (2016) for more information.

- Market option data contains noise from various sources which may lead to multimodal RND
- Theoretical RND lies in $[0, \infty]$ while the market option data can only estimate RND within bounds
- The estimation of RND suffers the 'curse of differentiation'.

Unlike the estimation of RND, how to separately recovery underlying real-world probabilities from option prices is a new topic. Prior to Ross (2015)¹¹, who proposes The Recovery Theorem (hereafter TRT), there is no model-free method that could uniquely recovery real-world probabilities of underlying from option prices¹². This break-through work seems to challenge the traditional option pricing framework under the risk neutral measure \mathbb{Q} and provide a new insight to re-examine or build all pricing, risk management frameworks under the recovered probability. However, until now, existing studies have reached no consensus on what information is recovered by Ross's theory. The original TRT provided by Ross has argued that the recovered probability is the real-world probability, namely the \mathbb{P} probability of the underlying asset. This argument is theoretically supported by Carr and Yu (2012), who provide an alternative way to derive TRT from a numeraire portfolio. However, Borovička et al. (2016) point that the since there is always a martingale component contained in the recovered probability, the recovered probability is between \mathbb{Q} and \mathbb{P} (They call this probability as natural probability).

Although researchers and practitioners have demonstrated a strong interest in using forward-looking information from option prices, two main problems are

¹¹The working paper is available online in 2012.

¹²To my best knowledge, there are only several studies try to solve this problem assuming particular stochastic discount factor(Jackwerth and Rubinstein (1996))

associated with extracting this information.

- How to extract a well-behaved RND from option prices?
- What information do I recovery from TRT, the real-world probability or natural probability?

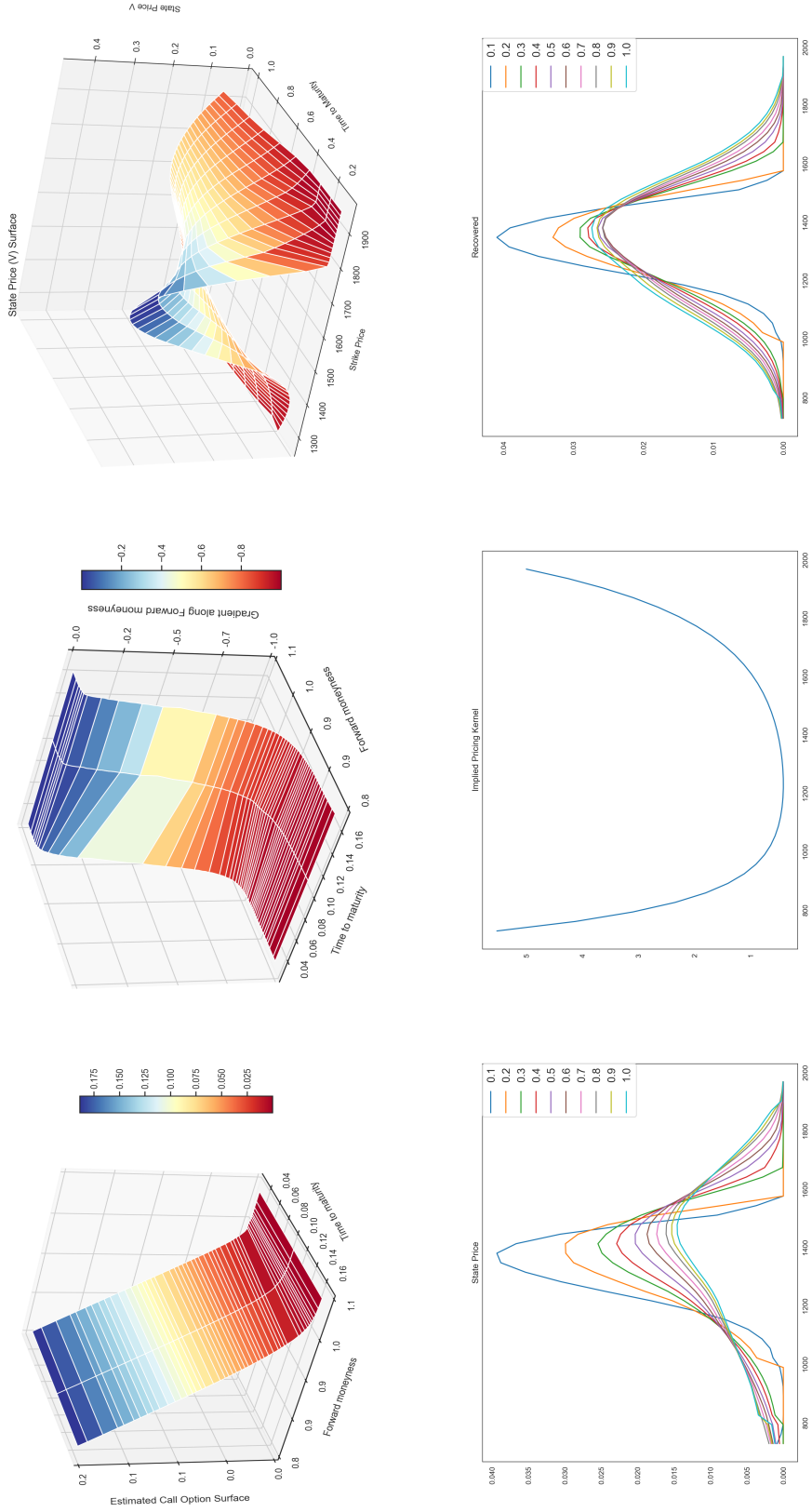
Motivated by these two problems and using a data-driven approach, in particular, I investigate the following questions:

- Can I estimate the RND using machine learning technique?, If the answer is yes, which technique could I use? (Chapter 3 and 5)
- How do I incorporate the no-arbitrage constraints in the machine learning framework? (Chapter 4 and 5)
- Compare to existing non-parametric method, whether my approach shows a better performance? (Chapter 5)
- Differ from previous discretization approach, if I examine TRT in a continuous time setting, what information do I recovered? (Chapter 6)

1.3 Findings and Contributions

Differing from many existing approaches that extract RND using specific parametric model, in this thesis, I try to solve the first problem from a machine learning perspective. In Chapter 5, I develop a data-driven estimator using a support vector machine(SVM). Including the bid-ask spread and no-arbitrage conditions, I modify the standard SVM into a linear programming framework. Compare with other methods, my approach yields four advantages. First, it is a fully nonparametric method and does not rely on the assumption of a prior

Figure 1.3.1: Key Results



Note: Figure 1.3.1 plots the key results in thesis. The top panel shows the extracted risk-neutral information: estimated call price surface, its first-order derivative and state price surface on 02/07/2013. The bottom panel displays the recovered results of Walden (2017)'s unbounded diffusion Recovery Theorem on 05/03/2008. The first two figures in top panel are plotted under forwards measure. In each subplot, selected time maturities (0.1 to 1.0) are indicated by different color.

distribution. Second, by setting trader's own error tolerance, it incorporates the information of bid-ask spread into an estimation framework. Third, it is a universal approach that incorporates all arbitrage-free constraints. Fourth, compared to the neural network, my approach could easily produce a smooth RND due to the explicit complexity control. The top panel of Figure 1.3.1 displays the estimation results of L_1 -SVM on July 2, 2013. Consistent with option pricing theory, my state price density estimator yields a unimodal, smooth and positive surface.

To solve the second problem, I empirically implement Walden (2017)'s unbounded diffusion Recovery Theorem using S&P 500 index option. As shown in the bottom panel of Figure 1.3.1, I find that the recovered probability exhibits thinner left tails than its risk-neutral counterparts. Using Audrino et al. (2015)'s quadratic loss function in my L_1 -SVM framework, I show that the quadratic loss function in state price estimation does not guarantee a well-recovered result. Finally, to further explain why The Recovery Theorem fails, I apply a bivariate unbounded recovery and confirm the existence of the martingale component in Ross's pricing kernel assumption.

To summarize, this thesis contributes to the previous studies in several three ways: First, I propose a new a data filter approach based on three principles: representative, accurate and no arbitrage in Chapter 4. I show that comparing with Zhang and Xiang (2008) and other studies, my filtered results contain less noise. Second, I contribute to the RND estimation studies by developing a more efficient linear programming support vector machine (L_1 -SVM) estimator. Incorporating all no-arbitrage restrictions and bid-ask spread, in Chapter 5, I show that my method establishes a somewhat better accuracy and is universally applicable. Third, I provide the first empirical evidence for applying Walden (2017)'s

unbounded diffusion recovery. Using the S&P 500 index option and synthetic data generated by Ornstein–Uhlenbeck (OU) process, I find consistent evidence on the recovered probability distributions with Jackwerth and Menner (2017). I show that the recovered probability is not the real-world probability.

1.4 Thesis Structure

This thesis contains seven chapters. Chapter 1 introduces the research background and gaps in existing studies. This chapter summarizes the main findings of the thesis. Chapter 2 gives a brief review of option theory, namely change of measure technique, Black-Scholes Model and no-arbitrage conditions for option price. In Chapter 3, I introduce two machine learning techniques and review their applications in finance. Chapter 4 shows the procedure for constructing option data panel and outlines the data filter rules. With the knowledge from proceeding chapters, I propose a new state price density estimator based on support vector machine (L_1 -SVM) in Chapter 5. In this chapter, I extend the standard SVM to incorporate the no-arbitrage conditions and compare its performance with another four models. Chapter 6 investigates the problem of what information is recovered from TRT. I empirically implement Ross (2015) in the single diffusion process and bivariate state variables case. Chapter 7 concludes and suggests the future research direction.

Chapter 2

A Brief Overview of Option Theory

People like the model because they can easily understand its assumptions. The model is often good as a first approximation, and if you can see the holes in the assumptions you can use the model in more sophisticated ways.

—Black (1990)

This chapter presents the necessary mathematical finance foundation for this thesis. In section 2.1, I start by introducing two fundamental theorems in mathematical finance. These two theorems are prerequisites to link the no-arbitrage pricing principle with option pricing. Under no-arbitrage principle, I model the option price as a stochastic process and derive the Black-Scholes formula. In the last section, I summarize the sufficient and necessary conditions for incorporating no-arbitrage directly on a call option price surface.

2.1 Fundamental Theorem in Mathematical Finance

2.1.1 The First Fundamental Theorem of Asset Pricing

Although it is challenged by behavioral finance literature(e.g. Barberis and Thaler (2003)), the no-arbitrage principle is still the most important principle

in financial modeling. The no-arbitrage principle arise from a basic belief about market behavior. Arbitrage is a trading strategy taking to make money when two or more securities are mis-priced relative to each other. When expressed in the mathematical finance literature, I connect the no-arbitrage principle with the probability measure. More precisely, if the market has no arbitrage opportunity, then there exists a unique equivalent martingale under which the underlying price process becomes a martingale.

Following Delbaen and Schachermayer (1994) and Dybvig and Ross (1989), let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. \mathcal{F} is a right continuous information filtration, \mathbb{Q} is the risk neutral probability and all possible outcomes are collected in Ω . S_t denotes a stochastic process ¹³.

Definition 1. (Martingale) Let $S = S_t$ for $0 < t < \infty$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, if for any bounded stop time s ($0 < s < t < \infty$), the $E(S_t | \mathcal{F}_s) = E(S_s)$, then S is a martingale

Roughly, this means when conditioned on all the information I know today \mathcal{F}_t , My future expectation of a martingale process S is equal to its current value. When pricing the option, since the asset price has captured all the information in the market (EMH), the current expectation is equal to the asset price discounted by the risk-free rate.

$$E^{\mathbb{Q}}\left[\frac{S(T)}{M(T)} \mid \mathcal{F}_t\right] = \frac{S(t)}{M(t)} \quad (2.1)$$

The $\frac{S}{M}$ is a martingale process, where M is called market money account in the concept of numeraire.

Definition 2. (numeraire) A numeraire can be any asset that has a positive price process.

¹³The popular choices in mathematical finance literature are Wiener process and Levy process.

In particular, market money account, currency exchange rate and zero-coupon bond are popular numeraire choices in the mathematical finance literature.

Theorem 1. (*The First Fundamental Theorem of Asset Pricing*) *A market model in probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is no-arbitrage, if and only if there exists a probability measure that is equivalent to \mathbb{P} .*

Proof: please refer to Delbaen and Schachermayer (1998)

Essentially, the First Fundamental Theorem of Asset Pricing implies that under no-arbitrage principle, I could find a unique probability measure that enables the price process to have a constant expectation.

2.1.2 Change of Measure

It is clear that in real-world probability \mathbb{P} , the underlying price process is not a martingale. To benefit from the properties of a martingale and use the no-arbitrage principle, I need change the original probability measure \mathbb{P} to another probability measure. This technique is called change of measure (unsurprisingly). For this, I first define the Radon-Nikodym Derivative.

Definition 3. (Radon-Nikodym Theorem) In probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a positive random variables Z and the probability measure \mathbb{Q} is equivalent to \mathbb{P} , if

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{2.2}$$

then Z is called the Radon-Nikodym derivative.

The Radon-Nikodym derivative can successfully change the measure when we know what we want to change into, but, in practice, the new probability measure is usually unknown. Therefore I consider Girsanov's Theorem.

Theorem 2. (*Girsanov's Theorem*) Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $L(t)$ is a adapted bounded process with $L(0) = 1$ $\varphi(s)$ is a positive process, if

$$L(t) = \exp\left(\int_0^t \varphi(s)dW^{\mathbb{P}}(t) - \frac{1}{2} \int_0^t \varphi^2(s)ds\right) \quad (2.3)$$

Where $W^{\mathbb{P}}(t)$ is a standard Brownian Motion under \mathbb{P} . The process $L(t)$ also is the unique solution to

$$dL(t) = \varphi(t)L(t)W^{\mathbb{P}}(t) \quad (2.4)$$

Then under a new probability \mathbb{Q} ,

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) - \int_0^t \varphi(s)ds \quad (2.5)$$

$$dW^{\mathbb{Q}} = dW^{\mathbb{P}} - \varphi_t dt \quad (2.6)$$

Proof: please refer to Wilmott (2006)

The interesting aspect of theorem is that when changing to a new measure, the diffusion term of the SDE is unchanged while the drift term is changed.

Theorem 3. (*change of numeraire*) If two different numeraire N^1 and N^2 and associated martingale measure \mathbb{Q}^{N^1} and \mathbb{Q}^{N^2} , using Radon-Nikodym derivative, the following relationship holds:

$$\frac{d\mathbb{Q}^{N^1}}{d\mathbb{Q}^{N^2}} = \frac{N^1(T)N^2(t)}{N^1(t)N^2(T)} \quad (2.7)$$

Proof. Since \mathbb{Q}^{N^1} and \mathbb{Q}^{N^2} are two martingale measures, according to definition 1, I obtain the following equations

$$E^{\mathbb{Q}^{N^1}} \left[\frac{S(T)}{N^1(T)} \middle| \mathcal{F} \right] = \frac{S(t)}{N^1(t)} \quad (2.8)$$

$$E^{\mathbb{Q}^{N^2}} \left[\frac{S(T)}{N^2(T)} \middle| \mathcal{F} \right] = \frac{S(t)}{N^2(t)} \quad (2.9)$$

Dividing these equations, I have

$$N^1(t) E^{\mathbb{Q}^{N^1}} \left[\frac{S(T)}{N^1(T)} \middle| \mathcal{F} \right] = N^2(t) E^{\mathbb{Q}^{N^2}} \left[\frac{S(T)}{N^2(T)} \middle| \mathcal{F} \right] \quad (2.10)$$

if I define $M(T) = \frac{S(T)}{N^1(T)}$,

$$\begin{aligned} E^{\mathbb{Q}^{N^1}} [M(T) | \mathcal{F}] &= \frac{N^2(t)}{N^1(t)} E^{\mathbb{Q}^{N^2}} \left[\frac{S(T)N^1(T)}{N^1(T)N^2(T)} \middle| \mathcal{F} \right] \\ &= E^{\mathbb{Q}^{N^2}} \left[M(T) \frac{N^1(T)N^1(t)}{N^1(t)N^2(T)} \middle| \mathcal{F} \right] \end{aligned} \quad (2.11)$$

Therefore, I can transform \mathbb{Q}^{N^1} to \mathbb{Q}^{N^2} using a Radon-Nikodym derivative defined by two numeraire assets. □

2.2 Option Pricing in a Nutshell

2.2.1 The Black-Scholes Model

Even though there has been some criticism of the Black-Scholes model, it is, without doubt, still the most widely used model in the derivative market. The Black-Scholes model provides a closed-form solution of the European option price and transforms the option pricing problem to solving partial differential equation(PDE).

If I make the following assumptions:

- The market has no-arbitrage.

2.2. Option Pricing in a Nutshell

- The market is frictionless: more precisely, there are no transaction costs on the underlying and the borrowing and lending interest rates are the same. All investors can immediately receive any available information.
- The underlying price follows a Geometric Brownian Motion(Top panel of Figure 2.2.1 shows an example of Geometric Brownian Motion prices paths)

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{P}}(t) \quad (2.12)$$

- The interest rate r is a function of t and volatility σ is constant.
- The underlying stock pays no dividends.

Let the $M(t)$ be market money account and $dM(t) = rM(t)dt$. According to definition 1, the risk-free discounted option price is a martingale, so

$$V(t, S) = M(t)E^{\mathbb{Q}}\left[\frac{V(T, S)}{M(T)} \middle| \mathcal{F}\right] \quad (2.13)$$

Using Ito's Lemma, I get

$$dV = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma SdW^{\mathbb{Q}} \quad (2.14)$$

If I define risk-free discounted option price as $\Pi \equiv \Pi(t, S)$

$$\Pi(t, S) = \frac{V(t, S)}{M(t)} = E^{\mathbb{Q}}\left[\frac{V(T, S)}{M(T)} \middle| \mathcal{F}\right] \quad (2.15)$$

Since the discounted option price depends on underlying price and time t , the change of portfolio is defined as

$$d\Pi = d\left(\frac{V}{M}\right) = \frac{1}{M}dV - \frac{V}{M^2}dM = \frac{1}{M}dV - r\frac{V}{M}dt \quad (2.16)$$

Therefore

$$\begin{aligned} d\Pi &= \frac{1}{M} \left(\left(\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S dW^{\mathbb{Q}} \right) - r \frac{V}{M} dt \\ &= \frac{1}{M} \left(\left(\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \right) dt + \frac{\sigma S}{M} dW^{\mathbb{Q}} \right) \end{aligned} \quad (2.17)$$

Since the risk-free discounted option is a martingale, therefore, it is drift-less.

This gives us

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (2.18)$$

The above equation is PDE of Black-Scholes. Consider the payoff condition of call option price

$$V_C(T, S) = \max(S(T) - K, 0) \quad (2.19)$$

I get a unique solution for the PDE for call option price (see bottom panel of Figure 2.2.1 for an example of call price surface using Black-Scholes)

$$\begin{aligned} V_C(t, S) &= sN(d_1) - Ke^{-r\tau}N(d_2) \\ d_1 &= \ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau \\ d_2 &= d_1 - \sigma\sqrt{\tau} \end{aligned} \quad (2.20)$$

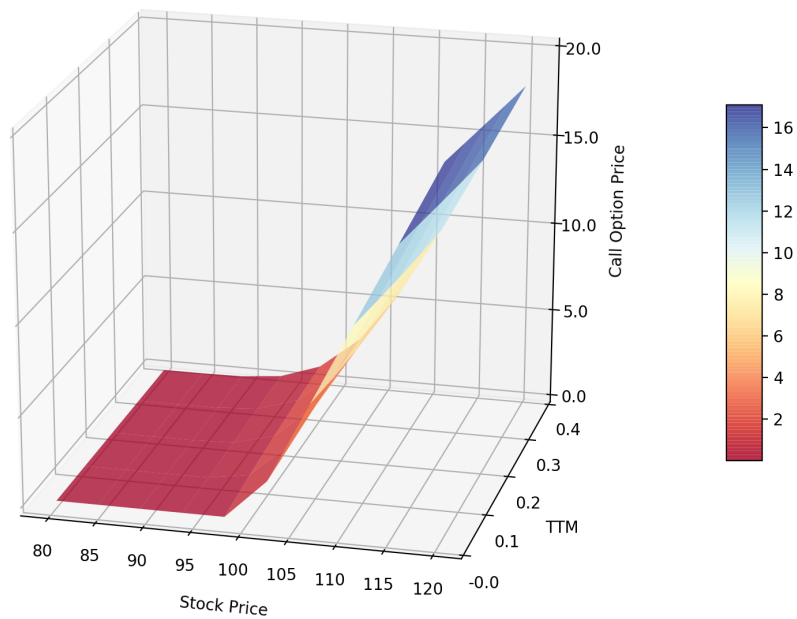
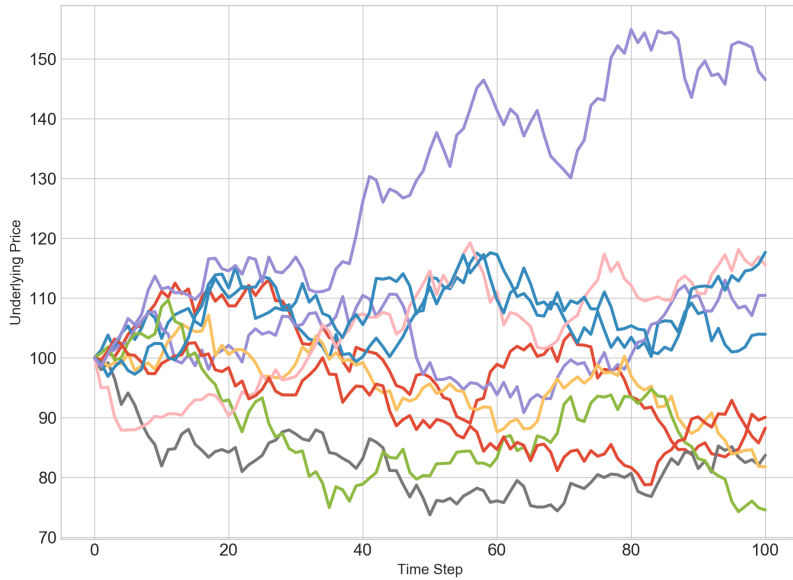
where $\tau = T - t$, $N(\cdot)$ is cumulative distribution function of a standard normal distribution. Using put-call parity, I can get the European put option price

$$V_C(t, S) = Ke^{-r\tau}N(-d_2) - SN(-d_1) \quad (2.21)$$

Several studies have relaxed the assumptions of Black-Scholes including assuming stochastic volatility (e.g. Heston (1993)), allowing different diffusion pro-

2.2. Option Pricing in a Nutshell

Figure 2.2.1: Geometric Brownian Motion Price Paths and Black-Scholes Call Price Surface



Note: The top panel plots the 10 stock price paths which follows the Geometric Brownian Motion. The price is plot against time step. I divide the time t into 100 steps. The current stock price is 100. The bottom panel displays the call option price surface based on the Black-Scholes model. The interest rate r is 0.01, constant volatility is 0.1 and the strike price is 100.

cess such as Levy process, extending to the commodity market (e.g.Black (1976)) and considering dividend yield (e.g.Merton (1973)).

2.2.2 Numerical Techniques

The objective of option pricing is to determine the “fair value” of the option. Generally following the idea of Black-Scholes, I could model the underlying with a specified diffusion process and derive the PDE as in the previous section. Then I could get the analytical solutions for option price by applying stochastic calculus and solving the PDE. However, analytical solutions are not always possible and the case may be more complicated for the exotic options. As a consequence, four groups of numerical methods are often employed. All four methods are good at some perspective such as accuracy or easily converge but none of them is suitable and efficient for all problems. More detailed description of four numerical methods can be found in Wilmott (2006) and Hull and Basu (2016). I briefly summarize the advantages and disadvantages of my methods as following: first, Lattice(tree) method is the most straightforward and easily understood approach. Although Cox et al. (1979) prove that tree methods will converge to the correct option values, the convergence rate is relatively poor. The second approach is The Finite Difference, which is introduced by Brennan and Schwartz (1977) and is a direct approach to approximate the solution of the PDE. This approach provides reliable results for early exercise and low dimensional problems and performs poorly for path-dependent and high dimensional cases. Third, the Monte Carlo method, which simulates the underlying price path by random sampling and uses the average as option price. Although relatively slow, it yields excellent results for high dimensional and path dependent problems. Last but not the least is Numerical Integration via the quadrature method, which is an effective technique

in both low and high dimensional problems. Mathematically, the idea behind the quadrature method (QUAD) is to approximate the areas under curves. In option pricing, Newton et al. (Andricopoulos et al. (2003, 2007); Chen et al. (2014)) present a series papers to apply QUAD in the classical Black-Scholes-Merton world, multi-asset and various underlying process.

2.2.3 Link to Equilibrium Pricing Model

As a matter of fact, the risk neutral pricing could connect to neoclassical finance with the no-arbitrage argument. According to Ross (2009), the no-arbitrage is equivalent to a positive linear pricing operator in asset pricing model.

Theorem 4. (*The Fundamental Theorem of Finance*) *The following three statements are equivalent:*

1. *No Arbitrage (NA)*
2. *The existence of a positive linear pricing operator that prices all assets*
3. *The existence of a (finite) optimal demand for some agent who prefers more to less*

Please refer to Dybvig and Ross (1989) for proof

The linear pricing operator is the necessary condition for setting up the equilibrium pricing model. The key idea of the equilibrium pricing model is that if the economy is characterized by a representative agent and no arbitrage, then the asset price is represented as a linear pricing operator multiplying future payoffs. The representative agent assumes that there exists a single agent in the economy who can represent the preferences of all investors. Strictly speaking, this is a strong and restrictive assumption. However, it implies a general result because it is independent of an individual's initial wealth. The general equilibrium pricing

model can be shown as

$$P_t = E_t[M_t Z_t] \quad (2.22)$$

where P_t is the asset price, Z_t is the future payoff and M_t is the linear pricing operator or stochastic discount factor(SDF). In the consumption-based capital asset pricing model developed by Lucas (1978), restricting the investor's consumption, the SDF is usually expressed as the marginal rate of substitution of consumption between different times. More specifically, $M_t = \beta \frac{u'(c_{t+1})}{u'(c_t)}$, c_{t+1} and c_t are the consumption at time $t+1$ and t , $u'(c_t)$ is marginal utility of consumption and β is the time preference discount factor. According to the model, I express the option price as expected value of stochastic discount factor weighted future payoffs. More formally, the option price at time t with payoff $Z(S_T)$ is

$$P_t = e^{-r\tau} E_t[M(S_T)Z(S_T)] = e^{-r\tau} \int_0^{\infty} M(S_T)Z(S_T)f(S_T)dS_T \quad (2.23)$$

Where S_T is a state variable and $f(S_T)$ is the payoff probability density under \mathbb{P} . To link the real-world probability \mathbb{P} with the risk neutral density, I rewrite the Equation (2.23) as:

$$P_t = e^{-r\tau} E_t^*[Z(S_T)] = e^{-r\tau} \int_0^{\infty} Z(S_T)p^*(S_T)dS_T \quad (2.24)$$

Where $p^*(S_T)$ is the state price density or risk neutral density, which is real-world probability \mathbb{P} discounted by stochastic discount factor. Compare Equation(2.24) with the option pricing equation under \mathbb{Q} (see Chapter 5 for more detail), I can conclude that risk neutral density is a product of the real-world probability and stochastic discount factor.

2.3 No Arbitrage Conditions

Following Merton (1973) and Carr and Madan (2005), I classify the no-arbitrage conditions on option prices into two types. Type 1 focusses on arbitrage between option and risk-free bond. Type 2 emphasizes the arbitrage between different strikes and maturities. Denote $B(t, T)$ as the price of zero bond with $B(T, T) = 1$, I summarize the no arbitrage conditions of option prices as follows

- Type 1: No arbitrage between option and risk-free bond

$$C(K, 0) = \max[0, S_T - K] \quad (2.25)$$

$$[0, S_t - KB(t, T)] \leq C(K, \tau) \leq S_t$$

$$C(K, \tau, S_t = 0) = 0$$

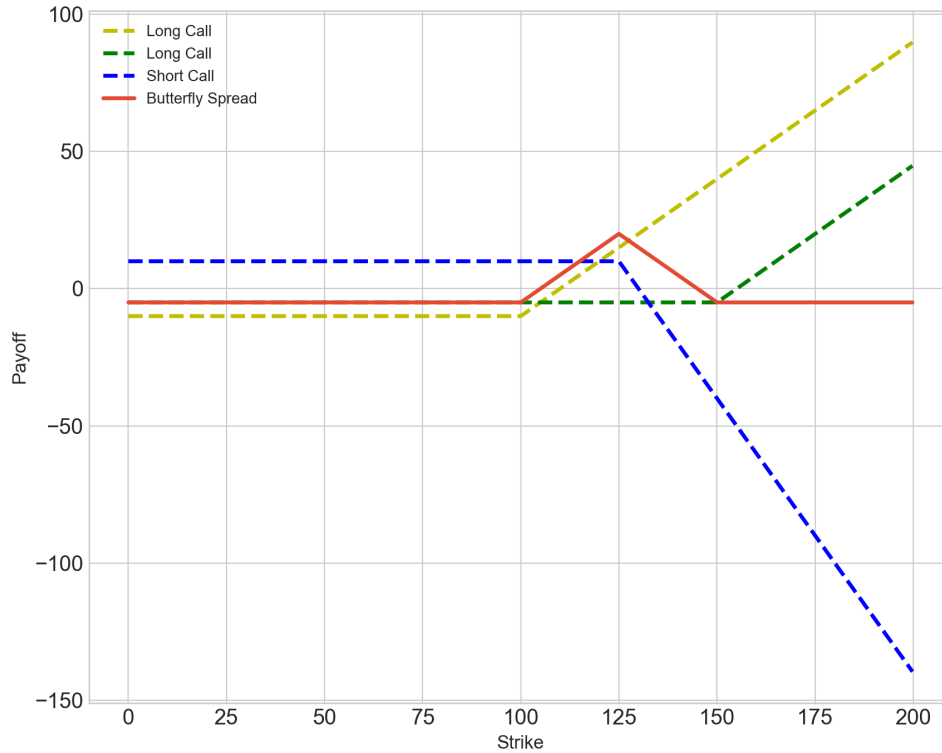
$$C(K, \tau) + KB(t, T) = P(K, \tau) + S_t$$

The first equation defines the payoff of call option. The second and third equations restrict the call option. The last equation states put call parity. According to section 6 in Merton (1973), if I use Black-Scholes to price the European call option and the volatility is large than zero, then the Type1 no arbitrage constraints hold. Actually, as shown in section 2.2, the Black-Scholes equation is derived from Merton's definition of no-arbitrage.

- Type 2: No-arbitrage between different strikes and maturities

For the type 2 no arbitrage constraints, I use Carr and Madan (2005)'s idea of static arbitrage, which means if the call price surface is free from butterfly and calendar spreads then it is sufficient to define an arbitrage-free option price surface.

Figure 2.3.1: Payoff of Butterfly Spread



Note: This figure shows the payoff of a butterfly spread, with $K_1 = 100, K_2 = 125$ and $K_3 = 150$

- Absence of butterfly-spread arbitrage (Convexity respect to strike price): consider three call options $C_1(K_1), C_2(K_2)$ and $C_3(K_3)$ with strike prices $K_1 < K_2 < K_3$. I can construct a butterfly spread whose payoff is always positive. For example, Figure 2.3.1 shows a butterfly spread with strike prices 100, 125 and 150.

$$B_f(K_1, K_2, K_3) = (K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3) \quad (2.26)$$

Clearly, the payoff is positive. Arbitrage is possible if I could construct a butterfly spread with a nonpositive price. Therefore, I need to restrict the map-

2.3. No Arbitrage Conditions

ping such that $K \rightarrow C(K, \tau)$ is convex. Mathematically, this implies that when I twice differentiate the call price to strike price, the second derivative of call price is always greater than or equal to zero.

$$\frac{\partial^2 C}{\partial K^2} \geq 0 \quad (2.27)$$

- Absence of calendar-spread arbitrage (increasing respect to time to maturity): Consider two maturities $T_1 < T_2$ and with any strike K^{T_1} and K^{T_2} , I define its forward value at maturity T as $F(T_1)$ and $F(T_2)$

$$\frac{K^{T_2}}{K^{T_1}} = \frac{F(T_2)}{F(T_1)} \quad (2.28)$$

If I construct a portfolio by buying a call option $C(K^{T_2}, T_2)$ and selling $\frac{D(T_2)F(T_2)}{D(T_1)F(T_1)}$ amount a call option $C(K^{T_1}, T_1)$, where $D(\cdot)$ is the value of zero bond expiring at T. The $D(T, T) = 1$. At time T_1 , if $S(T_1) > K^{T_1}$, I define the payoff of call option $C(K^{T_1}, T_1)$ as

$$S(T_1) - \frac{F(T_1)}{F(T_2)} K^{T_2} \quad (2.29)$$

Otherwise, the payoff of call option $C(K^{T_1}, T_1)$ is 0. The value of portfolio for $S(T_1) > K^{T_1}$ is

$$C(K^{T_2}, T_2) - \frac{D(T_2)F(T_2)}{D(T_1)F(T_1)} \left(S(T_1) - \frac{F(T_1)}{F(T_2)} K^{T_2} \right) \quad (2.30)$$

According to put-call parity, Equation (2.30) is equal to a put option with strike K^{T_2} and maturity T_2 . Since the value of portfolio for $S(T_1) \leq K^{T_1}$ is $C(K^{T_2}, T_2)$, therefore in either case, the portfolio is always positive. Expressed differently, $C(K^{T_2}, T_1) > C(K^{T_1}, T_1)$ with $T_2 > T_1$ for any strike

price. The call option price is increasing with time to maturity.

- Monotonicity respect to strike price:

$$-D(T) \leq \frac{\partial C}{\partial K}(K, T) \leq 0 \quad (2.31)$$

Consider I have two call option prices $C(K_1) < C(K_2)$, if I buy $C(K_1)$ and sell $C(K_2)$ ¹⁴, at the time t , I receive $C(K_2) - C(K_1) > 0$. If the payoff is not monotonically decreasing with respect to the strike price, then at the maturity, $C(K_2) - C(K_1) > 0$. This provides an opportunity for arbitrage. The left-hand inequality means that a small increase in the strike will cause a risk-free discounted decrease in the call price. The proof can be found in Hull and Basu (2016).

- Price bounds: I denote the S is the current price of the underlying and

$$D(T)F(T) = e^{-\delta\tau}$$

$$\max(0, (D(T)F(T)S(T) - D(T)K)) \leq C(K, T) \leq SD(T)F(T) \quad (2.32)$$

Where $D(T)K$ is the present value of strike, $D(T)F(T)S$ is the underlying price-adjusted dividend.

This right-hand inequality means, when adjusted for the dividend, the value call option is less than holding an underlying . If the value option call option is great than underlying, then there is an opportunity for arbitrage by buying the stock and selling the call option. The left-hand inequality implies that the value call option is great than a forward at the same strike. Let us consider a portfolio which contains a call option with strike K and time to maturity T and K amount zero-coupon bond. When the strike price is greater than K , the value of portfolio is $S(T)$, otherwise, the value of portfolio is K . Hence, the value of portfolio is

¹⁴buy low and sell high

2.3. No Arbitrage Conditions

always greater than $S(T)$. Following the no-arbitrage principle and discounting the value to today, the zero-coupon bond is worth $D(T)K$ and $D(T)F(T)S(T)$ is stock price discounting for the dividend. In addition, the call price cannot be negative. Therefore, the left-hand inequality holds.

- The final value of the option price is its payoff $C(K, 0) = \max(0, S - K)$

When the time to maturity is zero, I exercise the option right now, the value of option approaches its payoff. Since this condition is considered in type 1 constraints, please refer Merton(1973) and Hull and Basu (2016)for the detailed proof.

The above five conditions are summarized in The Theorem 2.1 in Roper (2010) from a mathematical martingale perspective(see Chapter 5 for more detail).

Chapter 3

Machine Learning in Finance

If you can't beat them, join them. Or maybe invest in them.

—Marsh (2018)

Machine learning has attracted a lot of attention over the past decade. From self-driving cars to speech/image recognition, from virtual assistants to cancer detection, it provides fruitful applications in many fields. In 2017, We witnessed an impressive breakthrough for artificial intelligence(AI)¹⁵. Google's AlphaGo, a machine learning program that plays the notoriously complex ancient board-game Go, quickly defeated top-ranked players. Training on millions of human professional games played in the past 3000 years, AlphaGo learned and mastered the game in the just 40 days¹⁶. The success of AlphaGo is a big leap forward for AI and assuredly it points towards the situation in the financial industry. Not surprisingly, investment banks, who have so much data for both buyers and sellers in the market, have already used machine learning to improve their performance. For example, JPMorgan¹⁷ and Citi¹⁸ use machine learning to make the competitive deals and develop trading strategies.

¹⁵Strictly speaking, machine learning is a subset of AI. Please see section 3.1 for more information

¹⁶see <https://deepmind.com/blog/alphago-zero-learning-scratch/> for more detail.

¹⁷see <https://www.bloomberg.com/news/articles/2018-08-20/inside-jpmorgan-a-bond-vet-builds-a-team-to-digitize-dealmaking> for more detail

¹⁸see <https://www.bloomberg.com/news/articles/2018-09-05/citi-s-credit-unit-starts-fintech-investment-group-under-zhang> for more detail.

There is no doubt that finance will be changed by artificial intelligence. However, how to incorporate machine learning to solve the real financial problems and how machine learning impacts the structure of the financial industry is of great interest to both academic researchers and practitioners. In this chapter, I briefly review algorithms and applications of two of the most popular machine learning techniques in finance. In Section 3.1, I introduce the basic concepts of machine learning and provide an overview of the neural network and support vector machine approaches. Section 3.2 explores how these two techniques can be applied in solving financial problems such as stock market forecasts, sentiment analysis, and bankruptcy prediction. Focusing on the options markets, I review important studies that use machine learning.

3.1 Machine Learning Overview

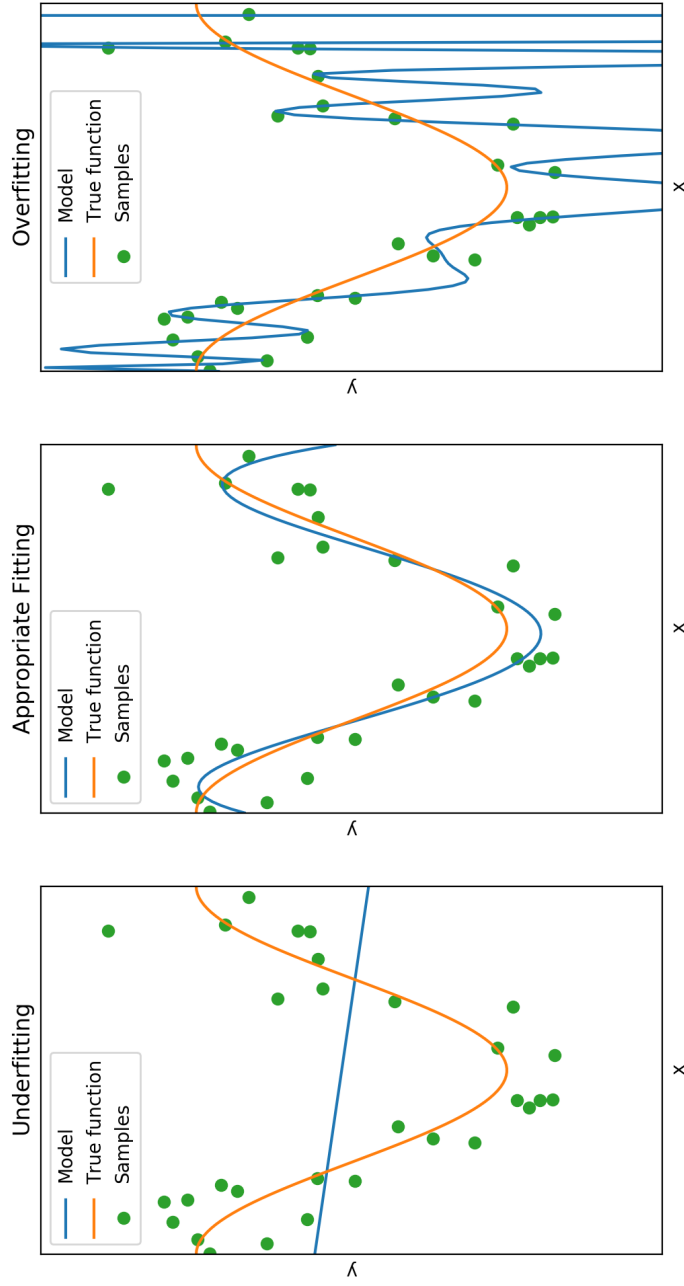
According to Stefanek (1987), AI is any computer system that exhibits elements of intelligence also shown in human behavior, such as understanding language, learning and reasoning. It includes broad approaches such as expert system, fuzzy system, neural network and support vector machine. Machine learning, as the most popular subset of AI, includes several statistical techniques that enable a machine to learn or complete a task. General speaking, “machine learning is using algorithms to extract information from input data and represent it in some type of model”(Patterson and Gibson (2017)). Based on the feature of training data, machine learning can be divided into two types: supervised and unsupervised. The objective of supervised learning is to find some type of model that maps inputs to the target value. Supervised learning is usually used in estimation and classification problems. In supervised learning, the training dataset clearly

labels the input data and the target value. Unsupervised learning starts with no information about the target value and is a self-adjusted algorithm which trains itself to discover patterns in data. An example of unsupervised learning is the The Beatles-style song written by Flow Machine of Sony Computer Science Laboratory (CSL)¹⁹.

According to Goodfellow et al. (2016), the machine learning framework generally contains four parts: the dataset, a loss function, an optimization algorithm and the estimated model. The dataset is usually split into training and testing subsets, with splitting done manually or via cross-validation. The loss function restricts model error and may be defined as the mean square error (MSE), cross-entropy loss or mean absolute error. The parameters of the estimated model are obtained from the training subset and the model's performance is evaluated using the testing subset. In practice, there is a trade-off between the quality of approximation and capability of generalization (Vapnik and Chervonenkis (2015)). That means, the learning algorithm may perform well in training but exhibit poor performance in prediction beyond the estimated dataset. From statistical learning theory, it can be shown that a model cannot achieve a perfect fit to the input data while being flexible enough to predict out-of-sample data. Vapnik and Chervonenkis (2015) argue that training error and the model complexity yield a U-shape relationship. This problem is known as bias-variance trade-off or Vapnik-Chervonenkis (VC) dimension in statistical learning theory. Figure 3.1.1 shows an example of trade-off between model complexity and generalization using different polynomial orders. Compared with the true function, the left panel that uses a linear function shows a low learning ability but high generalization ability. The right panel demonstrates that the 20th-order polynomial estimator has

¹⁹see <https://www.reuters.com/article/us-sony-algorithm-idUSKBN12H1ST> for more information.

Figure 3.1.1: Model Complexity and Generalization



Note: This figure reports an example of the polynomial fitting. The input contains 30 data points and the true function is $f(x) = \cos(2\pi x)$. The middle panel uses a first-order polynomial to estimate the problem. The right panel uses the 4th-order polynomial as the estimated model and the left panel uses the 20th-order polynomial to approximate the true function.

impressive learning ability but poor generalization ability. Usually, the left panel is regarded as under-fitting and the right panel is called over-fitting. Apparently, finding the optimal complexity point of the learning algorithm is not easy because it is determined by the input dataset and pre-defined training error.

3.1.1 Neural Networks in a Nutshell

The neural network is an important technique in machine learning. Inspired by the human nervous system, McCulloch and Pitts (1943) first describes mathematically a simple neural network based on a threshold logic algorithm. This work exhibits a parallel estimating architecture and is considered a milestone in Artificial Neural Network(ANN). Slightly later, Rosenblatt (1958) conducts the first valid neural network implementation on an IBM computer. Defining a “perceptron” structure, Rosenblatt (1958) successfully classifies a pattern set in the input using a single-layer perceptron. Follow this research direction, Minsky and Papert (1960) empirical test the learning ability of a perceptron structure. They show that no matter how long it is trained, the perceptron structure is incapable of learning a non-linear function. This limitation is resolved by Rumelhart et al. (1986) who extend the previous ‘single-layer’ network to multi-layers. By introducing so-called hidden layers in network structure, they show that a neural network can be a universal approximator and learn any function. However, because of the difficulty of training multi-layer network, although neural network performs well in prediction and classification tasks, it became unpopular for several years. The age of the neural network actually begins again in last decade or so. Hinton et al. (2006) rebrand the neural network as deep learning.

A standard neural network consists of three elements: neurons, layers and

connections between layers. Figure 3.1.2 illustrates a typical structure of a neural network. The input layer connects the input data x_i and the hidden layer. As shown in Figure 3.1.2, the neurons in the input layer receive input data and assign weights w_i plus a bias term b_i to them (gray lines in Figure 3.1.2). Taking these as input, the hidden layer that contains the transformation function (or activation function) maps a nonlinear transformation for weighted input. This layer is the most important and criticized part of the neural network. Since its values are not shown either in input and output data, mathematically, it is difficult to prove the resulting output in output layer is optimal. Obviously, in the prediction problem, the output layer only has one node but in the classification problem, the output layer contains many nodes (the number of nodes is equal to the number of classes). Figure 3.1.2 includes two hidden layers, therefore, the output from first hidden layer is further weighted by neurons and passed to the next hidden layer. The weights between different layers are updating repeatedly during the training process.

The neural network framework can be described as follows. Consider X $\{x_1, x_2, \dots, x_n\}$ as input vector, Y $\{y_1, y_2, \dots, y_n\}$ as output vector. f_1, f_2, \dots, f_l are the univariate transformation function for each hidden layer. The multilayer neural network can readily be defined as a chain structure $f(x) = f_l(\dots(f_2(f_1)))$. The length of the chain gives the depth of the neural network and the maximum number of neurons in hidden layers determines the width. If I express the problem in matrix form, the input data weighted by neurons defined as follows

$$\dot{Y} = X^T W + b \tag{3.1}$$

The three-layer neural network shown in Figure 3.1.2 is

$$\hat{Y} = f(X^T W + \bar{b})\tilde{W} + \tilde{b} \quad (3.2)$$

Where \hat{Y} is estimated vector, X is the input vector. $\{W, \tilde{W}\}$ is a weight matrix and $\{b, \tilde{b}\}$ bias terms. $f(\cdot)$ is the transformation function (or activation function) in each hidden layer. The common choices of $f(\cdot)$ are the polynomial function, radial basis function (RBF), spline function (see the next section for details) and sigmoid function²⁰. The weight matrix is initialized with random values or prior information before training the neural network. The process of training the neural network is to solve the optimization problem

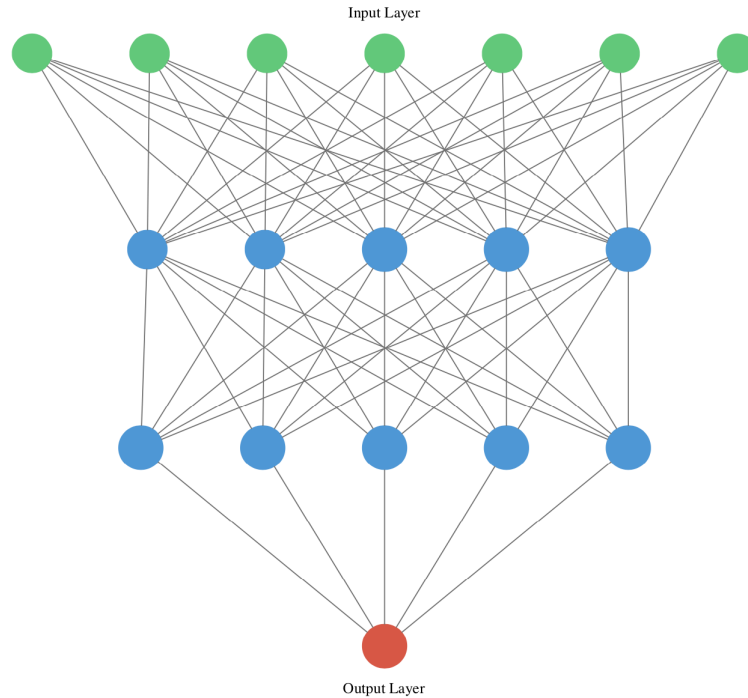
$$\operatorname{argmin}_{\Theta} \frac{1}{N} l(Y, \hat{Y}) = \operatorname{argmin}_{\Theta} \frac{1}{N} l(Y, f(X^T W + \bar{b})\tilde{W} + \tilde{b}) \quad (3.3)$$

Where $l(\cdot)$ is a loss function that measures the deviation between the predicted value and target value. $\Theta = \{W, \tilde{W}, b, \tilde{b}\}$ is the parameters set.

Based on Equation(3.3), I argue that training the network is an unconstrained nonlinear minimization problem. The goal of training the network is to find the weight matrix in Equation(3.3) which gives minimal error measure such as MSE . This is usually achieved using back-propagation optimization developed by Werbos (1974). Back-propagation optimization is a gradient descent method with constant step size and the procedure can be summarized in the following way. First, with the initialized weight matrix, calculate the gradient of the loss function with respect to each weight, such as $\frac{\partial |Y - \hat{Y}|}{\partial w_i}$ for mean absolute error. Second, adjust the weight according to the calculated error. If the error is increasing with weights, reduce weights; otherwise increase the weights. This adjustment is

²⁰These functions are reported in the Appendix.

Figure 3.1.2: Three-layer Neural Network Structure

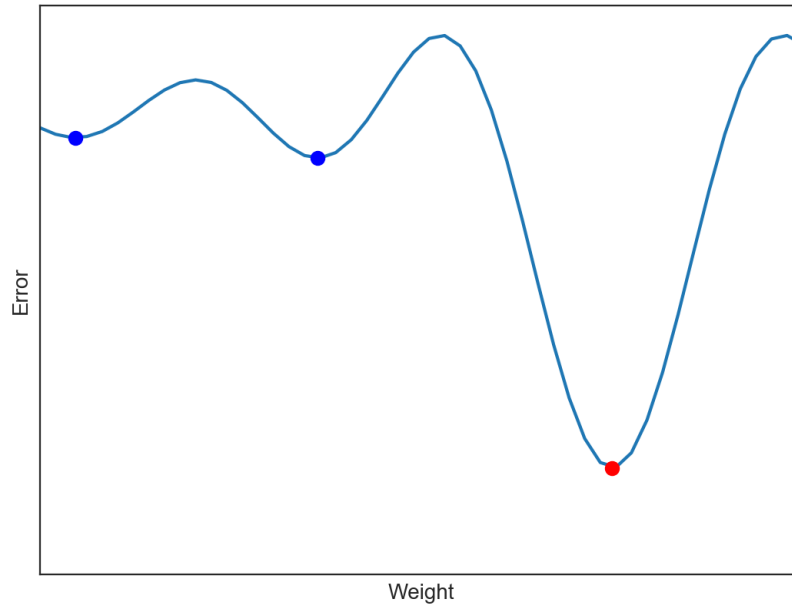


Note: This figure shows a structure of three-layer neural network. The green points are input layer, which contains 7 input features. The blue points represent the hidden layer that learning the input information. Gray lines denotes weights. The red point is output layer that gives the estimated value.

propagated backward; for instance, first correct the weights for the second hidden layer in Figure 3.1.2 and then the first hidden layer. After all weights are updated, the whole process starts over until the error meets the pre-determined threshold.

The neural network approach has been criticized for the stochastic weight initialization. The above optimization procedure generally leads to a minimum error. However, when the weight space exists and there are local and global minima, the optimal solution of the weight matrix is sensitive to the starting point. This effect is illustrated in Figure 3.1.3, where the blue points are local minima and the red point is global minimum. Simply, if the initial weight is between the first and second blue points, back-propagation moves the result to a local mini-

Figure 3.1.3: Weight Adjustment based on Back-propagation



Note: This figure shows the weight adjust. The x-axis is weight and the y-axis is the error between the target and the neural network predicted output. There are two local minimum and one global minimum in this search space. The blue points represent the local minimum and red point shows global minimum.

mum. If the initial weight is after the second blue point, the back-propagation finds the global minimum. Obviously, the global minimum is the preferred solution for Equation (3.3). However, with random initializing the starting weight matrix, there is no guarantee that guarantee the finding minimum is global. In particular, in the Figure 3.1.3, the probability of finding the minimum point is proportional to the distance between starting point to the relevant minimum points.

Another criticism of the neural network is that it is an ambiguous model. As shown in Figure 3.1.2, selecting the optimal model complexity, which is the degree in the polynomial estimated model, the number of knot for splines, is an

essential issue in machine learning. For the neural network, the optimal model complexity could be controlled by choosing the width and depth of the neural network, which determines the number of connections in the network. For a given problem, a neural network with limited connections tends towards underfitting while one with too many connections will over-fit. The optimal depth and width of a neural network are typically determined based on prior information or training dataset. Although Konishi and Kitagawa (2008) try to identify the optimal number of connections using AIC (Akaike Information Criterion), Ding et al. (2013) argue that AIC shows inconsistent performance and tends to overfit the model. Consequently, I argue there is no general principle for finding the optimal model complexity of the neural network and the performance of the network is highly dependent on network design.

3.1.2 Support Vector Machines in a Nutshell

The other important technique in machine learning is the support vector machine (SVM). In contrast with the neural network, SVM always yields global minima and has explicit model complexity control. The SVM is also a universal approximator (Hammer and Gersmann (2003)) and the optimal model complexity can be restricted via regularization. The idea of support vector machines originates in statistical learning theory, which deals with drawing effective statistical estimation from small samples. The initial theoretical work of SVM is developed by Vapnik and Lerner (1963) and extended to regression by Vapnik et al. (1997). The key idea of SVM is the support vector. As shown in Figure 5.3.1, only these points that lie on the margin (dash line) define the estimated model. Therefore, the estimated model is determined by number and weights of support vectors.

Define a high dimensional feature space as $\Psi \in R^N$ and there exists a function

ϕ that maps a random input vector $X \in R^d$ from R^d to R^N . In this space, if I consider the estimated function as a class of linear functions

$$F_{n,\Psi} = \{f : f(X) = w^T \phi(X) + b\} \quad (3.4)$$

where $w \in R^d$ and $b \in R^d$ is weight and bias respectively.

Even if the estimated function is in high-dimensional feature space, I do not need to treat the high-dimensional problem directly. Replacing the inner product of two data points X_k, X_l with the corresponding kernel $K(X_k, X_l)$, I can transform the estimated problem into a low-dimensional feature space. This is known as “kernel trick” and guaranteed by Mercer’s condition (Mercer (1909)).

Theorem 5. *(Mercer Condition) Let $K \in L^2(C)$ and $g \in L^2(C)$, where C is compact subset of R^d . The $K(t, z)$ describes an inner product of two feature space. If it is necessary and sufficient that*

$$\int \int_C K(t, z)g(t)g(z)dtdz \geq 0$$

where $t, z \in R^d$ and $g \in L^2(C)$, the continuous symmetric function K has an expansion

$$K(t, z) = \sum_{i=1}^{\infty} a_i \psi_i(t) \psi_i(z)$$

Proof: please refer to Mercer (1909)

Using the Mercer condition and defining $\phi(t) = \sqrt{a_i} \psi_i(t)$ and $\phi(z) = \sqrt{a_i} \psi_i(z)$, the kernel can be expressed as

$$K(t, z) = \phi(t) \phi(z) \quad (3.5)$$

Following Vapnik et al. (1997), the objective of SVM is to find a function that allows ε deviation from the target value (see Section 5.3 for more detail). Therefore, given the training data $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$, I minimize the empirical risk in the feature space

$$\min_{\{w,b\}} \frac{1}{n} \sum_{i=1}^n |Y_i - w^T \phi(X_i) - b|_\varepsilon \quad (3.6)$$

where $|\cdot|_\varepsilon$ is the ε insensitive loss function

$$|Y - F(X)|_\varepsilon = \begin{cases} 0 & |Y - F(X)| \leq \varepsilon \\ |Y - F(X)| - \varepsilon & \text{otherwise} \end{cases} \quad (3.7)$$

To ensure the estimated function is as flat as possible, I seek a small w that satisfies the Equation(3.6). If I consider minimizing²¹ the $\|w\|^2$, the optimization problem becomes

$$\min \frac{1}{2} \|w\|^2 \quad (3.8)$$

subject to

$$\begin{cases} Y_i - w^T \phi(X_i) - b \leq \varepsilon \\ w^T \phi(X_i) + b - Y_i \leq \varepsilon \end{cases}$$

By introducing the slack variables ξ_i and ξ_i^* and using Lagrange multipliers, Equation(3.6) can be transformed into following dual problem²²

$$\max_{\{\alpha, \alpha^*\}} - \frac{1}{2} \sum_{i,j=1}^n (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*) K(X_i, X_j) - \varepsilon \sum_{i=1}^n (\alpha_i - \alpha_i^*) + \sum_{i=1}^n Y_i (\alpha_i - \alpha_i^*) \quad (3.9)$$

²¹Please see Chapter 5 for the case of $|w|$

²²I recommend reader to refer Schölkopf and Smola (2002) for detailed derivation.

subject to

$$\sum_{i=1}^n (\alpha_i - \alpha_i^*) = 0$$

$$\alpha_i, \alpha_i^* \in [0, c]$$

where α_i and α_i^* are optimal solution of Equation(3.9), which can be considered as Quadratic Programming (QP) problem. The dual representation of the estimated model is

$$F(x) = \sum_{i=1}^n (\alpha_i - \alpha_i^*) K(x, X_i) + b \quad (3.10)$$

The data points with non-zero α_i are considered as support vectors. $K(x, X_i)$ is the symmetric kernel that satisfies the Mercer's condition.

Although there are many programming packages such as quadprog in Matlab and Cvxopt in Python to solve the QP problem, Andersen et al. (2011) find that their performance varies with their algorithms. To get rid of the influence of different algorithm, in this thesis, I further simplify the optimization problem into a linear programming (LP) problem (see Section 5.3) and use Python Cvxopt.

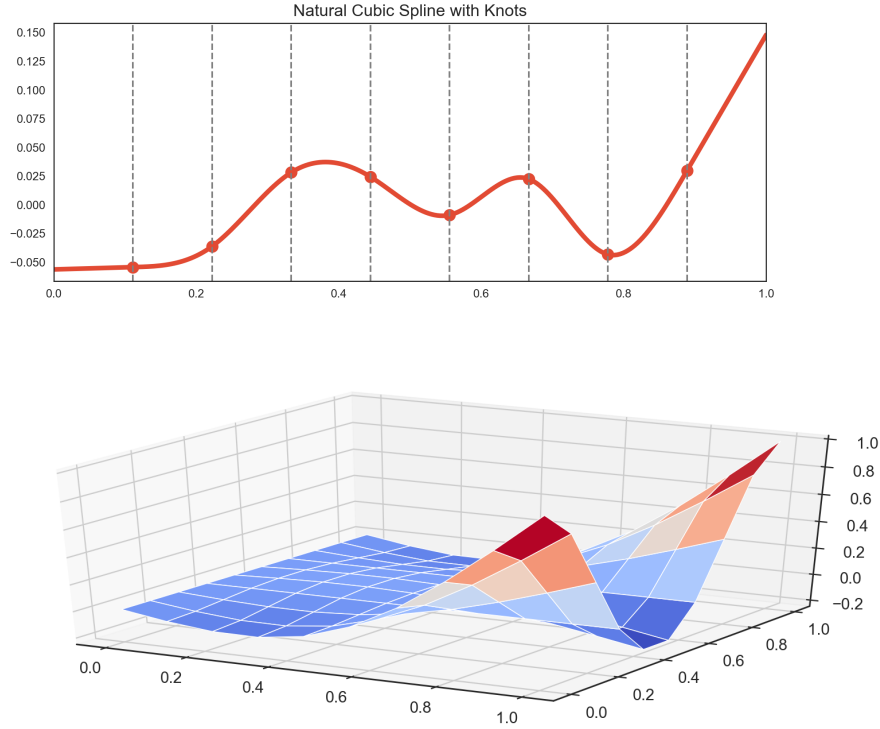
Since previous studies usually apply the spline function to approximate the call option price, following Birkhoff and de Boor (1965), I define the Spline and B-spline kernel as ²³:

Definition 4. (Spline) Consider a continuous function $s(x)$ with $x \in [a, b]$. If for each connection point $x_s \in [a, b]$ (or called knots), if a spline function's q-1 order derivatives exists, then this piecewise polynomial approximation $s(x)$ is called a spline function with order q.

The top panel of Figure 3.1.4 displays an example of a cubic spline which

²³Another kernels are introduced in the Appendix

Figure 3.1.4: Spline Kernel



Note: The top panel shows an example of cubic spline kernel. The knot points are shown as red points. The bottom panel plots a B-spline(or tensor product spline) kernel.

has the form $s(x) = \alpha_s x^3 + \beta_s x^2 + \gamma_s x + \varepsilon$ in interval $[0,1]$. The second-order derivatives of $s(x)$ exist in each knot point (red points). If I further define this second derivative as equal to zero, then the spline function is called a natural spline. Extending the univariate spline to the bivariate case, I define the B-spline as follows.

Let the $U = \{u_0, u_1 \dots u_h\}$ and $V = \{v_0, v_1 \dots v_h\}$ be the knot vectors in u and v direction and the knot points are $p_{i,j}$, where $0 \leq i \leq m$ and $0 \leq j \leq n$, the B-spline kernel with degree p in the u direction and the degree q in the v direction is defined as the tensor product of two spline functions

$$k(u, v) = \sum_{i=0}^m \sum_{j=0}^n c_{i,j} K_{i,p}(u) K_{j,q}(v) \quad (3.11)$$

where $K_{i,p}(u)$ and $K_{j,q}(v)$ are basis spline functions of degree p and q , respectively.

If I define the tensor product spline as matrices of $\mathbb{R}^{M_K \times N_K}$

$$K_{mn} = \begin{pmatrix} K(X_1)K(Y_1) & \dots & K(X_1)K(Y_n) \\ \vdots & \ddots & \vdots \\ K(X_m)K(Y_1) & \dots & K(X_m)K(Y_n) \end{pmatrix}$$

Then the matrix $\mathbb{K} \in \mathbb{R}^{(M_T \times M_K) \times (N_T \times N_K)}$ is simply defined as

$$K = \begin{pmatrix} K_{11} & \dots & K_{1N_T} \\ \vdots & \ddots & \vdots \\ K_{M_T1} & \dots & K_{M_T N_T} \end{pmatrix}$$

The bottom panel of Figure 3.1.4 shows an example of B-spline (or tensor product spline) kernel. As illustrated in the Figure 3.1.4, the B-spline kernel is a surface. Thus, compared with a univariate spline, it is a good way to avoid extrapolation in one direction when approximating a surface. Overall, following Bishop (2006), I compare the neural network and SVM in terms of ability in handling noisy data, processing large datasets, controlling model complexity, predictive accuracy and ease of operation in Table 3.1.1. Compared with a neural network, the support vector machine has high ability to handle noisy data, processing large data set and controlling model complexity. This is because, as shown in right subplot in Figure 5.3.1 and suggested by Vapnik et al. (1997), the estimated function is only influenced by support vectors. Therefore, without considering every data point, the SVM can easily process a large dataset and control model complexity. Furthermore, since the outliers do not affect either the shape or slope of the

Table 3.1.1: Characteristics of Neural Network and SVM

| | Neural Network | SVM |
|------------------------------|----------------|---------------|
| Ability to handle noisy data | poor | good |
| Process large data set | poor | good |
| Model complexity control | poor | good |
| Predictive accuracy | high | relative high |
| Ease of operation | difficult | easy |

estimated function, the SVM has high ability to handle noisy data. However, since the SVM model is highly kernel-dependent, the predictive accuracy is very sensitive to the selection of kernel function. Therefore, compared with a neural network, the SVM has relatively lower predictive accuracy.

3.2 Machine Learning in Finance

Although standard econometrics theory and parametric estimation methods dominated the finance literature for many years, motivated by the data-driven property of machine learning, various studies have applied machine learning in solving stock market forecasting, bankruptcy prediction and investor sentiment analysis. In this section, I provide a brief review focussing on the application of machine learning in time series forecasting and classification.

The stock market forecast problem can be regarded as a time series forecasting, which assumes that future outcomes are based (or partly based) on past observations. Forecasting time series using machine learning exhibits an attractive feature over standard econometrics tools since it is a data-driven approach. That said, even if the underlying relationship between input and output is difficult to specify, the machine learning approach can capture functional relationships. One of the successful applications of machine learning in the stock market is predicting stock volatility. GARCH-based volatility models stemming from Engle

and Granger (1987)²⁴, have long served as a benchmark for modeling past return movement and future volatility in finance. However, with the influence of unexpected news, the GARCH family framework may infer incorrect return-volatility relationships and lead to wrong trading strategies²⁵. Using the data-driven property and functional flexibility of machine learning, Donaldson and Kamstra (1997) and Pérez-cruz et al. (2003) use the neural network and support vector machine to estimate GARCH parameters respectively. They show that both techniques yield higher predicting ability than the common maximum likelihood estimated method. Following their studies, others investigate the performance of hybrid GARCH and machine learning models. For example, Cui et al. (2015) propose a new statistical arbitrage model based on a combination of a TGARCH model and a wavelet neural network. Using data from the Chinese metal futures market, they show that this hybrid model provides more accurate trading thresholds compared with the back-propagation neural network and simple TGARCH model.

The other application of machine learning in finance is classification. Essentially, the machine learning technique for classification is similar to logit regression, which labels the two possible dependent values as 0 and 1. However, thanks to its learning from examples, the machine learning technique can explore potentially “hidden” correlations among the predictive variables non-linear estimated function. Tam and Kiang (1992), estimate the bankruptcy problem and demonstrated the superiority of the neural network over discriminant analysis in terms of predictive accuracy and robustness. Later studies apply support vector machine (Chen et al. (2011)) and hybrid machine learning models (Chen (2014)) in the bankruptcy classification problem. In line with these studies and inspired

²⁴Engle received the Nobel price in 2003 for his contribution of “analyzing economic time series with time varying volatility”. Please see <https://www.nobelprize.org/prizes/economics/2003/engle/facts/> for more detail

²⁵This is called as asymmetry effect in literature.

by natural language processing, Luss and D'Aspremont (2015) and Pang et al. (2002) use machine learning to conduct sentiment analysis. Using newspaper articles and 10-K filings, they successfully estimate the relationship between investor sentiment and stock returns.

Studies applying machine learning in option markets started with Hutchinson et al. (1994). Using the S&P 500 index call option prices between 1987 and 1991, Hutchinson et al. (1994) apply two neural network models (the RBF network and multilayer perceptron) to pricing and hedging options. Considering the homogeneity of price and assuming constant volatility and the risk-free rate, they demonstrate that the neural network model outperforms the Black-Scholes (BS) formula. Subsequent studies further improve and extend this work by considering the stochastic volatility (Buchen and Kelly (1996)), including more input variables (e.g. interest rate (Qi and Maddala (1996))), investigating other markets (Bennell and Sutcliffe (2003)) and applying an alternative learning technique (Liang et al. (2009)).

From Equation(5.5), since risk neutral density is equal to the second derivative of the call option pricing function, the performance of estimation of state price density highly depends on the estimated option pricing model. In this thesis, I use the support vector machine to extract state price density (or risk neutral density) for two reasons. First, as noted in Section 3.1, the neural network has ambiguous model complexity control. The width and depth of the neural network, which determines the number of connections in the network, is selected from experience. Therefore, compared with SVM, the estimation model using a neural network may lead an over-fitting or under-fitting of the result. Second, based on the intuition of SVM (please see Chapter 5), the ε insensitive loss function in SVM naturally considers the bid-ask spread in the estimation model. However, as shown Table

3.2. *Machine Learning in Finance*

3.1.1, the support vector machine (SVM) technique is easily affected by noisy data; therefore I apply a relatively comprehensive data filter in Chapter 4.

Chapter 4

Option Data Panel

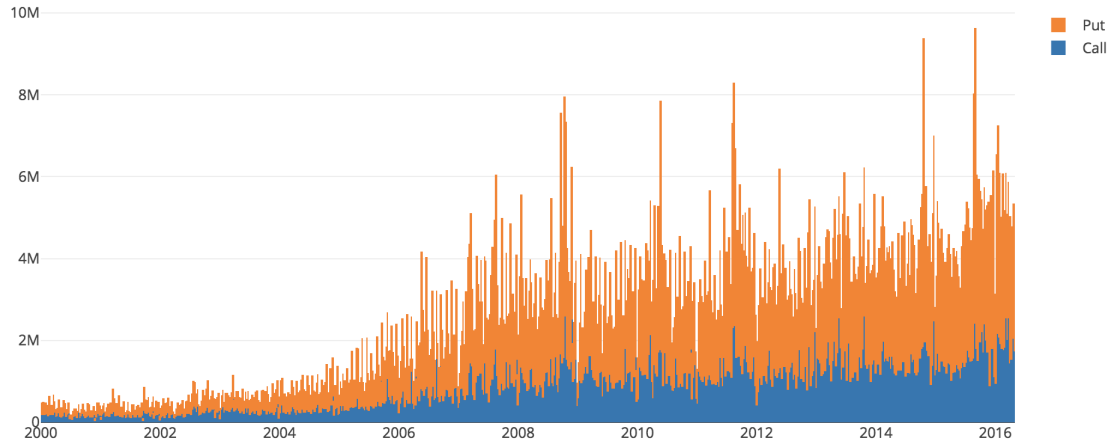
The empirical analysis of this thesis is based on a large daily option panel over 16 years. In section 4.1, I describe the dataset, including general characteristics and the reason of choosing S&P 500. Section 4.2 provide a detail description of data filter rules. Based representative, accurate and arbitrage-free principles, I develop a three levels of filter(data accuracy, liquidity and no arbitrage). Finally, in section 4.3, I show how to change the estimated framework from risk neutral measure \mathbb{Q} to forward measure \mathbb{Q}^T and derive the no-arbitrage conditions for call price under forward measure.

4.1 Data Overview

This thesis examines a large daily cross-sectional option panel from January 5, 2000 to April 30, 2016. These were the latest available data when chapter 5 was the written. I investigate closing Standard & Poor's 500 (S&P 500) option price from OpionMetrics via Wharton Research Data Services(WRDS). The S&P 500 index option (option symbol SPX) is traded in the Chicago Board of Trade's Option Exchange (CBOE) between 8:30 to 15:15 CST in every trading day. An option on the S&P 500 is cash settled²⁶ in the morning on the third Friday of

²⁶Option holder/writer pay the difference between strike and underlying price when the option expires

Figure 4.1.1: Daily Trading Volume of S&P 500 Index Option in my Sample



Note: This figure plots the daily trading volume of S&P 500 index option between January 5, 2000 to April 30, 2016

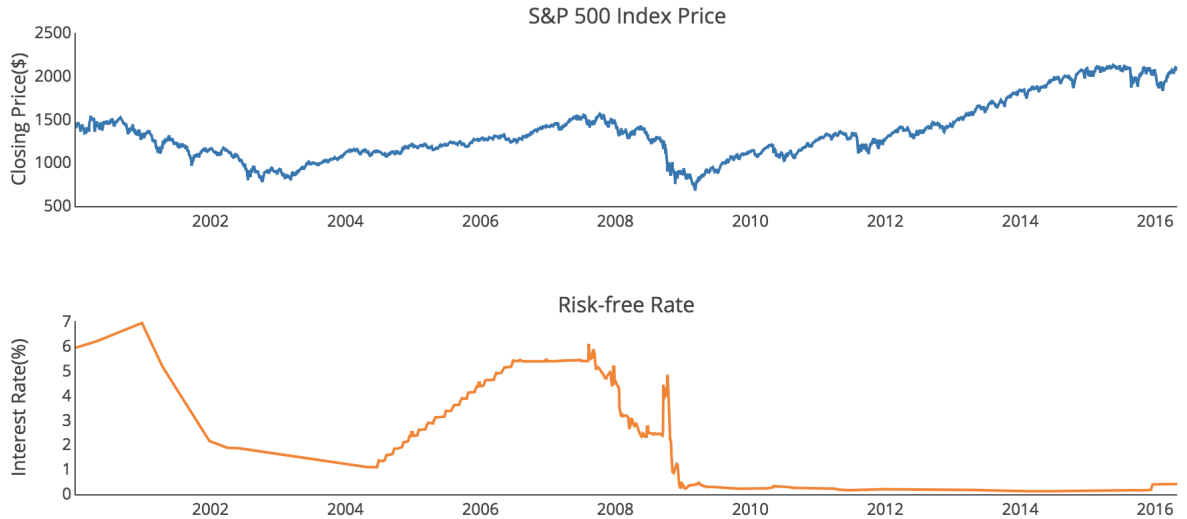
every month²⁷. There are up to 12 maturities available for each trading date and the strike interval is \$5. The underlying S&P 500 index is a value-weighted index that contains the top 500 companies according to market capitalization. It is usually treated as a leading benchmark of the American equity market. In the sample period, there are 915 firms in the S&P 500 index because the constituent firms are updated over time as market capitalization changes. The companies list is in Appendix. The S&P 500 index price, trading volume, risk-free rate are also downloaded from OptionMetrics.

Figure 4.1.1 depicts the trading volume of S&P 500 index options in my sample. It is evident that put options are more active than call options. The trading volume of put options has several peaks during the sample period. For example, one peak point is near the 2008 financial crisis; I suggest that this

²⁷See <http://www.cboe.com/products/stock-index-options-spx-rut-msci-ftse/s-p-500-index-options> for more details

4.1. Data Overview

Figure 4.1.2: Time Series of S&P 500 Index and Risk-free Rate



Note: This figure reports the monthly average of S&P 500 index and risk-free rate over the period January 5, 2000 to April 30, 2016. Both data are from OptionMetrics. The average dividend rate is 2.53% and the interest rate is 1.17% in this period.

overtrading of put options is caused by investor sentiment. Investors rushed to purchase put options so as to lock in current profits and protect themselves from the future market crash.

In Figure 4.1.2, I plot the monthly average of the S&P 500 index and the risk-free rate from 2000 to 2016. The S&P 500 index price dropped significantly in the 2008 financial crisis and the risk-free rate maintained a near zero value over the period from 2008 to 2016. The corresponding monthly average call price is plotted in Figure 5.4.1. As the S&P 500 includes a broad range of industries such as technology, retail, and financial services, its price is popular in previous literature as an indicator of consensus among market participants. Therefore, the corresponding S&P 500 index option reflects most investors' expectations about future market movements and the average attitude of market participants. That

said, economically, investors in the S&P 500 index comprise a good indicator of the representative agent.

4.2 Data Filter

In this section, I present the data filters rule and process. The option price selected in this thesis is inspired by three principles: representative, accurate and arbitrage-free. To infer the general properties of call option prices, a representative dataset should be estimated. I construct my dataset with daily European option written on the S&P 500 index as an economically representative market portfolio. I use only the out of the money (OTM) options because these have higher liquidity and therefore the quoted price are closer to theoretical prices. The OTM option is defined as $K < F_0$ ²⁸ for put option and $K > F_0$ for call option. To increase the data quality and assuming put-call parity holds

$$C - P = e^{-rt}(F - K) \tag{4.1}$$

Taking advantage of this relationship, I need to get the daily interest rate as close as possible to market to transform the OTM put prices to ITM call prices. There are two ways to deal with the issue.

1. Interpolation

The interest rate is intuitively determined by the interest rate yield curve; therefore, I can linear interpolate the curve to get the daily interest rate. For example, if the interest rate for 1-month maturity is 0.97%, 3-month maturity is

²⁸ F_0 is at the money(ATM) forward

0.95%, the interest rate for maturity at 24 days is computed as:

$$\begin{aligned} r &= r_{1mth} - (r_{3mth} - r_{1mth}) * \frac{30 - 24}{61} \\ &= 0.97\% - (0.95\% - 0.97\%) * \frac{30 - 24}{61} = 0.9720\% \end{aligned} \quad (4.2)$$

2. Simple Regression

If I use the put call parity relationship and consider a linear regression of at least 4 put-call option pairs, the put call parity of Equation (4.1) can be expressed as:

$$C_i - P_i = \alpha + \beta K_i \quad (4.3)$$

Where $\alpha = Fe^{-r\tau}$, $\beta = e^{-rt}$ and interest rate $r = \frac{-\ln(\beta)}{\tau}$, dividend yield is $\frac{-\ln(\frac{\alpha}{K})}{\tau}$. This approach enables us to back out the daily interest rate and dividend at the same time.

To ensure my dataset is close to the market, in this thesis, I adopt both approaches. I first linearly interpolate the yield curve to get daily interest rate because I can get zero coupon rate from OptionMetrics database. I use this as my interest rate. Subsequently, I use Equation (4.3) to compute the daily dividend. Last but not least, I modify the data filters of Constantinides et al. (2013) and apply three levels of filter designed to obtain accurate and arbitrage-free S&P 500 option prices.

1. Data Accuracy Filter

- Identical observations: I drop all duplicated observations. When observations have the same identical terms (start date, expiration date, strike, option type) but differ in price, I keep the quotes whose implied volatility is significantly away from its moneyness neighbors.

2. Liquidity Filter

- Zero bid price : I remove quotes with zero bid prices to avoid illiquid options.
- Zero volume: in the same spirit, I remove quotes with zero volume.
- Days to maturity <7 or >365 : I remove data with maturity less than 7 days or more than 1 year. According to Constantinides et al. (2013), quotes with shorter maturity will tend to move erratically and quotes with longer maturity lack volume. I modify this filter with longer durations²⁹ because, from Figure 4.2.1, S&P 500 options are still active before 1 year.
- Moneyness <0.8 or >1.2 : Constantinides et al. (2013) argue that option quotes in this range are thinly traded. As shown in Figure 4.2.1, I also observe this feature in my dataset. Therefore I remove option quotes with moneyness (the ratio of strike price to index price) below 0.8 or above 1.2.
- Implied volatility $<5\%$ or $>100\%$: I remove all quotes with implied volatility lower than 5% or higher than 100% because as suggested by Constantinides et al. (2013) , option quotes in this range can be considered as illiquidity.

3. No arbitrage Filter

- Negative Implied interest rate: I remove quotes with negative put-call parity implied interest rate. For each available date and maturity, I use put call pairs with at least six strike prices to calculate implied interest rate³⁰.
- Negative Time Value: I discard quotes that have negative time value. Option prices consist of two components: intrinsic value and time value. Option quotes with negative time value show little information about investor's

²⁹Constantinides et al.(2013) choose 0.5 years as days to maturity upper bound.

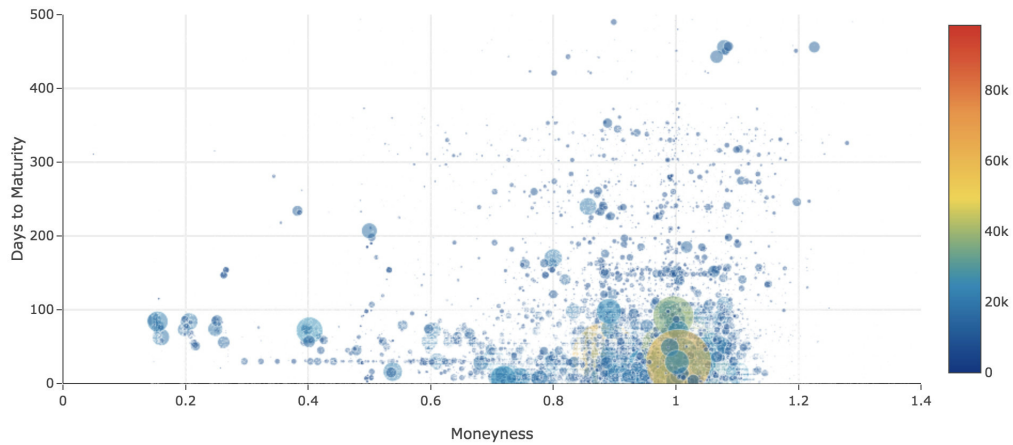
³⁰I use put call parity to get implied interest that minimize implied forward pricing errors.

expectation because time value representing the amount of risk premium that investor is willing to pay.

- IV filter: I remove quotes whose implied volatility is one standard deviation away from the average among peers. I define peer group by the different levels of moneyness. More precisely, for each date and maturity, I fit the log implied volatiles in entire sample via a quadratic curve (separately for call and put options). I compute the relative distance of all observed IV from fitted IV and then I truncate the fitted curve to bins of moneyness with a width of 0.05(0.8, 0.85, . . . , 1.2). After calculating the standard deviation for each moneyness bin, I discard quotes whose observed IV is one standard deviation apart from the fitted IV.
- Put-call parity filter: I remove any quote whose the put-call parity implied interest rate is more than one standard deviation away from the average among the peers. Peer group is defined as quotes with the same (date, time to maturity) pairs. I trim outliers in a similar way as with the IV filter. Specifically, I use the whole sample of distances of the put-call parity implied interest rates from the corresponding daily median implied interest rate to find the standard deviation of the corresponding distances.

Table 4.2.1 records the number of observations at each filtering level that are removed. Before the filters, I have a total of 8,261,170 observations. Level 1 filters remove 10 observations. The zero volume criteria in the Level 2 filters eliminate the most observations (5,433,167) and level 3 filters eliminate 8.3% of observations. The final number of observations in this thesis is 498,778, which is reasonable for three reasons. First, compared with Chiang et al. (2016), who also follows Constantinides et al. (2013) and gets total 404,822 observations on S&P

Figure 4.2.1: Trading Volume of S&P 500 on 2016



Note: This figure shows the trading volume of S&P 500 options. The data period is from 01 January 2016 to 30 April 2016.

500 options between 1996 and 2011, my dataset is relative large. Second, the No Arbitrage assumption is a strong assumption about option price observations in this thesis because the real traded option prices are determined in the real market, in which there are arbitrage opportunities. Third, as discussed in section 4.2, four No Arbitrage (NA) filters are supported by option pricing theory. Figure 4.2.1 plots the trading volume of the first 4-month in 2016 and the number of trading volume is indicated by color. Based on the liquidity filter, I remove the option whose days to maturity are greater than 365. Also, the trading volume for options with moneyness before 0.8 and days to maturity greater than 100 are scarce and I select the options between 0.8 and 1.2 moneyness and option that expire after 1 year.

I consider a sample of option price data on 27 August 2008 to illustrate the filter process and check the no-arbitrage property of the processed result. The

4.2. Data Filter

Table 4.2.1: Number of Observations

| | | Deleted | Remaining |
|-------------------|--------------------------------|-----------|-----------|
| Starting | Calls | | 4,130,624 |
| | Puts | | 4,130,546 |
| | All | | 8,261,170 |
| Accuracy Filters | Identical Terms | 10 | |
| | All | | 8,261,160 |
| Liquidity Filters | Zero Bid /Ask | 708,202 | |
| | Zero Volume | 5,433,167 | |
| | Days to Expiration Boundaries | 262,388 | |
| | Moneyness Boundaries | 254,545 | |
| | IV Boundaries | 417,832 | |
| | All | | 1,185,026 |
| NA validation | Negative Implied Interest Rate | 289,542 | |
| | Negative Time Value | 232 | |
| | IV Outlier | 173,725 | |
| | Implied Interest Rate Outlier | 222,749 | |
| | All | | 498,778 |
| OTM Options | | | 390,320 |

Note: Table 4.2.1 presents the number of observations after each filter. The sample period is from 01/01/2000 to 04/30/2016. The moneyness is defined as the ratio of strike price to index price. IV is implied volatility, NA is no arbitrage and OTM is out the money. Implied interest rate is calculated based on linear regression(Equation 4.3). The time value of option is defined in Equation(4.4) and (4.5).

raw option data and calculated call price are shown in Table 4.2.2. The forward price is \$1465.40 and the S&P 500 index price is \$1281.66, which means that the S&P 500 index dividend rate is higher than the risk-free interest rate. This is confirmed by statistics of my full dataset. The average dividend rate is 2.53% and the interest rate is 1.17%. Figure 4.2.2 displays the distribution of call prices after filters. I note that there are three maturities available on this trading date. Clearly, the options trading of deep OTM is very thin even for S&P 500 index options³¹. Although the dataset does not have a large coverage of call prices as high-frequency data, I argue my dataset is efficient enough to cover the call price surface. Compare with Chiang et al. (2016), who gets total 404,822 observations, my dataset is relative large. Also as shown in Figure 5.4.1, my dataset is large enough to plot a well-defined call price surface.

Furthermore, to compare my no arbitrage filter with Zhang and Xiang (2008), I plot the time value of call option price on 04/11/2003 in Figure 4.2.3 and check whether the dataset shows no arbitrage. The time value of an option is defined as

$$c_{time}(k, \tau) = c(k, \tau) - e^{-rt} \max(F_0 - K, 0) \quad (4.4)$$

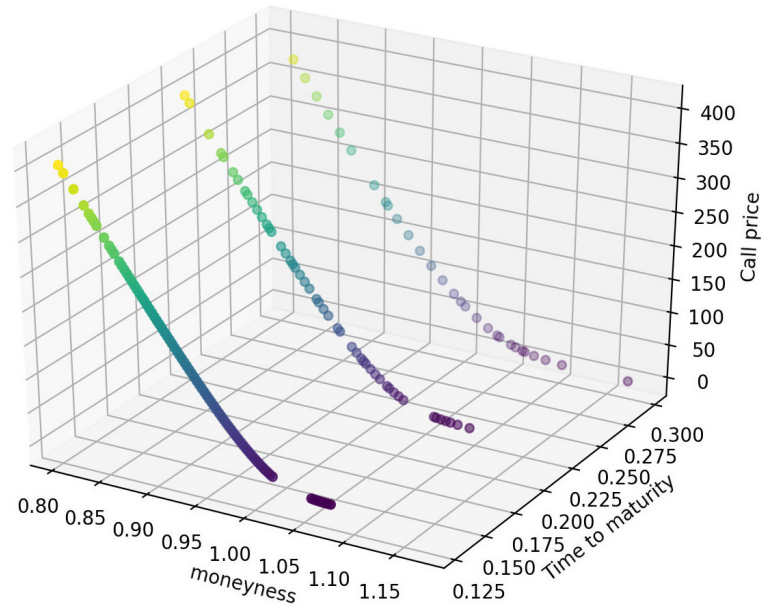
$$p_{time}(k, \tau) = p(k, \tau) - e^{-rt} \max(K - F_0, 0) \quad (4.5)$$

If the put call parity relationship in Equation (4.1) holds, this means that time value of call price $c_{time}(k, \tau)$ and put price $p_{time}(k, \tau)$ are equal. Compare Figure 4.2.3 with Figure 2 in Zhang and Xiang (2008), I argue that my dataset is arbitrage free and the time value of call and put prices almost coincide. The

³¹The deep of OTM option means the strike price is far away from current index price. Consequently, buying a deep OTM option is expecting that a extreme increase of index price.

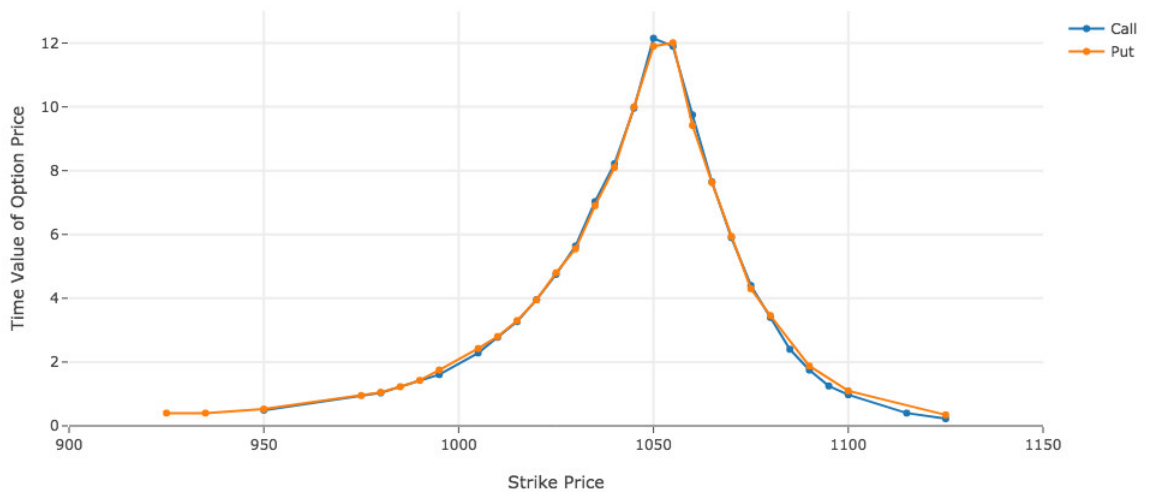
4.2. Data Filter

Figure 4.2.2: Distribution of Call Option Price after Filters on August 27, 2008



Note: This figure shows the S&P 500 index call option price across different strike and time to maturity on 27/08/2008. The S&P 500 index price is \$1281.66.

Figure 4.2.3: Time Value of S&P 500 Option Price on November 04, 2003



Note: This figure shows the time value of S&P 500 index call and put options on 04/11/2003. The days to maturity is 17 and the S&P 500 index price is \$1053.25.

4.3. Change of Measure for Estimated Framework

Table 4.2.2: Option Data Panel on August 27, 2008

| Options | Strike | Bid | Ask | Volume | Call Option Price |
|---------|---------|------|------|--------|-------------------------|
| Put | 1435 | 20 | 22.2 | 3177 | 51.40 |
| | 1440 | 20.9 | 23.5 | 7168 | 47.52 |
| | 1445 | 22.4 | 25 | 5322 | 44.03 |
| | 1450 | 24.2 | 25.3 | 25947 | 40.10 |
| | 1455 | 25.6 | 28.2 | 2594 | 37.26 |
| | 1460 | 27.5 | 29.9 | 3339 | 34.08 |
| | 1465 | 29.2 | 31.8 | 10440 | 30.89 |
| ATM | 1465.40 | | | | |
| Call | 1470 | 26.5 | 29.1 | 142 | 27.80 |
| | 1475 | 25 | 25.8 | 4931 | 25.40 |
| | 1480 | 22 | 23.5 | 2574 | 22.75 |
| | 1485 | 18.5 | 20.9 | 355 | 19.70 |
| | 1490 | 16.1 | 18.5 | 888 | 17.30 |
| | 1495 | 13.9 | 16.3 | 265 | 15.10 |
| | 1500 | 12.4 | 13.5 | 29721 | 12.95 |

Note: Table 4.2.2 shows the market data on 27/08/2007. The days to maturity is 24. The computed ATM forward price is \$1465.40 and S&P 500 index price is \$1281.66. To give reader a general idea, this table only presents a part of data.

difference in peak time value of call price in Zhang and Xiang (2008) is significantly higher than mine. That said, my dataset contains less noise than Zhang and Xiang (2008).

4.3 Change of Measure for Estimated Framework

4.3.1 Change to Forward Measure

To compare my L_1 -SVM estimator with other models in Chapter 5, I investigate the estimated call price problem under the forward measure (from \mathbb{Q} to \mathbb{Q}^T). Define $F(t, T)$ as the forward measure and $F(T, T) = 1$. According to

equation(2.11),

$$E^{\mathbb{Q}}[V(T)|\mathcal{F}] = E^{\mathbb{Q}^T}[V(T)\frac{M(t)P(T,T)}{M(T)P(t,T)}|\mathcal{F}] \quad (4.6)$$

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{e^{\int_t^T r(t)dt}}{F(t,T)} \quad (4.7)$$

Previous studies have different assumptions regarding interest rate and dividends. Some studies allow constant interest rate and some for a deterministic rate (see Chapter 5 Table 5.5.1 for summary). To compare existing estimators with the proposed estimator, I normalize the call price to eliminate the influence of interest rate and dividend. Recall Merton(1973)'s model, which includes dividends in the Black-Scholes framework. I define forward price $F(t,T) = se^{(r-\delta)\tau}$ and simplify the $e^{\int_t^T r(t)dt}$ as e^{rt} .

$$C(K, \tau) = se^{-\delta\tau} N(d_1) - Ke^{-r\tau} N(d_2) \quad (4.8)$$

$$\frac{C(K, \tau)e^{rt}}{se^{(r-\delta)\tau}} = \frac{se^{-\delta\tau} N(d_1)e^{rt}}{se^{(r-\delta)\tau}} - \frac{e^{rt}Ke^{-r\tau} N(d_2)}{se^{(r-\delta)\tau}} \quad (4.9)$$

$$\frac{C(K, \tau)}{se^{-\delta\tau}} = N(d_1) - \frac{Ke^{-r\tau} N(d_2)}{se^{-\delta\tau}} \quad (4.10)$$

If I define the forward moneyness $k = \frac{K}{F(t,T)} = \frac{K}{se^{(r-\delta)\tau}}$, then the Black-Scholes-Merton formula can transform to

$$\frac{C(K, \tau)e^{rt}}{F(t,T)} = N(d_1) - kN(d_2) \quad (4.11)$$

$$d_1 = \frac{\ln(\frac{s}{K}) + (r_{t,\tau} - \delta_\tau + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln(\frac{1}{k}) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \quad (4.12)$$

$$d_2 = d_1 - \sigma\sqrt{\tau} \quad (4.13)$$

Compare Equation(4.8) with Equation(4.11), the Equation(4.11) is equivalent to a European call option price with underlying is 1, interest rate r and dividend yield δ equal to 0. Hence, by transforming the estimating framework to forward measure, I eliminate the influence the interest rate and dividend yield.

4.3.2 No Arbitrage Conditions of Call Price under Forward Measure

There are two ways to restrict the arbitrage free call price. One is from the state price density perspective and the other is from the option strategies perspective (see Chapter 2 section 2.3). In this section, I show how to get the similar no arbitrage conditions of Roper (2010) under the forward measure. First, recall the no arbitrage conditions under the risk neutral measure \mathbb{Q} . The call price is the numerical integration of the payoff and risk neutral density (RND).

$$C(S_t, K, \tau, r, \delta) = e^{-r\tau} E_t^{\mathbb{Q}}[\max(S_T - K, 0)] = e^{-r\tau} \int_0^{\infty} \max(S_T - K, 0) q^*(S_T) dS_T \quad (4.14)$$

Following Equation(4.7) and define $k = \frac{K}{F(t,T)}$

$$C(S_t, k, \tau, r, \delta) = E_t^{\mathbb{Q}^T}[\max(\frac{S_T}{F(t,T)} - \frac{K}{F(t,T)}, 0)] = \int_0^{\infty} \max(\frac{S_T}{F(t,T)} - k, 0) q^{\mathbb{Q}^T}(S_T) dS_T \quad (4.15)$$

Take the first order derivative

$$\frac{\partial C(k, \tau)}{\partial k} = - \int_0^{\infty} q^{\mathbb{Q}^T}(S_T) dS_T = -(1 - F(k)) \quad (4.16)$$

4.3. Change of Measure for Estimated Framework

Where $F(k)$ is cumulative distribution function of the transition probability q^{Q^T} under forward measure

$$\Rightarrow \frac{\partial C(k, \tau)}{\partial k} = (F(k) - 1) \quad (4.17)$$

As $F(k)$ is always greater than zero:

$$\Rightarrow F(k) \geq 0$$

$$\Rightarrow \frac{\partial C(k, \tau)}{\partial k} \geq -1 \quad (4.18)$$

Since q^{Q^T} is transition probability, based on a generic property of probability density, it is non-negative.

$$\Rightarrow q^{Q^T}(S_T) \geq 0 \quad (4.19)$$

Based on Equation (4.16),

$$\Rightarrow \frac{\partial C(k, \tau)}{\partial k} \leq 0$$

Therefore, recalling Equation (4.18), I get

$$-1 \leq \frac{\partial C(k, \tau)}{\partial k} \leq 0 \quad (4.20)$$

Further differentiate the Equation (4.16),

$$\frac{\partial C(k, \tau)}{\partial k^2} = q^{Q^T}(S^T) \geq 0 \quad (4.21)$$

Since $C(S_t, k, \tau, r, \delta) = E_t^{Q^T} [\max(\frac{S_T}{F(t,T)} - \frac{K}{F(t,T)}, 0)]$, by Jensen's inequality (Jensen

(1906)), the max function is convex, thus

$$C(S_t, k, \tau, r, \delta) \geq \max(E_t^{Q^T}[(\frac{S_T}{F(t, T)} - k, 0)]) = \max(1 - k, 0) \quad (4.22)$$

This result can also be obtained by setting $D(T)F(T) = 1$ and $S(T) = 1$ in Equation(2.27) and (2.32). When $k = 0, C(S_t, k = 0, \tau, r, \delta) = E_t^{Q^T}[\frac{S_T}{F(t, T)}] = 1$. Intuitively, following Equation(4.19), the price bounds under forward measure is

$$0 < C(k, \tau) < 1 \quad (4.23)$$

Similarly, I can prove (C3) that $k \rightarrow \infty$, the option becomes worthless. From practical point of view, investor would not buy this option because it is impossible to exercise it. C(5) is easy to prove follow the convexity and monotonicity of call option price.

Following Ait-Sahalia and Duarte (2003) and Fengler and Hin (2015), I argue that using the homogeneity assumption, C(1),C(2) and C(4) are sufficient to restrict an arbitrage free call price surface. Therefore in Equation(5.30), I only consider incorporate C(1),C(2) and C(4) in machine learning framework $L_1 - SVM$.

4.4 Summary

In this chapter, I present a new approach of data filter based on three principles: representative, accurate and no arbitrage. Interestingly, as argue by Constantinides et al. (2013), previous studies seem relatively arbitrary because they only consider part of filters. For example, Bakshi et al. (2003) only consider the zero bid price and volatility arbitrage. Jiang and Tian (2005) and Polkovnichenko and

Zhao (2013) restrict the average of bid and ask price. Christoffersen et al. (2013) rule out the zero trading volume. My dataset include S&P 500 option prices January 5, 2000 to April 30, 2016. Since there are no market data for the daily dividend yield, I first use a linear regression to back out the dividend. Although my estimated framework under forward measure is independent from dividend and interest rate, I need them to calculate the forward price and filter dataset. After applying three levels of filter, my dataset ends with 390,320 quotes and 915 firms in sample period.

In addition, I change the measure of the estimated framework. Using Radon-Nikodym derivative, I show that under forward measure, the estimated framework is independent of dividend yield and interest rate. Also, in the last section, I provide the no arbitrage conditions of call option price from the state price density perspective.

Chapter 5

Universal Arbitrage-free Estimation of State Price Density

5.1 Introduction

“The future has to be based on more a dynamic belief in how markets work and how distributions unfold. Most of risk management technology is based on looking backwards not looking forwards and I do believe that there’s huge amounts of information in market prices, in particular in option market prices, about what the forward distribution of risks are, at least as gleaned by the market, and so risk management systems have to move in the direction of forward information which is contained in derivative contracts and not so much in looking in back.”

—Myron Scholes (2016)

Although both practitioners and academics are content with pricing securities based on models that make specific assumptions about the evolution of underlying prices (such as the Black-Scholes-Merton model), they also realize that these models do not always completely conform to the facts of the real world. For example, Rubinstein (1985) documented the phenomenon of the implied volatility smile before the crash of 1987, after which researchers detected pronounced

deviations from previous smile shapes³². The fact that the real world has richer information than is modeled seems obvious, but only recently has this perspective shifted from an assumed underlying process towards observed market prices and implied distributions.

Thanks to the availability of options data and greatly increased computational power, estimating state price density (SPD) using a data-driven approach has gained attention. Due to its forward-looking nature, this density provides information about market participants' expectations on the evolution of the underlying assets as well as their risk preferences. This is helpful in applications such as managing risk (Aït-Sahalia and Lo, 2000), selecting portfolio (Bali and Murray (2013) and DeMiguel et al. (2013)), studying policy events (Jondeau and Rockinger, 2000), and inferring the empirical stochastic discount factor (Jackwerth, 2000 and Bliss and Panigirtzoglou, 2004).

Broadly, the existing approaches for estimating the SPD can be roughly classified into two strands: parametric and non-parametric (see Jackwerth (1999), Yatchew and Härdle (2006) and Figlewski (2008) for comprehensive reviews). The parametric approach first specifies the SPD as a known distribution (e.g. lognormal distribution) with several unknown parameters then calibrates the unknown parameters by minimizing the discrepancy between the fitted and observed data³³. Three groups of literature have been developed to add flexibility to the fitting process. First are *the Expansion methods* which introduce correction terms to a known distribution function (e.g. Rosenberg (1998), Jondeau and Rockinger (2001) and Rompolis and Tzavalis (2009)). Second are *Generalized distribution methods* which use more flexible distribution functions with skewness and kurto-

³²According to Black and Scholes(1973), the implied volatility should be constant. In other word, when plot implied volatility against to strike, the implied volatility surface is flat.

³³Usually this can be achieved by assuming the underlying dynamics.

sis parameters (e.g. Corrado (2001), Lim et al. (2005) and Fabozzi et al. (2009)). Third are the *Mixture methods* which construct a distribution function as a combination of several simple functions (e.g. Melick and Thomas (1997) and Giacomini et al. (2008)). The non-parametric approach, instead of relying on a specific parametric form, interpolates the SPD between the points and selects the best fit from the possible distributions using set criteria. Specifically, *Kernel regression methods* construct the SPD estimator based on neighboring observations using a kernel function and selected bandwidth (e.g. Aït-Sahalia and Lo (1998)). *Curve fitting methods* fit the observed prices or implied volatility by least squares with flexible functions such as spline, and then derive the SPD via the result of Breeden and Litzenberger (1978). Additionally, there are approaches such as *the Maximum entropy methods* (e.g. Rockinger and Jondeau (2002)) and neural network (Ludwig (2015)), that do not properly fit into either strand and are non-parametric in nature.

Although a considerable number of papers has investigated SPD using various methods, Figlewski (2008) nevertheless argues that estimating SPD is still an open question and none of the techniques is clearly superior. As noted by Cont (1998), each method has drawbacks. The expansion and generalized distribution methods may lead to negative tails in SPD. The Mixture methods exhibit thin tails and Maximum entropy methods gives multimodal SPD estimates because they fail to consider constraints on the smoothness. While the kernel regression methods yield a smooth SPD, the estimator converges slowly and is sensitive to bandwidth choice. The curve fitting methods work well for interpolation but often break down when used for extrapolation. These difficulties are not surprising because estimating SPD poses five challenges. First, unlike theoretical options with continuous strike prices, the options are only traded at discrete strikes. For

example, strikes for the S&P 500 Index options are usually spaced \$5 apart. Second, Hentschel (2003) suggests that market option data contain noise from various sources, such as the non-synchronous prices. The index prices are measured fifteen minutes apart from option prices. This makes some noise sensitive methods unattractive. Also, there is an issue with matching market prices with average bid and ask prices. Although most papers³⁴ use these as true option prices, in practice, it is not obvious how to specify the true price since options are traded within the bid-ask spread. Third, since the SPD lies in $[0, +\infty]$, the range of observable data is insufficient to recovery the information in tails. Fourth, SPD should satisfy no-arbitrage constraints across maturities and strikes. For example, the estimated density should be positive and integrate to one. Finally, the estimation of SPD is afflicted by the “curse of differentiation”. This means the quality of the SPD estimator will be much worse than the quality of the option price estimator because the differentiation has an amplifying effect on local irregularities. The small irregularities in observed option prices can easily translate into serious irregularities in the SPD, including negative probabilities.

This chapter relies on a machine learning framework to address four challenges³⁵. I first apply a new data filter to eliminate the noise from market data and then propose a more effective approach to estimating the option price via support vector machine (SVM), a method based on statistical learning theory (Vapnik (2000), Schölkopf and Smola (2002)). Rather than attempting to solve the least squares problem, I use a ε insensitive loss function to incorporate bid-ask spread naturally and estimate the SPD by differentiating fitted call option prices

³⁴Except Figlewski (2008) considers a weighting function to incorporate the bid-ask spread, Monnier (2013) incorporates bid-ask spread constraints and Glaser and Heider (2012) use Gaussian random variables from bid and ask as input call price.

³⁵The third challenge involves extrapolate technique which is not discussed in this thesis, Please see Section 7.2.1 for more detail.

to avoid the “curse of differentiation”.

My approach improves upon other methods in several aspects. First, it is a fully nonparametric and has no strong assumption on the prior distribution compared with a parametric approach. Second, it naturally accommodates the information contained in the bid-ask spread by setting an up-bound predicted error, which allow traders to specify according to their own needs. Third, it is possible to incorporate all arbitrage-free constraints into SVM and give an arbitrage free estimator. Fourth, my approach could easily produce a smooth risk neutral density due to the explicit complexity control and has much better generalization capability compared to other non-parametric approaches. Finally, unlike neural network that need a large amount of input and output data to properly train the network, the SVM has a well-known ability of working well in small sample cases. Finally, this is a universal method. The cubic spline (Fengler (2009)) and tensor product estimator (Fengler and Hin (2015)) are special case of my method using cubic spline and B-spline kernel respectively.

The rest of chapter is organized as follows. Section 2 gives an overview of SPD as well as theoretical no-arbitrage constraints. Section 3 introduces briefly SVM and recasts it in no arbitrage context. An example of application is provided in section 4, where I show how to obtain reliable S&P 500 index options data and apply my approach to recovery a well defined call option prices. Section 5 outlines a framework to compare my approach with three other methods. Section 6 concludes.

5.2 SPD and Option Prices

The SPD is an important link and fundamental block in the economics. It can be obtained from either consumption-based asset pricing models ³⁶(such as Lucas(1978)) or the no-arbitrage asset pricing models. In the consumption-based asset pricing model, the SPD is characterized by marginal rates of substitution (often called stochastic discount factors). In the no-arbitrage asset pricing models, the SPD is referred as risk-neutral density(RND). Breeden and Litzenberger (1978) show that SPD can be obtained by differentiating the call option pricing function twice with respect to strike price.

5.2.1 SPD overview

Suppose I are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration (\mathcal{F}_t) , to which a stock price process S_t is adapted. According to the consumption based asset pricing model, the price of an asset today is equal to the net present value of future payoffs. In these models, the risk free rate r is used to map the future random payoffs to today and the current price can be calculated as the expected value of the stochastic discount factor weighted future payoffs. More formally, the asset price at time t with payoff $Z(S_T)$ is

$$P_t = e^{-rt} E_t[M(S_T)Z(S_T)] = e^{-rt} \int_0^{\infty} M(S_T)Z(S_T)f(S_T)dS_T \quad (5.1)$$

Where E_t denotes the expectation under real-world probability \mathbb{P} . The stochastic discount factor of Lucas (1978) can be represented as $M(S_T) = \frac{U'(C_{t+1})}{U'(C_t)}$ and

³⁶A consumption based asset pricing model focus on demand and supply and is a state dependent function. It usually is characterized by a representation of investor's preference and probability of future outcomes in the model. However, arbitrage-free model do not have a specific form. It considers the arbitrage possibly in the market and widely used in option pricing theory.

reflects the investor's willingness of substituting consumption. $Z(S_T)$ is the payoff at time T . S_T is a state variable that drives all the changes in the entire economy. $f(S_T)$ is the payoff probability density under \mathbb{P} . To link the real-world probability \mathbb{P} with the state price density, I rewrite the Equation (5.1) as:

$$P_t = e^{-r\tau} E_t^*[Z(S_T)] = e^{-r\tau} \int_0^{\infty} Z(S_T) p^*(S_T) dS_T \quad (5.2)$$

Where $p^*(S_T)$ is the state price density, which is real-world probability \mathbb{P} discounted by investor's preference. As a consequence, SPD is a product of the real-world probability and investor preferences³⁷.

The other way to construct the SPD is using no-arbitrage model and option pricing theory. By the fundamental theorem of asset pricing, no-arbitrage is equivalent to the existence of a martingale measure (risk-neutral probability measure \mathbb{Q} equivalent to \mathbb{P}) under which the discounted stock price process is a martingale. Therefore, the call price of an European option can be written as a discounted expectation of its terminal payoff.

$$C(S_t, K, \tau, r, \delta) = e^{-r\tau} E_t^{\mathbb{Q}}[(S_T - K)^+] = e^{-r\tau} \int_0^{\infty} \max(S_T - K, 0) q^*(S_T) dS_T \quad (5.3)$$

Where S_t is the underlying asset price at time t , $E_t^{\mathbb{Q}}$ denotes the risk-neutral expectation. $q^*(S_T)$ is the risk neutral probability density of underlying asset. K is the strike price, T is the expiry date and $\tau = T - t$ is time to maturity, δ is the corresponding dividend yield. Compare Equation (5.2) and Equation (5.3), $q^*(S_T)$ and $p^*(S_T)$ are equal. The state price density is also called risk neutral density in option pricing theory.

³⁷Indeed, the 'real-world probability' can be recovered from SPD estimator (see Chapter 6 for more information).

With respect to this Equation(5.3), Bredden and Lizenberger(1978) show that $q^*(S_T)$ can be obtained from the continuum of all European call option prices by differentiation of the strike twice. Differencing Equation(5.3) with respect to the strike price K . the risk neutral distribution $F(S_T)$ yields :

$$\frac{\partial C(S_t, K, \tau, r, \delta)}{\partial K} = -e^{-r\tau} \int_K^\infty q^*(S_T) dS_T = -e^{-r\tau} (1 - F(S_T)) \quad (5.4)$$

Further differentiating Equation (5.4) respect to strike price, the risk neutral density(or state price density) can be written as

$$q^*(K) = e^{r\tau} \frac{\partial^2 C(K, T)}{\partial K^2} \quad (5.5)$$

Since the state price density is the second derivatives of European call price function, the major step in estimating state price density is to interpolate a smooth and arbitrage free two dimensional call surface.

5.2.2 Practical Considerations

Although the mathematical relationship between call price function and SPD is clear, the practical implementation of calculating SPD involves three problems. First, it is difficult to find a best fitting parametric SPD estimator since it will be highly sensitive to assumptions on the six variables $C(S_t, K, \tau, r, \delta)$. For example, if the dynamics of the stock price S_t follows the arithmetic Brownian motion³⁸, then the SPD will be inconsistent with those following a classic geometric Brownian motion. Second, without any restrictions or assumptions on variables, there are too many variables to consider in the nonparametric estimator. From a non-

³⁸for example, it is reasonable to assumes the spread options whose underlying spread is positive follows the arithmetic brownian motion.

parametric statistics point of view, high dimensional regression is hardly able to achieve asymptotic consistency in practice. Third, both consumption-based asset pricing models and no-arbitrage asset pricing models assume the no-arbitrage condition, a set of no-arbitrage constraints is needed in estimator.

To overcome the first problem, in this chapter I change the risk neutral numeraire \mathbb{Q} to forward measure \mathbb{Q}^T ³⁹. From the first fundamental theorem of asset pricing, the absence of arbitrage means that there is a numeraire pair, (N, Q_N) , where a new probability measure can be introduced by choosing a different N . To be more specific, the Radon-Nikodym derivative that changes \mathbb{Q} to \mathbb{Q}^T can be introduced as:

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{e^{\int_t^T r(t)dt}}{F(t, T)} \quad (5.6)$$

where $F(t, T)$ is forward measure. This transformation enables the estimating framework in this chapter to consider a zero interest and zero dividend rate case and avoid comparing estimating methods on the ability to deal with input parameters (see Chapter 4 for detail). Further, this chapter applies the dimension reduction method of Aït-Sahalia and Lo (1998). By transforming an option on a stock to a option on future⁴⁰, I use the forward price $F(t, T) = S_t e^{(r-\delta)\tau}$ to represent the information of stock price S_t , interest rate r and dividend δ then the call price function can be converted into $C(F_{t,\tau}, K, \tau,)$. Also, by assuming the homogeneity of strike and asset price, the call price function can be reformulated as $C(k, \tau)$, where $k = \frac{K}{F(t, T)}$ is called forward moneyness. In conclusion, the call option price is estimated in the forward-money k and time to maturity τ space and the call price is changed to

³⁹see Jarrow(1987) and Geman et al(1995) for change of numerarie method. Also, Gope and Fries (2011)called this step as normalization of call price

⁴⁰According to Aït-Sahalia and Lo (1998), if the future and option have the same maturity, then the European option price on stock is equal to the European option on future.

$$C(k, \tau) = \frac{C(K, \tau)e^{r\tau}}{F(t, T)} \quad (5.7)$$

where $F(t, T)$ is forward price. $C(k, \tau)$ is called the pre-processed call price.

Finally, for ensuring that the estimating framework is independent of arbitrage, next I derive the no-arbitrage constraints for the pre-processed call price. The main results are summarized in Proposition 1 and 2.

Proposition 1. *Let $S_t > 0$. Define the function $C : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, such that if $C(K, \tau)$ satisfies following conditions*

(C1) (convexity in strike price K) $C(K, \tau)$ is convex function in K for all $\tau \geq 0$

(C2) (monotonicity in time to maturity τ) $C(K, \tau)$ is non-decreasing in τ for all $K \geq 0$

(C3) (The call price is limit as strike approach to infinity) $\lim_{K \rightarrow \infty} C(K, \tau) = 0$ for all τ

(C4) (Price bounds) for all $K \geq 0, \tau \geq 0$

$$\max(0, S - K) \leq C(K, \tau) \leq S \quad (5.8)$$

(C5) has expiry value $C(K, 0) = \max(S - K)$ for all K

Then

there exists a non-negative Markov martingale M_τ such that for all $K, \tau \geq 0$

$$C(K, \tau) = E[(M_T - K)^+ | \mathcal{F}_0] \quad (5.9)$$

that M is a non-negative martingale

Following the conditions of Roper (2010) and assuming that $r = \delta = 0$, I derive the no arbitrage conditions of pre-processed call prices as follows:

Proposition 2. *Under the pre-processed call prices framework, it is evident that (C1), (C2) and (C4) imply*

$$0 < C(k, \tau) < 1 \tag{5.10}$$

$$-1 \leq \frac{\partial C(k, \tau)}{\partial k} \leq 0 \tag{5.11}$$

$$\frac{\partial^2 C(k, \tau)}{\partial k^2} \geq 0 \tag{5.12}$$

see Chapter 4 section 4.3.2.

Existing studies have reached a consensus on the necessary and sufficient conditions to guarantee that option prices are arbitrage free and two papers offer a good insight into how to derive the no-arbitrage conditions of the call price surface. From the option strategy perspective, Carr and Madan (2005) show the idea of static arbitrage, which means there is no arbitrage opportunity on the option price surface. They show that excluding the opportunities of gaining from butterfly spread, calendar spread and other conditions are sufficient to define an arbitrage-free option price surface. From the classical mathematical finance perspective, Roper (2010) helpfully summaries the no-arbitrage condition for a call price surface based on the properties of a martingale. I sketch Theorem 2.1 of Roper (2010) in Proposition 8, then simplify the conditions for a pre-processed call price.

5.3 Support Vector Machine Framework

Suppose the market price data set is $\{(x_1(k, \tau), c_1(k, \tau)), \dots, (x_i(k, \tau), c_i(k, \tau)) \subset \mathbb{R}^d \times \mathbb{R}\}$, where i denotes the number of observations and d indexed the dimension of the input space. If I consider the (vectedored) call price $C(k, \tau)$, the call price

function approximation problem becomes finding a function f such that

$$C(k, \tau) = f(k, \tau) + \varepsilon \quad (5.13)$$

where $f(k, \tau)$ is an estimated function, and ε in an error term. In practice, as long as the estimated call price is in the bid ask range⁴¹ (or trader-specified error tolerance), I can consider the estimated price as precise. Thus, ideally a trader-specified error tolerance is preferable in the estimation framework. This intuition coincides with the key idea of support vector machine.

5.3.1 Support Vector Machine

Theoretically, the input call price $C(k, \tau)$ could be approximated by a linear combination of any continuous function. Put simply, I can use any continuous unknown function to connect the known call price dots.

$$f(k, \tau) = \sum_{i=1}^{\infty} \beta_i \phi_i(k) \phi_i(\tau) + b \quad (5.14)$$

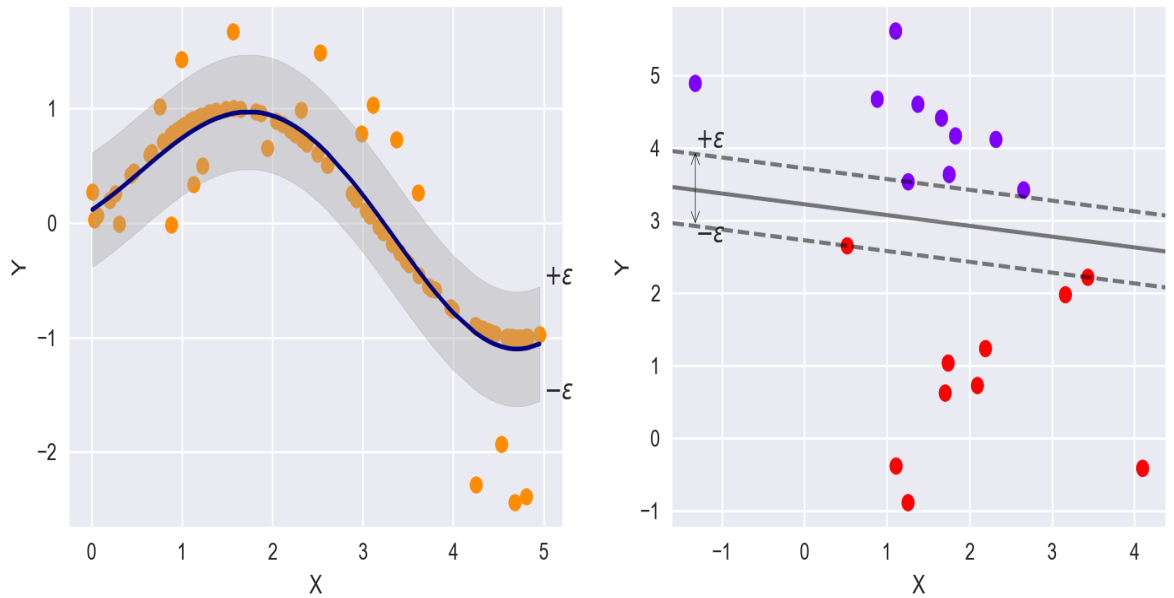
Where $\phi_i(\cdot)$ is basis function such as spline, polynomials, sigmoid function etc. β_i and b are associated coefficients. Motivated by statistical learning theory, the input call price $C(k, \tau)$ can be mapped into a linear feature space by a kernel function. I define the kernel function $K(k, \tau)$ as

$$K(k, \tau) = \phi_i(k) \phi_i(\tau) \quad (5.15)$$

The intuition of SVM is shown in Figure 5.3.1. The left subplot presents the original input space and the right subplot shows the input space after mapping

⁴¹Insider the [bid price, ask price] range.

Figure 5.3.1: Intuition of Support Vector Machine



Note: The left subplot illustrates the intuition of support vector machine. The right subplot presents the input space after mapping by a kernel function.

by a kernel function. As suggested by the left subplot, the estimated values are allowed have ε discrepancy from the actual call price, or put it another way, I accept the estimated call price inside the gray area⁴². After changing the estimation space via kernel function, as shown on the right in Figure 5.3.1, the estimated function is only influenced by the points near the dashed line. These points are called support vectors. Clearly, a small perturbation of data points away from the dashed line will not affect the slope and shape of the estimated function.

Consequently, I can numerically truncate the Equation (5.14) into

$$f(k, \tau) = \sum_{i \in SV}^N \alpha_i K(k, \tau) + b \quad (5.16)$$

⁴²The actual call prices are shown by orange points in the left subplot

5.3. Support Vector Machine Framework

Where the support vector defines as input pair of (k, τ) which has non-zero associated α and N is the number of support vectors. This remarkable feature turns an estimation problem of finding infinity coefficients β_i to specify a small number of α_i . Moreover, from a practical point of view, I wish the call price function to be explained as simply as possible. Mathematically, this means that $f(k, \tau)$ be as flat as possible and finding the smallest 'l₀ seminorm' of α_i , which has the form

$$\min \|\alpha_i\|_0 \tag{5.17}$$

However, Elad (2010) argues that a better way to address the 'l₀ seminorm' minimization problem is to minimize the l₁ norm because the current algorithm to solve the 'l₀ seminorm' problem is not efficient; thus in my support vector machine framework, my aim turns to finding

$$\min \|\alpha_i\|_1 \tag{5.18}$$

subject to

$$|f(k, \tau) - C(k, \tau)| < \varepsilon \tag{5.19}$$

Where ε can be controlled and bound depending on a trader's need. Furthermore, in practice, a perfect mapping from input data to linear feature space is unobtainable since the true values contain outliers and noise; hence I apply an ε insensitive loss function to penalize the deviation between estimated and true value. As I discussed before, only if a predicted value is outside the bid ask range

do I consider it as mis-priced. Formally, the loss function is

$$|\xi|_\varepsilon = \begin{cases} 0 & |\xi| \leq \varepsilon \\ |\xi| - \varepsilon & \text{otherwise} \end{cases} \quad (5.20)$$

The approximation problem is given by

$$\min_{(\alpha, b)} \|\alpha\|_1 + C \sum_{i=1}^N |f(k, \tau) - C(k, \tau)| \quad (5.21)$$

subject to

$$-\xi \leq \alpha \mathbf{K} + b - C(k, \tau) \leq \xi \quad (5.22)$$

Where \mathbf{K} is a vector of kernel function, the constant $C > 0$ balances the trade-off between the flatness of the estimated model and the amount of deviation allowed. ξ determines the error insensitive zone of the estimated model (the gray tube on the right panel of Figure1). If C is too large, then the objective function (19) is considered to minimize the empirical risk only that means only caring about how well the function approximates the input data. On the other hand, if the ξ is too large, I may get a flat estimated function. In my case, it may result in a noisy call price surface, which entails a multimodal state price density. Thus, the selection of C and ξ is highly important and is usually obtained from gridsearch and cross-validation technique in machine learning theory. In this chapter, I search C from $1e-3$ to $1e3$ and error tolerance ε from $[0, \frac{1}{4} \text{spread}]$.

Now I reformulate the estimated problem into matrix form and replacing $C(k, \tau)$ with y , it becomes

$$\min_{(\alpha, b, \xi, a)} \mathbf{1}^T \mathbf{a} + C \mathbf{1}^T \xi \quad (5.23)$$

subject to

$$\begin{aligned}
 -\xi &\leq \mathbf{K}\alpha + b\mathbf{1} - y \leq \xi \\
 -\mathbf{a} &\leq \alpha \leq \mathbf{a} \\
 0 &\leq \mathbf{1}\varepsilon \leq \xi
 \end{aligned} \tag{5.24}$$

Where \mathbf{a} is a vector of coefficients of $\|\alpha_i\|_1$.

The linear programming optimization program(5.23) and (5.24) serves as my basic call price SVM estimator. Since the call price function exhibits no arbitrage, further constraints are added in the next section.

5.3.2 No arbitrage Constrained L_1 -SVM

As has been seen in section 2.2, the arbitrage free pre-processed call price can be obtained through restrict the first or second order derivatives and output bound of price. In this section, I show that these constraints can be incorporated into the L_1 -SVM framework without changing its linear programming nature.

Remark 1. The form of the support vector machine implies that all derivatives of the estimated model are linear.

As noted in Equation (5.16), the estimated call price is linear in the parameters α , if I take first order derivative of the call price respect to the j th component k^j of $k \in \mathbb{R}^d$, the result is

$$\frac{\partial f(k)}{\partial k^j} = \sum_i \alpha_i \frac{\partial K(\mathbf{k}, k_i)}{\partial k^j} = r_1(x)^T \alpha \tag{5.25}$$

Where $r_1(x) = [\frac{\partial k(k_1)}{\partial k^j} \dots \frac{\partial k(k, k_i)}{\partial k^j} \dots \frac{\partial k(k, k_N)}{\partial k^j}]^T$. Similarity, if I take the k th order derivative of the call price respect to the j th component k^j of $k \in \mathbb{R}^d$, the result is still linear and only depends on the kernel function and input data

$$f^k(x) = r_k(x)^T \alpha \quad (5.26)$$

Where $r_k(x)$ is the coefficients for the k th order derivative. Let the corresponding value of call price write as vector \mathbf{Y}_k , The matrix form of monotonicity and convexity constraints $\Gamma_k(Z_k)$ is

$$\Gamma_k(Z_k) = \begin{bmatrix} r_k(x_1)^T & 0 \\ \dots & \dots \\ r_k(x_p)^T & 0 \\ \dots & \dots \\ r_k(x_{|Z_k|})^T & 0 \end{bmatrix} \quad (5.27)$$

$$Y_k(Z_k) = \begin{bmatrix} y_1^{(k)} \\ \dots \\ y_p^{(k)} \\ \dots \\ y_{|Z_k|}^{(k)} \end{bmatrix} \quad (5.28)$$

Where $Z_k = \{k_1, k_2, \dots, k_{|Z_k|}\}$ contains interesting points regarding derivative constraints. The choice of Z_k may be vary and depend on a trader's judgement. In my case, I argue that 0 should be included as $c(0, \tau) = \frac{e^{-r\tau}}{F^\tau}$. For the spline kernel, Z_k is called knot⁴³. Furthermore, consider the price bounds constraints as a restriction on the zero order derivative of the call price, I establish the general

⁴³Please referred Fengler and Hin (2015) for choosing ideal knots using Akaike Information Criterion(AIC).

no arbitrage constrained SVM

$$\min_{(\alpha, b, \xi, a)} \mathbf{1}^T \mathbf{a} + C \mathbf{1}^T \xi + \sum_{k=1}^{n_{sc}} \lambda \mathbf{1}^T z_k \quad (5.29)$$

subject to

$$\begin{aligned} -\xi &\leq \mathbf{K}\alpha + b\mathbf{1} - y \leq \xi \\ -\mathbf{a} &\leq \alpha \leq \mathbf{a} \\ 0 &\leq \mathbf{1}\varepsilon \leq \xi \\ \Gamma_k(Z_k)\theta - Y_k(Z_k) &\leq z_k \end{aligned} \quad (5.30)$$

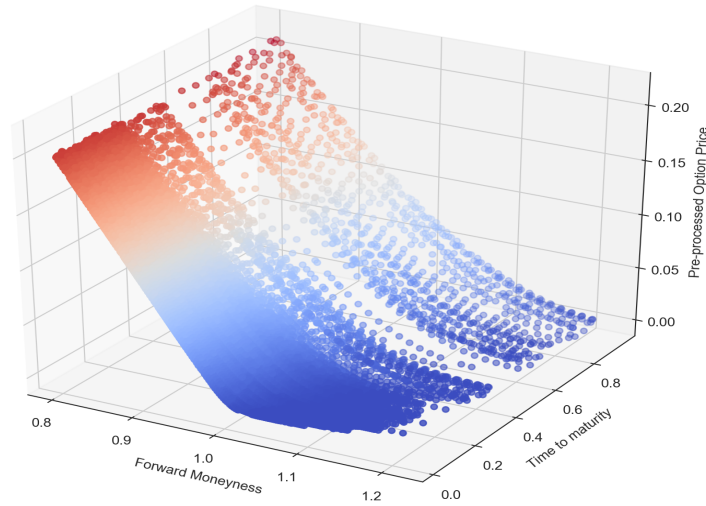
5.4 Empirical Analysis

5.4.1 Data Description

To illustrate the effectiveness of my method, I use daily prices for S&P 500 index options from OptionMetrics. The S&P 500 index is taken as a good indicator for the U.S. market portfolio, and the corresponding options are therefore expected to infer the consensus between market participants. The dataset includes closing index price and interest rate. As regards the option price, I collect the bid and ask price, trading volume and strike price. The dataset run from January 5, 2000 to April 30, 2016, which yields 4,025 trading dates and 412 expiration dates. The average daily trading volume of each contract is 1321.35. My dataset contains total 8,261,170 quotes with time to maturity varying from 7 days to 365 days. I use the average of the bid and ask price as 'true' call price, but I set ε in Equation (5.30) as a quarter of bid and ask spread to consider the information within the bid and ask range.

Unlike a parametric method, my non-parametric L_1 -SVM estimator is data-driven and requires arbitrage-free input data. Therefore, my dataset poses three

Figure 5.4.1: Scatter Plot of Pre-processed Call Option Price



Note: This figure plots the pre-processed call price from 01/01/2016 to 31/04/2016. The x-axis is forward moneyness and y-axis is time to maturity.

challenges. First, the option prices are imprecise because tick sizes, bid-ask spread, and non-synchronicity of index and option prices constitute a source of error (Hentschel (2003)). The true trade price is not always centered between the bid and ask prices. Second, there are no observable data for the daily dividend yield. Third, in the money (ITM) options are less liquid than out the money (OTM) options, with a potential impact of differential liquidity on prices.

To ensure that I have reliable option quotes and solve the above challenges, I apply three levels of the filter in Chapter 4 to eliminate the influence of liquidity and errors. I first remove identical observations from the OptionMetrics database then select liquidity data using certain criteria. Finally, I delete outliers based on the value of implied volatility and implied interest rate. To solve the second challenge and increase data quality, I first linearly interpolate the interest rate and use the put-call parity relationship to derive the implied dividend (see Chapter 4

Table 5.4.1: Descriptive Statistics of S&P 500 OTM Options Data

| Panel A: Descriptive Statistics | | | | | | | | | | | |
|---------------------------------|---------|-----------|--------|-------------|---------|---------|---------|---------|---------|--|-----|
| Variables | Mean | Std. Dev. | Min | Percentiles | | | | | | | Max |
| | | | | 5% | 10% | 50% | 90% | 95% | | | |
| Call Price (\$) | 13.31 | 15.13 | 0.08 | 0.28 | 0.50 | 8.15 | 33.90 | 43.75 | 146.3 | | |
| Put Price (\$) | 12.77 | 14.31 | 0.08 | 0.50 | 0.93 | 7.60 | 32.20 | 42.05 | 151.50 | | |
| Implied Volatility σ (%) | 20.62 | 8.39 | 6.46 | 10.30 | 11.46 | 19.19 | 31.10 | 35.69 | 87.11 | | |
| Strike price (K) | 1537.75 | 377.62 | 550 | 930 | 1030 | 1540 | 2030 | 2110 | 2500 | | |
| Days(τ) | 53.82 | 41.47 | 7 | 14 | 18 | 44 | 95 | 127 | 365 | | |
| Index Price (S) | 1601.16 | 389.84 | 676.53 | 944.89 | 1091.60 | 1606.28 | 2079.51 | 2098.48 | 2130.82 | | |
| Trading Volume (V) | 1321.35 | 3690.84 | 1 | 2 | 4 | 110 | 3581 | 6698 | 157542 | | |

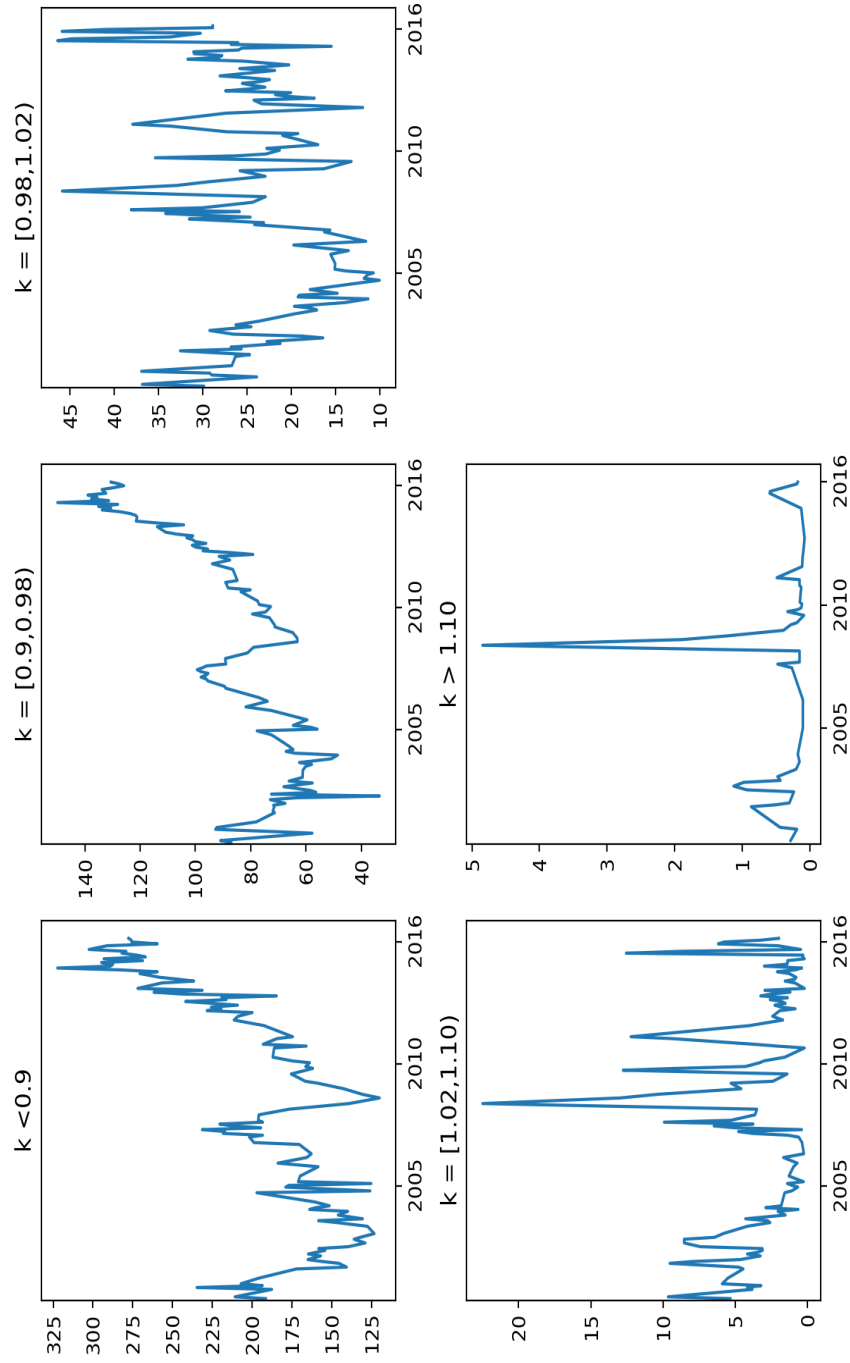
| Panel B. Options by Forward Moneyness-Maturity | | | | | |
|--|-----|-------|--------------|--------------|--------------|
| Forward Moneyness K/F | DTM | ITM | ATM | OTM | DOTM |
| | | <0.90 | [0.90, 0.98) | [0.98, 1.02) | [1.02, 1.10) |

| Average Call Prices (\$) | | | | | |
|----------------------------------|--------|--------|-------|-------|-------|
| Short-term $\tau \in (7, 60]$ | 237.04 | 30.30 | 6.60 | 2.05 | |
| Medium-term $\tau \in (60, 180]$ | 243.74 | 50.02 | 15.35 | 4.60 | |
| Long-term $\tau \in (180, 365]$ | 249.55 | 144.49 | 85.76 | 42.38 | 12.96 |

| Average Implied Volatility σ (%) | | | | |
|---|-------|-------|-------|-------|
| Short-term $\tau \in (7, 60]$ | 29.96 | 16.05 | 15.54 | 22.36 |
| Medium-term $\tau \in (60, 180]$ | 25.26 | 16.75 | 14.51 | 16.82 |
| Long-term $\tau \in (180, 365]$ | 22.59 | 17.51 | 15.70 | 14.39 |

Note: Table 5.4.1 summaries descriptive statistics of my S&P 500 option dataset (total 390,320 observations). The sample period is from January 5, 2000 to April 30, 2016. The call price is calculated as average of bid and ask price. The OTM put prices are translated into ITM call prices by put and call parity. Std.Dev denotes the standard deviation from call option price.

Figure 5.4.2: Monthly Average of 1-month Call Option Prices over Different Forward-moneyness



Note: This figure plots the time series of call price between 2000 and 2016 across different forward moneyness groups. k is forward-moneyness.

for details). Over the sample period, the average risk-free rate and implied dividends were 1.17% and 2.53%, respectively. After calculating the implied continuous dividend, I compute the Black-Scholes implied volatility using the Bisection approach. Finally, I obtain 390,320 observations⁴⁴. The resulting processed call prices are free of assumptions on interest rate and dividend yield.

Figure 5.4.1 shows the pre-processed call price of final dataset from 01/01/2016 to 31/04/2016. Clearly, my final dataset yields a well-defined call option price surface. Panel A in Table 5.4.1 presents descriptive statistics for my sample of S&P 500 daily prices and Panel B reports the statistics of call prices and implied volatility by time to maturity and forward-moneyness groups. As expected, implied volatility shows a clear smile pattern and call price decreases as forward moneyness increases. Figure 5.4.2 compares the behavior of the monthly average of 1 month S&P 500 option prices over different forward moneyness. Consistent with Barletta et al. (2017), not surprisingly, the ITM options exhibit non-stationary behavior, whereas the OTM options are stationary. Barletta et al. (2017) further explain this as the price of ITM option mainly associated with the underlying index since holding a price of the call option is equal to holding an underlying index strike price approaching zero. In other words, the behavior of the ITM option price is similar to the S&P 500 trend to some extent.

5.4.2 Performance Measures

When comparing the proposed L_1 -SVM method with other nonparametric methods (Fengler (2009) and Fengler and Hin (2015)), I assess with regard to three aspects: accuracy of estimation, running speed and smoothness of call price surface with respect to forward-moneyness and time to maturity. Although the mean

⁴⁴This is a reasonable number compares with Chiang etc(2016) who get total 404,822 observations using the same filters on S&P 500 options between 1996 and 2011.

squared error (MSE) is a general accuracy indicator, in this thesis, I use relative distance to measure the goodness of fit. This is because the total estimated number of my L_1 -SVM method is different from (Fengler (2009) and Fengler and Hin (2015)). As proposed in Section 5.4, I use a 5 fold cross-validation process to determine the optimal parameter and calculate the estimated error, and so the final estimated number equals the number of its training subset. Put simply, if I choose 5 fold cross-validation, the number of total estimated option price only accounts 20% of the whole data size. Considering this size effect, I use a relative error to measure the estimated goodness.

- **Accuracy**

I define the relative error as relative distance

$$Relative \quad Distance = \sqrt{\sum_{n=1}^N \left(\frac{C(k, \tau) - \hat{C}(k, \tau)}{C(k, \tau)} \right)^2} \quad (5.31)$$

where $C(k, \tau)$ is pre-processed call price from market and $\hat{C}(k, \tau)$ estimated by different methods. N is the number of total estimated option price.

- **Smoothness**

I use the absolute value of second order derivative for both dimensions to measure the smoothness of the surface. I use a numerical differentiation approach, applying the central finite difference approximation.

$$\frac{\partial^2 \hat{C}(k, \tau)}{\partial k^2} \approx \frac{C(k_{m+1}, \tau) - 2C(k_m, \tau) + C(k_{m-1}, \tau)}{(k_m - k_{m-1})(k_{m+1} - k_m)} \quad (5.32)$$

$$\frac{\partial^2 \hat{C}(k, \tau)}{\partial \tau^2} \approx \frac{C(k, \tau_{m+1}) - 2C(k, \tau_m) + C(k, \tau_{m-1})}{(\tau_m - \tau_{m-1})(\tau_{m+1} - \tau_m)} \quad (5.33)$$

Where $1 < m < N$. The smaller the second order derivatives, the smoother the call option price surface is.

5.4.3 Empirical Results

Using the pre-processed S&P 500 option prices, I first compare the proposed L_1 -SVM method using a radial basis function (RBF) and a spline kernel⁴⁵. The algorithm for the full estimating process is summarized in Algorithm 1. As noted in Section 3.1, the algorithm's performance is determined by the parameter choice for the kernel function. I apply a grid search with 5-fold cross validation⁴⁶ to find the optimal parameters that minimize Equation(5.28). Table 5.4.3 reports the mean relative distance of specific kernel functions. I test the univariate and bivariate approximation using RBF and cubic spline kernel in 3 sub-periods (the period before the financial crisis, financial crisis period and the period after the financial crisis). In each case, I report the relative distance of L_1 -SVM. The cubic spline and tensor product B-spline function is introduced in Section 3.1.2.

I expect intuitively that the relative distance for the financial crisis period is higher because the financial market is highly volatile in this period. Therefore, the L_1 -SVM needs additional effort to fit these observations. However, as shown in Table 5.4.3, when comparing the relative distance of the L_1 -SVM across different periods, the relative distance turns out to be decreased in the financial crisis period for both kernels in univariate and bivariate cases. This suggests that my L_1 -SVM method is not affected by volatility in the market. The smallest relative distance is obtained by using the univariate cubic spline kernel. Also, consistent with Fengler and Hin (2015), I find that the tensor product kernel (or

⁴⁵The kernel function is reported in Section 3.1.2 and Appendix.

⁴⁶I use the Python library GridSearchCV and Cvxopt to program the code and the laptop conducts the codes is an Intel Core i7 and 2.9 GHz.

Algorithm 1: Approximation call price surface use support vector machine

- (1) Initial ;
- Input:** Observed forward moneyness $\kappa_i, i = 1, \dots, N$
Observed maturity $\tau_i, i = 1, \dots, N$
Observed European call price $c_i(\kappa_i, \tau_i)$
- a. Applying the three levels data filters in Chapter 4
 - b. Transform option prices $c_i(K_i, \tau_i)$ to pre-processed call prices (under forward measure) $c_i(\kappa_i, \tau_i)$
 - c. Estimate and compare call price under forward measure
 - d. Transform the estimated call price under risk neutral measure Q
 - e. Estimate the state price density
- Output:** Estimated call price $c_i(\hat{\kappa}, \hat{\tau})$

Estimate the call price under forward measure $c_i(\hat{\kappa}, \hat{\tau})$

- (1) Randomly split data D into 5 "folds" of equal size: D_1, D_2, \dots, D_5
 - (2) For $i = 1, \dots, 5$,
$$\min_{(\alpha, b, \xi, a)} 1^T \alpha + C 1^T \xi + \sum \lambda 1^T z_k$$

subject to Equation(5.30)

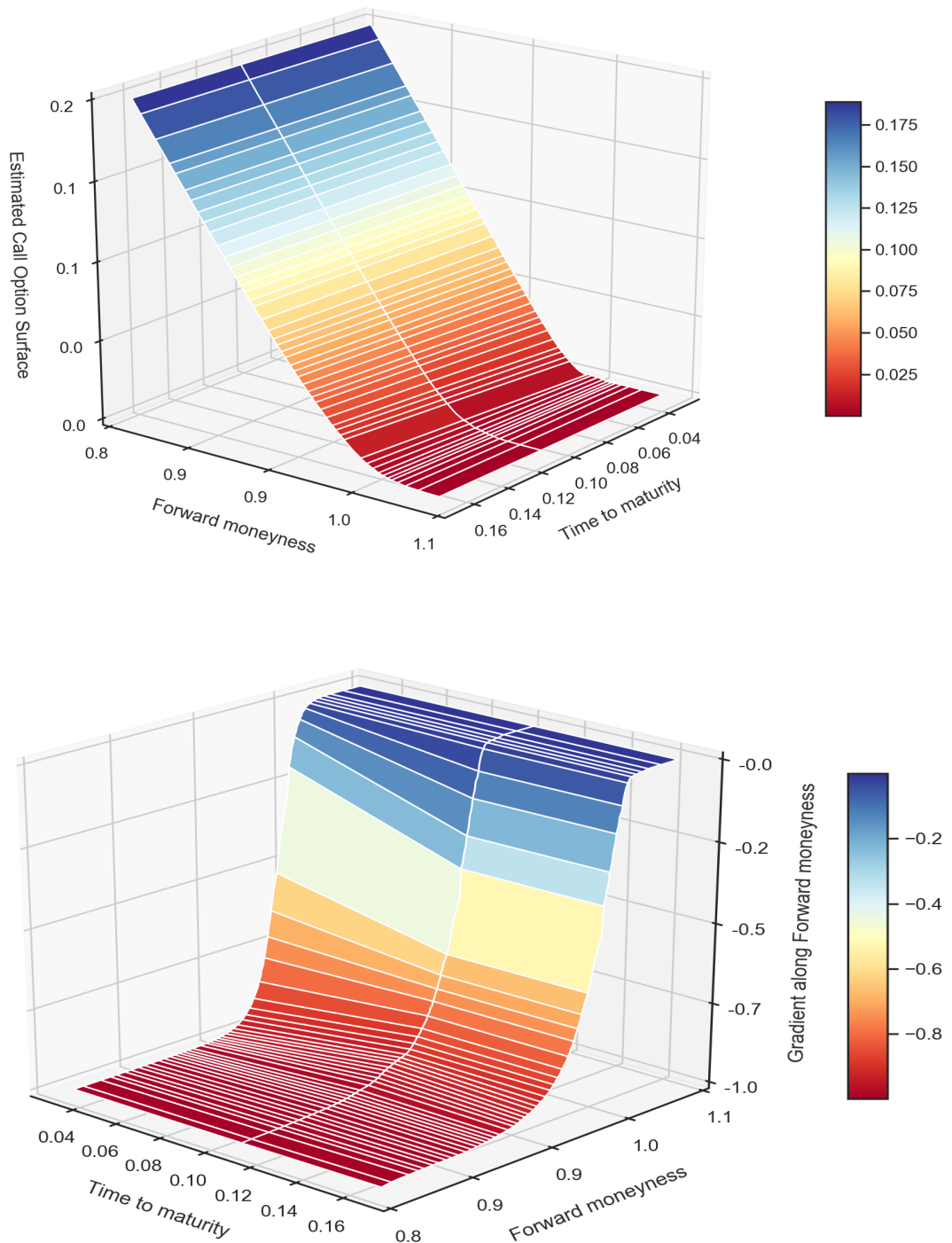
fit above model use defined parameters set of C, ε and γ (for different kernel parameters)
 - (3) Return optimal parameters of C, ε and γ , which lead the minimal test error
 - (4) Use optimal parameters to fit the call option price function
-

called surface estimator in Fengler and Hin (2015)) seems to underperform the univariate kernel (a slice by slice approach). In my result, the relative distances for the bivariate kernel approximation are 2.732 and 2.359 for RBF and cubic spline in the period after the financial crisis respectively, which is significantly higher than the corresponding univariate case. Fengler and Hin (2015) argue that without considering calendar-spread arbitrage (Equation(2.28)-(3.30)), the surface estimator does not improve the fitting quality. In other words, without using calendar-spread constraints, using a surface estimator only increases the fitting difficulty since it uses a base surface to fit the call option price surface.

Figure 5.4.3 displays an example of the estimated arbitrage free call option

5.4. Empirical Analysis

Figure 5.4.3: Estimated Call Option Price and Its First-order Derivative



Note: Figure 5.4.3 plots the estimated call price and its first-order derivative on 02/07/2013. The index price is 1614.08. The top panel shows the estimated call price. The bottom panel displays the first-order derivative of estimated call price respect to forward-moneyness.

5.5. Comparison of nonparametric methods

Table 5.4.3: Empirical Results with RBF and Spline Kernel

| Period | Univariate | | Bivariate | |
|-----------|------------|--------------|-----------|--------------|
| | RBF | Cubic Spline | RBF | Cubic Spline |
| 2000-2007 | 4.664 | 2.894 | 5.114 | 3.086 |
| 2008-2009 | 3.934 | 1.737 | 3.945 | 2.645 |
| 2010-2016 | 2.805 | 1.032 | 2.732 | 2.359 |

Note: This table compares the performance of L_1 -SVM of radial basis function(RBF) and cubic spline kernel. For each kernel, I compare the relative distance of the univariate and bivariate case. Univariate refers as using a kernel function of forward-moneyness to fit each maturity slice. Bivariate refers as using a tensor product kernel function of forward-moneyness and maturity to the pre-processed call price surface.

price and its first order derivative. Consistent with Equation(5.10) and (5.11), the call option price under the forward measure is greater than 0 and less than 1. Its first-order derivative monotonically increases with forward-moneyness. If I reverse the change of measure (Equation(5.6), this first-order derivative under risk neutral density is called delta, which measures the sensitivity of option price to change in the underlying value.

5.5 Comparison of nonparametric methods

In the previous sections, I test the proposed method using different kernels and show the cubic spline kernel yields the smallest relative distance. To assess my machine learning based framework with another nonparametric methods, I first summarize the differences between these nonparametric methods into two aspects:

- Assumptions of input variables: as shown in Section 5.2.2, the call price function can be expressed as $C(S_t, K, \tau, r, \delta)$, in which the variables S_t, K and τ can be easily obtained from the market while r and δ are difficult

to calibrate. Without using the forward measure as in this thesis, previous studies have different assumptions regards these two variables (such as Glaser and Heider (2012) assume they are both constant).

- Choice of approximation method includes two important parts: choice of kernel function and kernel dimension.
 - Various kernel functions have been applied to approximate the call option price surface in previous studies such as low order polynomial (Kundu et al. (2016)), radial basis function (Lai (2011)) and cubic spline kernel (Fengler (2009)). In this chapter, since the smallest relative distance is obtained by using a cubic spline kernel, I compare my method with Fengler (2009).
 - In terms of kernel dimension, Fengler and Hin (2015) is the only study that uses the bivariate kernel approximation. In contrast with this thesis, they incorporate the no-arbitrage constraints in the control net of the tensor product B-spline and solve a quadratic programming to fit the call option price surface.

To compare my L_1 -SVM method with Fengler (2009) and Fengler and Hin (2015), I first summarize the difference between two methods in Table 5.5.1. Since my estimated framework is independent of interest rate and dividend yield, the performances of these three models are mainly determined by their optimization procedure. I briefly review the key optimization function of Fengler (2009) and Fengler and Hin (2015) as follows.

- Maturity Slice by Slice (Fengler (2009))

Dividing the call option price surface into several maturity slices, Fengler (2009)

proposes a method that estimates the call option price using a cubic spline function. To make the call option price surface as smooth as possible, Fengler (2009) first develops a smooth technique in implied volatility space. After obtaining the smoothed data, he transforms them back to the option price space. Using the no-arbitrage conditions on the option price, he simplifies the estimation problem as follow:

For each maturity

$$\min_{g_\tau} \sum_{m=1}^M w_m [c(k_m, \tau) - g(k_m)]^2 + \lambda \int_{k_1}^{k_m} (g''_\tau(k))^2 dx \quad (5.34)$$

Where $g(k_m)$ is spline series, $c(k_m, \tau)$ is the observed call price. $g''(k)$ is a regularization term introduced by Green and Silverman (1994). The optimization problem Equation(5.31) is solved with respect to the no-arbitrage conditions. I present the matrix form of final estimation problem in Algorithm 2⁴⁷.

- Two dimensional Tensor Product Kernel (Fengler and Hin (2015))

Fengler and Hin (2015) extends the earlier method using a two univariate spline kernel. This method represents the call option price surface as a linear combination of tensor product B-spline surface. By minimizing a penalized least squares , the estimated problem becomes

$$\min_{\theta} \frac{1}{N} \sum_{i=1}^N (c(\kappa_i, \tau_i) - s(\kappa_i, \tau_i))^2 + \lambda_N |\theta|^2 \quad (5.35)$$

where $s(\kappa_i, \tau_i)$ is tensor product spline, $c(\kappa_i, \tau_i)$ is the observed call price. θ is vector of the tensor product B-spline coefficients. Without directly considering no-arbitrage in the quadratic programming framework, Fengler and Hin (2015) establish no-arbitrage conditions on an artificial kernel surface (which is called

⁴⁷Please refer to Fengler (2009) for more detail

the control net of the B-spline). This approach involves a complex knot search and relocate process, which is highly time consuming. I present the matrix form of this method in Algorithm 3⁴⁸.

5.5.1 Comparison Results and State Price Density

Table 5.5.2 reports the comparison results for these three methods. Not surprisingly, my L_1 -SVM method shows similar result to those of Fengler (2009) because they both apply a univariate cubic spline kernel. This similar result strongly indicates that my L_1 -SVM method is a universal approach. Previous studies that use different kernel and optimization procedures can be replicated by my method. As shown in the table, compared to Fengler (2009), although my L_1 -SVM method is slow, it shows a somewhat better accuracy. I argue that this can be attributed to my comprehensive data filter approach, simple linear programming framework, incorporating the bid-ask spread information or all of them.

- Analysis of computation time

In Figure 5.5.1, I show the evolution of computation time during the estimated period. It is evident that Fengler and Hin (2015)'s running time far exceeds the others. As shown in Algorithm 3, this is because it searches simultaneous optimizations for forward-moneyness and maturity. Unlike the other two methods, which are completely unfazed by the financial crisis period, I observe a significant peak of Fengler and Hin (2015) around the financial crisis period, which implies that Fengler and Hin (2015) may need additional effort to calculate more volatile data. In the recent period, my method and Fengler (2009) both highlight two notable peaks, and, as both methods are based on the maturity slice approach, I argue that these may be caused by the market microstructure; for example,

⁴⁸Please refer to Fengler and Hin (2015) for more detail

possible price manipulation through controlling order size, or my data filter does not eliminate all possible arbitrary data. I suggest this for future research.

- Analysis of relative distance

For the relative distance, as shown in Tale 5.1.2, the mean of relative distance for Fengler and Hin (2015), Fengler (2009) and my L_1 -SVM method is 18.734, 2.368 and 2.363 respectively. My L_1 -SVM method produces the smallest relative distance and Fengler and Hin (2015) displays the worst performance with the mean of relative distance almost 8 times higher. Figure 5.5.2 shows the changes of relative distance over time. My L_1 -SVM method shows the similar pattern with Fengler (2009) but with slightly lower value.

- Analysis of smoothness

When comparing the smoothness of call price surface in the forward-moneyness direction, my L_1 -SVM method stands out by having a stable result. As illustrated in Figure 5.5.3, without considering the data around 2016, my L_1 -SVM method exhibits stable behavior with a lower absolute second-order derivative with respect to forward-moneyness. Another interesting comparison among the three methods is to compare the surface smoothness in the time to maturity direction: it can be seen in Table 5.1.2 that there are no significant differences in their ability to interpolate across time. Except the max value and standard deviation, my method shows the same result as Fengler (2009). This finding provides empirical evidence for Fengler and Hin (2015)'s simulation result, which shows the calendar

Algorithm 2: Fengler(2009): Approximation of the call price surface with cubic spline

(1) Calculate pre-estimated call price surface;

Input: Observed forward moneyness level κ_i , $i = 1, \dots, N$

Observed maturity level τ_i , $i = 1, \dots, N$

Observed Black Schole implied volatility $\sigma_i(\kappa_i, \tau_i)$

a. Define a regular grid as $\hat{\kappa} \times \hat{\tau} \subseteq [\kappa_i, \tau_i]$ with 100 grid points

b. Get pre-estimated implied volatility

$$\sigma_i(\hat{\kappa}, \hat{\tau}) = \text{Interpolate}(\sigma_i(\kappa_i, \tau_i))$$

Output: Estimated call price $c_i(\hat{\kappa}, \hat{\tau})$

(2) Fitted the call price surface slice by slice (from last to the first maturity), and solve the quadratic program. For t_m , solve

$$Q = \begin{bmatrix} \frac{1}{h_1} & 0 & 0 & 0 & 0 \\ (-\frac{1}{h_1} - \frac{1}{h_2}) & \frac{1}{h_2} & 0 & 0 & 0 \\ \frac{1}{h_2} & (-\frac{1}{h_2} - \frac{1}{h_3}) & \dots & \dots & 0 \\ 0 & \frac{1}{h_3} & \dots & \dots & 0 \\ \vdots & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{h_{M-2}} \\ 0 & 0 & 0 & 0 & (-\frac{1}{h_{M-2}} - \frac{1}{h_{M-1}}) \\ 0 & 0 & 0 & 0 & \frac{1}{h_{M-1}} \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{1}{3}(h_1 + h_2) & \frac{1}{6}h_2 & 0 & 0 & 0 \\ \frac{1}{6}h_2 & \frac{1}{3}(h_2 + h_3) & \frac{1}{6}h_3 & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{6}h_{M-3} & \frac{1}{3}(h_{M-3} + h_{M-2}) & \frac{1}{6}h_{M-2} \\ 0 & 0 & 0 & \frac{1}{6}h_{M-2} & \frac{1}{3}(h_{M-2} + h_{M-1}) \end{bmatrix}$$

Define $A = (Q, -R^T)$ and set $W_M = \text{diag}(w_1, w_2, \dots, w_M)$, where

$$h_m = \kappa_{m+1} - \kappa_m$$

$$B = \begin{pmatrix} W_M & 0 \\ 0 & \gamma \end{pmatrix}$$

For each time to maturity:

Set: $x = (g^T, \gamma^T)^T$ and $y = (w_1 c(\hat{\kappa}_1, \hat{\tau}_1), \dots, w_M c(\hat{\kappa}_M, \hat{\tau}_M), 0, \dots, 0)^T$

$$\min_x -y^T + \frac{1}{2} X^T B X$$

Subject to $A^T X = 0$

$$\hat{c}(\hat{\kappa}, \hat{\tau}) = g_{\tau}^*$$

Algorithm 3: Fengler and Hin(2013): Approximation of the call price surface with tensor product B spline

(1) Initial ;

Input: Observed forward moneyness $\kappa_i, i = 1, \dots, N$

Observed maturity $\tau_i, i = 1, \dots, N$

Smoothing parameter λ

Observed European forward call price $c_i(\kappa_i, \tau_i)$

Output: Estimated call price $c_i(\hat{\kappa}, \hat{\tau})$

(2) Fitted the call price surface using tensor product of spline kernel

a. Define the tensor product kernel as

$$B = \begin{bmatrix} (B_{0,p_1}(\kappa_1)) \dots (B_{q_1,p_1}(\kappa_1)) \otimes (B_{0,p_1}(\tau_1)) \dots (B_{q_1,p_1}(\tau_1)) \\ (B_{0,p_1}(\kappa_N)) \dots (B_{q_1,p_1}(\kappa_N)) \otimes (B_{0,p_1}(\tau_N)) \dots (B_{q_1,p_1}(\tau_N)) \end{bmatrix}$$

Where p_1 and p_2 are the order of b spline kernel

b. Define a regular grid as $\hat{\kappa} \times \hat{\tau} \subseteq [\kappa_i, \tau_i]$ with 100 grid points

c. Search and Relocate the knots in grid that have minimum Akaike Information Criterion

The search boundary is v and the new knot as ϵ^{add} , the original knots is ϵ^0 and the AIC

value of original knots is AIC_0

While the knots in grid do:

$$\epsilon^{add} = \operatorname{argmin}(AIC(\epsilon^0 \cup v))$$

Until

$$AIC(\epsilon^{add} \cup \epsilon^0) > AIC_0$$

(3) Set optimization problem as $D = B^T B + \lambda I$

$$B^T = \begin{pmatrix} c_1(\kappa_1, \tau_1) \\ c_1(\kappa_N, \tau_N) \end{pmatrix}$$

Solve the quadratic program

$$\min(0.5\theta^T D\theta - \theta^T d)$$

subject to constraints on coefficients of b spline kernel (reference paper for more detail)

Finally

$$\hat{c}(\hat{\kappa}, \hat{\tau}) = \sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \theta_{j_1, j_2} B_{j_1, p_1}(\kappa) B_{j_2, p_2}(\tau)$$

5.5. Comparison of nonparametric methods

Table 5.5.1: Summary of Compared Models

| Model | Interest rate | Dividend | Kernel | Scope |
|---------------------------|---------------|---------------|----------------------------|--------------------------------|
| Fengler(2009) | Deterministic | Deterministic | Cubic Spline | Maturity Slice (Univariate) |
| Fengler and Hin (2013) | Deterministic | Deterministic | Tensor product B-spline | Global (Bivariate) |

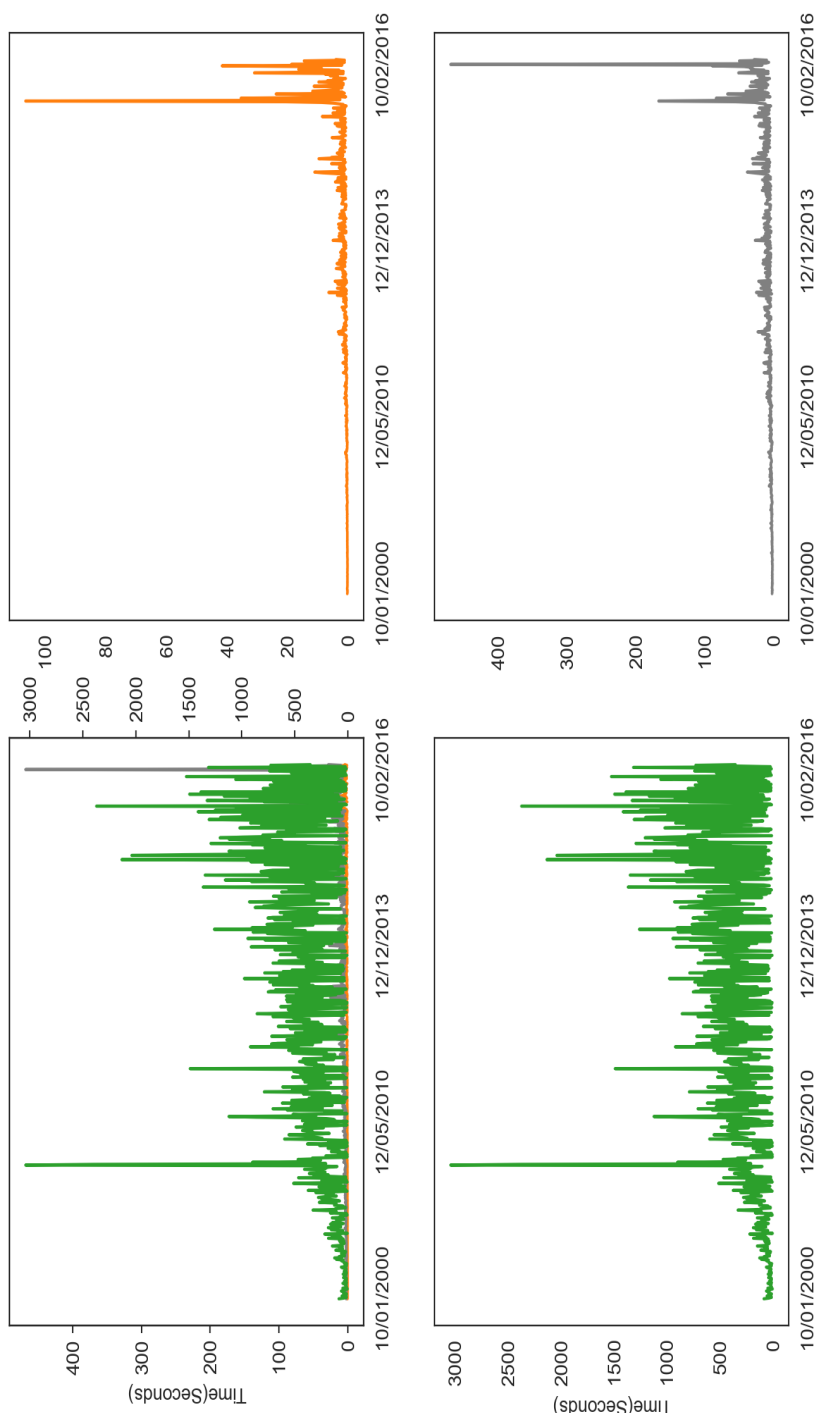
Note: this table compares two nonparametric estimation methods for assumption of interest rate and dividend, choice of kernel in my estimation framework.

Table 5.5.2: Empirical Results for Estimating Call Option Price

| | | Fengler (2009) | L_1 -SVM | Fengler (2015) |
|------------------------|------|----------------|------------|----------------|
| Time | Min | 0.093 | 1.259 | 4.579 |
| | Mean | 1.708 | 8.771 | 331.934 |
| | Max | 106.481 | 468.83 | 3037.909 |
| | Std | 5.043 | 19.680 | 305.849 |
| Relative Distance | Min | 0.314 | 0.312 | 1.164 |
| | Mean | 2.368 | 2.363 | 18.734 |
| | Max | 10.132 | 10.135 | 367.530 |
| | Std | 1.933 | 1.937 | 27.119 |
| Smoothness(Moneyiness) | Min | 0.060 | 0.004 | 0.082 |
| | Mean | 35.250 | 33.369 | 43.633 |
| | Max | 4406.122 | 2854.166 | 1197.856 |
| | Std | 171.411 | 184.698 | 97.831 |
| Smoothness(TTM) | Min | 0.000 | 0.000 | 0.000 |
| | Mean | 0.006 | 0.006 | 0.010 |
| | Max | 0.116 | 0.117 | 0.231 |
| | Std | 0.011 | 0.016 | 0.020 |

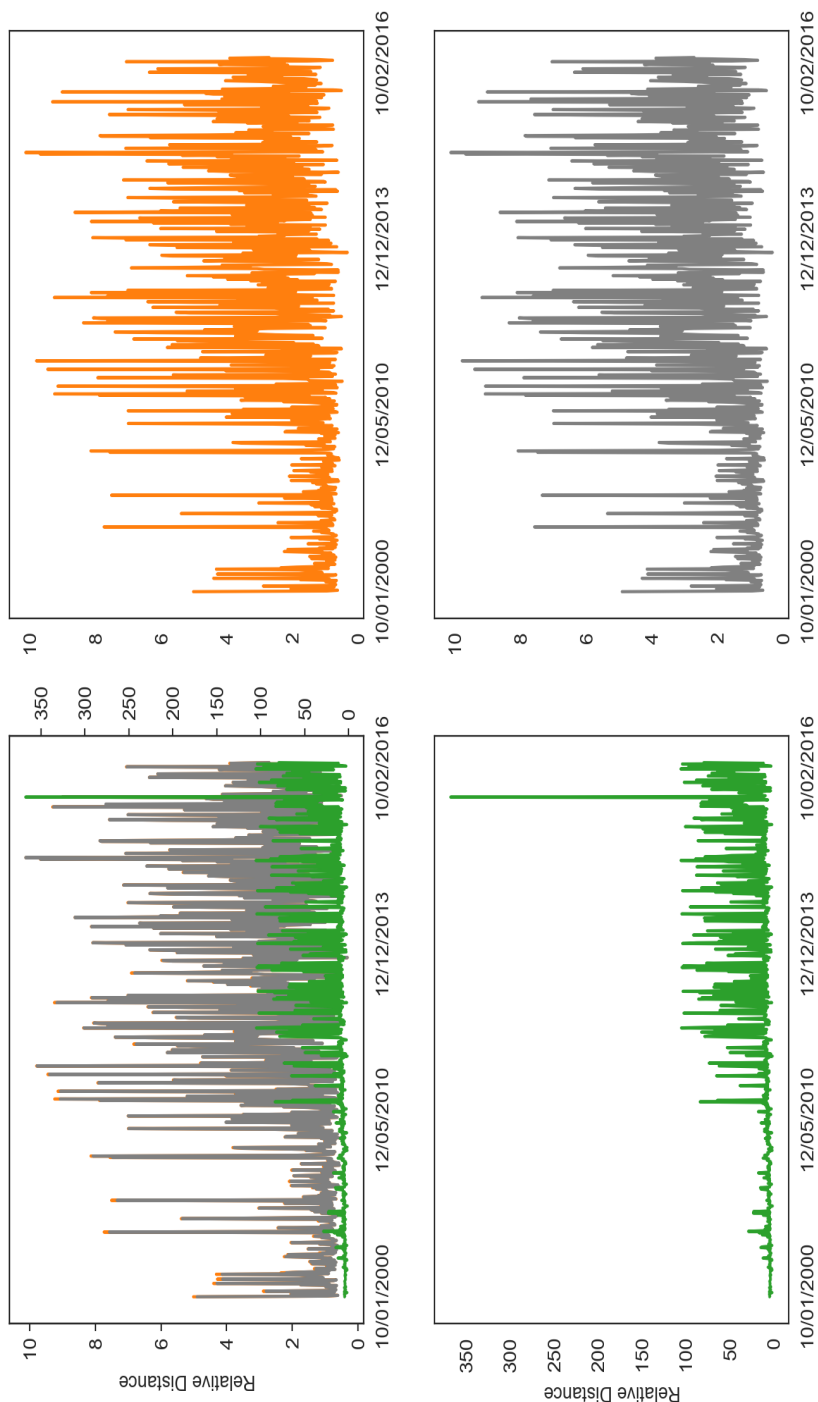
Note: this table reports the performance of three compared methods. Time refers as the computation time. Relative distance and smoothness (forward-moneyiness and maturity direction) is defined in Section 5.5.1. Std represents standard deviation. I use Python's Cvxopt library to solve the optimization and the laptop conducts the codes is an Intel Core i7 and 2.9 GHz.

Figure 5.5.1: Running Time Comparison



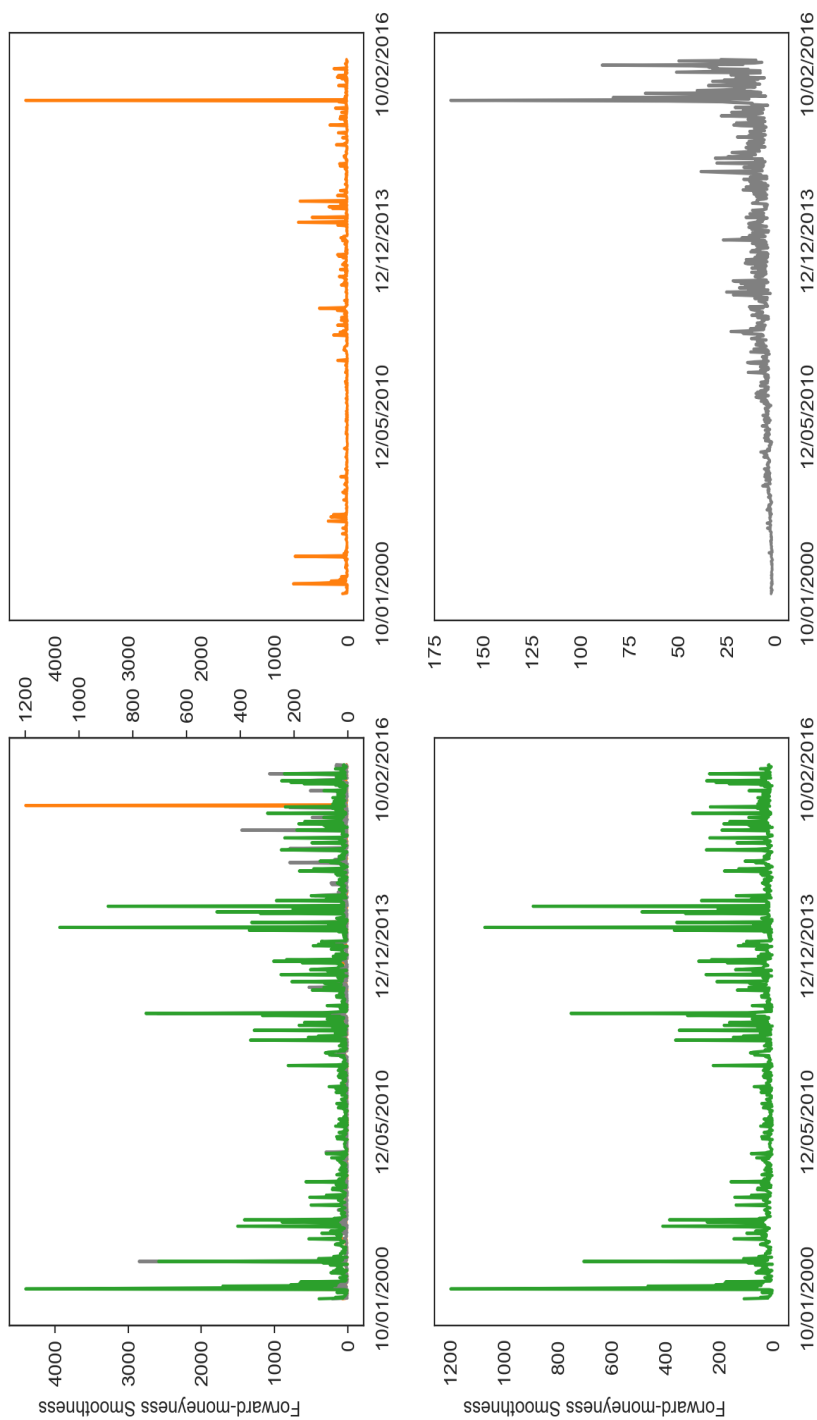
Note: this figure displays the computation time of three methods. The top left panel plots all results. The green line represents Fengler and Hin (2015). The orange line represents Fenler (2009) and the gray line represents the result of L_1 -SVM.

Figure 5.5.2: Relative Distance Comparison



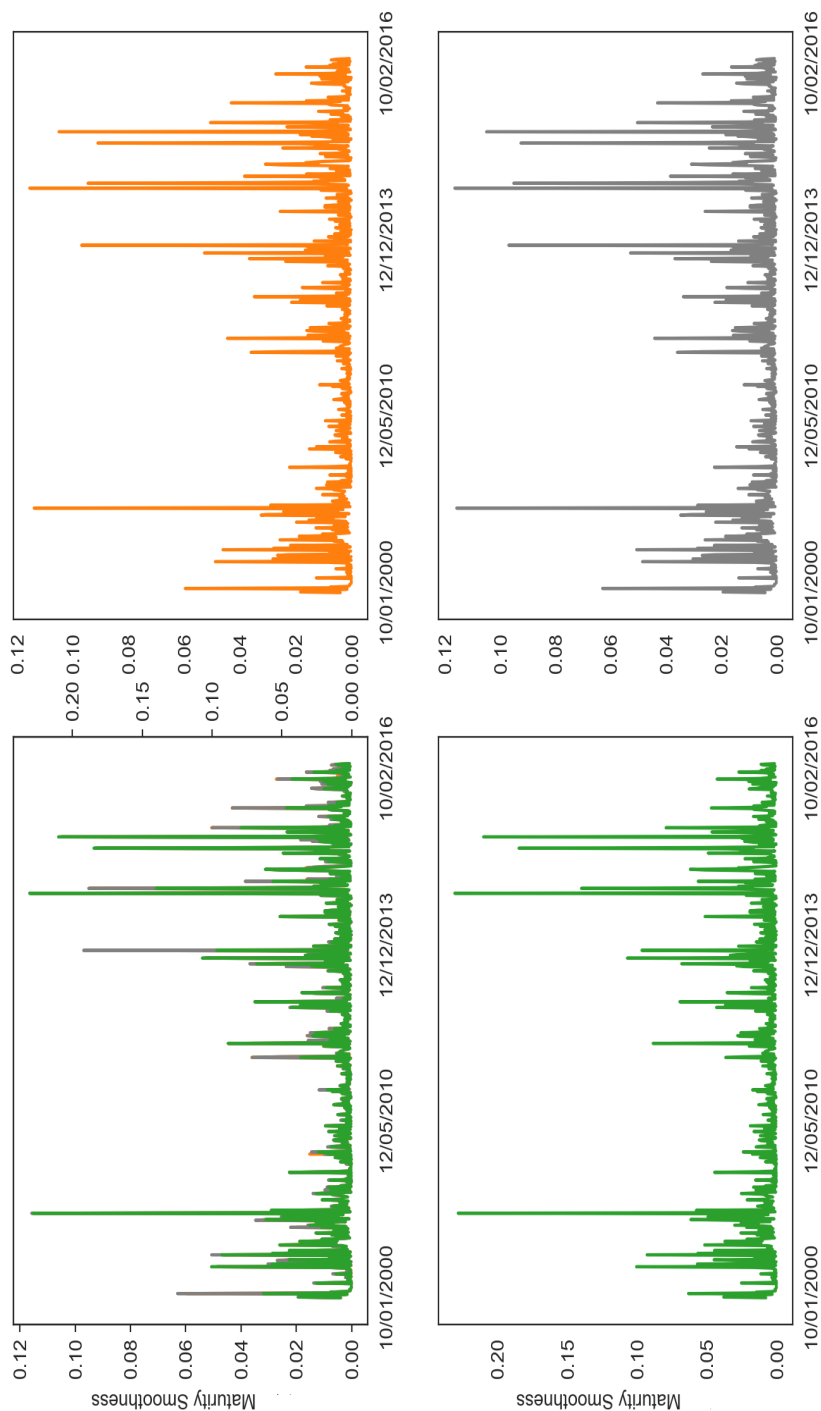
Note: this figure shows the relative distance of three methods. The top left panel plots all results. The green line represents Fengler and Hin (2015). The orange line represents Fengler (2009) and the gray line represents the result of L_1 -SVM.

Figure 5.5.3: Forward-moneyness Smoothness Comparison



Note: this figure reports the smoothness (in forward-moneyness direction) of three methods. The top left panel plots all results. The green line represents Fengler and Hin (2015). The orange line represents Fengler (2009) and the gray line represents the result of L_1 -SVM.

Figure 5.5.4: Maturity Smoothness Comparison



Note: this figure reports the smoothness (in maturity direction) of three methods. The top left panel plots all results in one figure. The green line represents Fengler and Hin (2015). The orange line represents Fengler (2009) and the gray line represents the result of L_1 -SVM.

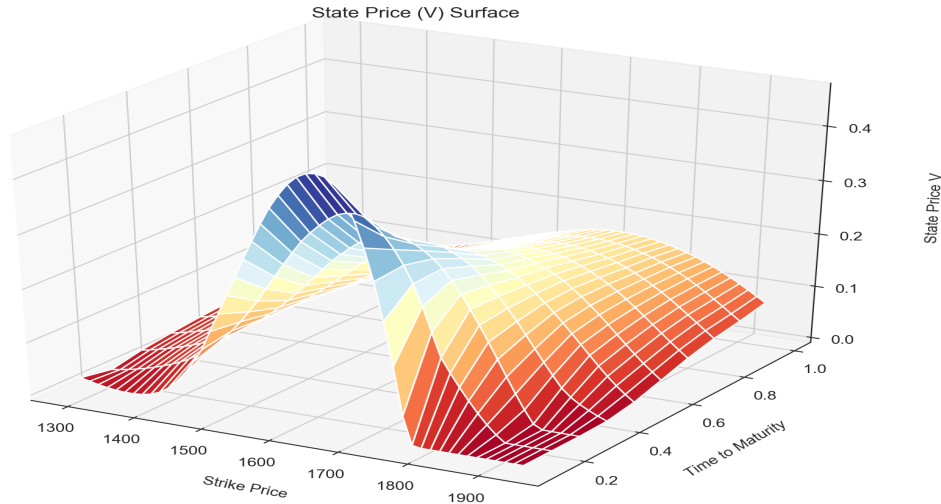
spread no-arbitrage condition is a weak constraint. Overall, my method displays a relatively smooth call price surface.

From the above analysis, my L_1 -SVM method provides a universal framework that incorporates previous studies by using different kernel function in Equation(5.14). Previous studies with univariate and bivariate kernels can both be replicated in my linear programming framework. Although my method requires additional time to search the optimal parameters, as shown in Table 5.5.2, it improves the estimation accuracy and surface smoothness in the forward-moneyness direction. Therefore, in this chapter, I use the estimated call option price from L_1 -SVM to extract the state price density (SPD). Since the above comparison is under the forward measure, based on Equation(5.6), I first transform the estimated option price back to risk neutral measure and then calculate SPD based on Equation(5.5). Figure 5.5.5 shows the extracted SPD on 02 July, 2013. As expected, the SPD is unimodal, smooth and positive. The SPD becomes small for far OTM and far ITM option.

5.6 Summary

In this chapter, I investigate the problem of estimating risk neutral information (SPD or RND) from option price. I find estimation of SPD from option prices faces five challenges: (a) in Breeden and Litzenberger (1978)'s estimation equation, the strike price is continuous while strike prices in real markets are discrete; (b) market data contain noise that may lead coarse and multimodal SPD; (c) theoretically SPD starts from 0 and extends to infinity while the market option data can only estimate SPD within certain bounds; (d) the estimated call option price surface should incorporate no-arbitrage constraints; (e) the estimation of

Figure 5.6.1: State Price Density on July 2, 2013



Note: Figure 5.5.5 plots the state price surface on 02/07/2013. The index price is 1614.08. The state price density obtained as second order derivatives of call price function respect to strike price.

SPD suffers the 'curse of differentiation'.

After briefly review existing parametric and non-parametric methods, I find that none of the methods is superior and has successfully solved all five challenges. More specifically, the parametric method is not flexible to satisfied all no-arbitrage constraints and thus leads to under-fitting the real market data. The non-parametric method shows good performance in approximating the surface but is sensitive to the pre-determined parameters. Therefore, differing from these two methods, I propose a new machine learning approach to estimate the call option price surface. Compared with parametric and non-parametric methods, machine learning has two advantages. First, since it is a data-driven approach, machine learning exhibits good performance in solving constrained optimization problems. Second, it is not sensitive to pre-determined parameters because the optimal value of these parameters are chosen using the gridsearch technique during training.

5.6. Summary

Based on empirical studies, although most use the average of the bid and ask price as fair option price, the true transaction option price lies in a range. To take this into account, I develop a data-driven approach L_1 -SVM based on standard support vector machine(SVM)⁴⁹, which incorporates the information in the bid-ask spread in pre-defining error tolerance in a loss function. As shown in Section 5.3.2, my L_1 -SVM method is sufficiently flexible to consider different models and all arbitrage-free constraints.

Empirically comparing my L_1 -SVM method with other non-parametric methods using S&P 500 index options, I show that my method is accurate and smooth. It is easy to implement and can be universally applied by choosing different kernel functions. Previous studies that use cubic spline, low-order polynomial and tensor product spline estimation method can all be replicated in my framework.

⁴⁹See Section 3.1.2 for more detail.

Chapter 6

Extracting Natural Probabilities from Option Prices: The Empirical Ross Recovery Theorem

All models are wrong, but some are useful.

—Box (1976)

6.1 Introduction

Traditionally, practitioners and academics have been content with forecasting returns based on historical data, where future returns are assumed to be drawn from specific models or distributions. Although this standard econometrics approach yields relative a good performance in predicting stock volatility, it fails to capture extreme movements in the financial markets. History has clearly revealed that rare events in the financial markets occur with a low probability which, from a statistical perspective, lies in the tail of a probability distribution(Liu (2014)). When I use historical data to infer future returns, these extreme values are likely to be treated as outliers. Hence, valid inferences on future returns can only be obtained by incorporating the tail dependence(Poon et al. (2003)) and adding additional restrictions.

Is it possible that I can obtain a valid inference without using historical data? Researchers have investigated this issue in the derivatives market. With payoffs extending into future, financial derivatives naturally offer a prism through which any market's perception of future returns can be revealed. The hope that this revealed information can be used to forecast underlying future returns has fascinated academics and practitioners. While there exists a rich equity option market and a literature about how to extract risk-neutral probabilities from option prices (see Chapter 5 and Anagnou et al. (2002) for a review), previous studies do not consider directly extracting real-world probabilities from observed option prices. This is for two reasons. First, classic option pricing theory is based on a risk-neutral valuation framework (Black and Scholes (1973); Merton (1973)), in which there is no role for the dynamics of the underlying under a real-world measure \mathbb{P} . If the option is valued without knowing the underlying dynamics under \mathbb{P} , how can this information be inferred from option prices? It seems impossible to recover real-world probabilities from option prices. Second, as suggested by Jackwerth (2000) and shown in Section 5.2.1, in each state of the world, the relationship between risk neutral price and real-world probability and risk aversion is shown as below.

$$\text{risk neutral probability} = \text{stochastic discount factor} \times \text{real world probability}$$

Apparently, from this relationship, there is an infinite combination of stochastic discount factor and the real-world probability that yield the same risk neutral probability. Economically, this means when I extract the risk-neutral probabilities from the prices of options on the S&P 500, I find the risk-neutral probability of, for example, a 25% drop in one month to be higher than the probability calculated from historical stock returns. Since the risk-neutral probability is the

real probability adjusted for the stochastic discount factor, this result indicates either that the market forecasts a higher probability of a stock decline (implied by real-world probability) than has occurred historically or the market requires a very high risk premium (implied by stochastic discount factor) to insure against a decline. Without knowing which is the case, it is impossible to separate the two and infer the market's forecast of the real-world probability. Therefore, in order to estimate real-world probabilities from option prices, the stochastic discount factor (or pricing kernel in asset pricing literature) should be known. However, there is no widely accepted measure of the stochastic discount factor in previous studies, this therefore leads to controversial recovered results.

Recently, without appealing to historical data and assuming an investor's risk preferences, Ross (2015) poses "The Recovery Theorem (TRT)", which enables the recovery of real-world probability and stochastic discount factor together from Arrow-Debreu state prices (Arrow (1964)) implied by observed option prices. This is a remarkable study and the result relies on two insights: (a) the stochastic discount factor is transition independent; (b) there is a time homogeneous, finite state Markov chain process driving all changes in the economy. These two assumptions together allow for a unique recovery of real-world probability.

Following Ross's approach, several studies attempt to generalize TRT by relaxing these two key assumptions. As pointed out by Huang and Shaliastovich (2014), current economic conditions may influence future wealth and, therefore, the stochastic discount factor implied utility in Ross (2015)'s first assumption is incorporated with Kreps and Porteus (1978)'s recursive preferences. By considering these, they extend TRT to an Epstein-Zin type utility based framework, in which when wealth to consumption ratio is given, recovered pricing kernel captures the investor's preference for timing uncertainty.

Additionally, several recent papers focus on extending the second assumption of TRT from discrete time to continuous and from bounded state space to unbounded. Replacing the discrete-time Markov chain with a continuous time recurrent Borel right process, Qin and Linetsky (2016) prove that TRT is possible when the stochastic discount factor is a positive semi-martingale function. This semi-martingale stochastic discount factor setting is further extended and linked to Hansen and Scheinkman (2009) by Qin and Linetsky (2017). Changing the risk neutral measure to T-forward measure, Qin and Linetsky (2017) extend TRT without the Markovian assumption. Clearly, TRT should extend to continuous time since the real financial market is in continuous time, but whether I need TRT in unbounded state space is debatable. Some researchers argue this is not necessary because even if the true economy state space is unbounded, I always truncate the space because I have a limited number of observations. However, I argue that the extension of TRT is required for two reasons. First, although there is no conclusion about whether the economy state space is bounded or unbounded, for robustness, it is necessary to extend Ross (2015) to an unbounded diffusion setting. Second, even though I assume the state space is bounded, it is impossible to determine the upper bound of economy state space. Take the S&P 500 index as an example, no one can know a correct upper bound .

The study of the boundary condition of TRT is initiated by Dubynskiy and Goldstein (2013), who prove the boundaries of state space are essential for separating risk aversion and real-world probability. However, they argue the assumed boundaries which, in order to apply TRT, fundamentally change the solution of real-world probability. This result is confirmed by Walden (2017), who shows that boundaries only concern the state dynamics instead of the pricing kernel. In contrast to Dubynskiy and Goldstein (2013)'s finding of boundaries, by re-

laxing the bound restriction, Walden (2017) shows that recovery is possible in an unbounded diffusion-type state space when the diffusion process is recurrent and the sufficient but not necessary condition for deriving unbounded is a mean reverting diffusion process.

Instead of recovering from the Arrow-Debreu state prices perspective, Carr and Yu (2012) consider an alternative set of sufficient conditions to derive TRT. Relying on restricting the dynamics of the numeraire portfolio, they show the unique real-world probability can be recovered using the regular Sturm-Liouville theory. This study provides an insight into recovering real-world probability with a preference-free approach. Along the line of this perspective, Park (2016) further supports Walden (2017) by proving recovery under a recurrent and mean reverting diffusion process is possible under Carr and Yu (2012)'s framework.

Although previous studies have successfully extended TRT, Borovička et al. (2016) point out that there is a theoretical flaw in TRT, arguing that Ross' first assumption is a special case of Hansen and Scheinkman (2009), in which the martingale component in the stochastic discount factor factorization equals one. They suggest that this special case only holds in an economic environment where the highest return asset is the long bond. In particular, they prove that, under the general stochastic discount factor factorization, what is recovered via the Perron-Frobenius theory is a distorted probability since the martingale component serves as the change of measure. This distorted probability reveals the market's long-term perspective. This distorted probability is also discussed in Alvarez and Jermann (2005), Hansen and Scheinkman (2009) and Qin and Linetsky (2017).

Empirical studies implementing TRT in stock options and bond futures options markets show conflicting results. In the context of the fixed income market, Qin et al. (2016) and Bakshi et al. (2017) test the potential misspecification of

TRT in Borovička et al. (2016)'s long-term factorization framework. Qin et al. (2016) conduct a hypothesis test for Ross's first assumption on the US Treasury market. They show that the recovered measure is different from the estimated real-world probability measure. The difference between these measures can be explained by the instantaneous volatility of the martingale component. Similarly, using the 30-year Treasury bond futures option, Bakshi et al. (2017) demonstrate that the martingale component of the stochastic discount factor exhibits stochastic behavior. These two studies confirm Borovička et al. (2016)'s factorization of the stochastic discount factor and show Ross's assumption (the special case of martingale component equal to one) is violated in the long bond market⁵⁰ market. In the equity option market, using S&P 500 index option, Jackwerth and Menner (2017) empirically evaluate the short-term forecast ability of recovered probability. They show that statistically the 1-month realized returns are not drawn from recovered probability. However, using a neural network estimation framework, Audrino et al. (2015) show that the recovered distribution of S&P 500 index option contains predictive information. A trading strategy that uses recovered moment significantly outperforms one using this risk neutral information.

Focusing on two conflicting results in the equity option market, I argue there are two issues with previous studies. First, they both use discretization to implement TRT. As shown by Tran and Xia (2015) and Dubynskiy and Goldstein (2013), the discretized recovery result is sensitive to the number of states and bounds on the state space. As a result, without comparing the two methods in the same number of states and bounds on the state space, it is difficult to conclude whether or not the recovered result has predictive ability. Second, Jackwerth and Menner (2017) argue that the recovered result of Audrino et al. (2015) can be at-

⁵⁰Long bond refers as zero coupon bond with long maturity.

tributed to the loss function (or penalty term) in the neural network, but, without directly theoretical proving and empirically testing, it is hard to argue whether the failure of TRT in Jackwerth and Menner (2017) is due to theoretical flaws in TRT or their implemented technique.

To provide empirical evidence for the recovery problem, following Walden (2017), this chapter implements TRT in a continuous time setting. To my knowledge, this is the first study to investigate the empirical validity of TRT in continuous time. Without incurring discretization error, this chapter makes two contributions. First, I complete Walden (2017)'s unbounded diffusion TRT framework by providing the empirical evidence. Using Audrino et al. (2015)'s quadratic loss function in my L_1 -SVM framework, my study shows that using a quadratic loss function in state price estimation does not guarantee a well-recovered result. To further investigate whether the distorted recovery result is caused by the implementation procedure associated error or the flaws of TRT, I simulate option data using the Ornstein–Uhlenbeck (OU) process and define a specified $m(x)$ function. In line with Jackwerth and Menner (2017) and my empirical study using the S&P 500, I find that the recovered probability is different from the OU process generated real-world probability. Second, I contribute to the empirical studies on multivariate TRT. Although Ross (2015) and Walden (2017) both show that TRT can be applied in univariate and multivariate settings, the empirical studies are scarce. Sanford (2017) is the only empirical study that tests Ross (2015)'s recovery theorem in a multivariate Markov chain. Including volatility and underlying price as state variable, they show that the forecast results from multivariate TRT are far superior to univariate TRT. However, this result does not directly prove that the recovered probability is the real-world probability. In this chapter, using the theoretical result from Walden (2017) Section 3.4, I directly calibrate

the two-dimensional consumption model of Bansal and Yaron (2004). I confirm that the recovered probability is not the real-world probability.

The road map of this chapter is as follows. In Section 2, I provide a theoretical review of three key studies of TRT. Section 3 introduces the implementation procedure of TRT and how to back out key input parameters from option prices. I test the TRT in unbound diffusion using S&P 500 index option price and Ornstein–Uhlenbeck (OU) process generated synthetic data. Section 4 provides an example of bivariate TRT using Bansal and Yaron (2004)’s consumption model. In Section 5, I discuss the reasons that why TRT fails and Section 6 concludes.

6.2 The Recovery Theorem

The basic objective of TRT is to extract the real-world measure from the risk neutral measure. Until Ross (2015), this aim has always been viewed as impossible because it is hard to disentangle real-world probability from option implied information. As shown in Section 2.23 and Section 6.1, the risk neutral probability is a product of two components: (a) real-world probability, and (b) the stochastic discount factor. In general, option prices do not offer sufficient information to separate them. Surprisingly, Ross (2015) proposes TRT, which is a new way to recover the real-world probability from the risk neutral measure under certain assumptions. In this section, I provide a theoretical review of three key studies of TRT. I illustrate the basic assumptions and key recovery procedure of TRT under discrete and continuous time settings.

6.2.1 Discrete Recovery

6.2.1.1 Basic Framework

To successfully recover the real-world probability, Ross (2015) makes two main assumptions. First, he assumes a discrete-time economy world, in which the driver of uncertainty follows a finite time homogeneous Markov chain process. This assumption implies two essential characteristics for the uncertainty driver X : (a) X is bounded and lower and upper limits exist. (b) the future state of X only depends on the current state. Second, the stochastic discount factor (or pricing kernel) is transition-independent, in other words, the utility for a representative agent in the market is state-independent and additively separable. Using these two assumptions, Ross (2015) shows that the recovered real world probability problem transforms to an eigenvalue analysis of Arrow Debreu state prices.

Let us consider an economy world with a finite number of states M and time horizons N . The risk neutral information implied by Arrow Debreu state price S is defined as:

$$S = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m,1} & s_{m,2} & \cdots & s_{m,n} \end{bmatrix}$$

As the S is driven by a Markov chain process, I define the next state Arrow Debreu state price S_{n+1} only depending on current state S_n . I express this relationship as $S_{n+1} = \mathbf{Q}S_n$, where \mathbf{Q} is an $M \times M$ transition matrix denoted by:

$$Q = \begin{bmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m,1} & q_{m,2} & \cdots & q_{m,m} \end{bmatrix}$$

The transition matrix Q is the key to implementing Ross's Recovery. The rows of Q denote the current state and the columns represent the next potential state, so the sum of each row of Q will be one. If I define the real-world transition matrix as P , where

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m,1} & p_{m,2} & \cdots & p_{m,m} \end{bmatrix}$$

My aim is to recovery P from Q , which is uniquely determined by Arrow Debreu state price S ⁵¹. Under the general equilibrium pricing model (see Section 2.2.3), this two matrixes are connected with the pricing kernel. Assume the pricing kernel matrix is given as:

$$\Phi = \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,n} \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{m,1} & \varphi_{m,2} & \cdots & \varphi_{m,m} \end{bmatrix}$$

Using the transition independent assumption, the pricing kernel from state i to j can be written in the following form:

⁵¹Ross(2015) assumes the Arrow Debreu state price S is known. Please see Chapter 5 for the estimation of this matrix.

$$\Phi_{i,j} = \frac{q_{i,j}}{p_{i,j}} = \delta \frac{h(i)}{h(j)} \quad (6.1)$$

Where $h(\cdot)$ is a positive function. δ is the time discount factor. If I define the diagonal matrix as

$$D = \begin{bmatrix} h(1) & 0 & \cdots & 0 \\ 0 & h(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(3) \end{bmatrix}$$

Replacing $h(\cdot)$ with D , then Equation(6.1) can be written as:

$$Q = P\Phi = \delta D^{-1}D \quad (6.2)$$

Rearranging this equation yields

$$\delta^{-1}QD^{-1} = D^{-1}1 \quad (6.3)$$

Setting $z = D^{-1}$, Ross(2015) further simplifies Equation(6.3) as an eigenvalue analysis problem

$$Qz = \delta z \quad (6.4)$$

Since the transition matrix Q is non-negative, applying the Perron-Frobenius Theorem(Meyer (2000)), Ross(2015) shows unique recovery of the real world probability (matrix F) and the pricing kernel (matrix D up to a positive scaling).

Theorem 6. (*Perron-Frobenius Theorem*) *If all elements in a square matrix A is a non-negative matrix, the following statements hold:*

1. *A has a nonnegative real eigenvalue and the largest eigenvalue dominates the absolute values of all other eigenvalues.*

2. the following problem has an unique pair of eigenvalue and eigenvector

$$Ax = \lambda x$$

Where x is a vector and λ is a scalar. The elements in the eigenvector are also strictly positive.

6.2.1.2 Discussion

While Ross(2015) elegant proves the recovery of the real-world probability from risk neutral measure is possible, I argue that the framework of TRT has two groups of limitations (see Table 6.2.1 for summary). In contrast with the most option pricing models, which are build on continuous time with a diffusion process, Ross (2015) assumes a discrete and bounded economy world. Undoubtedly, a continuous time setting may more appropriate as the real financial market is continuous. However, whether the state space must unbounded is unclear. In Ross's framework, the information of state bound is included in Arrow Debreu state price matrix S . The first row in S shows the information of low bound and the last row in Q presents the upper bound. As a result, the choice of bound is determined in the value of S , which further influences Q and the recovered results. This influence of boundaries is also mathematically proved by Dubynskiy and Goldstein (2013) and Tran and Xia (2015).

Moreover, Ross's assumption of investor's utility is also criticized by fellow researchers (Carr and Yu (2012); Borovička et al. (2016)) and challenged by empirical work on the stock market (Mehra and Prescott (1985)). In fact, the utility function itself is difficult to capture and measure. For example, how should I decide on the form of the utility function? According to the Expected Utility Theory, the utility function is expressed as probability weighted average of

Table 6.2.1: Limitations of Ross's Recovery Theorem

| | Problem | Remark |
|-----------------------------|---|--|
| Assumption Drawbacks | Discrete Time | Potentially problematic as the real world is in continuous time |
| | Boundaries, the best and worst state must be known State independent and additively time separable utility Can only forecast stock index, which is a proxy of entire holding of representative agent. Could not provide forecast for assets, whose underlying is zero net supply | Walden (2017) argues, even if the true state space is bounded, it may even be unknown This assumption rules out habit formation and additive of utility may be applied with restriction In other word, assets, which could not reflect holding of representative agent, cannot be forecast. |
| Implementation Restrictions | Could not provide forecast for cash-settled derivative, whose underlying is not tradable. For example, underlying is temperature | This assumption is also related to representative agent, which requires underlying is positive net supply assets. Hence, assets such as Volatility Index (VIX) and future could not be forecast. Because the $h(\cdot)$ function in Ross(2015) links to investor's marginal rates of consumption. |

contingent utility. However, based on Prospect Theory, utility is a weighted related probability and a non-linear function, since people overweight small probability. The choice of utility function becomes extremely complex since Ross's assumptions involve time-additivity and transition independence. The transition independence of utility, means the utility of a representative agent only depends on final and current state and rules out both an economic satiation effect and habit persistence in utility function (Carr and Yu (2012)). The satiation effect, put simply, is the diminishing marginal utility effect. Habit persistence, or habit formation, captures the effect that an increase in current consumption increases its marginal utility in the next time period. since Ross's framework is build on the assumption that there exist a representative agent, the implementation of TRT is restricted. It can only be applied to forecasting an index asset because it is a proxy for the representative agent's holding. Besides, it could not be used to forecast the underlying, which is zero net supply and not tradable.

6.2.2 Continuous Recovery

To derive a general version of TRT and loosen the assumption of Ross, Carr and Yu (2012) and Walden (2017) demonstrate that similar separation can be achieved in a bounded and unbounded continuous time setting. In contrast to using the risk neutral measure as the Arrow Debreu state price, Carr and Yu (2012) accomplish the recovery task using the properties of a numeraire portfolio. As introduced by Long (1990), in a market with no arbitrage opportunities, there exists a portfolio under which the deflated asset price evolves at a constant expected rate. When examining Ross's assumption in the numeraire portfolio context, Carr and Yu (2012) replaces Ross's assumption about the utility with the restrictions on the driver X . They show that unique recovery is possible in a time-homogeneous

diffusion setting when the boundary conditions are known.

6.2.2.1 Bounded Recovery

Consider an economy with probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathbb{P} is a unknown real world probability. Assume the risk-less asset (or money market account) is defined as

$$dS_{0t} = r_t S_{0t} dt \quad (6.5)$$

with a initial condition $S_{00} = 1$. There are no arbitrage opportunities in the market, therefore, according to the fundamental asset pricing theorem in Section 2.1, a unique risk neutral measure \mathbb{Q} exists. Under \mathbb{Q} , the security price adjusted by risk free rate is a martingale

$$E^{\mathbb{Q}}\left[\frac{S_{iT}}{S_{0T}}\right] = \frac{S_{it}}{S_{0t}} \quad (6.6)$$

Using the numeraire portfolio maps the risk neutral measure \mathbb{Q} to real-world measure \mathbb{P} , Carr and Yu (2012) denote the relationship between numeraire portfolio L_t and money market account as

$$L_t \equiv \frac{S_{0t}}{M_t} \quad (6.7)$$

Assume a univariate uncertainty process X exists in the economy and drives all the observables, such that, $r(x, t) \equiv r(x), \sigma(x, t) \equiv \sigma(x)$. X satisfies the stochastic differential equation

$$dX_t = \beta(X_t)dt + \alpha(X_t)dW_t^{\mathbb{Q}} \quad (6.8)$$

Where $\beta(x)$ and $\alpha^2(x)$ are the known drift and variance function respectively, and $W_t^{\mathbb{Q}}$ is a Brownian motion under \mathbb{Q} . The infinitesimal generator \mathcal{G} of X is

defined by

$$\mathcal{G} = \frac{\partial}{\partial t} + \frac{\alpha^2(x)}{2} \frac{\partial^2}{\partial x^2} + \beta(x) \frac{\partial}{\partial x} \quad (6.9)$$

Directly apply the Girsanov's Theorem in Section 2.1.2 to change the measure of X_t , the dynamics of X_t under \mathbb{P} is

$$dX_t = (\beta(X_t) + \sigma_t(X_t)\alpha(X_t))dt + \alpha(X_t)dW_t^{\mathbb{P}} \quad (6.10)$$

Based on Equation(6.7), the \mathbb{P} dynamics of the numeraire portfolio L_t can be derived by Ito's lemma, which yields

$$\frac{dL_t}{L_t} = (r(X_t) + \sigma^2(X_t))dt + \sigma(X_t)dW_t^{\mathbb{P}} \quad (6.11)$$

Where r_t is the short rate, σ_t^2 is the instantaneous variance and $W_t^{\mathbb{P}}$ is a Brownian motion under \mathbb{P} . From Equation(6.11), it is clear that the risk premium of the numeraire portfolio L_t is determined by its variance σ_t^2 . In other words, if I can determine the volatility process of L_t , I can recover the \mathbb{P} dynamics of the numeraire portfolio. If I apply Ito's formula to $L_t \equiv L(X_t, t)$ and substitute Equation(6.9), then the volatility of L_t is

$$\sigma(X) = \alpha(X) \frac{\partial}{\partial x} \ln L(X, t) \quad (6.12)$$

Rearrange Equation(6.12) and take the exponentiation for both sides, the value of numeraire portfolio can be expressed by two separated terms

$$L(X, t) = \rho(X)\gamma(t) \quad (6.13)$$

Where $\rho(x) = e^{\int_t^x \frac{\sigma(y)}{\alpha(y)} dy}$ and $\gamma(t) = e^{f(t)}$, $f(t)$ is the constant of integration and

$\rho(\cdot)$ and $\gamma(\cdot)$ are positive function. Substituting Equation(6.12) into the diffusion generator (Equation(6.9)), it implies

$$\frac{\alpha^2(x)}{2} \frac{\rho''(x)}{\rho(x)} + \beta(x) \frac{\rho'(x)}{\rho(x)} - r(x) = \frac{\gamma'(t)}{\gamma(t)} \quad (6.14)$$

This equation can hold only if both sides are constant and so finding the unique solution of this problem fits into the the regular Sturm Liouville theorem. The general solution of this eigenfunction analysis problem yields the value of the numeraire portfolio having the form

$$L(X, t) = \rho(X)e^{rt} \quad (6.15)$$

Hence, imposing the restrictions on the \mathbb{P} dynamics of the numeraire portfolio, Carr and Yu (2012) provide an alternative way to extract the real world measure. I argue that the restrictions of numeraire portfolio in Carr and Yu (2012) fundamentally connect to the utility assumptions of Ross (2015). As suggested by Kelly (1956) and Platen (2006), the numeraire portfolio has the optimal growth rate, which means the investors in the portfolio aim to maximize the logarithm of their terminal wealth. This is consistent with Ross (2015)'s framework, in which the representative agent aims to maximize his utility. However, the empirical implication of Carr and Yu (2012) is difficult because the numeraire portfolio is either not actually traded nor can it be constructed in the real market.

6.2.2.2 Unbounded Recovery

Inspired by Carr and Yu (2012), Walden (2017) further extends the recovery into the unbounded diffusion. Instead of restricting the numeraire portfolio, Walden (2017) extends Ross (2015)'s TRT framework on Arrow Debreu securities. In this

section, I summarize the Walden (2017) approach and show how it is related to Ross (2015).

Similar to Carr and Yu (2012), Walden (2017) considers an economy with probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathbb{P} is a unknown real world probability. Suppose there is a state variable X that drives all the observations

$$dX_t = \mu(X_t)dt + \sigma(X_t)dw \quad (6.16)$$

Mathematically, to ensure there are strong solutions in any interval of Equation(6.16), Walden (2017) assumes that μ and σ are continuously differentiable. For any x and y , there are exists constant C_1, C_2 and C_3 , such that $\frac{|\mu(x)-\mu(y)|}{|x-y|} \leq C_1$, $\frac{|\sigma(x)-\sigma(y)|}{|x-y|} \leq C_3$ and $0 < C_2 < \sigma(x)$.

The transition density function $f^t(x, y) = \frac{\partial F^t}{\partial y}$ associated with the above diffusion satisfies the Fokker-Planck equation, therefore

$$\frac{\partial f^t}{\partial t} = \mathcal{L}f^t \quad (6.17)$$

$$f^0(x, y) = \sigma_x(y) \quad (6.18)$$

Similar to Equation(5.3), in this framework, the price of call option $C^t(K)$ is written as

$$C^t(K) = \int_K^\infty (y - K) \frac{m(y)}{m(x)} f^t(x, y) dy \quad (6.19)$$

Based on Equation(6.16), the Arrow Debreu state prices that inferred from call option prices $C^t(K)$ are defined as

$$p^t(x, y) = e^{-\rho t} \frac{m(y)}{m(x)} f^t(x, y) \quad (6.20)$$

This definition of Arrow Debreu state prices corresponds with Equation(6.1) in

Ross's framework. In the terminology of Ross (2015), the pricing kernel is transition independent. For convenience, I define the function $q(x) = \frac{m'(x)}{m(x)}$ and $z(x) = \frac{1}{m(x)}$. Clearly, $q(x)$ is related to the representative agent's utility in Ross's framework.

Similar to Ross (2015), I assume the Arrow Debreu security price is known, which means in Equation(6.20) $p^t(x, y)$ is known. As proven by Walden (2017), I can infer the underlying parameters ρ , $\mu(x)$, $\sigma(x)$ and $m(x)$ from $p^t(x, y)$. More formally, the recovery problem of unbounded diffusion process becomes one of solving the fundamental ODE

$$z'' + \frac{k}{D}z' + \frac{\lambda - r}{D}z = 0 \tag{6.21}$$

Where $D = \frac{\sigma^2(x)}{2}$, $z' = \frac{-q(x)}{m(x)}$ and $z'' = -\frac{1}{m(x)}(q(x)' - q(x)^2)$. r is the short risk-free rate. $\lambda = \rho$. $k(x) = \mu(x) + 2q(x)D(x)$. In fact, Equation (6.21) is similar to the eigenvector formulation of Ross (2015). Although I can recover $m(x)$ using Equation (6.21), more conditions are needed to guarantee uniqueness. Consistent with Carr and Yu (2012), Walden (2017) finds that the recovery is only related to the dynamics of the state variable.

Proposition 3. *The necessary and sufficient conditions for recovery of $m(x)$ are*

$$\int_{-\infty}^0 e^{-\int_0^x \frac{\mu(s)}{D(s)} ds} dx = \infty$$

$$\int_0^{\infty} e^{-\int_0^x \frac{\mu(s)}{D(s)} ds} dx = \infty$$

Overall, as shown above, Walden (2017) provides a general version of Ross (2015) in an unbounded continuous setting. With restrictions on the behavior of the state variable, he shows the recovery is possible in unbounded diffusion.

6.3 Empirically Recovery in Continuous Time

Following Walden (2017), I empirically investigate TRT in unbounded diffusion setting. The procedure for applying the TRT is shown in Figure 6.3.1. The first step for TRT in continuous time is extract state price from observed market price. In this chapter, I use L_1 -SVM state price density estimator to extract state price. Compare with other parametric or nonparametric methods that are reviewed in Chapter 5, my estimator shows two advantages. First, as shown in Chapter 5, my L_1 -SVM estimator is comparatively accurate and smooth. Second, applying my L_1 -SVM estimator will allow us to examine the issues whether the quadratic penalty term (or loss function) in machine learning framework will influence recovered result. To apply the quadratic loss function, I modify the Equation(5.21) as

$$\min_{(\alpha,b)} \frac{1}{2} \|\alpha_i\|^2 + C \sum_{i=1}^N |f(k, \tau) - C(k, \tau)| \quad (6.22)$$

According to Section 3.1.2, this problem can be transformed to a dual optimization problem

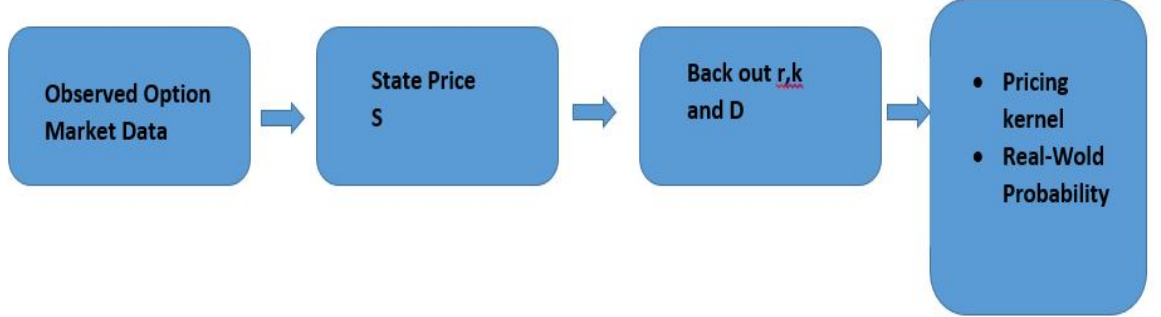
$$\max_{\{\alpha, \alpha^*\}} - \frac{1}{2} \sum_{i,j=1}^n (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*) K(X_i, X_j) - \varepsilon \sum_{i=1}^n (\alpha_i - \alpha_i^*) + \sum_{i=1}^n Y_i (\alpha_i - \alpha_i^*) \quad (6.23)$$

subject to

$$\sum_{i=1}^n (\alpha_i - \alpha_i^*) = 0$$

Following Audrino et al. (2015) and use Equation(6.23), I remove the estimated results that violate three no arbitrage conditions (Equation(2.27), Equation(2.31) and Equation(2.32)) during modeling. I suggest this standard support

Figure 6.3.1: The Recovery Procedure



Note: This figure gives the procedure for applying TRT in unbounded diffusion in practice. The S represents the state price. r is the short risk-free rate. k and D are defined in Section 6.2.2.2

vector machine can be estimated using Python's scikit-learn library. The second step of recovery problem is back out r , k and D from option prices, which has illustrated by Walden (2017) as follow:

Proposition 4. *For all $t \in (0, T)$, If the second order derivative of call option prices is defined as $V(t, y) = p^t(x_0, y)$. Then, for each y and $t > 0$, this second order derivative has following relationship*

$$V_t = D(y)V_{yy} + \alpha_1(y)V_y + \alpha_0(y)V$$

where

$$\alpha_1(y) = 2D'(y) - k(y)$$

$$\alpha_0(y) = D''(y) - k'(y) - r(y)$$

Where V_t, V_{yy} and V_y are the partial derivatives of V, D and k are defined in Section 6.2.2.2. $r(y) = \rho - q(y)\mu(y) - (q'(y) + q(y))D(y)$ and $q(x) = \frac{m'(x)}{m(x)}$

Please refer Walden (2017) Section 3.8 for proof

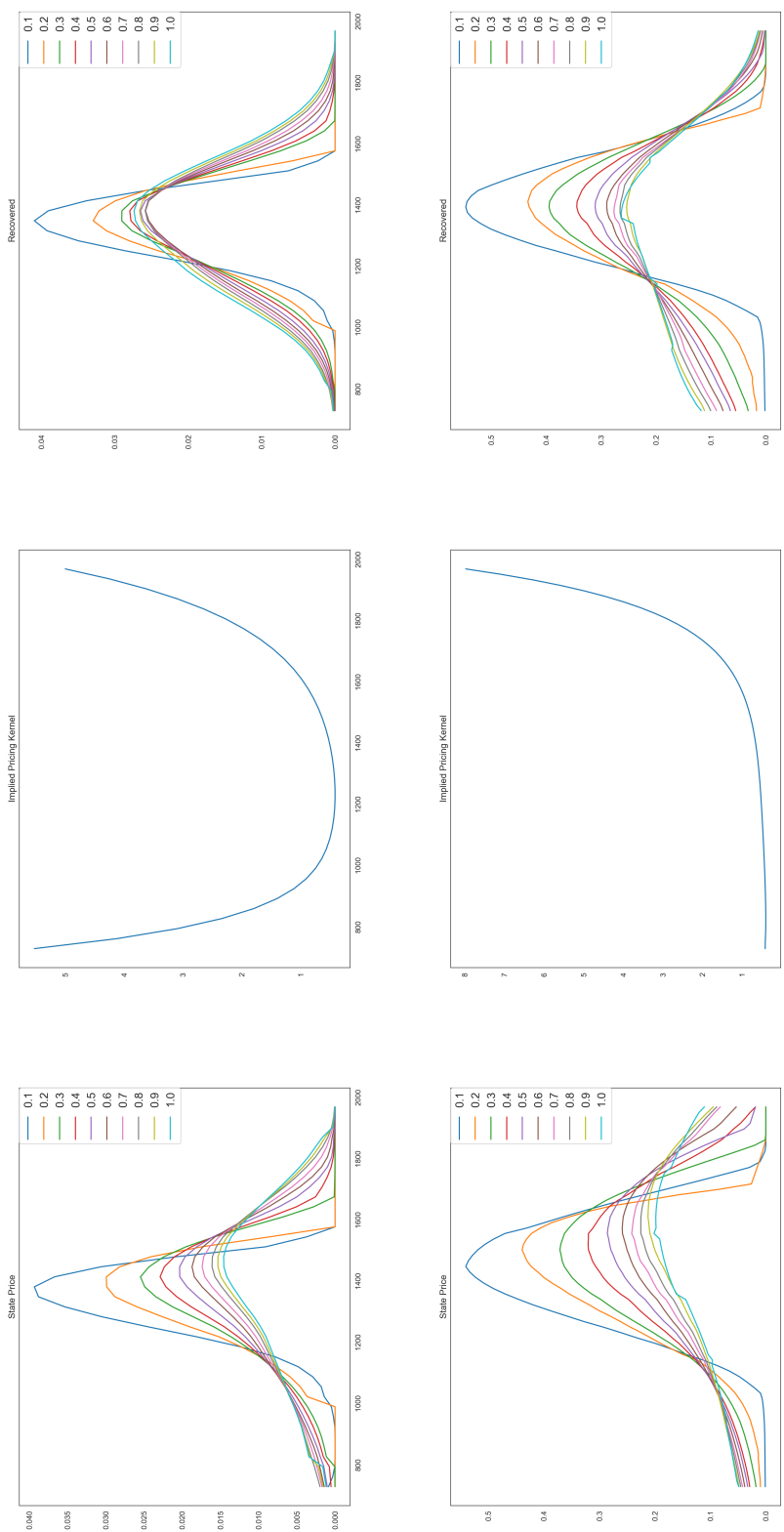
6.3.1 S&P 500 Index Option Example

In this chapter, I estimate the state price using the same dataset with Chapter 5. The dataset includes daily observations of S&P 500 index option from January 5, 2000 to April 30, 2016. I only use the OTM option data and apply the three level data filter proposed in Chapter 4. I estimate the state price using a standard quadratic programming support vector machine (Equation(6.23)). The description of the whole dataset is shown in Table 5.4.1. In Figure 6.3.2, I report two recovered results. The left panels show the state prices estimated from Equation(6.23). The right panels show the recovered physical distributions and middle panels report corresponding pricing kernels. Theoretically, the pricing kernel of S&P 500 index should monotonically decrease with state variable, in my case, the forward-moneyness. However, the empirical studies (such as Jackwerth (2000), Rosenberg and Engle (2002) and Constantinides et al. (2013)) show that the pricing kernel may increase locally. In my results, as shown in Figure 6.3.2, I do not find a persistent pricing shape kernel. Most surprisingly, instead of decreasing with forward-moneyness, the pricing kernel of bottom panel in Figure 6.3.2 even increase with it.

Specifically, to compare my continuous recovery with Ross (2015), in Table 6.3.1, I present the descriptive statistics of S&P 500 Options on April 27, 2011⁵². There are total 62 observations and the time to maturity from 7 to 365 days. Figure 6.3.3 illustrates the estimation results from Equation(6.23). Consistent with Equation(2.31), the call option surface is decreasing across the strike price and the state price is unimodal, smooth and positive. Figure 6.3.4 shows the corresponding partial derivatives of state price in Proposition 4. The recovered pricing kernel and probability are reported in Figure 6.3.5.

⁵²This is the same date utilized in Ross (2015).

Figure 6.3.2: State Price, Recovered Pricing Kernel and Probability



Note: The first column compares the recovered state price from Equation(6.23). The second column illustrates the recovered pricing kernels, which is defined as the ratio of the state price to the real world probability. In each subplot, selected time maturities (0.1 to 1.0) are indicated by different color. The third column shows the recovered probability density. The top panel shows the recovered results on 05/03/2008 and the bottom panel reports the recovered results on 12/03/2011.

Table 6.3.1: Descriptive Statistics of S&P 500 Options on April 27, 2011

| Variables | Mean | Std. Dev. | Min | Percentiles | | | | | Max |
|---------------------------------|---------|-----------|--------|-------------|---------|---------|---------|---------|---------|
| | | | | 5% | 10% | 50% | 90% | 95% | |
| Call Price (\$) | 13.31 | 15.13 | 0.08 | 0.28 | 0.50 | 8.15 | 33.90 | 43.75 | 146.3 |
| Put Price (\$) | 12.77 | 14.31 | 0.08 | 0.50 | 0.93 | 7.60 | 32.20 | 42.05 | 151.50 |
| Implied Volatility σ (%) | 20.62 | 8.39 | 6.46 | 10.30 | 11.46 | 19.19 | 31.10 | 35.69 | 87.11 |
| Strike price(K) | 1289.51 | 103.35 | 1100 | 930 | 1030 | 1540 | 2030 | 2110 | 1475 |
| Days(τ) | 53.82 | 41.47 | 7 | 14 | 18 | 44 | 95 | 127 | 365 |
| Index Price (S) | 1601.16 | 389.84 | 676.53 | 944.89 | 1091.60 | 1606.28 | 2079.51 | 2098.48 | 2130.82 |
| Trading Volume (V) | 1321.35 | 3690.84 | 1 | 2 | 4 | 110 | 3581 | 6698 | 157542 |

Note: Table 6.1 summaries descriptive statistics of S&P 500 Options on April 27, 2011. The total number of observations is 62. The call price is calculated as average of bid and ask price. The OTM put prices are translated into ITM call prices by put and call parity. Std.Dev denotes the standard deviation from call option price.

In contrast to Figure 11 in Ross (2015), I do not find a strictly decreasing pricing kernel. My recovered pricing kernel exhibits a U shape, which provides new evidence to Rosenberg and Engle (2002) using a forward-looking information. Overall, I conclude I can recover the probability using Walden (2017) but what information do I recovery is not clear.

To test whether the recovered probability is drawn from real-world probability. Following Walden (2017), I test following null hypothesis using Berkowitz (2001) test, which has better accuracy in small samples Bliss and Panigirtzoglou (2004).

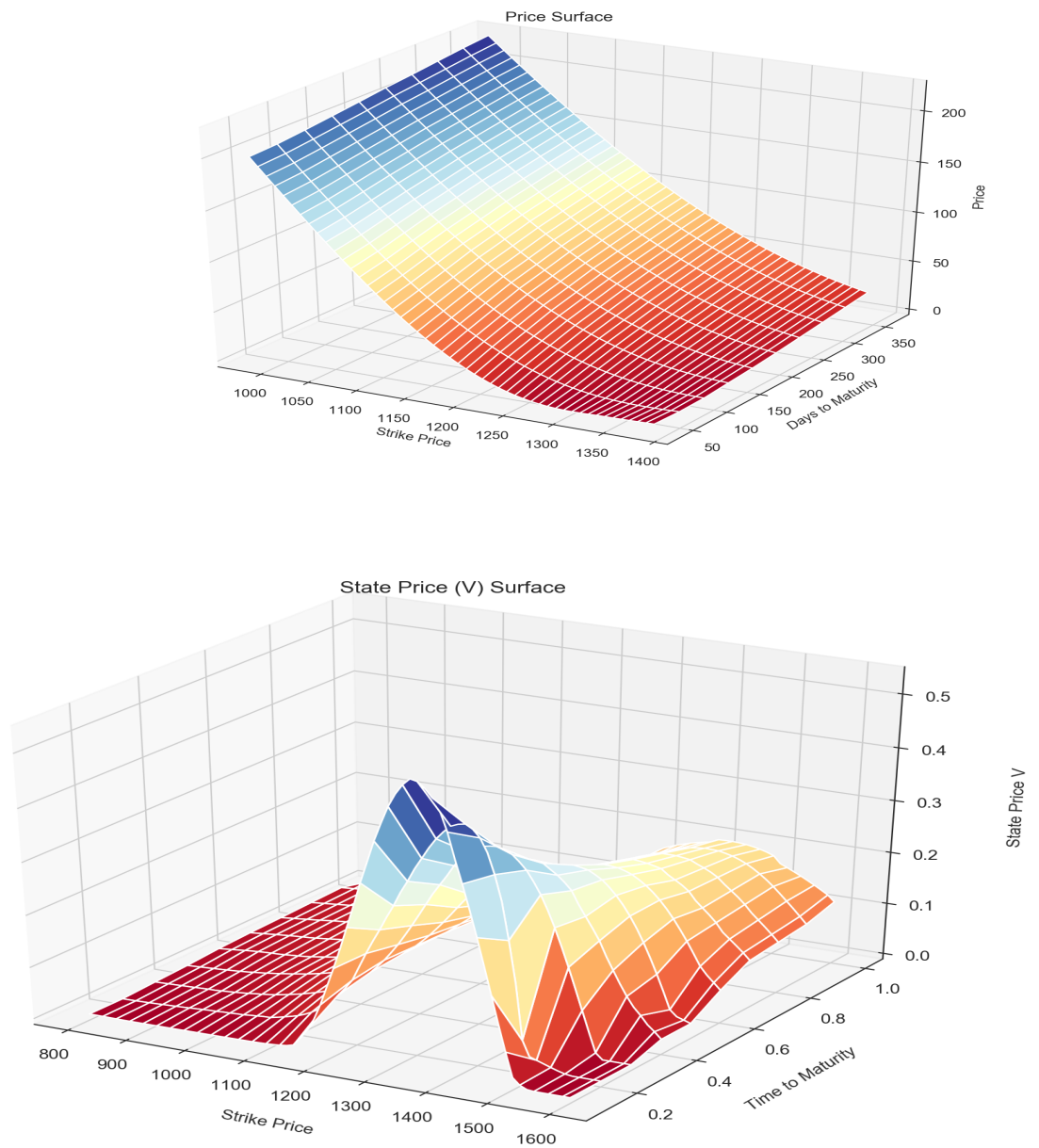
H_0 : Future returns are drawn from the recovered probability distribution.

In particular, I test this hypothesis with a monthly time horizon. I test whether the 1-month future realized S&P 500 index return is drawn from its real-world counterparts. If null hypothesis holds, I concludes that the recovered probability is the real-world probability. The Recovery Theorem is well specified or defined. The test procedure can be summarized as follow.

- First, I estimate the recovered probability p_i following procedure in Figure 6.3.1.
- Second, for each option, I find all 1-month horizon⁵³ options and their start and expiration dates.
- Third, I record the return between these start and expiration dates as 1-month realized return R_i .
- Finally, test the hypothesis whether R_i is drawn from p_i using Berkowitz test.

⁵³This means the days to maturity is 30. In my sample, I accept days to maturity in[21,33] as 1-month horizon.

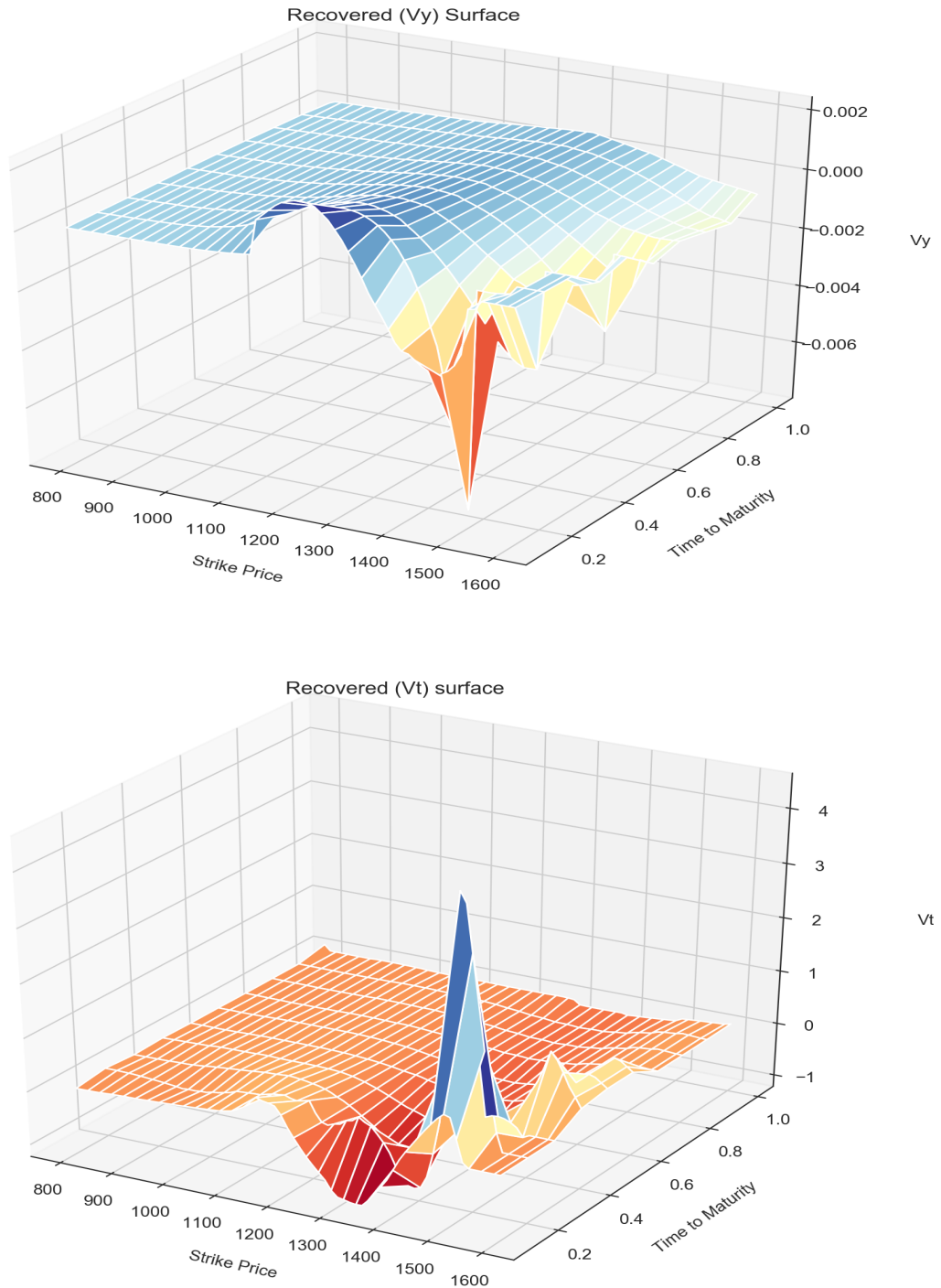
Figure 6.3.3: Estimated Price and State Price Surface on April 27, 2011



Note: Figure 6.3.4 shows the call option price and state price surface and on 27/04/2011. The top panel shows the call option price surface. The bottom panel displays the state price surface.

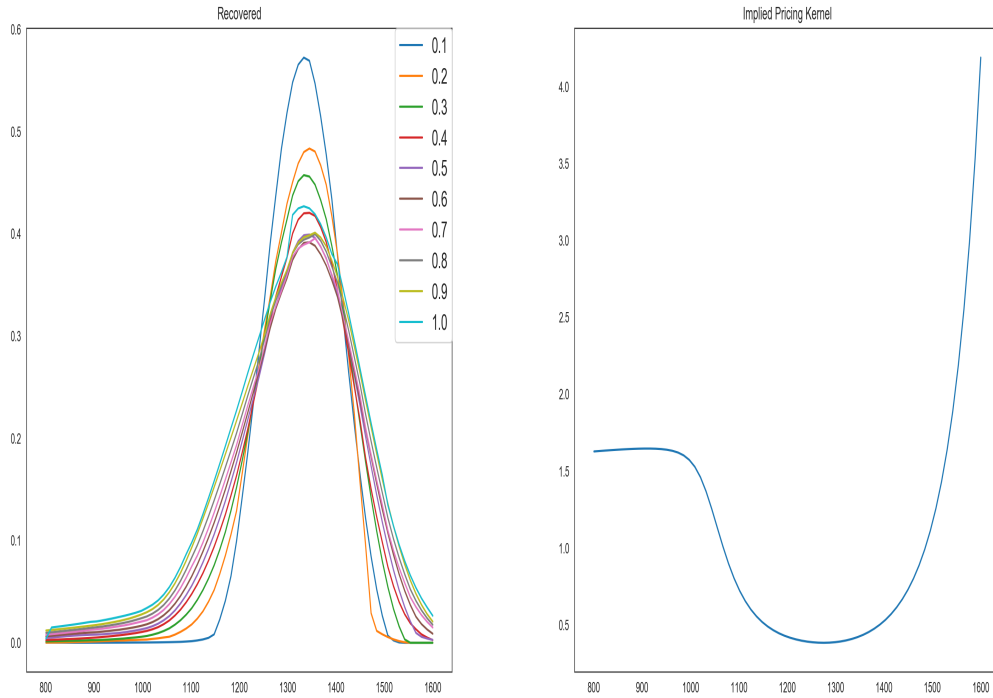
6.3. Empirically Recovery in Continuous Time

Figure 6.3.4: Partial Derivative of State Price Surface on April 27, 2011



Note: Figure 6.3.4 shows the associated partial derivative of state price surface on 27/04/2011. The top panel plots its first order derivative respect to the strike price. The bottom panel displays the first-order derivative of state price surface with respect to time to maturity.

Figure 6.3.5: Recovered Probability and Pricing Kernel on April 27, 2011



Note: Figure 6.3.5 plots the recovered results of S&P 500 index option on April 27, 2011. The right panel shows the recovered pricing kernel and the left panel reports the recovered probability density.

Table 6.3.3: Test Result of Recovered Probability

| Period | Berkowitz(p-value) |
|-----------|--------------------|
| 2000-2007 | 0.001 |
| 2008-2009 | 0.005 |
| 2010-2016 | 0.001 |

Note: This table present test results for whether the 1-month realized return are drawn from recovered probability.

The test results are reported in Table 6.4. Similar to Walden (2017), my results strongly reject the null hypothesis.

6.3.2 Ornstein–Uhlenbeck Example

To investigate whether the distorted recovery result is caused by error associated with implementation procedure or comes from a theoretical flaw in TRT then, following Walden (2017), I consider the state evolving with the Ornstein–Uhlenbeck (OU) process

$$dX = \theta(a - X)dt + \sigma dw \quad (6.24)$$

where θ and σ are greater than zero. Since the OU process tends to drift towards its mean, it is generally used to model the mean reverting behavior of underlying in mathematical finance. As I can always define a new state variable $\dot{X} = X - a$, assuming $a = 0$, I further simplify Equation (6.24) as follow

$$dX = \theta(-X)dt + \sigma dw \quad (6.25)$$

Comparing above equation with Equation (6.16), I define $\mu = -\theta x$. According to Proposition 3, I check whether recovery is possible for this diffusion. Since $\frac{\mu}{D} = -\frac{\theta}{\sigma}x$, then I get $e^{-\int_0^x \frac{\mu(s)}{D(s)} ds} = e^{\int_0^x \frac{\theta}{\sigma} s ds} = e^{\frac{\theta}{\sigma}x}$. Therefore, two conditions in Proposition 3 are satisfied. The recovery is possible for this unbounded diffusion.

To test whether the recovered result is affected by the estimation of Arrow Debreu state prices, in this section, instead of extracted state prices, I use simulated state prices based on Equation (6.20) directly. The empirical procedure can be summarized as follows.

- Generate the OU probability density under \mathbb{P} measure by assuming $\mu = 1.1$, $\sigma = 0.3$, $\theta = 1$ and the initial underlying price $S_0 = x = 1900$.

- Get Arrow Debreu state prices by assuming $m(x) = 1 + x^2$ and using Equation(6.20).
- Back out r , k and D from the Arrow Debreu state prices
- Obtain the recovered probability

In the top panel of Figure 6.3.6, I plot my simulated option prices. The x-axis is the strike price and the y-axis is the option price. The strike price starts from 50 to 4000 and time to maturity is from 0.1 to 0.8. There are total 5400 option prices in my sample. Consistent with the real market option data plotted in Figure 5.4.1, the option prices reflect the decreasing behavior as I expect. The ITM option is more expensive than OTM option. Using the definition of r and k in Proposition 3, I obtain

$$r = \rho + \frac{2x^2\theta - \sigma^2}{1 + x^2} \quad (6.26)$$

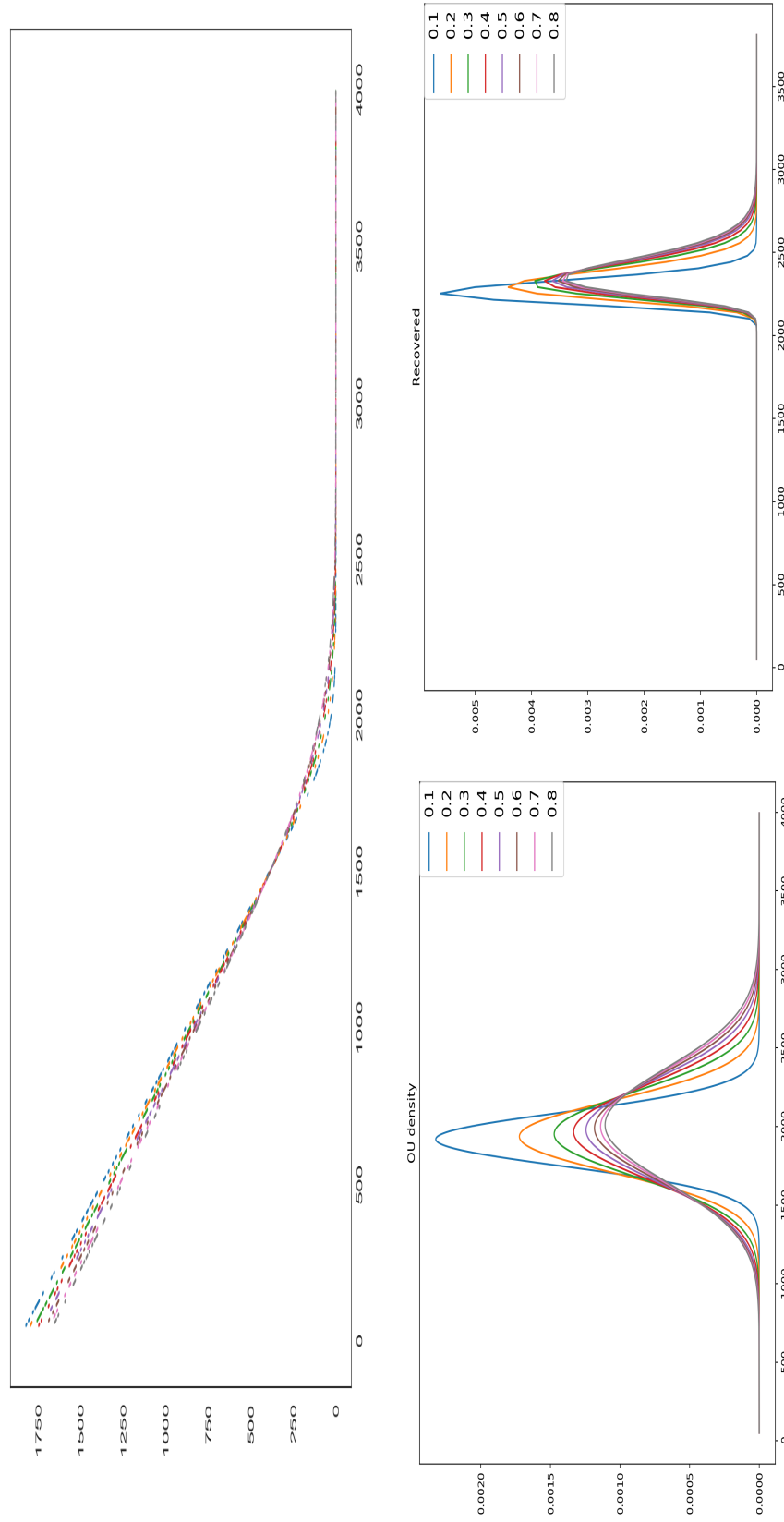
$$k = -x\theta + \frac{2x\sigma^2}{1 + x^2} \quad (6.27)$$

The fundamental ODE problem Equation(6.21) becomes

$$z'' + \frac{x}{D} \left(\frac{2\sigma^2}{1 + x^2} - \theta \right) z' + \frac{1}{D} \left(\lambda - \rho - \frac{2x^2\theta - \sigma^2}{1 + x^2} \right) z = 0 \quad (6.28)$$

Solving this equation, I plot the recovered probability in bottom panel of Figure 6.3.6. I can see that the recovered probability density is significantly different from the OU density distribution under \mathbb{P} . The recovered distribution exhibits fatter tails than the real world distribution. Except for the options with 0.1 time to maturity, the recovered distributions with other times to maturity

Figure 6.3.6: Results for Ornstein–Uhlenbeck Process



Note: this figure plots the simulated data and recovered result for Ornstein–Uhlenbeck process. The top panel plots the simulated call option prices. There are total 5400 option prices in my dataset. The bottom panel shows the recovered density and the real OU density under P measure. The index price is 1900.

are significantly higher than the real-world probability. Consequently, instead of associated state price estimation error, the distorted recovery result is caused by a theoretical flaw in TRT.

6.4 Reasons for Distorted Recovery

6.4.1 Theoretical Explanation

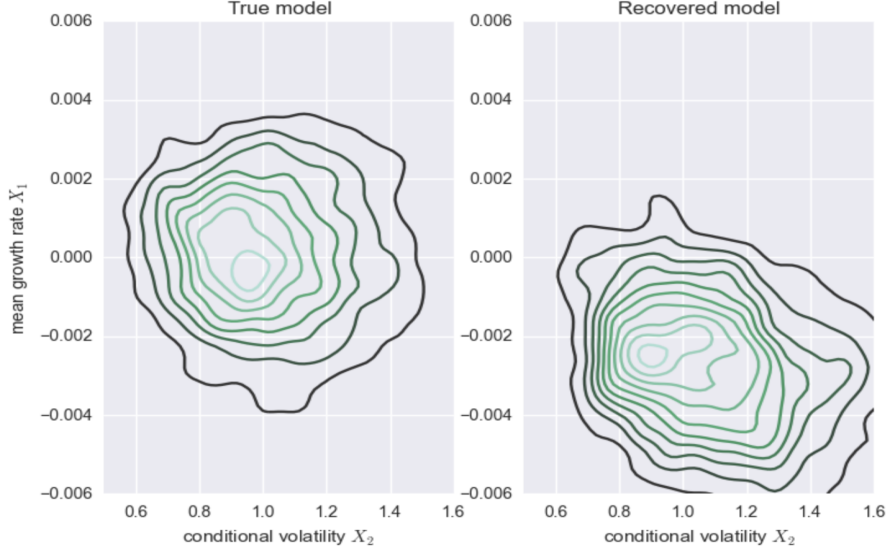
As criticized by Borovička et al. (2016), Ross's Recovery Theorem would lead a misspecified pricing kernel, which further results in a distorted real-world probability. They demonstrate that although Ross's first assumption (transition independence of pricing kernel) mathematically delivers a unique recovery, it is inconsistent with stochastic discount factor decomposition of Alvarez and Jermann (2005). The transition independence assumption implies that there is a martingale component associated with the stochastic discount factor and the recovered probability from Ross (2015) actually reflects a long term expectation.

To illustrate this issue, let us consider a discrete time environment with probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There exists a state process X_t that is strictly stationary and time-homogenous. If I define the stochastic discount factor process as S_t , then based on Equation(2.2.2), at each time t , the price of claims that receive payoff Z_{t+n} at $t+n$, is given by

$$E\left[\frac{S_{t+n}}{S_t} Z_{t+n} | X_t\right] \tag{6.29}$$

According to Hansen and Scheinkman (2009), if I further restrict $Z_{t+n} =$

Figure 6.4.1: Recovery for Bivariate State Variables



Note: This figure shows the true and recovered joint distribution of state vector (mean growth rate and stochastic volatility). The parameters of the model is $a = -0.021$, $\sigma_{\bar{X}_1} = [0 \ 0.00034 \ 0]'$ and $\sigma_{\bar{X}_2} = [0 \ 0 \ -0.038]'$

$\psi(X_{t+n})$ and introduce a collection of linear pricing operators as $\{M_n\}$, I can get

$$M_n \psi(x) = E\left[\frac{S_{t+n}}{S_t} \psi(X_{t+n})\right] \quad (6.30)$$

Using the Perron-Frobenius Theorem and defining Φ is the positive eigenfunction/eigenvector and ρ is associated eigenvalue, I can further simplify the pricing operator M_n as⁵⁴

$$M_n \phi = \rho^n \phi \quad (6.31)$$

This means I that can decompose the pricing kernel into two parts, namely permanent component $M_t^P = \rho^{-t} M_t \phi(X_t)$ and transitory component $M_t^T = \rho^t \phi(X_t)$ (Alvarez and Jermann (2005) and Hansen and Scheinkman (2009)). There-

⁵⁴Please see Hansen and Scheinkman (2009) and Hansen (2013) for more detail.

fore the SDF can be expressed as

$$\Phi = \frac{S_{t+n}}{S_t} = \frac{M_{t+n}^P}{M_t^P} \frac{M_{t+n}^T}{M_t^T} \quad (6.32)$$

where

$$\frac{M_{t+n}^P}{M_t^P} = \rho^{-1} \frac{M_{t+n} \phi(X_{t+n})}{M_t \phi(X_t)} \quad (6.33)$$

$$\frac{M_{t+n}^T}{M_t^T} = \rho \frac{\phi(X_t)}{\phi(X_{t+n})} \quad (6.34)$$

If I compare Equation (6.32) with Equation (6.1), apparently Ross (2015) assumes the transitory component [Equation (6.34)] is equal to 1]. This is an unrealistic assumption Alvarez and Jermann (2005) and means the asset that can achieve the highest return is the long bond. Hansen and Scheinkman (2009) further shows that this transitory component is actually a martingale component, which defines a new probability that reflects long-term risk adjustments. As a result, due this martingale component, using Ross's TRT, I always get a distorted result.

6.4.2 Martingale Extraction using Bivariate Recovery

To test whether there is a martingale component in the pricing kernel, in this section, I apply Walden (2017) to Bansal and Yaron (2004)'s two-dimensional consumption dynamic model. Assume the state variables X_1 and X_2 evolve according to

$$dX_1 = -aX_2 dt + \sigma_{X_1} dw_{X_1} \quad (6.35)$$

$$dX_2 = -X_1 dt + \sigma_{X_2} w_{X_2} \quad (6.36)$$

where X_1 is growth rate and X_2 is stochastic volatility, w_{X_1} and w_{X_2} are indepen-

dent Brownian motion processes. Based on Equations (6.33) and (6.20), I express the pricing kernel as $\Lambda_t = e^{-\rho t \frac{m(X_t, Y_t)}{m(X_0, Y_0)}}$. If I assume the aggregate consumption $C_t = e^{X_t}$, then the pricing kernel can be expressed as $\Lambda_t = e^{\rho t \frac{e^{-r X_t}}{e^{-r Y_t}}}$, where γ is the risk aversion coefficient for a representative agent with power utility. Using this assumption and based on Equation(6.20), the recovery problem becomes one of solving the following PDE

$$\frac{\sigma_{X_1}^2}{2} z_{X_1 X_1} + \frac{\sigma_{X_2}^2}{2} z_{X_2 X_2} - X_2 z_{X_2} - (\alpha X_2 + \gamma \sigma_{X_1}^2) z_{X_1} + (\lambda - \rho + \gamma \alpha X_2 + \gamma^2 \frac{\sigma_{X_1}^2}{2}) z = 0 \quad (6.37)$$

In Section 3.4 of Walden (2017), he mathematically proves that this PDF has a unique positive solution. Thus, I can directly recover X_1 and X_2 . The solutions of this PDE are related to discrete martingale component extraction in Borovička et al. (2016) Section 5. Following Borovička et al. (2016) and Hansen et al. (2007), if the recovered X_1 and X_2 are identical to the pre-determined X_1 and X_2 , this means that the transitory component in Equation(6.34) is equal to 1 and there is no martingale component in the pricing kernel. To compare the recovered result and true state variables, using the same input parameters a , σ_{X_1} and σ_{X_2} in Borovička et al. (2016), I plot the true joint stationary distribution of the state vector $[X_1, X_2]$ in Figure 6.4.1. It can be seen that the recovered distribution shares similar characteristics with the true distribution. However, the recovered distribution exhibits a lower mean growth rate X_1 and a higher conditional volatility X_2 . This result indicates that there exists a martingale component that defines a new probability and under which the distribution of state variables is distorted. The mathematical proof of the new set state variables can be found in Appendix of Borovička et al. (2016).

6.5 Summary

In this chapter, I study the second main problem addressed in the thesis: what is recovered from The Recovery Theorem (TRT). Although previous studies by Audrino et al. (2015) and Jackwerth and Menner (2017) have explored this problem in the equity option market, I find their studies pose two issues: first, they both focus on discrete TRT by Ross (2015), which is sensitive to the number of states and boundaries of state space. Second, the error associated with state price estimation procedure may lead to a misspecified recovery result.

In order to solve these issues, I empirically implement Walden (2017)'s unbounded diffusion Recovery Theorem using S&P 500 index options. In particular, I test whether 1-month realized S&P 500 returns are drawn from recovered probability. My results strongly reject the null hypothesis and confirm Jackwerth and Menner (2017)'s finding using discrete TRT. To further analyze whether the distorted recovery result is due to state price estimation associated error or a theoretical flaw in TRT, I use simulated state prices generated by the Ornstein–Uhlenbeck (OU) process. As expected by Borovička et al. (2016), the recovered probability still differs from the real-world probability. In Section 6.3, I explore theoretical reason that why The Recovery Theorem fails. Also, instead of using a discrete TRT, I provide alternative evidence for the existence of a martingale component in eigenvector/eigenfunction decomposition of the stochastic discount factor using bivariate unbounded recovery.

This study contributes to previous empirical studies of the recovery problem in several aspects. First, to the best of my knowledge, this is the first empirical study of the recovery of real-world probability with an unbounded diffusion process. Without the influence of state space boundaries, I complete Walden (2017)'s unbounded diffusion TRT framework by providing empirical evidence and show

that recovery is possible with the unbounded diffusion process. Second, using an extended quadratic SVM, I provide the empirical evidence for Jackwerth and Menner (2017). I show that using a quadratic loss function in state price estimation does not result in well specified recovery results. Third, I contribute to the empirical studies on multivariate TRT. I empirically implement a bivariate recovery, which provides additional unbounded diffusion evidence for Borovička et al. (2016).

Overall, although The Recovery Theorem starting with Ross (2015) (followed by Carr and Yu (2012), Walden (2017) and other researchers) fails to extract the real-world probability, I nevertheless argue it is a remarkable work and leaves many opportunities for further research. Even if at this early stage I can not discard some unrealistic assumptions in TRT, I strongly believe that some future work will solve it.

Chapter 7

Conclusions, Limitations and Future Research

Not everything that counts can be counted, and not everything that
can be counted counts.

—Cameron (1963)⁵⁵

7.1 Summary

This thesis focuses on extract forward-looking information from option prices. In particular, I attempt to extract risk neutral and real-world information from option prices. Risk neutral information is related to the theoretical prices of options and real-world information reflects the information associated with the dynamics of the underlying. To extract the risk neutral information, I apply a linear programming support vector machine L_1 -SVM framework. Using S&P 500 index options, I show that this L_1 -SVM framework is a universal approach and comparatively accurate and smooth. Moreover, using a modified L_1 -SVM framework, I further demonstrate how to back out the real-world information from option prices. I empirically investigate The Recovery Theorem (TRT) in an unbounded diffusion setting. Consist with previous empirical study by Jackwerth

⁵⁵This quote is often attributed to Albert Einstein.

and Menner (2017) and theoretical criticism of Borovička et al. (2016), I demonstrate that the recovered probability from TRT is not the real-world probability.

The thesis contains 7 Chapters. Chapter 1 introduces the research background, motivation and key contributions. I briefly summarize the history of options, noting the introduction of electronic trading platforms and creation of more option products as the options markets have become more active and important. Thus, how to obtain and use the information from option markets has been a fundamental concern of both scholars and practitioners. In this thesis, I focus on how to extract accurate information from an option market and the use this information for future research. Specifically, I investigate two problems: (a) how to extract a well-behaved risk neutral density from option prices and (b) whether the information extracted from The Recovery Theorem is the real-world measure.

Chapter 2 provides a brief review of related option pricing and general asset pricing. I illustrate how the risk neutral and real-world measures are related from mathematical finance and asset pricing perspectives. In Section 2.3, I provide a comprehensive review of no arbitrage conditions for call option prices. An introduction to machine learning and its applications in finance is given in Chapter 3, where two machine learning techniques: the neural network and the support vector machine are presented. Comparing their characteristics in terms of ability to handle noisy data, processing large datasets, controlling model complexity, predictive accuracy and ease of operation, I argue the support vector machine is more suitable for extracting the risk neutral density from option prices.

Chapter 4 develops a data filter approach based on three principles: representative, accurate and no arbitrage. In Section 4.2, I provide a detail description of data filter rules and discuss the reason for choosing to work with S&P 500 in-

dex options. Comparing my filtered result with Zhang and Xiang (2008), I show that my dataset contains less noise. Finally, to compare my L_1 -SVM with other non-parametric methods, I change the estimated framework under the forward measure using the Radon-Nikodym derivative. This transformation enables us to compare all the models on common ground and treat interest rate and dividends as zero.

In light of the above techniques and the theorem, in Chapter 5, I specify how to incorporate no-arbitrage conditions into the linear programming support vector framework. Compared with other methods, I show that previous studies that use different interpolation techniques can all be incorporated in my framework by choosing a different kernel function. Also, taking advantage of the error insensitive range in the support vector machine, my framework naturally considers the information in bid-ask spread. Using 16 years of daily S&P 500 index options, I show that my method establishes a somewhat better accuracy and is stability smooth in the forward-moneyness direction.

In Chapter 6, following Walden (2017), I provide the first empirical study of recovery of real-world probability with the unbounded diffusion process. Modifying my support vector machine framework to consider the quadratic penalty term, I document that using a quadratic loss function in state price estimation does not guarantee a well-recovered result. This distorted result may be caused by either the implementation error or by a theoretical flaw in TRT. To show which is the case, I apply TRT using simulated data from an Ornstein–Uhlenbeck (OU) process. In line with my finding in S&P 500 index options, I note the recovered results always differ from the real-world probability. Following Borovička et al. (2016)’s criticism of Ross (2015), in Section 6.3, I further show the existence of a martingale component in a two-dimensional unbounded diffusion setting.

7.2 Limitations and Future Work

7.2.1 Limitations

Although my study has successfully shown how to extract the risk neutral density and natural probability from option prices, I argue it has some limitations.

First, a data limitation. Although, as shown in Figure 5.4.1, my dataset is large enough to construct a well-defined surface, I argue that high-frequency data are preferred for machine learning approach. This is because, as discussed in Section 5.4.2, the dataset always needs to be split during the learning process.

Second, I fail to address the risk neutral density tail extrapolation problem. As suggested in Section 5.1, I identify five challenges that are imposed in extracting RND. The third issue, that RND should theoretically lie in $[0, +\infty]$, is not appropriately investigated in L_1 -SVM framework. In Equation (5.27)-(5.28), I only consider the zero point in my framework because the Python Cvxopt library produces NaN when I use infinity as input. I suggest that future study could apply a different library to solve this issue.

Third, in Chapter 6, while I show empirically that the recovered probability differs from the real-world probability, I argue this does not necessarily mean I should discard the forecasting ability of recovered probability. The proof of the existence of a martingale component using bivariate TRT only statistically implies these two probabilities are not the same. However, it is still possible that partial real-world information is recovered by TRT. How to prove this argument and identify this partial information is an interesting research direction.

7.2.2 Future Research

Clearly, building on Breeden and Litzenberger (1978) and Ross (2015) 's theoretical foundation, various future researches can be explored in order to get option implied information. Based on the content of this thesis, I recommend following research directions.

- It is worth extending the study to more complex options such as American options. This is because, even though the European option is widely traded in the market, in some cases investors prefer to use other types of option to express their future view. For example, the most traded crude oil option traded on the CME is American-style.
- Although the recovered probability is distorted, it still sheds light on applying new extracted probability in risk management and portfolio selection. Given the limited forecast ability of historical data, the estimation of tail risk from past returns has been criticized by researchers. The forward-looking information implied from options, which more closely represents a market's perspective on the future, may yield better performance. I suggest the future research can start with reconstructing the standard risk measure value at the risk (VAR) using implied probability. However, as discussed in Carr and Yu (2012), one issue that should be borne in mind is that the recovered probability reflects the market's view about the future but the market is not always right. Incorrect views obviously would lead incorrect extracted measures.

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Appendix

Kernel Function

Let X and Y are input data, the parameter γ determines the amplitude on the function. The popular choice of kernel function $K(X, Y)$ are listed as following:

- Polynomial kernel with degree c :

$$K(X, Y) = (\gamma X^T Y + \gamma)^c$$

- Linear kernel:

$$K(X, Y) = X^T Y$$

- Radial basis function(RBF) kernel:

$$K(X, Y) = \exp(-\|X - Y\|^2 / 2\sigma^2)$$

- Sigmoid kernel:

$$K(X, Y) = \tanh(\gamma \cdot X^T Y + r)$$

Selected Python Code

Selected Code for Data Cleaning

```
import numpy as np
import pandas as pd
import datetime
import matplotlib.pyplot as plt
import dateutil
import re

from pandas import DataFrame
from scipy.interpolate import interp1d
from scipy.stats import norm
from scipy.linalg import lstsq

# 1. import option data
df_option = pd.read_csv('/Users/Desktop/original data copy/
    spx2000-20160430.csv')
# Note that the strike prices are out by a factor of 1,000! So
    let's fix them: df_option.strike_price = df_option.
    strike_price/1000
# format date
df_option['date'] = pd.to_datetime(df_option['date'],format = '%m
    /%d/%Y')
df_option['exdate'] = pd.to_datetime(df_option['exdate'],format =
    '%m/%d/%Y')
df_option['days']=(df_option['exdate']-df_option['date']).dt.days

# 2. import index data
df_index = pd.read_csv('/Users/chelsea/Desktop/all thesis/
    original data copy/chapter1data/spxunderlying2000-2016.csv')
df_index['date'] = pd.to_datetime(df_index['date'],format = '%m/%
    d/%Y') df_index = df_index.rename(columns = {'close':'
```

Selected Python Code

```
index_price', 'volume': 'underlying_volume'})

# final dataset
df = pd.merge(df_option, df_index)
df['moneyness'] = df['strike_price'] / df['index_price']
df['mid_price'] = 0.5 * (df['best_bid'] + df['best_offer'])

# drop identical observation
print(len(df.drop_duplicates(subset=['date', 'exdate', 'strike_price', 'cp_flag', 'best_bid', 'best_offer'], keep='first')))

# drop Identical except price
df_i = df.drop_duplicates(subset=['date', 'exdate', 'strike_price', 'cp_flag'], keep='first')
len(df_i)

# Liquidity Filter
df_zero = df_i[(df_i['best_bid'] != 0) & (df_i['best_offer'] != 0) & (df_i['volume'] != 0)]
df_days = df_zero[(df_zero['days'] >= 7) & (df_zero['days'] <= 365)]
df_f = df_days[(df_days['moneyness'] >= 0.8) & (df_days['moneyness'] <= 1.2)]
```

Selected Code for L_1 -SVM

```
import numpy as np
import pandas as pd
import sympy as sp
from sympy import exp
from sympy.utilities.lambdify import lambdify
from cvxopt import matrix, solvers
solvers.options['show_progress'] = False
from sklearn.model_selection import train_test_split
from sklearn.model_selection import GridSearchCV, cross_val_score, KFold
from sklearn.base import BaseEstimator, RegressorMixin
from sklearn.preprocessing import StandardScaler
from sklearn.metrics import make_scorer

# kernels
def linear_ker(x, y):
    return np.dot(x, y)

def poly_ker(x, y, p=3):
    return (1 + np.dot(x, y)) ** p

def rbf_ker(x, y, sigma):
    return exp(-np.dot(x-y, x-y) / (2 * (sigma ** 2)))

def sigmoid_ker(x, y, p1=1, p2=2):
    return sp.tanh(p1*np.dot(x, y) + p2)

## differential function
def diff_fun(zinput, k, ker, xj, vars):
    '''
    k is order
```

```

ker:kernel function choice
zinput:knots in spline
xj:jth compoent of X
vars: sigma if rbf_kernel
'''
sizeX = len(xj)
sizeinput = len(zinput)
sx = sp.symbols('x0:%d' % sizeX)
x0 = sp.symbols('x0') # change to x1, whether results change???
difftmp = np.empty([sizeinput, sizeX], dtype=object)

for i in np.arange(sizeinput):
    for j in np.arange(sizeX):
        difftmp[i][j] = ker(zinput[i], sx[j], vars)

diff_fun = sp.diff(sp.Matrix(difftmp), x0, k)
dffun = diff_fun.col(0)
yf = lambdify(x0, dffun, 'numpy')
res = yf(xj)
n = res.shape[0]
taok = res.reshape(n, -1)

return taok

# Optimazation

class mySVR(BaseEstimator, RegressorMixin):
    '''
    Support vector regression and No arbitarge constraints
    '''
    def __init__(self, e=0.25, ker=rbf_ker, nsplit=2, staus='Non_Cons', C=100,

```

Selected Python Code

```
        lam0=50,lam1=50,lam2=50,sigma=0.707,tol = 1e-10):
self.e = e
self.ker = ker
self.nsplitt = nsplitt
self.staus = staus
self.C = C
self.lam0 = lam0
self.lam1 = lam1
self.lam2 = lam2
self.sigma = sigma
self.tol = tol

def fit(self, X, y):
    '''
    my SVR estimator
    y-output vector
    x-input vector
    ker is kernel, choices:[linear_ker,poly_ker,rbf_ker,
                           sigmoid_ker,tensorproduct,b-spline]
    C,e,lam0 are trade-off parameters
    set e 1/4(bid-ask spread)
    C=1,tol =1e-10 tolerance for support vector detection
    nsplitt-no. of splited fwd points
    staus: whether add NA constraints or not. Flase is Non_Cons,Ture = Cons
    '''
    n = len(X)
    K = kernelmatrix(X, self.ker, self.sigma)
    y = y.reshape(len(y), -1)

    # define the linear programing
    if self.staus == 'Non_Cons':
```



```
c = matrix(np.vstack((np.zeros((n, 1)), 0, (1/n)*self.C
                      * np.ones((n, 1)), (1/n)*np.ones((n, 1)))), tc='d')
```

S&P 500 Index Sample List

| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|--------------------------------|------------------|------------------|------------|------------------|---|
| 1 | ABT | ABBOTT LABORATORIES | 31-Mar-1964 | | 19023 | 2834 | Pharmaceutical Preparations |
| 2 | HON | HONEYWELL INTERNATIONAL INC | 31-Mar-1964 | | 19023 | 9997 | |
| 3 | AEP | AMERICAN ELECTRIC POWER CO | 31-Mar-1964 | | 19023 | 4911 | Electric Services |
| 4 | BA | BOEING CO | 31-Mar-1964 | | 19023 | 3721 | Aircraft |
| 5 | BMY | BRISTOL-MYERS SQUIBB CO | 31-Mar-1964 | | 19023 | 2834 | Pharmaceutical Preparations |
| 6 | CPB | CAMPBELL SOUP CO | 31-Mar-1964 | | 19023 | 2030 | Canned, Frozen & Preserved Fruit, Veg & Food Specialties |
| 7 | CAT | CATERPILLAR INC | 31-Mar-1964 | | 19023 | 3531 | Construction Machinery & Equip |
| 8 | CVX | CHEVRON CORP | 31-Mar-1964 | | 19023 | 2911 | Petroleum Refining |
| 9 | KO | COCA-COLA CO | 31-Mar-1964 | | 19023 | 2086 | Bottled & Canned Soft Drinks & Carbonated Waters |
| 10 | CL | COLGATE-PALMOLIVE CO | 31-Mar-1964 | | 19023 | 2844 | Perfumes, Cosmetics & Other Toilet Preparations |
| 11 | ED | CONSOLIDATED EDISON INC | 31-Mar-1964 | | 19023 | 4931 | Electric & Other Services Combined |
| 12 | GLW | CORNING INC | 31-Mar-1964 | | 19023 | 3679 | Electronic Components, NEC |
| 13 | DE | DEERE & CO | 31-Mar-1964 | | 19023 | 3523 | Farm Machinery & Equipment |

Continued on next page

S&P 500 Index Sample List

| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 14 | DTE | DTE ENERGY CO | 31-Mar-1964 | | 19023 | 4931 | Electric & Other Services |
| 15 | D | DOMINION ENERGY INC | 31-Mar-1964 | | 19023 | 4911 | Combined |
| 16 | DWDP | DOWDUPONT INC | 31-Mar-1964 | | 19023 | 2821 | Electric Services |
| 17 | ETN | EATON CORP PLC | 31-Mar-1964 | | 19023 | 3620 | Plastic Materials, Synth Resins & Nonvulcan Elastomers |
| 18 | XOM | EXXON MOBIL CORP | 31-Mar-1964 | | 19023 | 2911 | Electrical Industrial Apparatus |
| 19 | F | FORD MOTOR CO | 31-Mar-1964 | | 19023 | 3711 | Petroleum Refining |
| 20 | GD | GENERAL DYNAMICS CORP | 31-Mar-1964 | | 19023 | 3721 | Motor Vehicles & Passenger Car Bodies |
| 21 | GE | GENERAL ELECTRIC CO | 31-Mar-1964 | | 19023 | 9997 | Aircraft |
| 22 | GT | GOODYEAR TIRE & RUBBER CO | 31-Mar-1964 | | 19023 | 3011 | Tires & Inner Tubes |
| 23 | HAL | HALLIBURTON CO | 31-Mar-1964 | | 19023 | 1389 | Oil & Gas Field Services, NEC |
| 24 | HSY | HERSHEY CO | 31-Mar-1964 | | 19023 | 2060 | Sugar & Confectionery Products |
| 25 | IBM | INTL BUSINESS MACHINES CORP | 31-Mar-1964 | | 19023 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 26 | IP | INTL PAPER CO | 31-Mar-1964 | | 19023 | 2631 | Paperboard Mills |
| 27 | K | KELLOGG CO | 31-Mar-1964 | | 19023 | 2040 | Grain Mill Products |
| 28 | KMB | KIMBERLY-CLARK CORP | 31-Mar-1964 | | 19023 | 2621 | Paper Mills |
| 29 | KR | KROGER CO | 31-Mar-1964 | | 19023 | 5411 | Retail-Grocery Stores |
| 30 | SPGI | S&P GLOBAL INC | 31-Mar-1964 | | 19023 | 7323 | |

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S&P 500 Index Sample List

| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------|------------------|------------------|------------|------------------|---|
| 31 | CVS | CVS HEALTH CORP | 31-Mar-1964 | | 19023 | 5912 | Retail-Drug Stores and Proprietary Stores |
| 32 | MRK | MERCK & CO | 31-Mar-1964 | | 19023 | 2834 | Pharmaceutical Preparations |
| 33 | ETR | ENERGY CORP | 31-Mar-1964 | | 19023 | 4911 | Electric Services |
| 34 | MMM | 3M CO | 31-Mar-1964 | | 19023 | 2670 | Converted Paper & Paperboard |
| 35 | MSI | MOTOROLA SOLUTIONS INC | 31-Mar-1964 | | 19023 | 3663 | Prods (No Containers/Boxes) Radio & TV Broadcasting & Communications Equipment |
| 36 | NSC | NORFOLK SOUTHERN CORP | 31-Mar-1964 | | 19023 | 4011 | Railroads, Line-Haul Operating |
| 37 | XEL | XCEL ENERGY INC | 31-Mar-1964 | | 19023 | 4931 | Electric & Other Services Combined |
| 38 | FE | FIRSTENERGY CORP | 31-Mar-1964 | | 19023 | 4911 | Electric Services |
| 39 | PPG | PPG INDUSTRIES INC | 31-Mar-1964 | | 19023 | 2851 | Paints, Varnishes, Lacquers, Enamels & Allied Prods |
| 40 | PCG | PG&E CORP | 31-Mar-1964 | | 19023 | 4931 | Electric & Other Services Combined |
| 41 | SRE | SEMPRA ENERGY | 31-Mar-1964 | | 19023 | 4931 | Electric & Other Services Combined |
| 42 | PEP | PEPSICO INC | 31-Mar-1964 | | 19023 | 2080 | Beverages |
| 43 | PFE | PFIZER INC | 31-Mar-1964 | | 19023 | 2834 | Pharmaceutical Preparations |
| 44 | EXC | EXELON CORP | 31-Mar-1964 | | 19023 | 4911 | Electric Services |
| 45 | MO | ALTRIA GROUP INC | 31-Mar-1964 | | 19023 | 2111 | Cigarettes |

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S&P 500 Index Sample List

| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 46 | COP | CONOCOPHILLIPS | 31-Mar-1964 | | 19023 | 1311 | Crude Petroleum & Natural Gas |
| 47 | PG | PROCTER & GAMBLE CO | 31-Mar-1964 | | 19023 | 2840 | Soap, Detergents, Cleaning Preparations, Perfumes, Cosmetics |
| 48 | PEG | PUBLIC SERVICE ENTRP GRP INC | 31-Mar-1964 | | 19023 | 4931 | Electric & Other Services |
| 49 | RTN | RAYTHEON CO | 31-Mar-1964 | | 19023 | 3812 | Combined |
| 50 | ROK | ROCKWELL AUTOMATION | 31-Mar-1964 | | 19023 | 3620 | Search, Detection, Navigation, Guidance, Aeronautical Sys |
| 51 | EIX | EDISON INTERNATIONAL | 31-Mar-1964 | | 19023 | 4911 | Electrical Industrial Apparatus |
| 52 | SO | SOUTHERN CO | 31-Mar-1964 | | 19023 | 4911 | Electric Services |
| 53 | TXN | TEXAS INSTRUMENTS INC | 31-Mar-1964 | | 19023 | 3674 | Semiconductors & Related Devices |
| 54 | UNP | UNION PACIFIC CORP | 31-Mar-1964 | | 19023 | 4011 | Railroads, Line-Haul Operating |
| 55 | UTX | UNITED TECHNOLOGIES CORP | 31-Mar-1964 | | 19023 | 3724 | Aircraft Engines & Engine Parts |
| 56 | WY | WEYERHAEUSER CO | 31-Mar-1964 | | 19023 | 2400 | Lumber & Wood Products (No Furniture) |
| 57 | WHR | WHIRLPOOL CORP | 31-Mar-1964 | | 19023 | 3630 | Household Appliances |
| 58 | XRX | XEROX CORP | 31-Mar-1964 | | 19023 | 3577 | Computer Peripheral Equipment, NEC |
| 59 | D2 | CONSOLIDATED NATURAL GAS CO | 31-Mar-1964 | 30-Jan-2000 | 19023 | 4923 | Natural Gas Transmission & Distribution |

Continued on next page

S&P 500 Index Sample List

| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 60 | FWLT | FOSTER WHEELER AG | 31-Mar-1964 | 30-Jan-2000 | 19023 | 1600 | Heavy Construction Other Than Bldg Const - Contractors |
| 61 | ARC.3 | ATLANTIC RICHFIELD CO | 31-Mar-1964 | 17-Apr-2000 | 19023 | 2911 | Petroleum Refining |
| 62 | RLM.1 | REYNOLDS METALS CO | 31-Mar-1964 | 4-May-2000 | 19023 | 3334 | Primary Production of Aluminum |
| 63 | CBS.2 | CBS CORP -OLD | 31-Mar-1964 | 4-May-2000 | 19023 | 4833 | Television Broadcasting Stations |
| 64 | NC | NACCO INDUSTRIES -CL A | 31-Mar-1964 | 4-Jun-2000 | 19023 | 1221 | Bituminous Coal & Lignite Surface Mining |
| 65 | CSR.1 | CENTRAL & SOUTH WEST CORP | 31-Mar-1964 | 15-Jun-2000 | 19023 | 4911 | Electric Services |
| 66 | WLA | WARNER-LAMBERT CO | 31-Mar-1964 | 20-Jun-2000 | 19023 | 2834 | Pharmaceutical Preparations |
| 67 | MZIAQ | MILACRON INC | 31-Mar-1964 | 28-Jun-2000 | 19023 | 3559 | Special Industry Machinery, NEC |
| 68 | GTE.1 | GTE CORP | 31-Mar-1964 | 2-Jul-2000 | 19023 | 4813 | Telephone Communications (No Radiotelephone) |
| 69 | GAPTQ | GREAT ATLANTIC & PAC TEA CO | 31-Mar-1964 | 29-Aug-2000 | 19023 | 5411 | Retail-Grocery Stores |
| 70 | OC | OWENS CORNING | 31-Mar-1964 | 1-Oct-2000 | 19023 | 3290 | Abrasive, Asbestos & Misc Nonmetallic Mineral Prods |
| 71 | BFO.2 | BESTFOODS | 31-Mar-1964 | 2-Oct-2000 | 19023 | 2030 | Canned, Frozen & Preserved Fruit, Veg & Food Specialties |
| 72 | UCM.2 | UNICOM CORP | 31-Mar-1964 | 22-Oct-2000 | 19023 | 4911 | Electric Services |
| 73 | NI2 | COLUMBIA ENERGY GROUP | 31-Mar-1964 | 1-Nov-2000 | 19023 | 4923 | Natural Gas Transmission & Distribution |
| 74 | 3ACKH | ARMSTRONG HOLDINGS INC | 31-Mar-1964 | 16-Nov-2000 | 19023 | 3089 | Plastics Products, NEC |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|----------------------|------------------|------------------|------------|------------------|---|
| 75 | 3BHMSQ | BETHLEHEM STEEL CORP | 31-Mar-1964 | 10-Dec-2000 | 19023 | 3312 | Steel Works, Blast Furnaces & Rolling Mills (Coke Ovens) |
| 76 | CCK | CROWN HOLDINGS INC | 31-Mar-1964 | 10-Dec-2000 | 19023 | 3411 | Metal Cans |
| 77 | GRA | GRACE (W R) & CO | 31-Mar-1964 | 10-Dec-2000 | 19023 | 2810 | Industrial Inorganic Chemicals |
| 78 | VO.3 | SEAGRAM CO LTD | 31-Mar-1964 | 10-Dec-2000 | 19023 | 3652 | Phonograph Records & Prerecorded Audio Tapes & Disks |
| 79 | PRDCQ | PRIMARY PDC INC | 31-Mar-1964 | 11-Dec-2000 | 19023 | 3861 | Photographic Equipment & Supplies |
| 80 | DOW2 | UNION CARBIDE CORP | 31-Mar-1964 | 6-Feb-2001 | 19023 | 2860 | Industrial Organic Chemicals |
| 81 | OAT. | QUAKER OATS CO | 31-Mar-1964 | 2-Aug-2001 | 19023 | 2040 | Grain Mill Products |
| 82 | TKR | TIMKEN CO | 31-Mar-1964 | 6-Aug-2001 | 19023 | 3562 | Ball & Roller Bearings |
| 83 | OKE | ONEOK INC | 31-Mar-1964 | 29-Aug-2001 | 19023 | 4923 | Natural Gas Transmission & Distribution |
| 84 | TX.2 | TEXACO INC | 31-Mar-1964 | 9-Oct-2001 | 19023 | 2911 | Petroleum Refining |
| 85 | ENRNQ | ENRON CORP | 31-Mar-1964 | 29-Nov-2001 | 19023 | 5172 | Wholesale-Petroleum & Petroleum Products (No Bulk Stations) |
| 86 | HM | HOMESTAKE MINING | 31-Mar-1964 | 16-Dec-2001 | 19023 | 1040 | Gold and Silver Ores |
| 87 | FMC | FMC CORP | 31-Mar-1964 | 1-Jan-2002 | 19023 | 2870 | Agricultural Chemicals |
| 88 | SHLD | SEARS HOLDINGS CORP | 31-Mar-1964 | 16-Jan-2002 | 19023 | 5311 | Retail-Department Stores |
| 89 | MEA.1 | MEAD CORP | 31-Mar-1964 | 29-Jan-2002 | 19023 | 2621 | Paper Mills |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|--------------------------------|------------------|------------------|------------|------------------|--|
| 90 | NMK | NIAGARA MOHAWK HOLDINGS INC | 31-Mar-1964 | 31-Jan-2002 | 19023 | 4931 | Electric & Other Services |
| 91 | AL.1 | ALCAN INC | 31-Mar-1964 | 21-Jul-2002 | 19023 | 3350 | Combined Rolling Drawing & Extruding of Nonferrous Metals |
| 92 | INCLF | INCO LTD | 31-Mar-1964 | 21-Jul-2002 | 19023 | 3330 | Primary Smelting & Refining of Nonferrous Metals |
| 93 | RDPL | ROYAL DUTCH PETROLEUM NV | 31-Mar-1964 | 21-Jul-2002 | 19023 | 2911 | Petroleum Refining |
| 94 | UN | UNILEVER NV | 31-Mar-1964 | 21-Jul-2002 | 19023 | 2840 | Soap, Detergents, Cleaning Preparations, Perfumes, Cosmetics |
| 95 | TRW.1 | TRW INC | 31-Mar-1964 | 11-Dec-2002 | 19023 | 3714 | Motor Vehicle Parts & Accessories |
| 96 | AAL | AMERICAN AIRLINES GROUP INC | 31-Mar-1964 | 13-Mar-2003 | 19023 | 4512 | Air Transportation, Scheduled |
| 97 | HBC1 | HSBC FINANCE CORP | 31-Mar-1964 | 30-Mar-2003 | 19023 | 6141 | Personal Credit Institutions |
| 98 | PHA.1 | PHARMACIA CORP | 31-Mar-1964 | 15-Apr-2003 | 19023 | 2834 | Pharmaceutical Preparations |
| 99 | WINN | WINN-DIXIE STORES INC | 31-Mar-1964 | 2-Dec-2004 | 19023 | 5411 | Retail-Grocery Stores |
| 100 | CR | CRANE CO | 31-Mar-1964 | 19-Dec-2004 | 19023 | 3490 | Miscellaneous Fabricated Metal Products |
| 101 | S.1 | SEARS ROEBUCK & CO | 31-Mar-1964 | 27-Mar-2005 | 19023 | 5311 | Retail-Department Stores |
| 102 | UCL | UNOCAL CORP | 31-Mar-1964 | 10-Aug-2005 | 19023 | 1311 | Crude Petroleum & Natural Gas |
| 103 | MAY.2 | MAY DEPARTMENT STORES CO | 31-Mar-1964 | 28-Aug-2005 | 19023 | 5311 | Retail-Department Stores |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 104 | G-1 | GILLETTE CO | 31-Mar-1964 | 2-Oct-2005 | 19023 | 3420 | Cutlery, Handtools & General Hardware |
| 105 | T-2 | AT&T CORP | 31-Mar-1964 | 20-Nov-2005 | 19023 | 4813 | Telephone Communications (No Radiotelephone) |
| 106 | GP.1 | GEORGIA-PACIFIC CORP | 31-Mar-1964 | 19-Dec-2005 | 19023 | 2600 | Papers & Allied Products |
| 107 | DAN | DANA INC | 31-Mar-1964 | 2-Mar-2006 | 19023 | 3714 | Motor Vehicle Parts & Accessories |
| 108 | MYG.1 | MAYTAG CORP | 31-Mar-1964 | 2-Apr-2006 | 19023 | 3630 | Household Appliances |
| 109 | NAV | NAVISTAR INTERNATIONAL CORP | 31-Mar-1964 | 19-Dec-2006 | 19023 | 3711 | Motor Vehicles & Passenger Car Bodies |
| 110 | PGL.1 | PEOPLES ENERGY CORP | 31-Mar-1964 | 21-Feb-2007 | 19023 | 4924 | Natural Gas Distribution |
| 111 | PD | PHHELPS DODGE CORP | 31-Mar-1964 | 19-Mar-2007 | 19023 | 3330 | Primary Smelting & Refining of Nonferrous Metals |
| 112 | 0033A | ENERGY FUTURE HOLDINGS CORP | 31-Mar-1964 | 9-Oct-2007 | 19023 | 4911 | Electric Services |
| 113 | WWY | WRIGLEY (WM) JR CO | 31-Mar-1964 | 6-Oct-2008 | 19023 | 2060 | Sugar & Confectionery Products |
| 114 | UIS | UNISYS CORP | 31-Mar-1964 | 10-Nov-2008 | 19023 | 7373 | Services-Computer Integrated Systems Design |
| 115 | HPC | HERCULES INC | 31-Mar-1964 | 13-Nov-2008 | 19023 | 2890 | Miscellaneous Chemical Products |
| 116 | GM | GENERAL MOTORS CO | 31-Mar-1964 | 2-Jun-2009 | 19023 | 3711 | Motor Vehicles & Passenger Car Bodies |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 117 | IR | INGERSOLL-RAND PLC | 31-Mar-1964 | 30-Jun-2009 | 19023 | 3585 | Air-Cond & Warm Air Heatg Equip & Comm & Inhl Refrig Equip |
| 118 | CBE | COOPER INDUSTRIES PLC | 31-Mar-1964 | 8-Sep-2009 | 19023 | 3640 | Electric Lighting & Wiring Equipment |
| 119 | WYE | WYETH | 31-Mar-1964 | 15-Oct-2009 | 19023 | 2834 | Pharmaceutical Preparations |
| 120 | SGP | SCHERING-PLOUGH | 31-Mar-1964 | 3-Nov-2009 | 19023 | 2834 | Pharmaceutical Preparations |
| 121 | BRK3 | BURLINGTON NORTHERN SANTA FE | 31-Mar-1964 | 15-Feb-2010 | 19023 | 4011 | Railroads, Line-Haul Operating |
| 122 | KODK | EASTMAN KODAK CO | 31-Mar-1964 | 19-Dec-2010 | 19023 | 3577 | Computer Peripheral Equipment, NEC |
| 123 | ITT | ITT INC | 31-Mar-1964 | 31-Oct-2011 | 19023 | 3561 | Pumps & Pumping Equipment |
| 124 | CEG | CONSTELLATION ENERGY GRP INC | 31-Mar-1964 | 13-Mar-2012 | 19023 | 4931 | Electric & Other Services Combined |
| 125 | GR | GOODRICH CORP | 31-Mar-1964 | 30-Jul-2012 | 19023 | 3728 | Aircraft Parts & Auxiliary Equipment, NEC |
| 126 | KHC | KRAFT HEINZ CO | 31-Mar-1964 | 6-Jun-2013 | 19023 | 2030 | Canned, Frozen & Preserved Fruit, Veg & Food Specialties |
| 127 | JCP | PENNEY (J C) CO | 31-Mar-1964 | 1-Dec-2013 | 19023 | 5311 | Retail-Department Stores |
| 128 | BEAM | BEAM INC | 31-Mar-1964 | 30-Apr-2014 | 19023 | 2085 | Paperboard Mills |
| 129 | MWV | MEADWESTVACO CORP | 31-Mar-1964 | 1-Jul-2015 | 19023 | 2631 | Paperboard Mills |
| 130 | AA-3 | ALCOA INC | 31-Mar-1964 | 31-Oct-2016 | 19023 | 3720 | Aircraft & Parts |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|--------------------------|------------------|------------------|------------|------------------|--|
| 131 | PBI | PITNEY BOWES INC | 31-Mar-1964 | 28-Feb-2017 | 19023 | 3579 | Office Machines, NEC |
| 132 | BHI | BAKER HUGHES INC | 31-Mar-1964 | 4-Jul-2017 | 19023 | 1381 | Drilling Oil & Gas Wells |
| 133 | DD | DU PONT (E I) DE NEMOURS | 31-Mar-1964 | 31-Aug-2017 | 19023 | 2820 | Plastic Material, Synth Resin/Rubber, Cellulos (No Glass) |
| 134 | SHW | SHERWIN-WILLIAMS CO | 30-Jun-1964 | | 18932 | 2851 | Paints, Varnishes, Lacquers, Enamels & Allied Prods |
| 135 | TWX.1 | TIME WARNER INC-OLD | 30-Jun-1964 | 15-Jan-2001 | 18932 | 7812 | Services-Motion Picture & Video Tape Production |
| 136 | CMI | CUMMINS INC | 31-Mar-1965 | | 18658 | 3510 | Engines & Turbines |
| 137 | EMR | EMERSON ELECTRIC CO | 31-Mar-1965 | | 18658 | 3823 | Industrial Instruments For Measurement, Display, and Control |
| 138 | SLB | SCHLUMBERGER LTD | 31-Mar-1965 | | 18658 | 1389 | Oil & Gas Field Services, NEC |
| 139 | TMC | TIMES MIRROR CO -SER A | 31-Dec-1965 | 12-Jun-2000 | 18383 | 2711 | Newspapers: Publishing or Publishing & Printing |
| 140 | EFU.1 | EASTERN ENTERPRISES | 30-Sep-1966 | 1-Nov-2000 | 18110 | 4924 | Natural Gas Distribution |
| 141 | TEK.1 | TEKTRONIX INC | 31-Mar-1967 | 15-Nov-2007 | 17928 | 3825 | Instruments For Meas & Testing of Electricity & Elec Signals |
| 142 | AVP | AVON PRODUCTS | 30-Jun-1967 | 22-Mar-2015 | 17837 | 2844 | Perfumes, Cosmetics & Other Toilet Preparations |
| 143 | CSX | CSX CORP | 30-Sep-1967 | | 17745 | 4011 | Railroads, Line-Haul Operating |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|--------------------------------|------------------|------------------|------------|------------------|--|
| 144 | SML.1 | SPRINGS INDUSTRIES -CL A | 30-Sep-1967 | 11-Dec-2000 | 17745 | 2211 | Broadwoven Fabric Mills, Cotton |
| 145 | DAL | DELTA AIR LINES INC | 31-Mar-1968 | 18-Aug-2005 | 17562 | 4512 | Air Transportation, Scheduled |
| 146 | BGG | BRIGGS & STRATTON | 30-Jun-1968 | 2-Apr-2001 | 17471 | 3510 | Engines & Turbines |
| 147 | MDP | MEREDITH CORP | 30-Jun-1968 | 3-Jan-2011 | 17471 | 2721 | Periodicals: Publishing or Publishing & Printing |
| 148 | CLX | CLOROX CO/DE | 31-Mar-1969 | | 17197 | 2842 | Specialty Cleaning, Polishing and Sanitation Preparations |
| 149 | GIS | GENERAL MILLS INC | 31-Mar-1969 | | 17197 | 2040 | Grain Mill Products |
| 150 | SUN.1 | SUNOCO INC | 31-Mar-1969 | 4-Oct-2012 | 17197 | 2911 | Petroleum Refining |
| 151 | NEM | NEWMONT MINING CORP | 30-Jun-1969 | | 17106 | 1040 | Gold and Silver Ores |
| 152 | 4622B | CERIDIAN CORP | 30-Jun-1969 | 1-Apr-2001 | 17106 | 8721 | Retail-Eating Places |
| 153 | MCD | MCDONALD'S CORP | 30-Jun-1970 | | 16741 | 5812 | Paper Mills |
| 154 | CHA.3 | CHAMPION INTERNATIONAL CORP | 30-Jun-1970 | 18-Jun-2000 | 16741 | 2621 | Sawmills & Planing Mills, General |
| 155 | PCH | POTLATCHDELTAIC CORP | 30-Jun-1970 | 8-Jul-2001 | 16741 | 2421 | Hotels & Motels |
| 156 | HLT | HILTON WORLDWIDE HOLDINGS | 30-Jun-1970 | 24-Oct-2007 | 16741 | 7011 | Wholesale-Paper & Paper Products |
| 157 | OMX | OFFICEMAX INC | 30-Jun-1970 | 22-Jun-2008 | 16741 | 5110 | Pharmaceutical Preparations |
| 158 | LLY | LILLY (ELI) & CO | 31-Dec-1970 | | 16557 | 2834 | Perfumes, Cosmetics & Other |
| 159 | ACV.1 | ALBERTO-CULVER CO | 31-Mar-1971 | 16-Nov-2006 | 16467 | 2844 | Toilet Preparations |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------|------------------|------------------|------------|------------------|---|
| 160 | MDR | MCDERMOTT INTL INC | 30-Sep-1971 | 19-Aug-2003 | 16284 | 3730 | Ship & Boat Building & Repairing |
| 161 | BAX | BAXTER INTERNATIONAL INC | 30-Sep-1972 | | 15918 | 2834 | Pharmaceutical Preparations |
| 162 | BDX | BECTON DICKINSON & CO | 30-Sep-1972 | | 15918 | 3841 | Surgical & Medical Instruments & Apparatus |
| 163 | BC | BRUNSWICK CORP | 31-Mar-1973 | 22-Jun-2008 | 15736 | 3510 | Engines & Turbines |
| 164 | RSHCQ | RS LEGACY CORP | 31-Mar-1973 | 30-Jun-2011 | 15736 | 5731 | Retail-Radio, TV & Consumer Electronics Stores |
| 165 | JNJ | JOHNSON & JOHNSON | 30-Jun-1973 | | 15645 | 2834 | Pharmaceutical Preparations |
| 166 | GPC | GENUINE PARTS CO | 31-Dec-1973 | | 15461 | 5013 | Wholesale-Motor Vehicle Supplies & New Parts |
| 167 | FLTWQ | FLEETWOOD ENTERPRISES INC | 30-Jun-1974 | 30-Jan-2000 | 15280 | 3716 | Motor Homes |
| 168 | HPQ | HP INC | 31-Dec-1974 | | 15096 | 3570 | Computer & office Equipment |
| 169 | TNB | THOMAS & BETTS CORP | 31-Dec-1974 | 2-Aug-2004 | 15096 | 3640 | Electric Lighting & Wiring Equipment |
| 170 | LPX | LOUISIANA-PACIFIC CORP | 31-Dec-1974 | 9-Nov-2006 | 15096 | 2400 | Lumber & Wood Products (No Furniture) |
| 171 | ABI.3 | APPLIED BIOSYSTEMS INC | 31-Dec-1974 | 23-Nov-2008 | 15096 | 2835 | In Vitro & In Vivo Diagnostic Substances |
| 172 | WMB | WILLIAMS COS INC | 31-Mar-1975 | | 15006 | 4922 | Natural Gas Transmission |
| 173 | MKG | MALLINCKRODT INC | 31-Mar-1975 | 17-Oct-2000 | 15006 | 3842 | Orthopedic, Prosthetic & Surgical Appliances & Supplies |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------|------------------|------------------|------------|------------------|---|
| 174 | BCR | BARD (C.R.) INC | 30-Jun-1975 | 2-Jan-2018 | 14915 | 3841 | Surgical & Medical Instruments & Apparatus |
| 175 | RAD | RITE AID CORP | 31-Dec-1975 | 26-Jul-2000 | 14731 | 5912 | Retail-Drug Stores and Proprietary Stores |
| 176 | KRI | KNIGHT-RIDDER INC | 31-Dec-1975 | 27-Jun-2006 | 14731 | 2711 | Newspapers: Publishing or Publishing & Printing |
| 177 | TGNA | TEGNA INC | 31-Dec-1975 | 1-Jun-2017 | 14731 | 4833 | Television Broadcasting Stations |
| 178 | IFF | INTL FLAVORS & FRAGRANCES | 31-Mar-1976 | | 14640 | 2860 | Industrial Organic Chemicals |
| 179 | BDK | BLACK & DECKER CORP | 31-Mar-1976 | 14-Mar-2010 | 14640 | 3540 | Metalworkg Machinery & Equipment |
| 180 | AET | AETNA INC | 30-Jun-1976 | | 14549 | 6324 | Hospital & Medical Service Plans |
| 181 | AXP | AMERICAN EXPRESS CO | 30-Jun-1976 | | 14549 | 6141 | Personal Credit Institutions |
| 182 | CI | CIGNA CORP | 30-Jun-1976 | | 14549 | 6324 | Hospital & Medical Service Plans |
| 183 | JPM | JPMORGAN CHASE & CO | 30-Jun-1976 | | 14549 | 6020 | |
| 184 | TAP | MOLSON COORS BREWING CO | 30-Jun-1976 | | 14549 | 2082 | Malt Beverages |
| 185 | DIS | DISNEY (WALT) CO | 30-Jun-1976 | | 14549 | 4888 | |
| 186 | DUK | DUKE ENERGY CORP | 30-Jun-1976 | | 14549 | 4931 | Electric & Other Services |
| 187 | NEE | NEXTERA ENERGY INC | 30-Jun-1976 | | 14549 | 4911 | Combined |
| 188 | LNC | LINCOLN NATIONAL CORP | 30-Jun-1976 | | 14549 | 6311 | Electric Services |
| 189 | BAC | BANK OF AMERICA CORP | 30-Jun-1976 | | 14549 | 6020 | Life Insurance |
| 190 | WFC | WELLS FARGO & CO | 30-Jun-1976 | | 14549 | 6020 | |

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|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 191 | JPM.Z | MORGAN (J P) & CO | 30-Jun-1976 | 1-Jan-2001 | 14549 | 6020 | |
| 192 | AGC.1 | AMERICAN GENERAL CORP | 30-Jun-1976 | 29-Aug-2001 | 14549 | 6311 | Life Insurance |
| 193 | SPC.2 | ST PAUL COS | 30-Jun-1976 | 1-Apr-2004 | 14549 | 6331 | Fire, Marine & Casualty Insurance |
| 194 | JP.1 | JEFFERSON-PILOT CORP | 30-Jun-1976 | 2-Apr-2006 | 14549 | 6311 | Life Insurance |
| 195 | MEL.3 | MELLON FINANCIAL CORP | 30-Jun-1976 | 1-Jul-2007 | 14549 | 6020 | |
| 196 | SAF | SAFECO CORP | 30-Jun-1976 | 22-Sep-2008 | 14549 | 6331 | Fire, Marine & Casualty Insurance |
| 197 | BUD.2 | ANHEUSER-BUSCH COS INC | 30-Jun-1976 | 18-Nov-2008 | 14549 | 2082 | Malt Beverages |
| 198 | CB.1 | CHUBB CORP | 30-Jun-1976 | 18-Jan-2016 | 14549 | 6331 | Fire, Marine & Casualty Insurance |
| 199 | TGT | TARGET CORP | 31-Dec-1976 | | 14365 | 5331 | Retail-Variety Stores |
| 200 | INTC | INTEL CORP | 31-Dec-1976 | | 14365 | 3674 | Semiconductors & Related Devices |
| 201 | NSM.2 | NATIONAL SEMICONDUCTOR CORP | 31-Dec-1976 | 25-Sep-2011 | 14365 | 3674 | Semiconductors & Related Devices |
| 202 | CTX.2 | CENTEX CORP | 31-Mar-1977 | 18-Aug-2009 | 14275 | 1531 | Operative Builders |
| 203 | DJ | DOW JONES & CO INC | 30-Jun-1978 | 13-Dec-2007 | 13819 | 2711 | Newspapers: Publishing or Publishing & Printing |
| 204 | BMS | BEMIS CO INC | 30-Sep-1978 | 4-Dec-2014 | 13727 | 2670 | Converted Paper & Paperboard Prods (No Containers/Boxes) |
| 205 | TXT | TEXTRON INC | 31-Dec-1978 | | 13635 | 3721 | Aircraft |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 206 | ATI | ALLEGHENY TECHNOLOGIES INC | 31-Dec-1978 | 1-Jul-2015 | 13635 | 3350 | Rolling Drawing & Extruding of Nonferrous Metals |
| 207 | THC | TENET HEALTHCARE CORP | 31-Mar-1979 | 17-Apr-2016 | 13545 | 8062 | Services-General Medical & Surgical Hospitals, NEC |
| 208 | VFC | VF CORP | 30-Jun-1979 | | 13454 | 2320 | Men's & Boys' Furnishings, Work Clothing, & Allied Garments |
| 209 | WBA | WALGREENS BOOTS ALLIANCE INC | 31-Dec-1979 | | 13270 | 5912 | Retail-Drug Stores and Proprietary Stores |
| 210 | DNB | DUN & BRADSTREET CORP | 31-Dec-1979 | 2-Oct-2000 | 13270 | 7323 | |
| 211 | AIG | AMERICAN INTERNATIONAL GROUP | 31-Mar-1980 | | 13179 | 6331 | Fire, Marine & Casualty Insurance |
| 212 | FLR | FLUOR CORP | 31-Mar-1980 | | 13179 | 1600 | Heavy Construction Other Than Bldg Const - Contractors |
| 213 | PNU.1 | PHARMACIA & UPJOHN INC | 30-Jun-1980 | 2-Apr-2000 | 13088 | 2834 | Pharmaceutical Preparations |
| 214 | FDX | FEDEX CORP | 31-Dec-1980 | | 12904 | 4513 | Air Courier Services |
| 215 | PCAR | PACCAR INC | 31-Dec-1980 | | 12904 | 3711 | Motor Vehicles & Passenger Car Bodies |
| 216 | NRTLQ | NORTEL NETWORKS CORP | 31-Dec-1980 | 21-Jul-2002 | 12904 | 9995 | Non-Operating Establishments |
| 217 | ADP | AUTOMATIC DATA PROCESSING | 31-Mar-1981 | | 12814 | 7374 | Services-Computer Processing & Data Preparation |
| 218 | SFA.1 | SCIENTIFIC-ATLANTA INC | 31-Mar-1981 | 26-Feb-2006 | 12814 | 3663 | Radio & TV Broadcasting & Communications Equipment |

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|-----|--------|----------------------------|------------------|------------------|------------|------------------|--|
| 219 | GWV | GRAINGER (W W) INC | 30-Jun-1981 | | 12723 | 5000 | Wholesale-Durable Goods |
| 220 | MAS | MASCO CORP | 30-Jun-1981 | | 12723 | 3430 | Heating Equip, Except Elec & Warm Air; & Plumbing Fixtures |
| 221 | DXC | DXC TECHNOLOGY COMPANY | 30-Jun-1981 | 30-Nov-2015 | 12723 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 222 | ADM | ARCHER-DANIELS-MIDLAND CO | 29-Jul-1981 | | 12694 | 2070 | Fats & Oils |
| 223 | H.2 | HARCOURT GENERAL INC | 30-Sep-1981 | 27-Jun-2001 | 12631 | 2731 | Books: Publishing or Publishing & Printing |
| 224 | HSB | HILLSHIRE BRANDS CO | 30-Sep-1981 | 28-Jun-2012 | 12631 | 2013 | Sausages & Other Prepared Meat Products |
| 225 | MAT | MATTEL INC | 31-Mar-1982 | | 12449 | 3942 | Dolls & Stuffed Toys |
| 226 | WEN.2 | WENDY'S INTERNATIONAL INC | 31-Mar-1982 | 29-Sep-2008 | 12449 | 5812 | Retail-Eating Places |
| 227 | BAC2 | MERRILL LYNCH & CO INC | 30-Jun-1982 | 1-Jan-2009 | 12358 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 228 | WMT | WALMART INC | 31-Aug-1982 | | 12296 | 5331 | Retail-Variety Stores |
| 229 | SNA | SNAP-ON INC | 30-Sep-1982 | | 12266 | 3420 | Cutlery, Handtools & General Hardware |
| 230 | SWK | STANLEY BLACK & DECKER INC | 30-Sep-1982 | | 12266 | 3540 | Metalworkg Machinery & Equipment |
| 231 | ROH | ROHM AND HAAS CO | 30-Sep-1982 | 1-Apr-2009 | 12266 | 2821 | Plastic Materials, Synth Resins & Nonvulcan Elastomers |
| 232 | BF.B | BROWN FORMAN CORP | 31-Oct-1982 | | 12235 | 2085 | |

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|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 233 | AAPL | APPLE INC | 30-Nov-1982 | | 12205 | 3663 | Radio & TV Broadcasting & Communications Equipment |
| 234 | OXY | OCCIDENTAL PETROLEUM CORP | 31-Dec-1982 | | 12174 | 1311 | Crude Petroleum & Natural Gas |
| 235 | R | RYDER SYSTEM INC | 31-Dec-1982 | 18-Jun-2017 | 12174 | 7510 | Services-Auto Rental & Leasing (No Drivers) |
| 236 | EC:2 | ENGELHARD CORP | 31-Jul-1983 | 5-Jun-2006 | 11962 | 2810 | Industrial Inorganic Chemicals |
| 237 | CAG | CONAGRA BRANDS INC | 31-Aug-1983 | | 11931 | 2000 | Food and Kindred Products |
| 238 | LB | L BRANDS INC | 30-Sep-1983 | | 11901 | 5621 | Retail-Women's Clothing Stores |
| 239 | 1231B | TOYS R US INC | 30-Sep-1983 | 21-Jul-2005 | 11901 | 5945 | Retail-Hobby, Toy & Game Shops |
| 240 | VZ | VERIZON COMMUNICATIONS INC | 30-Nov-1983 | | 11840 | 4812 | Radiotelephone Communications |
| 241 | T | AT&T INC | 30-Nov-1983 | | 11840 | 4812 | Radiotelephone Communications |
| 242 | BLS | BELLSOUTH CORP | 30-Nov-1983 | 3-Jan-2007 | 11840 | 4813 | Telephone Communications (No Radiotelephone) |
| 243 | Q:2 | QWEST COMMUNICATION INTL INC | 30-Nov-1983 | 31-Mar-2011 | 11840 | 4813 | Telephone Communications (No Radiotelephone) |
| 244 | LOW | LOWE'S COMPANIES INC | 29-Feb-1984 | | 11749 | 5211 | Retail-Lumber & Other Building Materials Dealers |
| 245 | PHM | PULTEGROUP INC | 30-Apr-1984 | | 11688 | 1531 | Operative Builders |
| 246 | HES | HESS CORP | 31-May-1984 | | 11657 | 1311 | Crude Petroleum & Natural Gas |
| 247 | AMD | ADVANCED MICRO DEVICES | 30-Jun-1984 | 22-Sep-2013 | 11627 | 3674 | Semiconductors & Related Devices |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 248 | LMT | LOCKHEED MARTIN CORP | 31-Jul-1984 | | 11596 | 3760 | Guided Missiles & Space Vehicles & Parts |
| 249 | RRD | DONNELLEY (R R) & SONS CO | 31-Jul-1984 | 11-Dec-2012 | 11596 | 2750 | Commercial Printing |
| 250 | HAS | HASBRO INC | 30-Sep-1984 | | 11535 | 3944 | Games, Toys & Children's Vehicles (No Dolls & Bicycles) |
| 251 | BLI | BALL CORP | 31-Oct-1984 | | 11504 | 3411 | Metal Cans |
| 252 | SMS.2 | SHARED MEDICAL SYSTEMS CORP | 31-Oct-1984 | 7-Jun-2000 | 11504 | 7373 | Services-Computer Integrated Systems Design |
| 253 | KMG.1 | KERR-MCGEE CORP | 31-Oct-1984 | 10-Aug-2006 | 11504 | 1311 | Crude Petroleum & Natural Gas |
| 254 | KATE | KATE SPADE & CO | 31-Oct-1984 | 1-Dec-2008 | 11504 | 2300 | Apparel & Other Finished Prods of Fabrics & Similar Matl |
| 255 | RML.1 | RUSSELL CORP | 30-Nov-1984 | 11-Dec-2000 | 11474 | 2253 | Knit Outerwear Mills |
| 256 | ABS.1 | ALBERTSON'S INC | 30-Nov-1984 | 1-Jun-2006 | 11474 | 5411 | Retail-Grocery Stores |
| 257 | ANDW | ANDREW CORP | 30-Nov-1984 | 1-Oct-2006 | 11474 | 3357 | Drawing & Insulating of Nonferrous Wire |
| 258 | NYT | NEW YORK TIMES CO -CL A | 30-Nov-1984 | 19-Dec-2010 | 11474 | 2711 | Newspapers: Publishing or Publishing & Printing |
| 259 | SVU | SUPERVALU INC | 31-Jan-1985 | 30-Apr-2012 | 11412 | 5141 | Wholesale-Groceries, General Line (merchandise) |
| 260 | RDC | ROWAN COMPANIES PLC | 31-Jan-1985 | 18-Aug-2014 | 11412 | 1381 | Drilling Oil & Gas Wells |
| 261 | APD | AIR PRODUCTS & CHEMICALS INC | 30-Apr-1985 | | 11323 | 2810 | Industrial Inorganic Chemicals |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------|------------------|------------------|------------|------------------|--|
| 262 | NUE | NUCOR CORP | 30-Apr-1985 | | 11323 | 3312 | Steel Works, Blast Furnaces & Rolling Mills (Coke Ovens) |
| 263 | PKI | PERKINELMER INC | 31-May-1985 | | 11292 | 3826 | Laboratory Analytical Instruments |
| 264 | EP2 | EL PASO CGP CO | 31-May-1985 | 29-Jan-2001 | 11292 | 4922 | Natural Gas Transmission |
| 265 | NOC | NORTHROP GRUMMAN CORP | 30-Jun-1985 | | 11262 | 3812 | Search, Detection, Navigation, Guidance, Aeronautical Sys |
| 266 | CNP | CENTERPOINT ENERGY INC | 31-Jul-1985 | | 11231 | 4931 | Electric & Other Services |
| 267 | TJX | TJX COMPANIES INC | 30-Sep-1985 | | 11170 | 5651 | Combined |
| 268 | DOV | DOVER CORP | 31-Oct-1985 | | 11139 | 3585 | Retail-Family Clothing Stores |
| 269 | PH | PARKER-HANNIFIN CORP | 30-Nov-1985 | | 11109 | 3490 | Air-Cond & Warm Air Heatg Equip & Comm & Incl Refrig Equip |
| 270 | LDG | LONGS DRUG STORES CORP | 30-Nov-1985 | 1-Jul-2001 | 11109 | 5912 | Miscellaneous Fabricated Metal Products |
| 271 | TRCO | TRIBUNE MEDIA CO | 31-Jan-1986 | 20-Dec-2007 | 11047 | 4833 | Retail-Drug Stores and Proprietary Stores |
| 272 | ITW | ILLINOIS TOOL WORKS | 28-Feb-1986 | | 11019 | 3560 | Television Broadcasting Stations |
| 273 | DLX | DELUXE CORP | 30-Apr-1986 | 19-Dec-2004 | 10958 | 2780 | General Industrial Machinery & Equipment |
| | | | | | | | Blankbooks, Looseleaf Binders & Bookbinding & Related Work |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|----------------------------|------------------|------------------|------------|------------------|--|
| 274 | JCI | JOHNSON CONTROLS INTL PLC | 31-May-1986 | | 10927 | 3585 | Air-Cond & Warm Air Heatg Equip & Comm & Inld Refrig Equip |
| 275 | FJ | FORT JAMES CORP | 31-May-1986 | 23-Nov-2000 | 10927 | 2621 | Paper Mills |
| 276 | 6583B | BAUSCH & LOMB HLDGS -REDH | 31-May-1986 | 28-Oct-2007 | 10927 | 2834 | Pharmaceutical Preparations |
| 277 | GPS | GAP INC | 31-Aug-1986 | | 10835 | 5651 | Retail-Family Clothing Stores |
| 278 | JWN | NORDSTROM INC | 31-Aug-1986 | | 10835 | 5651 | Retail-Family Clothing Stores |
| 279 | MDT | MEDTRONIC PLC | 31-Oct-1986 | | 10774 | 3845 | Electromedical & Electrotherapeutic Apparatus |
| 280 | DDS | DILLARDS INC -CL A | 31-Oct-1986 | 21-Oct-2008 | 10774 | 5311 | Retail-Department Stores |
| 281 | HRB | BLOCK H & R INC | 30-Nov-1986 | | 10744 | 7200 | Services-Personal Services |
| 282 | SYU | SYSCO CORP | 31-Dec-1986 | | 10713 | 5140 | Wholesale-Groceries & Related Products |
| 283 | SCI | SERVICE CORP INTERNATIONAL | 31-Dec-1986 | 15-Mar-2000 | 10713 | 7200 | Services-Personal Services |
| 284 | NSI.1 | NATIONAL SERVICE INDS INC | 31-Dec-1986 | 2-Dec-2001 | 10713 | 7200 | Services-Personal Services |
| 285 | 3UAIHQ | US AIRWAYS GROUP INC-OLD | 31-Dec-1986 | 14-May-2002 | 10713 | 4512 | Air Transportation, Scheduled |
| 286 | TIN | TEMPLE-INLAND INC | 31-Dec-1986 | 30-Dec-2007 | 10713 | 2631 | Paperboard Mills |
| 287 | RBK | REEBOK INTERNATIONAL LTD | 31-Jan-1987 | 31-Jan-2006 | 10682 | 3021 | Rubber & Plastics Footwear |
| 288 | UST.1 | UST INC | 30-Apr-1987 | 5-Jan-2009 | 10593 | 2100 | Tobacco Products |
| 289 | CA | CA INC | 31-Jul-1987 | | 10501 | 7372 | Services-Prepackaged Software |
| 290 | MMC | MARSH & MCLENNAN COS | 31-Aug-1987 | | 10470 | 6411 | Insurance Agents, Brokers & Service |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------------|------------------|------------------|------------|------------------|--|
| 291 | PDG | PLACER DOME INC | 31-Aug-1987 | 21-Jul-2002 | 10470 | 1040 | Gold and Silver Ores |
| 292 | AVY | AVERY DENNISON CORP | 31-Dec-1987 | | 10348 | 2670 | Converted Paper & Paperboard |
| 293 | PLL | PALL CORP | 31-Dec-1987 | 30-Aug-2015 | 10348 | 3569 | Prods (No Containers/Boxes) |
| 294 | IKN | IKON OFFICE SOLUTIONS | 29-Feb-1988 | 28-Jun-2000 | 10288 | 5040 | General Industrial Machinery & Equipment, NEC |
| 295 | CPQ.2 | COMPAQ COMPUTER CORP | 29-Feb-1988 | 5-May-2002 | 10288 | 3571 | Wholesale-Professional & Commercial Equipment & Supplies |
| 296 | HD | HOME DEPOT INC | 31-Mar-1988 | | 10257 | 5211 | Electronic Computers |
| 297 | PNC | PNC FINANCIAL SVCS GROUP INC | 30-Apr-1988 | | 10227 | 6020 | Retail-Lumber & Other Building Materials Dealers |
| 298 | ONE.1 | BANK ONE CORP | 30-Apr-1988 | 30-Jun-2004 | 10227 | 6020 | |
| 299 | C | CITIGROUP INC | 31-May-1988 | | 10196 | 6199 | Finance Services |
| 300 | STI | SUNTRUST BANKS INC | 31-May-1988 | | 10196 | 6020 | |
| 301 | GDW | GOLDEN WEST FINANCIAL CORP | 30-Jun-1988 | 1-Oct-2006 | 10166 | 6035 | Savings Institution, Federally Chartered |
| 302 | WOR | WORTHINGTON INDUSTRIES | 31-Jul-1988 | 19-Dec-2004 | 10135 | 3310 | Steel Works, Blast Furnaces & Rolling & Finishing Mills |
| 303 | MIL.2 | MILLIPORE CORP | 31-Jul-1988 | 14-Jul-2010 | 10135 | 3826 | Laboratory Analytical Instruments |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|---|
| 304 | FNMA | FANNIE MAE | 31-Aug-1988 | 10-Sep-2008 | 10104 | 6111 | Federal & Federally Sponsored Credit Agencies |
| 305 | NKE | NIKE INC | 30-Nov-1988 | | 10013 | 3021 | Rubber & Plastics Footwear |
| 306 | FBF | FLEETBOSTON FINANCIAL CORP | 30-Nov-1988 | 31-Mar-2004 | 10013 | 6020 | |
| 307 | ECL | ECOLAB INC | 31-Jan-1989 | | 9951 | 2842 | Specialty Cleaning, Polishing and Sanitation Preparations |
| 308 | WB.3 | WACHOVIA CORP | 28-Feb-1989 | 1-Jan-2009 | 9923 | 6020 | |
| 309 | KBH | KB HOME | 31-Mar-1989 | 20-Dec-2009 | 9892 | 1531 | Operative Builders |
| 310 | NWL | NEWELL BRANDS INC | 30-Apr-1989 | | 9862 | 3990 | Miscellaneous Manufacturing Industries |
| 311 | TMK | TORCHMARK CORP | 30-Apr-1989 | | 9862 | 6311 | Life Insurance |
| 312 | CCTYQ | CIRCUIT CITY STORES INC | 31-May-1989 | 30-Mar-2008 | 9831 | 5731 | Retail-Radio, TV & Consumer Electronics Stores |
| 313 | JOSEA | JOSTENS INC | 31-Jul-1989 | 9-May-2000 | 9770 | 3911 | Jewelry, Precious Metal |
| 314 | TYC | TYCO INTERNATIONAL PLC | 31-Jul-1989 | 16-Mar-2009 | 9770 | 3669 | Communications Equipment, NEC |
| 315 | S | SPRINT CORP | 31-Jul-1989 | 8-Jul-2013 | 9770 | 4812 | Radiotelephone Communications |
| 316 | ORCL | ORACLE CORP | 31-Aug-1989 | | 9739 | 7372 | Services-Prepackaged Software |
| 317 | ASH | ASHLAND GLOBAL HOLDINGS INC | 31-Aug-1989 | 13-Nov-2008 | 9739 | 2820 | Plastic Material, Synth Resin/Rubber, Cellulos (No Glass) |
| 318 | GAS.2 | NICOR INC | 31-Aug-1989 | 12-Dec-2011 | 9739 | 4924 | Natural Gas Distribution |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|----------------------------|------------------|------------------|------------|------------------|--|
| 319 | DUK8 | PROGRESS ENERGY INC | 31-Aug-1989 | 1-Jul-2012 | 9739 | 4911 | Electric Services |
| 320 | AM.1 | AMERICAN GREETINGS -CL A | 31-Oct-1989 | 2-May-2004 | 9678 | 2771 | Greeting Cards |
| 321 | STJ | ST JUDE MEDICAL INC | 30-Nov-1989 | 4-Jan-2017 | 9648 | 3845 | Electromedical & Electrotherapeutic Apparatus |
| 322 | ADSK | AUTODESK INC | 1-Dec-1989 | | 9647 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 323 | CZR | CAESARS ENTERTAINMENT CORP | 1-Feb-1990 | 28-Jan-2008 | 9585 | 7990 | Services-Miscellaneous Amusement & Recreation |
| 324 | CTB | COOPER TIRE & RUBBER CO | 1-May-1990 | 17-Jul-2006 | 9496 | 3011 | Tires & Inner Tubes |
| 325 | PBY | PEP BOYS-MANNY MOE & JACK | 1-Aug-1990 | 2-Apr-2000 | 9404 | 5531 | Retail-Auto & Home Supply Stores |
| 326 | 5938B | BIOMET INC | 1-Aug-1990 | 11-Jul-2007 | 9404 | 3842 | Orthopedic, Prosthetic & Surgical Appliances & Supplies |
| 327 | AZA.1 | ALZA CORP | 11-Dec-1990 | 24-Jun-2001 | 9272 | 2834 | Pharmaceutical Preparations |
| 328 | MRO | MARATHON OIL CORP | 1-May-1991 | | 9131 | 1311 | Crude Petroleum & Natural Gas |
| 329 | X | UNITED STATES STEEL CORP | 1-May-1991 | 1-Jul-2014 | 9131 | 3312 | Steel Works, Blast Furnaces & Rolling Mills (Coke Ovens) |
| 330 | AEE | AMEREN CORP | 19-Sep-1991 | | 8990 | 4931 | Electric & Other Services Combined |
| 331 | NOVL | NOVELL INC | 1-Oct-1991 | 27-Apr-2011 | 8978 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 332 | GLK.1 | GREAT LAKES CHEMICAL CORP | 1-Nov-1991 | 4-Jul-2005 | 8947 | 2890 | Miscellaneous Chemical Products |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------------|------------------|------------------|------------|------------------|--|
| 333 | AMGN | AMGEN INC | 2-Jan-1992 | | 8885 | 2836 | Biological Products, (No Diagnostic Substances) |
| 334 | FMCC | FEDERAL HOME LOAN MORTG CORP | 2-Jan-1992 | 10-Sep-2008 | 8885 | 6111 | Federal & Federally Sponsored Credit Agencies |
| 335 | AGN.2 | ALLERGAN INC | 7-Feb-1992 | 22-Mar-2015 | 8849 | 2834 | Pharmaceutical Preparations |
| 336 | KRB | MBNA CORP | 1-Apr-1992 | 2-Jan-2006 | 8795 | 6020 | |
| 337 | PX | PRAXAIR INC | 1-Jul-1992 | | 8704 | 2810 | Industrial Inorganic Chemicals |
| 338 | JAVA | SUN MICROSYSTEMS INC | 1-Aug-1992 | 28-Jan-2010 | 8673 | 3571 | Electronic Computers |
| 339 | IPG | INTERPUBLIC GROUP OF COS | 1-Oct-1992 | | 8612 | 7311 | Services-Advertising Agencies |
| 340 | RAL. | RALSTON PURINA CO | 1-Aug-1993 | 12-Dec-2001 | 8308 | 2040 | Grain Mill Products |
| 341 | HCA | HCA HEALTHCARE INC | 1-Sep-1993 | 19-Nov-2006 | 8277 | 8062 | Services-General Medical & Surgical Hospitals, NEC |
| 342 | COST | COSTCO WHOLESALE CORP | 1-Oct-1993 | | 8247 | 5399 | Retail-Misc General Merchandise Stores |
| 343 | WB.1 | WACHOVIA CORP-OLD | 1-Oct-1993 | 3-Sep-2001 | 8247 | 6020 | |
| 344 | ABX | BARRICK GOLD CORP | 1-Nov-1993 | 21-Jul-2002 | 8216 | 1040 | Gold and Silver Ores |
| 345 | BR.2 | BURLINGTON RESOURCES INC | 1-Nov-1993 | 2-Apr-2006 | 8216 | 1311 | Crude Petroleum & Natural Gas |
| 346 | CSCO | CISCO SYSTEMS INC | 1-Dec-1993 | | 8186 | 3576 | Computer Communications Equipment |
| 347 | CMCSA | COMCAST CORP | 1-Dec-1993 | 18-Nov-2002 | 8186 | 4841 | Cable & Other Pay Television Services |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------|------------------|------------------|------------|------------------|--|
| 348 | EMN | EASTMAN CHEMICAL CO | 1-Jan-1994 | | 8155 | 2821 | Plastic Materials, Synth Resins & Nonvulcan Elastomers |
| 349 | KEY | KEYCORP | 1-Mar-1994 | | 8096 | 6020 | |
| 350 | UNM | UNUM GROUP | 1-Mar-1994 | | 8096 | 6321 | Accident & Health Insurance |
| 351 | MSFT | MICROSOFT CORP | 1-Jun-1994 | | 8004 | 7372 | Services-Prepackaged Software |
| 352 | LUV | SOUTHWEST AIRLINES | 1-Jul-1994 | | 7974 | 4512 | Air Transportation, Scheduled |
| 353 | UNH | UNITEDHEALTH GROUP INC | 1-Jul-1994 | | 7974 | 6324 | Hospital & Medical Service Plans |
| 354 | CBS | CBS CORP | 1-Sep-1994 | | 7912 | 4888 | |
| 355 | SIAL | SIGMA-ALDRICH CORP | 1-Sep-1994 | 18-Nov-2015 | 7912 | 2833 | Medicinal Chemicals & Botanical Products |
| 356 | MU | MICRON TECHNOLOGY INC | 27-Sep-1994 | | 7886 | 3674 | Semiconductors & Related Devices |
| 357 | FDC | FIRST DATA CORP | 27-Sep-1994 | 24-Sep-2007 | 7886 | 7374 | Services-Computer Processing & Data Preparation |
| 358 | NCC | NATIONAL CITY CORP | 27-Sep-1994 | 1-Jan-2009 | 7886 | 6020 | |
| 359 | DUK6 | CINERGY CORP | 25-Oct-1994 | 2-Apr-2006 | 7858 | 4911 | Electric Services |
| 360 | AT-2 | ALLTEL CORP | 21-Dec-1994 | 18-Nov-2007 | 7801 | 4812 | Radiotelephone Communications |
| 361 | SGICQ | SILICON GRAPHICS INC | 18-Jan-1995 | 20-Jun-2000 | 7773 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 362 | GPU. | GPU INC | 9-Feb-1995 | 6-Nov-2001 | 7751 | 4911 | Electric Services |
| 363 | BSX | BOSTON SCIENTIFIC CORP | 24-Feb-1995 | | 7736 | 3845 | Electromedical & Electrotherapeutic Apparatus |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 364 | AMAT | APPLIED MATERIALS INC | 16-Mar-1995 | | 7716 | 3559 | Special Industry Machinery, NEC |
| 365 | BK | BANK OF NEW YORK MELLON CORP | 31-Mar-1995 | | 7701 | 6020 | |
| 366 | CAR | AVIS BUDGET GROUP INC | 1-May-1995 | 31-Jul-2006 | 7670 | 7510 | Services-Auto Rental & Leasing (No Drivers) |
| 367 | L | LOEWS CORP | 10-May-1995 | | 7661 | 6331 | Fire, Marine & Casualty Insurance |
| 368 | ETS | ENTERASYS NETWORKS INC | 22-May-1995 | 5-Aug-2001 | 7649 | 3576 | Computer Communications Equipment |
| 369 | DRI | DARDEN RESTAURANTS INC | 30-May-1995 | | 7641 | 5812 | Retail-Eating Places |
| 370 | TLAB | TELLABS INC | 3-Jul-1995 | 20-Dec-2011 | 7607 | 3661 | Telephone & Telegraph Apparatus |
| 371 | HBC2 | HSBC USA INC | 5-Jul-1995 | 2-Jan-2000 | 7605 | 6020 | Fire, Marine & Casualty Insurance |
| 372 | ALL | ALLSTATE CORP | 13-Jul-1995 | | 7597 | 6331 | Fire, Marine & Casualty Insurance |
| 373 | FCX | FREEMONT-MCMORAN INC | 31-Jul-1995 | | 7579 | 1000 | Metal Mining |
| 374 | WLL.1 | WILLAMETTE INDUSTRIES | 18-Aug-1995 | 10-Feb-2002 | 7561 | 2621 | Paper Mills |
| 375 | MS | MORGAN STANLEY | 22-Sep-1995 | | 7526 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 376 | USB.1 | U S BANCORP/DE-OLD | 23-Oct-1995 | 26-Feb-2001 | 7495 | 6020 | |
| 377 | UMG.Z | MEDIAONE GROUP INC | 1-Nov-1995 | 15-Jun-2000 | 7486 | 4841 | Cable & Other Pay Television Services |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 378 | M | MACY'S INC | 8-Nov-1995 | | 7479 | 5311 | Retail-Department Stores |
| 379 | PPL | PPL CORP | 27-Nov-1995 | | 7460 | 4911 | Electric Services |
| 380 | CMA | COMERICA INC | 1-Dec-1995 | | 7456 | 6020 | |
| 381 | HUM | HUMANA INC | 1-Dec-1995 | | 7456 | 6324 | Hospital & Medical Service Plans |
| 382 | LSI.1 | LSI CORP | 14-Dec-1995 | 7-May-2014 | 7443 | 3674 | Semiconductors & Related Devices |
| 383 | HIG | HARTFORD FINANCIAL SERVICES | 20-Dec-1995 | | 7437 | 6331 | Fire, Marine & Casualty Insurance |
| 384 | COMS | 3COM CORP | 2-Jan-1996 | 27-Jul-2000 | 7424 | 3576 | Computer Communications Equipment |
| 385 | FITB | FIFTH THIRD BANCORP | 8-Mar-1996 | | 7358 | 6020 | |
| 386 | EMC | EMC CORP/MA | 28-Mar-1996 | 7-Sep-2016 | 7338 | 3572 | Computer Storage Devices |
| 387 | GIC.3 | GENERAL INSTRUMENT CORP | 1-Apr-1996 | 5-Jan-2000 | 7334 | 3663 | Radio & TV Broadcasting & Communications Equipment |
| 388 | MCIP | MCI INC | 1-Apr-1996 | 14-May-2002 | 7334 | 4813 | Telephone Communications (No Radiotelephone) |
| 389 | AON | AON PLC | 23-Apr-1996 | | 7312 | 6411 | Insurance Agents, Brokers & Service |
| 390 | TUP | TUPPERWARE BRANDS CORP | 31-May-1996 | 24-Mar-2004 | 7274 | 3089 | Plastics Products, NEC |
| 391 | MTG | MGIC INVESTMENT CORP/WI | 19-Jul-1996 | 30-Oct-2008 | 7225 | 6351 | Surety Insurance |
| 392 | SEG.2 | SEAGATE TECHNOLOGY-OLD | 16-Aug-1996 | 21-Nov-2000 | 7197 | 3572 | Computer Storage Devices |
| 393 | 7732B | DELL TECHNOLOGIES INC | 6-Sep-1996 | 28-Oct-2013 | 7176 | 3571 | Electronic Computers |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 394 | LU | LUCENT TECHNOLOGIES INC | 1-Oct-1996 | 30-Nov-2006 | 7151 | 7373 | Services-Computer Integrated Systems Design |
| 395 | UPR.1 | UNION PACIFIC RESOURCES GRP | 16-Oct-1996 | 16-Jul-2000 | 7136 | 1311 | Crude Petroleum & Natural Gas |
| 396 | IMS | IMS HEALTH HOLDINGS INC | 4-Nov-1996 | 28-Feb-2010 | 7117 | 8700 | Services-Engineering, Accounting, Research, Management |
| 397 | MBI | MBIA INC | 3-Dec-1996 | 20-Dec-2009 | 7088 | 6351 | Surety Insurance |
| 398 | GDT | GUIDANT CORP | 19-Dec-1996 | 23-Apr-2006 | 7072 | 3841 | Surgical & Medical Instruments & Apparatus |
| 399 | TMO | THERMO FISHER SCIENTIFIC INC | 2-Jan-1997 | | 7058 | 3826 | Laboratory Analytical Instruments |
| 400 | AZO | AUTOZONE INC | 2-Jan-1997 | | 7058 | 5531 | Retail-Auto & Home Supply Stores |
| 401 | EHC | ENCOMPASS HEALTH CORP | 8-Jan-1997 | 20-Mar-2003 | 7052 | 8060 | Services-Hospitals |
| 402 | CNO | CNO FINANCIAL GROUP INC | 15-Jan-1997 | 24-Jul-2002 | 7045 | 6321 | Accident & Health Insurance |
| 403 | PTC | PTC INC | 3-Apr-1997 | 2-Jan-2007 | 6967 | 7372 | Services-Prepackaged Software |
| 404 | ADBE | ADOBE SYSTEMS INC | 6-May-1997 | | 6934 | 7372 | Services-Prepackaged Software |
| 405 | CAH | CARDINAL HEALTH INC | 27-May-1997 | | 6913 | 5122 | Wholesale-Drugs, Proprietaries & Druggists' Sundries |
| 406 | SCHW | SCHWAB (CHARLES) CORP | 2-Jun-1997 | | 6907 | 6282 | Investment Advice |
| 407 | PVN | PROVIDIAN FINANCIAL CORP | 11-Jun-1997 | 2-Oct-2005 | 6898 | 6020 | |
| 408 | CFC.3 | COUNTRYWIDE FINANCIAL CORP | 18-Jun-1997 | 30-Jun-2008 | 6891 | 6162 | Mortgage Bankers & Loan Correspondents |

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|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 409 | EFX | EQUIFAX INC | 19-Jun-1997 | | 6890 | 7323 | |
| 410 | WAMUQ | WASHINGTON MUTUAL INC | 2-Jul-1997 | 29-Sep-2008 | 6877 | 6035 | Savings Institution, Federally Chartered |
| 411 | APA | APACHE CORP | 28-Jul-1997 | | 6851 | 1311 | Crude Petroleum & Natural Gas |
| 412 | APC | ANADARKO PETROLEUM CORP | 28-Jul-1997 | | 6851 | 1311 | Crude Petroleum & Natural Gas |
| 413 | PGR | PROGRESSIVE CORP-OHIO | 4-Aug-1997 | | 6844 | 6331 | Fire, Marine & Casualty Insurance |
| 414 | OI | OWENS-ILLINOIS INC | 4-Aug-1997 | 10-Dec-2000 | 6844 | 3221 | Glass Containers |
| 415 | HBAN | HUNTINGTON BANCSHARES | 6-Aug-1997 | | 6842 | 6020 | |
| 416 | MIR.1 | MIRAGE RESORTS INC | 7-Aug-1997 | 31-May-2000 | 6841 | 7990 | Services-Miscellaneous Amusement & Recreation |
| 417 | STT | STATE STREET CORP | 18-Aug-1997 | | 6830 | 6020 | |
| 418 | IHRTQ | IHEARTMEDIA INC | 1-Sep-1997 | 30-Jul-2008 | 6816 | 4832 | Radio Broadcasting Stations |
| 419 | KLAC | KLA-TENCOR CORP | 15-Sep-1997 | | 6802 | 3827 | Optical Instruments & Lenses |
| 420 | YUM | YUM BRANDS INC | 7-Oct-1997 | | 6780 | 5812 | Retail-Eating Places |
| 421 | SNV | SYNOVUS FINANCIAL CORP | 28-Nov-1997 | 1-Jan-2008 | 6728 | 6020 | |
| 422 | BBT | BB&T CORP | 4-Dec-1997 | | 6722 | 6020 | |
| 423 | CINF | CINCINNATI FINANCIAL CORP | 18-Dec-1997 | | 6708 | 6331 | Fire, Marine & Casualty Insurance |
| 424 | OMC | OMNICOM GROUP | 24-Dec-1997 | | 6702 | 7311 | Services-Advertising Agencies |
| 425 | LEHMQ | LEHMAN BROTHERS HOLDINGS INC | 12-Jan-1998 | 16-Sep-2008 | 6683 | 6211 | Security Brokers, Dealers & Flotation Companies |

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|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 426 | BIG | BIG LOTS INC | 20-Jan-1998 | 13-Feb-2013 | 6675 | 5331 | Retail-Variety Stores |
| 427 | SUB.2 | SUMMIT BANCORP | 28-Jan-1998 | 28-Feb-2001 | 6667 | 6020 | |
| 428 | NTRS | NORTHERN TRUST CORP | 30-Jan-1998 | | 6665 | 6020 | |
| 429 | SEE | SEALED AIR CORP | 1-Apr-1998 | | 6604 | 3081 | Unsupported Plastics Film & Sheet |
| 430 | NXTL | NEXTEL COMMUNICATIONS INC | 1-Apr-1998 | 14-Aug-2005 | 6604 | 4812 | Radiotelephone Communications |
| 431 | AFS | ASSOCIATES FIRST CAP -CL A | 8-Apr-1998 | 3-Dec-2000 | 6597 | 6141 | Personal Credit Institutions |
| 432 | GTW | GATEWAY INC | 27-Apr-1998 | 31-Jul-2006 | 6578 | 3571 | Electronic Computers |
| 433 | BEN | FRANKLIN RESOURCES INC | 29-Apr-1998 | | 6576 | 6282 | Investment Advice |
| 434 | MAR | MARRIOTT INTL INC | 21-May-1998 | | 6554 | 7011 | Hotels & Motels |
| 435 | COF | CAPITAL ONE FINANCIAL CORP | 1-Jul-1998 | | 6513 | 6141 | Personal Credit Institutions |
| 436 | BSC.1 | BEAR STEARNS COMPANIES INC | 1-Jul-1998 | 1-Jun-2008 | 6513 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 437 | NAVI | NAVIENT CORP | 1-Jul-1998 | 4-Jun-2018 | 6513 | 6111 | Federal & Federally Sponsored Credit Agencies |
| 438 | DG | DOLLAR GENERAL CORP | 16-Jul-1998 | 8-Jul-2007 | 6498 | 5331 | Retail-Variety Stores |
| 439 | WM | WASTE MANAGEMENT INC | 17-Jul-1998 | | 6497 | 4953 | Refuse Systems |
| 440 | EDS. | ELECTRONIC DATA SYSTEMS CORP | 11-Aug-1998 | 26-Aug-2008 | 6472 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 441 | KSS | KOHL'S CORP | 14-Aug-1998 | | 6469 | 5311 | Retail-Department Stores |
| 442 | RF | REGIONS FINANCIAL CORP | 28-Aug-1998 | | 6455 | 6020 | |

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S&P 500 Index Sample List

| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|--------------------------------|------------------|------------------|------------|------------------|---|
| 443 | NGH.2 | NABISCO GROUP HOLDINGS CORP | 16-Sep-1998 | 11-Dec-2000 | 6436 | 2052 | Cookies & Crackers |
| 444 | HCR | MANOR CARE INC | 28-Sep-1998 | 8-Nov-2007 | 6424 | 8051 | Services-Skilled Nursing Care Facilities |
| 445 | PAYX | PAYCHEX INC | 1-Oct-1998 | | 6421 | 8721 | |
| 446 | UPC. | UNION PLANTERS CORP | 1-Oct-1998 | 30-Jun-2004 | 6421 | 6020 | |
| 447 | BMC | BMC SOFTWARE INC | 1-Oct-1998 | 10-Sep-2013 | 6421 | 7372 | Services-Prepackaged Software |
| 448 | AES | AES CORP | 2-Oct-1998 | | 6420 | 4991 | Co-generation Services & Small Power Producers |
| 449 | PSFT. | PEOPLESOFT INC | 2-Oct-1998 | 28-Dec-2004 | 6420 | 7372 | Services-Prepackaged Software |
| 450 | SPLS | STAPLES INC | 7-Oct-1998 | 17-Sep-2017 | 6415 | 5110 | Wholesale-Paper & Paper Products |
| 451 | CCE | COCA-COLA EUROPEAN PARTNERS | 8-Oct-1998 | 30-May-2016 | 6414 | 2086 | Bottled & Canned Soft Drinks & Carbonated Waters |
| 452 | NCE.1 | NEW CENTURY ENERGIES INC | 3-Nov-1998 | 20-Aug-2000 | 6388 | 4931 | Electric & Other Services Combined |
| 453 | SWY | SAFEWAY INC | 13-Nov-1998 | 26-Jan-2015 | 6378 | 5411 | Retail-Grocery Stores |
| 454 | DHR | DANAHER CORP | 18-Nov-1998 | | 6373 | 3826 | Laboratory Analytical Instruments |
| 455 | PCS.1 | SPRINT PCS GROUP | 24-Nov-1998 | 22-Apr-2004 | 6367 | 4812 | Radiotelephone Communications |
| 456 | GCL | CARNIVAL CORP / PLC (USA) | 22-Dec-1998 | | 6339 | 4400 | Water Transportation |
| 457 | SLR | SOLETRON CORP | 31-Dec-1998 | 1-Oct-2007 | 6330 | 3672 | Printed Circuit Boards |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-------------------------------|------------------|------------------|------------|------------------|--|
| 458 | USB | U S BANCORP | 4-Jan-1999 | | 6326 | 6020 | |
| 459 | CPWR.1 | COMPUWARE CORP | 4-Jan-1999 | 2-Jan-2012 | 6326 | 7372 | Services-Prepackaged Software |
| 460 | TWX | TIME WARNER INC | 4-Jan-1999 | 19-Jun-2018 | 6326 | 4888 | |
| 461 | MCK | MCKESSON CORP | 13-Jan-1999 | | 6317 | 5122 | Wholesale-Drugs, Proprietaries & Druggists' Sundries |
| 462 | SOTR | SOUTHTRUST CORP | 1-Mar-1999 | 31-Oct-2004 | 6270 | 6020 | |
| 463 | ASO.1 | AMSOUTH BANCORPORATION | 10-Mar-1999 | 5-Nov-2006 | 6261 | 6020 | |
| 464 | CTL | CENTURLINK INC | 25-Mar-1999 | | 6246 | 4813 | Telephone Communications (No Radiotelephone) |
| 465 | KSU | KANSAS CITY SOUTHERN | 5-Apr-1999 | 12-Jul-2000 | 6235 | 4011 | Railroads, Line-Haul Operating |
| 466 | AGN | ALLERGAN PLC | 12-Apr-1999 | | 6228 | 2834 | Pharmaceutical Preparations |
| 467 | CMS | CMS ENERGY CORP | 3-May-1999 | | 6207 | 4931 | Electric & Other Services Combined |
| 468 | AFL | AFLAC INC | 28-May-1999 | | 6182 | 6321 | Accident & Health Insurance |
| 469 | APTIV | APTIV PLC | 28-May-1999 | 10-Oct-2005 | 6182 | 3714 | Motor Vehicle Parts & Accessories |
| 470 | PWJ. | PAINE WEBBER GROUP | 4-Jun-1999 | 5-Nov-2000 | 6175 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 471 | WLP.1 | WELLPOINT HEALTH NETWORKS INC | 9-Jun-1999 | 30-Nov-2004 | 6170 | 6324 | Hospital & Medical Service Plans |
| 472 | PGN3 | FLORIDA PROGRESS CORP | 22-Jun-1999 | 30-Nov-2000 | 6157 | 4911 | Electric Services |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|---|
| 473 | ODP | OFFICE DEPOT INC | 24-Jun-1999 | 19-Dec-2010 | 6155 | 5940 | Retail-Miscellaneous Shopping Goods Stores |
| 474 | NTAP | NETAPP INC | 25-Jun-1999 | | 6154 | 3572 | Computer Storage Devices |
| 475 | BBY | BEST BUY CO INC | 30-Jun-1999 | | 6149 | 5731 | Retail-Radio, TV & Consumer Electronics Stores |
| 476 | VMC | VULCAN MATERIALS CO | 1-Jul-1999 | | 6148 | 1400 | Mining & Quarrying of Nonmetallic Minerals (No Fuels) |
| 477 | QCOM | QUALCOMM INC | 22-Jul-1999 | | 6127 | 3674 | Semiconductors & Related Devices |
| 478 | ADCT | ADC TELECOMMUNICATIONS INC | 2-Aug-1999 | 1-Jul-2007 | 6116 | 3661 | Telephone & Telegraph Apparatus |
| 479 | AW | ALLIED WASTE INDUSTRIES INC | 2-Aug-1999 | 4-Dec-2008 | 6116 | 4953 | Refuse Systems |
| 480 | COC1 | CONOCO INC | 9-Aug-1999 | 2-Sep-2002 | 6109 | 2911 | Petroleum Refining |
| 481 | LXK | LEXMARK INTL INC -CL A | 13-Aug-1999 | 30-Sep-2012 | 6105 | 3577 | Computer Peripheral Equipment, NEC |
| 482 | TOS.1 | TOSCO CORP | 20-Sep-1999 | 17-Sep-2001 | 6067 | 2911 | Petroleum Refining |
| 483 | GLBC | GLOBAL CROSSING LTD | 29-Sep-1999 | 9-Oct-2001 | 6058 | 4813 | Telephone Communications (No Radiotelephone) |
| 484 | SXCL | STEEL EXCEL INC | 1-Oct-1999 | 13-May-2001 | 6056 | 1389 | Oil & Gas Field Services, NEC |
| 485 | BBBY | BED BATH & BEYOND INC | 1-Oct-1999 | 25-Jul-2017 | 6056 | 5700 | Retail-Home Furniture, Furnishings & Equipment Stores |
| 486 | PNW | PINNACLE WEST CAPITAL CORP | 4-Oct-1999 | | 6053 | 4911 | Electric Services |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 487 | ADI | ANALOG DEVICES | 12-Oct-1999 | | 6045 | 3674 | Semiconductors & Related Devices |
| 488 | TROW | PRICE (T. ROWE) GROUP | 13-Oct-1999 | | 6044 | 6282 | Investment Advice |
| 489 | LEG | LEGGETT & PLATT INC | 18-Oct-1999 | | 6039 | 2510 | Household Furniture |
| 490 | EP | EL PASO CORP | 26-Oct-1999 | 24-May-2012 | 6031 | 4922 | Natural Gas Transmission |
| 491 | CMVT | COMVERSE TECHNOLOGY INC | 27-Oct-1999 | 31-Jan-2007 | 6030 | 7372 | Services-Prepackaged Software |
| 492 | PTV | PACTIV CORP | 5-Nov-1999 | 16-Nov-2010 | 6021 | 3089 | Plastics Products, NEC |
| 493 | XLNX | XILINX INC | 8-Nov-1999 | | 6018 | 3674 | Semiconductors & Related Devices |
| 494 | TER | TERADYNE INC | 15-Nov-1999 | 22-Dec-2013 | 6011 | 3825 | Instruments For Meas & Testing of Electricity & Elec Signals |
| 495 | 0573B | QUINTILES TRANSNATIONAL CORP | 16-Nov-1999 | 25-Sep-2003 | 6010 | 8731 | Services-Commercial Physical & Biological Research |
| 496 | CTXS | CITRIX SYSTEMS INC | 1-Dec-1999 | | 5995 | 7372 | Services-Prepackaged Software |
| 497 | MOLX | MOLEX INC | 1-Dec-1999 | 9-Dec-2013 | 5995 | 3678 | Electronic Connectors |
| 498 | OK.2 | OLD KENT FINANCIAL CORP | 2-Dec-1999 | 1-Apr-2001 | 5994 | 6020 | |
| 499 | AABA | ALTABA INC | 8-Dec-1999 | 18-Jun-2017 | 5988 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 500 | RIG | TRANSOCEAN LTD | 31-Dec-1999 | 18-Dec-2008 | 5965 | 1381 | Drilling Oil & Gas Wells |
| 501 | NCR | NCR CORP | 4-Jan-2000 | 30-Sep-2007 | 5961 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 502 | YNR.1 | YOUNG & RUBICAM INC | 6-Jan-2000 | 2-Oct-2000 | 5959 | 7311 | Services-Advertising Agencies |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------------|------------------|------------------|------------|------------------|---|
| 503 | HOG | HARLEY-DAVIDSON INC | 31-Jan-2000 | | 5934 | 3751 | Motorcycles, Bicycles & Parts |
| 504 | CNXT.1 | CONEXANT SYSTEMS INC | 31-Jan-2000 | 25-Jun-2002 | 5934 | 3674 | Semiconductors & Related Devices |
| 505 | BGEN | BIOGEN INC-OLD | 31-Jan-2000 | 12-Nov-2003 | 5934 | 2836 | Biological Products, (No Diagnostic Substances) |
| 506 | TSG.2 | SABRE HOLDINGS CORP -CL A | 16-Mar-2000 | 1-Apr-2007 | 5889 | 7373 | Services-Computer Integrated Systems Design |
| 507 | VRTS.1 | VERITAS SOFTWARE CORP | 3-Apr-2000 | 4-Jul-2005 | 5871 | 7372 | Services-Prepackaged Software |
| 508 | LLTC | LINEAR TECHNOLOGY CORP | 3-Apr-2000 | 12-Mar-2017 | 5871 | 3674 | Semiconductors & Related Devices |
| 509 | ALTR.1 | ALTERA CORP | 18-Apr-2000 | 28-Dec-2015 | 5856 | 3674 | Semiconductors & Related Devices |
| 510 | SAPE | SAPIENT CORP | 5-May-2000 | 12-May-2002 | 5839 | 7373 | Services-Computer Integrated Systems Design |
| 511 | SEBL | SIEBEL SYSTEMS INC | 5-May-2000 | 31-Jan-2006 | 5839 | 7372 | Services-Prepackaged Software |
| 512 | MXIM | MAXIM INTEGRATED PRODUCTS | 10-May-2000 | 26-Sep-2007 | 5834 | 3674 | Semiconductors & Related Devices |
| 513 | APCC. | AMERICAN POWER CONVERSION CP | 1-Jun-2000 | 14-Feb-2007 | 5812 | 3620 | Electrical Industrial Apparatus |
| 514 | A | AGILENT TECHNOLOGIES INC | 5-Jun-2000 | | 5808 | 3826 | Laboratory Analytical Instruments |
| 515 | SBUX | STARBUCKS CORP | 8-Jun-2000 | | 5805 | 5812 | Retail-Eating Places |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------|------------------|------------------|------------|------------------|--|
| 516 | CVG | CONVERGYS CORP | 13-Jun-2000 | 20-Dec-2009 | 5800 | 7389 | Services-Business Services, NEC |
| 517 | MEDI | MEDIMMUNE INC | 16-Jun-2000 | 31-May-2007 | 5797 | 2836 | Biological Products, (No Diagnostic Substances) |
| 518 | CF.6 | CHARTER ONE FINANCIAL INC | 19-Jun-2000 | 31-Aug-2004 | 5794 | 6020 | Special Industry Machinery, NEC |
| 519 | NVLS.1 | NOVELLUS SYSTEMS INC | 19-Jun-2000 | 4-Jun-2012 | 5794 | 3559 | Retail-Jewelry Stores |
| 520 | TIF | TIFFANY & CO | 21-Jun-2000 | 1-Jul-2007 | 5792 | 5944 | Printed Circuit Boards |
| 521 | SANM | SANMINA CORP | 21-Jun-2000 | 2-Jan-2006 | 5792 | 3672 | Motor Vehicle Parts & Accessories |
| 522 | VC | VISTEON CORP | 29-Jun-2000 | 3-Jan-2006 | 5784 | 3714 | Services-Prepackaged Software |
| 523 | MERQ | MERCURY INTERACTIVE CORP | 29-Jun-2000 | 31-Jan-2016 | 5784 | 7372 | Semiconductors & Related Devices |
| 524 | BRCM | BROADCOM CORP | 3-Jul-2000 | 22-Nov-2011 | 5780 | 3674 | Investment Advice |
| 525 | JNS | JANUS CAPITAL GROUP INC | 13-Jul-2000 | 3-Jun-2001 | 5770 | 6282 | Finance Lessors |
| 526 | CIT.1 | CIT GROUP INC-OLD | 17-Jul-2000 | 22-Dec-2013 | 5766 | 6172 | Computer Communications Equipment |
| 527 | VIAV | VIAMI SOLUTIONS INC | 27-Jul-2000 | 13-Aug-2002 | 5756 | 3576 | Electronic Computers |
| 528 | PALM | PALM INC | 28-Jul-2000 | 26-Aug-2007 | 5755 | 4931 | Electric & Other Services Combined |
| 529 | KSE | KEYSPAN CORP | 21-Aug-2000 | 25-Oct-2007 | 5731 | 4931 | Crude Petroleum & Natural Gas |
| 530 | DVN | DEVON ENERGY CORP | 30-Aug-2000 | | 5722 | 1311 | Radio & TV Broadcasting & Communications Equipment |
| 531 | 5933B | AVAYA INC | 2-Oct-2000 | | 5689 | 3663 | |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------------|------------------|------------------|------------|------------------|---|
| 532 | MCO | MOODY'S CORP | 3-Oct-2000 | | 5688 | 7323 | |
| 533 | DYN | DYNEGY INC | 3-Oct-2000 | 20-Dec-2009 | 5688 | 1311 | Crude Petroleum & Natural Gas |
| 534 | KG | KING PHARMACEUTICALS INC | 3-Oct-2000 | 19-Dec-2010 | 5688 | 2834 | Pharmaceutical Preparations |
| 535 | NBR | NABORS INDUSTRIES LTD | 18-Oct-2000 | 22-Mar-2015 | 5673 | 1381 | Drilling Oil & Gas Wells |
| 536 | PWER | POWER-ONE INC | 23-Oct-2000 | 13-Mar-2005 | 5668 | 3679 | Electronic Components, NEC |
| 537 | NI | NISOURCE INC | 2-Nov-2000 | | 5658 | 4932 | Gas & Other Services Combined |
| 538 | EOG | EOG RESOURCES INC | 2-Nov-2000 | | 5658 | 1311 | Crude Petroleum & Natural Gas |
| 539 | BVSN | BROADVISION INC | 6-Nov-2000 | 3-Sep-2001 | 5654 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 540 | HOT | STARWOOD HOTELS&RESORTS WRLD | 17-Nov-2000 | 22-Sep-2016 | 5643 | 7011 | Hotels & Motels |
| 541 | FRX | FOREST LABORATORIES -CL A | 22-Nov-2000 | 30-Jun-2014 | 5638 | 2834 | Pharmaceutical Preparations |
| 542 | CHIR | CHIRON CORP | 24-Nov-2000 | 19-Apr-2006 | 5636 | 2834 | Pharmaceutical Preparations |
| 543 | CPN | CALPINE CORP | 1-Dec-2000 | 1-Dec-2005 | 5629 | 4991 | Co-generation Services & Small Power Producers |
| 544 | RHI | ROBERT HALF INTL INC | 5-Dec-2000 | | 5625 | 7363 | Services-Help Supply Services |
| 545 | INTU | INTUIT INC | 11-Dec-2000 | | 5619 | 7372 | Services-Prepackaged Software |
| 546 | MET | METLIFE INC | 11-Dec-2000 | | 5619 | 6311 | Life Insurance |
| 547 | SBL.2 | SYMBOL TECHNOLOGIES | 11-Dec-2000 | 9-Jan-2007 | 5619 | 3577 | Computer Peripheral Equipment, NEC |
| 548 | AMBC | AMBAC FINANCIAL GROUP INC | 11-Dec-2000 | 10-Jun-2008 | 5619 | 6351 | Surety Insurance |
| 549 | AYE | ALLEGHENY ENERGY INC | 11-Dec-2000 | 27-Feb-2011 | 5619 | 4911 | Electric Services |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 550 | SYK | STRYKER CORP | 12-Dec-2000 | | 5618 | 3842 | Orthopedic, Prosthetic & Surgical Appliances & Supplies |
| 551 | VTSS | VITESSE SEMICONDUCTOR CORP | 12-Dec-2000 | 20-Aug-2002 | 5618 | 3674 | Semiconductors & Related Devices |
| 552 | KMI | KINDER MORGAN INC | 12-Dec-2000 | 30-May-2007 | 5618 | 4923 | Natural Gas Transmission & Distribution |
| 553 | QLGC | QLOGIC CORP | 12-Dec-2000 | 17-Jan-2011 | 5618 | 3570 | Computer & office Equipment |
| 554 | AMCC | APPLIED MICRO CIRCUITS CORP | 2-Jan-2001 | 17-May-2006 | 5597 | 3674 | Semiconductors & Related Devices |
| 555 | NE | NOBLE CORP PLC | 16-Jan-2001 | 26-Mar-2009 | 5583 | 1381 | Drilling Oil & Gas Wells |
| 556 | JBL | JABIL INC | 30-Jan-2001 | 4-Nov-2014 | 5569 | 3672 | Printed Circuit Boards |
| 557 | 5952B | UNIVISION COMMUNICATIONS INC | 7-Feb-2001 | 28-Mar-2007 | 5561 | 4833 | Television Broadcasting Stations |
| 558 | FTR | FRONTIER COMMUNICATIONS CORP | 27-Feb-2001 | 19-Mar-2017 | 5541 | 4813 | Telephone Communications (No Radiotelephone) |
| 559 | CTAS | CINTAS CORP | 1-Mar-2001 | | 5539 | 2320 | Men's & Boys' Furnishings, Work Clothing, & Allied Garments |
| 560 | FISV | FISERV INC | 2-Apr-2001 | | 5507 | 7374 | Services-Computer Processing & Data Preparation |
| 561 | CE.2 | CONCORD EFS INC | 2-Apr-2001 | 26-Feb-2004 | 5507 | 6099 | Functions Related To Depository Banking, NEC |
| 562 | GEN.3 | GENON ENERGY INC | 3-Apr-2001 | 16-Jul-2003 | 5506 | 4911 | Electric Services |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|----------------------------|------------------|------------------|------------|------------------|---|
| 563 | PBG | PEPSI BOTTLING GROUP INC | 14-May-2001 | 28-Feb-2010 | 5465 | 2086 | Bottled & Canned Soft Drinks & Carbonated Waters |
| 564 | MWW | MONSTER WORLDWIDE INC | 4-Jun-2001 | 18-Dec-2011 | 5444 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 565 | ZION | ZIONS BANCORPORATION | 25-Jun-2001 | | 5423 | 6020 | Life Insurance |
| 566 | JHF | HANCOCK JOHN FINL SVCS INC | 28-Jun-2001 | 28-Apr-2004 | 5420 | 6311 | Aircraft Parts & Auxiliary Equipment, NEC |
| 567 | COL | ROCKWELL COLLINS INC | 2-Jul-2001 | | 5416 | 3728 | Radiotelephone Communications |
| 568 | AWE.2 | AT&T WIRELESS SERVICES INC | 9-Jul-2001 | 26-Oct-2004 | 5409 | 4812 | Semiconductors & Related Devices |
| 569 | PMCS | PMC-SIERRA INC | 3-Aug-2001 | 1-Jul-2007 | 5384 | 3674 | Retail-Variety Stores |
| 570 | FDO | FAMILY DOLLAR STORES | 6-Aug-2001 | 8-Jul-2015 | 5381 | 5331 | Orthopedic, Prosthetic & Surgical Appliances & Supplies |
| 571 | ZBH | ZIMMER BIOMET HOLDINGS INC | 7-Aug-2001 | | 5380 | 3842 | Wholesale-Drugs, Proprietaries & Druggists' Sundries |
| 572 | ABC | AMERISOURCEBERGEN CORP | 30-Aug-2001 | | 5357 | 5122 | Telephone & Telegraph Apparatus |
| 573 | CIEN | CIENA CORP | 30-Aug-2001 | 20-Dec-2009 | 5357 | 3661 | Surety Insurance |
| 574 | XL | XL GROUP LTD | 4-Sep-2001 | | 5352 | 6351 | Services-Miscellaneous |
| 575 | IGT.1 | INTL GAME TECHNOLOGY | 4-Sep-2001 | 22-Jun-2014 | 5352 | 7990 | Amusement & Recreation |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 576 | IMNX | IMMUNEX CORP | 21-Sep-2001 | 16-Jul-2002 | 5335 | 2836 | Biological Products, (No Diagnostic Substances) |
| 577 | EOP | EQUITY OFFICE PROPERTIES TR | 10-Oct-2001 | 11-Feb-2007 | 5316 | 6798 | Real Estate Investment Trusts |
| 578 | TE | TECO ENERGY INC | 10-Oct-2001 | 30-Jun-2016 | 5316 | 4931 | Electric & Other Services Combined |
| 579 | HMA | HEALTH MANAGEMENT ASSOC | 7-Nov-2001 | 1-Mar-2007 | 5288 | 8062 | Services-General Medical & Surgical Hospitals, NEC |
| 580 | NVDA | NVIDIA CORP | 30-Nov-2001 | | 5265 | 3674 | Semiconductors & Related Devices |
| 581 | EQR | EQUITY RESIDENTIAL | 3-Dec-2001 | | 5262 | 6798 | Real Estate Investment Trusts |
| 582 | GENZ | GENZYME CORP | 14-Dec-2001 | 3-Apr-2011 | 5251 | 2836 | Biological Products, (No Diagnostic Substances) |
| 583 | JNY | JONES GROUP INC | 17-Dec-2001 | 3-Mar-2009 | 5248 | 2330 | Women's, Misses', and Juniors Outerwear |
| 584 | WAT | WATERS CORP | 2-Jan-2002 | | 5232 | 3826 | Laboratory Analytical Instruments |
| 585 | PCL | PLUM CREEK TIMBER CO INC | 17-Jan-2002 | 21-Feb-2016 | 5217 | 2400 | Lumber & Wood Products (No Furniture) |
| 586 | CB | CHUBB LTD | 30-Jan-2002 | 17-Jul-2008 | 5204 | 6331 | Fire, Marine & Casualty Insurance |
| 587 | RATL | RATIONAL SOFTWARE CORP | 1-Feb-2002 | 23-Feb-2003 | 5202 | 7370 | Services-Computer Programming, Data Processing, Etc. |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 588 | MI.1 | MARSHALL & ILSLEY CORP | 11-Feb-2002 | 5-Jul-2011 | 5192 | 6020 | |
| 589 | FHN | FIRST HORIZON NATIONAL CORP | 6-May-2002 | 23-Jun-2013 | 5108 | 6020 | |
| 590 | TT.2 | TRANE INC | 13-May-2002 | 5-Jun-2008 | 5101 | 3585 | Air-Cond & Warm Air Heatg Equip & Comm & Incl Refrig Equip |
| 591 | BJS.1 | BJ SERVICES CO | 15-May-2002 | 29-Apr-2010 | 5099 | 1389 | Oil & Gas Field Services, NEC |
| 592 | APOL | APOLLO EDUCATION GROUP INC | 15-May-2002 | 30-Jun-2013 | 5099 | 8200 | Services-Educational Services |
| 593 | SPG | SIMON PROPERTY GROUP INC | 26-Jun-2002 | | 5057 | 6798 | Real Estate Investment Trusts |
| 594 | NFB | NORTH FORK BANCORPORATION | 17-Jul-2002 | 30-Nov-2006 | 5036 | 6020 | |
| 595 | UPS | UNITED PARCEL SERVICE INC | 22-Jul-2002 | | 5031 | 4210 | Trucking & Courier Services (No Air) |
| 596 | EA | ELECTRONIC ARTS INC | 22-Jul-2002 | | 5031 | 7372 | Services-Prepackaged Software |
| 597 | EBAY | EBAY INC | 22-Jul-2002 | | 5031 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 598 | GS | GOLDMAN SACHS GROUP INC | 22-Jul-2002 | | 5031 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 599 | PRU | PRUDENTIAL FINANCIAL INC | 22-Jul-2002 | | 5031 | 6311 | Life Insurance |
| 600 | PFG | PRINCIPAL FINANCIAL GRP INC | 22-Jul-2002 | | 5031 | 6282 | Investment Advice |
| 601 | 0139A | SUNGARD DATA SYSTEMS INC | 22-Jul-2002 | 11-Aug-2005 | 5031 | 7372 | Services-Prepackaged Software |
| 602 | ANTM | ANTHEM INC | 25-Jul-2002 | | 5028 | 6324 | Hospital & Medical Service Plans |
| 603 | MON | MONSANTO CO | 14-Aug-2002 | 6-Jun-2018 | 5008 | 100 | Agricultural Production-Crops |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|----------------------------|------------------|------------------|------------|------------------|--|
| 604 | TRV | TRAVELERS COS INC | 21-Aug-2002 | | 5001 | 6331 | Fire, Marine & Casualty Insurance |
| 605 | RAI | REYNOLDS AMERICAN INC | 4-Sep-2002 | 25-Jul-2017 | 4987 | 2111 | Cigarettes |
| 606 | CMCSA | COMCAST CORP | 19-Nov-2002 | | 4911 | 4841 | Cable & Other Pay Television Services |
| 607 | DGX | QUEST DIAGNOSTICS INC | 12-Dec-2002 | | 4888 | 8071 | Services-Medical Laboratories |
| 608 | AN | AUTONATION INC | 24-Feb-2003 | 7-Aug-2017 | 4814 | 5500 | Retail-Auto Dealers & Gasoline Stations |
| 609 | AIV | APARTMENT INVST & MGMT CO | 14-Mar-2003 | | 4796 | 6798 | Real Estate Investment Trusts |
| 610 | MKC | MCCORMICK & CO INC | 21-Mar-2003 | | 4789 | 2090 | Miscellaneous Food Preparations & Kindred Products |
| 611 | SYMC | SYMANTEC CORP | 31-Mar-2003 | | 4779 | 7372 | Services-Prepackaged Software |
| 612 | FII | FEDERATED INVESTORS INC | 16-Apr-2003 | 1-Jan-2013 | 4763 | 6282 | Investment Advice |
| 613 | PLD | PROLOGIS INC | 17-Jul-2003 | | 4671 | 6798 | Real Estate Investment Trusts |
| 614 | MHS | MEDCO HEALTH SOLUTIONS INC | 20-Aug-2003 | 3-Apr-2012 | 4637 | 5912 | Retail-Drug Stores and Proprietary Stores |
| 615 | ESRX | EXPRESS SCRIPTS HOLDING CO | 26-Sep-2003 | | 4600 | 5912 | Retail-Drug Stores and Proprietary Stores |
| 616 | BIIB | BIOGEN INC | 13-Nov-2003 | | 4552 | 2836 | Biological Products, (No Diagnostic Substances) |
| 617 | MTB | M & T BANK CORP | 27-Feb-2004 | | 4446 | 6020 | |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 618 | CMX.1 | CAREMARK RX INC | 25-Mar-2004 | 22-Mar-2007 | 4419 | 5912 | Retail-Drug Stores and Proprietary Stores |
| 619 | ETFC | E TRADE FINANCIAL CORP | 1-Apr-2004 | | 4412 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 620 | ACS | AFFILIATED COMPUTER SERVICES | 2-Apr-2004 | 7-Feb-2010 | 4411 | 7374 | Services-Computer Processing & Data Preparation |
| 621 | MYL | MYLAN NV | 23-Apr-2004 | | 4390 | 2834 | Pharmaceutical Preparations |
| 622 | VLO | VALERO ENERGY CORP | 29-Apr-2004 | | 4384 | 2911 | Petroleum Refining |
| 623 | HSP | HOSPIRA INC | 3-May-2004 | 2-Sep-2015 | 4380 | 2834 | Pharmaceutical Preparations |
| 624 | GILD | GILEAD SCIENCES INC | 1-Jul-2004 | | 4321 | 2836 | Biological Products, (No Diagnostic Substances) |
| 625 | STD2 | SANTANDER HOLDINGS USA INC | 1-Jul-2004 | 29-Jan-2009 | 4321 | 6020 | |
| 626 | FSH.2 | FISHER SCIENTIFIC INTL INC | 3-Aug-2004 | 9-Nov-2006 | 4288 | 5040 | Wholesale-Professional & Commercial Equipment & Supplies |
| 627 | TPR | TAPESTRY INC | 1-Sep-2004 | | 4259 | 3100 | Leather & Leather Products |
| 628 | CIT | CIT GROUP INC | 27-Oct-2004 | 26-Jul-2009 | 4203 | 6172 | Finance Lessors |
| 629 | LH | LABORATORY CP OF AMER HLDGS | 1-Nov-2004 | | 4198 | 8071 | Services-Medical Laboratories |
| 630 | LLL | L3 TECHNOLOGIES INC | 1-Dec-2004 | | 4168 | 3812 | Search, Detection, Navigation, Guidance, Aeronautical Sys |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|---------------------------------|------------------|------------------|------------|------------------|--|
| 631 | BX1 | FREESCALE SEMICONDUCTOR INC | 3-Dec-2004 | 3-Dec-2006 | 4166 | 3674 | Semiconductors & Related Devices |
| 632 | FOXA | TWENTY-FIRST CENTURY FOX INC | 20-Dec-2004 | | 4149 | 4888 | |
| 633 | BBVA1 | BBVA COMPASS BANCSHARES INC | 20-Dec-2004 | 6-Sep-2007 | 4149 | 6020 | |
| 634 | ASN | ARCHSTONE INC-REDH | 20-Dec-2004 | 7-Oct-2007 | 4149 | 6798 | Real Estate Investment Trusts |
| 635 | XTO | XTO ENERGY INC | 29-Dec-2004 | 27-Jun-2010 | 4140 | 1311 | Crude Petroleum & Natural Gas |
| 636 | NOV | NATIONAL OILWELL VARCO INC | 14-Mar-2005 | | 4065 | 3533 | Oil & Gas Field Machinery & Equipment |
| 637 | SHLD | SEARS HOLDINGS CORP | 28-Mar-2005 | 4-Sep-2012 | 4051 | 5311 | Retail-Department Stores |
| 638 | STZ | CONSTELLATION BRANDS | 5-Jul-2005 | | 3952 | 2082 | Malt Beverages |
| 639 | DHI | D R HORTON INC | 5-Jul-2005 | | 3952 | 1531 | Operative Builders |
| 640 | WFT | WEATHERFORD INTL PLC | 22-Jul-2005 | 25-Feb-2009 | 3935 | 1381 | Drilling Oil & Gas Wells |
| 641 | TSN | TYSON FOODS INC -CL A | 11-Aug-2005 | | 3915 | 2011 | Meat Packing Plants |
| 642 | VNO | VORNADO REALTY TRUST | 12-Aug-2005 | | 3914 | 6798 | Real Estate Investment Trusts |
| 643 | MUR | MURPHY OIL CORP | 15-Aug-2005 | 25-Jul-2017 | 3911 | 1311 | Crude Petroleum & Natural Gas |
| 644 | PSA | PUBLIC STORAGE | 19-Aug-2005 | | 3907 | 6798 | Real Estate Investment Trusts |
| 645 | CVH | COVENTRY HEALTH CARE INC | 30-Aug-2005 | 8-May-2013 | 3896 | 6324 | Hospital & Medical Service Plans |
| 646 | AMP | AMERIPRISE FINANCIAL INC | 3-Oct-2005 | | 3862 | 6211 | Security Brokers, Dealers & Flotation Companies |
| 647 | LEN | LENNAR CORP | 4-Oct-2005 | | 3861 | 1531 | Operative Builders |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 648 | PDCO | PATTERSON COMPANIES INC | 11-Oct-2005 | 18-Mar-2018 | 3854 | 5047 | Wholesale-Medical, Dental & Hospital Equipment & Supplies |
| 649 | AMZN | AMAZON.COM INC | 21-Nov-2005 | | 3813 | 5961 | Retail-Catalog & Mail-Order Houses |
| 650 | GNW | GENWORTH FINANCIAL INC | 2-Dec-2005 | 17-Nov-2015 | 3802 | 6311 | Life Insurance |
| 651 | SSP | EW SCRIPPS -CL A | 20-Dec-2005 | 30-Jun-2008 | 3784 | 4833 | Television Broadcasting Stations |
| 652 | VIAB | VIACOM INC | 3-Jan-2006 | | 3770 | 4833 | Television Broadcasting Stations |
| 653 | WFM | WHOLE FOODS MARKET INC | 3-Jan-2006 | 28-Aug-2017 | 3770 | 5411 | Retail-Grocery Stores |
| 654 | EL | LAUDER (ESTEE) COS INC -CL A | 5-Jan-2006 | | 3768 | 2844 | Perfumes, Cosmetics & Other Toilet Preparations |
| 655 | VRSN | VERISIGN INC | 1-Feb-2006 | | 3741 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 656 | HAR | HARMAN INTERNATIONAL INDS | 1-Feb-2006 | 15-Mar-2017 | 3741 | 3651 | Household Audio & Video Equipment |
| 657 | BRL | BARR PHARMACEUTICALS INC | 27-Feb-2006 | 22-Dec-2008 | 3715 | 2834 | Pharmaceutical Preparations |
| 658 | CHK | CHESAPEAKE ENERGY CORP | 3-Mar-2006 | 18-Mar-2018 | 3711 | 1311 | Crude Petroleum & Natural Gas |
| 659 | KIM | KIMCO REALTY CORP | 3-Apr-2006 | | 3680 | 6798 | Real Estate Investment Trusts |
| 660 | BXP | BOSTON PROPERTIES INC | 3-Apr-2006 | | 3680 | 6798 | Real Estate Investment Trusts |
| 661 | GOOGL | ALPHABET INC | 3-Apr-2006 | | 3680 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 662 | DF | DEAN FOODS CO | 3-Apr-2006 | 23-May-2013 | 3680 | 2020 | Dairy Products |
| 663 | SNDK | SANDISK CORP | 20-Apr-2006 | 12-May-2016 | 3663 | 3572 | Computer Storage Devices |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|---|
| 664 | LM | LEGG MASON INC | 24-Apr-2006 | 1-Dec-2016 | 3659 | 6282 | Investment Advice |
| 665 | EQ | EMBARQ CORP | 18-May-2006 | 30-Jun-2009 | 3635 | 4813 | Telephone Communications (No Radiotelephone) |
| 666 | JNPR | JUNIPER NETWORKS INC | 2-Jun-2006 | | 3620 | 3576 | Computer Communications Equipment |
| 667 | CBH.1 | COMMERCE BANCORP INC/NJ | 6-Jun-2006 | 30-Mar-2008 | 3616 | 6020 | |
| 668 | CNX | CNX RESOURCES CORPORATION | 28-Jun-2006 | 3-Mar-2016 | 3594 | 1311 | Crude Petroleum & Natural Gas |
| 669 | WIN | WINDSTREAM HOLDINGS INC | 18-Jul-2006 | 6-Apr-2015 | 3574 | 4813 | Telephone Communications (No Radiotelephone) |
| 670 | RLGY | REALOGY HOLDINGS CORP | 1-Aug-2006 | 9-Apr-2007 | 3560 | 6531 | Real Estate Agents & Managers (For Others) |
| 671 | WYND | WYNDHAM DESTINATIONS INC | 1-Aug-2006 | 30-May-2018 | 3560 | 6531 | Real Estate Agents & Managers (For Others) |
| 672 | CME | CME GROUP INC | 11-Aug-2006 | | 3550 | 6200 | Security & Commodity Brokers, Dealers, Exchanges & Services |
| 673 | WU | WESTERN UNION CO | 2-Oct-2006 | | 3498 | 6099 | Functions Related To Depository Banking, NEC |
| 674 | SH | SMITH INTERNATIONAL INC | 2-Oct-2006 | 26-Aug-2010 | 3498 | 2890 | Miscellaneous Chemical Products |
| 675 | CELG | CELGENE CORP | 6-Nov-2006 | | 3463 | 2834 | Pharmaceutical Preparations |
| 676 | FIS | FIDELITY NATIONAL INFO SVCS | 10-Nov-2006 | | 3459 | 7374 | Services-Computer Processing & Data Preparation |
| 677 | CBRE | CBRE GROUP INC | 10-Nov-2006 | | 3459 | 6500 | Real Estate |

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|-----|--------|----------------------------|------------------|------------------|------------|------------------|---|
| 678 | CTSH | COGNIZANT TECH SOLUTIONS | 17-Nov-2006 | | 3452 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 679 | BTU | PEABODY ENERGY CORP | 20-Nov-2006 | 21-Sep-2014 | 3449 | 1220 | Bituminous Coal & Lignite Mining |
| 680 | IAC | IAC/INTERACTIVECORP | 1-Dec-2006 | 20-Aug-2008 | 3438 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 681 | STR | QUESTAR CORP | 1-Dec-2006 | 30-Jun-2010 | 3438 | 4923 | Natural Gas Transmission & Distribution |
| 682 | DTV.2 | DIRECTV | 4-Dec-2006 | 28-Jul-2015 | 3435 | 4841 | Cable & Other Pay Television Services |
| 683 | TEX | TEREX CORP | 20-Dec-2006 | 5-Nov-2008 | 3419 | 3530 | Construction, Mining & Materials Handling Machinery & Equip |
| 684 | SE.7 | SPECTRA ENERGY CORP | 3-Jan-2007 | 27-Feb-2017 | 3405 | 4923 | Natural Gas Transmission & Distribution |
| 685 | ESV | ENSCO PLC | 4-Jan-2007 | 22-Dec-2009 | 3404 | 1381 | Drilling Oil & Gas Wells |
| 686 | AVB | AVALONBAY COMMUNITIES INC | 10-Jan-2007 | | 3398 | 6798 | Real Estate Investment Trusts |
| 687 | RL | RALPH LAUREN CORP | 2-Feb-2007 | | 3375 | 2320 | Men's & Boys' Furnishings, Work Clothing, & Allied Garments |
| 688 | VAR | VARIAN MEDICAL SYSTEMS INC | 12-Feb-2007 | | 3365 | 3845 | Electromedical & Electrotherapeutic Apparatus |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 689 | HCBK | HUDSON CITY BANCORP INC | 15-Feb-2007 | 2-Nov-2015 | 3362 | 6035 | Savings Institution, Federally Chartered |
| 690 | WEC3 | INTEGRYS HOLDING INC | 22-Feb-2007 | 30-Jun-2015 | 3355 | 4931 | Electric & Other Services |
| 691 | CHRW | C H ROBINSON WORLDWIDE INC | 2-Mar-2007 | | 3347 | 4213 | Combined |
| 692 | HST | HOST HOTELS & RESORTS INC | 20-Mar-2007 | | 3329 | 6798 | Trucking (No Local) |
| 693 | DDR | DDR CORP | 23-Mar-2007 | 29-Mar-2009 | 3326 | 6798 | Real Estate Investment Trusts |
| 694 | ANF | ABERCROMBIE & FITCH -CL A | 29-Mar-2007 | 22-Dec-2013 | 3320 | 5651 | Retail-Family Clothing Stores |
| 695 | MDLZ | MONDELEZ INTERNATIONAL INC | 2-Apr-2007 | | 3316 | 2000 | Food and Kindred Products |
| 696 | AIZ | ASSURANT INC | 10-Apr-2007 | | 3308 | 6351 | Surety Insurance |
| 697 | SUNEQ | SUNEDISON INC | 31-May-2007 | 18-Dec-2011 | 3257 | 3620 | Electrical Industrial Apparatus |
| 698 | PCP | PRECISION CASTPARTS CORP | 1-Jun-2007 | 31-Jan-2016 | 3256 | 3728 | Aircraft Parts & Auxiliary Equipment, NEC |
| 699 | DFS | DISCOVER FINANCIAL SVCS | 2-Jul-2007 | | 3225 | 6141 | Personal Credit Institutions |
| 700 | GGP | GGP INC | 2-Jul-2007 | 12-Nov-2008 | 3225 | 6798 | Real Estate Investment Trusts |
| 701 | COV | COVIDIEN PLC | 2-Jul-2007 | 4-Jun-2009 | 3225 | 3845 | Electromedical & Electrotherapeutic Apparatus |
| 702 | TEL | TE CONNECTIVITY LTD | 2-Jul-2007 | 25-Jun-2009 | 3225 | 3678 | Electronic Connectors |
| 703 | ACAS | AMERICAN CAPITAL LTD | 9-Jul-2007 | 3-Mar-2009 | 3218 | 6797 | |
| 704 | AKAM | AKAMAI TECHNOLOGIES INC | 12-Jul-2007 | | 3215 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 705 | JEF | JEFFERIES FINANCIAL GRP INC | 27-Aug-2007 | | 3169 | 2011 | Meat Packing Plants |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 706 | MCHP | MICROCHIP TECHNOLOGY INC | 7-Sep-2007 | | 3158 | 3674 | Semiconductors & Related Devices |
| 707 | ICE | INTERCONTINENTAL EXCHANGE | 26-Sep-2007 | | 3139 | 6200 | Security & Commodity Brokers, Dealers, Exchanges & Services |
| 708 | ANDV | ANDEAVOR | 27-Sep-2007 | | 3138 | 2911 | Petroleum Refining |
| 709 | TDC | TERADATA CORP | 1-Oct-2007 | 18-Jun-2017 | 3134 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 710 | EXPE | EXPEDIA GROUP INC | 2-Oct-2007 | | 3133 | 4700 | Transportation Services |
| 711 | NBL | NOBLE ENERGY INC | 8-Oct-2007 | | 3127 | 1311 | Crude Petroleum & Natural Gas |
| 712 | EXPD | EXPEDITORS INTL WASH INC | 10-Oct-2007 | | 3125 | 4731 | Arrangement of Transportation of Freight & Cargo |
| 713 | NYX | NYSE EURONEXT | 25-Oct-2007 | 12-Nov-2013 | 3110 | 6200 | Security & Commodity Brokers, Dealers, Exchanges & Services |
| 714 | JEC | JACOBS ENGINEERING GROUP INC | 26-Oct-2007 | | 3109 | 1600 | Heavy Construction Other Than Bldg Const - Contractors |
| 715 | TIE | TITANIUM METALS CORP | 29-Oct-2007 | 23-Dec-2012 | 3106 | 3350 | Rolling Drawing & Extruding of Nonferrous Metals |
| 716 | POM | PEPCO HOLDINGS INC | 9-Nov-2007 | 29-Mar-2016 | 3095 | 4911 | Electric Services |
| 717 | MTW | MANITOWOC CO | 16-Nov-2007 | 31-Aug-2009 | 3088 | 3530 | Construction, Mining & Materials Handling Machinery & Equip |
| 718 | AMT | AMERICAN TOWER CORP | 19-Nov-2007 | | 3085 | 6798 | Real Estate Investment Trusts |

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|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 719 | GME | GAMESTOP CORP | 14-Dec-2007 | 24-Apr-2016 | 3060 | 5734 | Retail-Computer & Computer Software Stores |
| 720 | RRC | RANGE RESOURCES CORP | 21-Dec-2007 | 17-Jun-2018 | 3053 | 1311 | Crude Petroleum & Natural Gas |
| 721 | GHC | GRAHAM HOLDINGS CO | 31-Dec-2007 | 21-Sep-2014 | 3043 | 8200 | Services-Educational Services |
| 722 | TSS | TOTAL SYSTEM SERVICES INC | 2-Jan-2008 | | 3041 | 7389 | Services-Business Services, NEC |
| 723 | CAM | CAMERON INTERNATIONAL CORP | 29-Jan-2008 | 3-Apr-2016 | 3014 | 3533 | Oil & Gas Field Machinery & Equipment |
| 724 | HCP | HCP INC | 31-Mar-2008 | | 2952 | 6798 | Real Estate Investment Trusts |
| 725 | PM | PHILIP MORRIS INTERNATIONAL | 31-Mar-2008 | | 2952 | 2111 | Cigarettes |
| 726 | ISRG | INTUITIVE SURGICAL INC | 2-Jun-2008 | | 2889 | 3845 | Electromedical & Electrotherapeutic Apparatus |
| 727 | SWN | SOUTHWESTERN ENERGY CO | 6-Jun-2008 | 3-Apr-2017 | 2885 | 4923 | Natural Gas Transmission & Distribution |
| 728 | LO | LORILLARD INC | 11-Jun-2008 | 11-Jun-2015 | 2880 | 2111 | Cigarettes |
| 729 | COG | CABOT OIL & GAS CORP | 23-Jun-2008 | | 2868 | 1311 | Crude Petroleum & Natural Gas |
| 730 | MEE | MASSEY ENERGY CO | 23-Jun-2008 | 1-Jun-2011 | 2868 | 1220 | Bituminous Coal & Lignite Mining |
| 731 | AKS | AK STEEL HOLDING CORP | 1-Jul-2008 | 18-Dec-2011 | 2860 | 3312 | Steel Works, Blast Furnaces & Rolling Mills (Coke Ovens) |
| 732 | SNI | SCRIPPS NETWORKS INTERACTIVE | 1-Jul-2008 | 6-Mar-2018 | 2860 | 4833 | Television Broadcasting Stations |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 733 | MA | MASTERCARD INC | 18-Jul-2008 | | 2843 | 6099 | Functions Related To Depository Banking, NEC |
| 734 | DVA | DAVITA INC | 31-Jul-2008 | | 2830 | 8090 | Services-Misc Health & Allied Services, NEC |
| 735 | IVZ | INVESCO LTD | 21-Aug-2008 | | 2809 | 6282 | Investment Advice |
| 736 | CF | CF INDUSTRIES HOLDINGS INC | 27-Aug-2008 | | 2803 | 2870 | Agricultural Chemicals |
| 737 | FAST | FASTENAL CO | 15-Sep-2008 | | 2784 | 5000 | Wholesale-Durable Goods |
| 738 | CRM | SALESFORCE.COM INC | 15-Sep-2008 | | 2784 | 7372 | Services-Prepackaged Software |
| 739 | HRS | HARRIS CORP | 22-Sep-2008 | | 2777 | 3812 | Search, Detection, Navigation, Guidance, Aeronautical Sys |
| 740 | PXD | PIONEER NATURAL RESOURCES CO | 24-Sep-2008 | | 2775 | 1311 | Crude Petroleum & Natural Gas |
| 741 | APH | AMPHENOL CORP | 30-Sep-2008 | | 2769 | 3678 | Electronic Connectors |
| 742 | FLS | FLOWSERVE CORP | 2-Oct-2008 | | 2767 | 3561 | Pumps & Pumping Equipment |
| 743 | DPS | DR PEPPER SNAPPLE GROUP INC | 7-Oct-2008 | | 2762 | 2086 | Bottled & Canned Soft Drinks & Carbonated Waters |
| 744 | NDAQ | NASDAQ INC | 22-Oct-2008 | | 2747 | 6200 | Security & Commodity Brokers, Dealers, Exchanges & Services |
| 745 | WEC | WEC ENERGY GROUP INC | 31-Oct-2008 | | 2738 | 4931 | Electric & Other Services |
| 746 | SJM | SMUCKER (JM) CO | 6-Nov-2008 | | 2732 | 2033 | Combined Canned, Fruits, Veg, Preserves, Jams & Jellies |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-------------------------------|------------------|------------------|------------|------------------|---|
| 747 | PBCT | PEOPLE'S UNITED FINL INC | 13-Nov-2008 | | 2725 | 6020 | |
| 748 | XRAY | DENTSPLY SIRONA INC | 14-Nov-2008 | | 2724 | 3843 | Dental Equipment & Supplies |
| 749 | WYNN | WYNN RESORTS LTD | 14-Nov-2008 | | 2724 | 7990 | Services-Miscellaneous |
| 750 | CEPH | CEPHALON INC | 17-Nov-2008 | 16-Oct-2011 | 2721 | 2834 | Amusement & Recreation |
| 751 | SRCL | STERICYCLE INC | 19-Nov-2008 | | 2719 | 4955 | Pharmaceutical Preparations |
| 752 | LIFE.3 | LIFE TECHNOLOGIES CORP | 24-Nov-2008 | 23-Jan-2014 | 2714 | 2836 | Hazardous Waste Management |
| 753 | DNB | DUN & BRADSTREET CORP | 2-Dec-2008 | 4-Apr-2017 | 2706 | 7323 | Biological Products, (No Diagnostic Substances) |
| 754 | RSR | REPUBLIC SERVICES INC | 5-Dec-2008 | | 2703 | 4953 | Refuse Systems |
| 755 | EQT | EQT CORP | 19-Dec-2008 | | 2689 | 4923 | Natural Gas Transmission & Distribution |
| 756 | MFE | MCAFFEE INC | 23-Dec-2008 | 28-Feb-2011 | 2685 | 7372 | Services-Prepackaged Software |
| 757 | SCG | SCANA CORP | 2-Jan-2009 | | 2675 | 4931 | Electric & Other Services |
| 758 | FLIR | FLIR SYSTEMS INC | 2-Jan-2009 | | 2675 | 3812 | Combined |
| 759 | OI | OWENS-ILLINOIS INC | 2-Jan-2009 | 1-Dec-2016 | 2675 | 3221 | Search, Detection, Navigation, Guidance, Aeronautical Sys |
| 760 | IRM | IRON MOUNTAIN INC | 6-Jan-2009 | | 2671 | 6798 | Glass Containers |
| 761 | WELL | WELLTOWER INC | 30-Jan-2009 | | 2647 | 6798 | Real Estate Investment Trusts |
| 762 | DO | DIAMOND OFFSHORE DRILLING INC | 26-Feb-2009 | 2-Oct-2016 | 2620 | 1381 | Real Estate Investment Trusts |
| | | | | | | | Drilling Oil & Gas Wells |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 763 | HRL | HORMEL FOODS CORP | 4-Mar-2009 | | 2614 | 2011 | Meat Packing Plants |
| 764 | VTR | VENTAS INC | 4-Mar-2009 | | 2614 | 6798 | Real Estate Investment Trusts |
| 765 | ES | EVERSOURCE ENERGY | 17-Mar-2009 | | 2601 | 4931 | Electric & Other Services |
| 766 | ORLY | O'REILLY AUTOMOTIVE INC | 27-Mar-2009 | | 2591 | 5531 | Combined Retail-Auto & Home Supply |
| 767 | TWC | TIME WARNER CABLE INC | 30-Mar-2009 | 17-May-2016 | 2588 | 4841 | Stores Cable & Other Pay Television |
| 768 | DNR | DENBURY RESOURCES INC | 2-Apr-2009 | 22-Mar-2015 | 2585 | 1311 | Crude Petroleum & Natural Gas |
| 769 | FTI.1 | FMC TECHNOLOGIES INC | 5-Jun-2009 | 16-Jan-2017 | 2521 | 3533 | Oil & Gas Field Machinery & Equipment |
| 770 | ATGE | ADTALEM GLOBAL EDUCATION INC | 9-Jun-2009 | 30-Sep-2012 | 2517 | 8200 | Services-Educational Services |
| 771 | PCS | METROPCS COMMUNICATIONS INC | 30-Jun-2009 | 30-Apr-2013 | 2496 | 4812 | Radiotelephone Communications |
| 772 | WDC | WESTERN DIGITAL CORP | 1-Jul-2009 | | 2495 | 3572 | Computer Storage Devices |
| 773 | PWR | QUANTA SERVICES INC | 1-Jul-2009 | | 2495 | 1731 | Electrical Work |
| 774 | RHT | RED HAT INC | 27-Jul-2009 | | 2469 | 7372 | Services-Prepackaged Software |
| 775 | FMC | FMC CORP | 19-Aug-2009 | | 2446 | 2870 | Agricultural Chemicals |
| 776 | CFN | CAREFUSION CORP | 1-Sep-2009 | 17-Mar-2015 | 2433 | 3841 | Surgical & Medical Instruments & Apparatus |

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|-----|--------|---------------------------|------------------|------------------|------------|------------------|--|
| 777 | ARG | AIRGAS INC | 9-Sep-2009 | 22-May-2016 | 2425 | 5084 | Wholesale-Industrial Machinery & Equipment |
| 778 | FSLR | FIRST SOLAR INC | 16-Oct-2009 | 19-Mar-2017 | 2388 | 3674 | Semiconductors & Related Devices |
| 779 | BKNG | BOOKING HOLDINGS INC | 6-Nov-2009 | | 2367 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 780 | ROST | ROSS STORES INC | 21-Dec-2009 | | 2322 | 5651 | Retail-Family Clothing Stores |
| 781 | V | VISA INC | 21-Dec-2009 | | 2322 | 6099 | Functions Related To Depository Banking, NEC |
| 782 | LDOS | LEIDOS HOLDINGS INC | 21-Dec-2009 | 22-Sep-2013 | 2322 | 7373 | Services-Computer Integrated Systems Design |
| 783 | CLF | CLEVELAND-CLIFFS INC | 21-Dec-2009 | 1-Apr-2014 | 2322 | 1000 | Metal Mining |
| 784 | MJN | MEAD JOHNSON NUTRITION CO | 21-Dec-2009 | 18-Jun-2017 | 2322 | 2020 | Dairy Products |
| 785 | ROP | ROPER TECHNOLOGIES INC | 23-Dec-2009 | | 2320 | 3823 | Industrial Instruments For Measurement, Display, and Control |
| 786 | NRG | NRG ENERGY INC | 29-Jan-2010 | | 2283 | 4911 | Electric Services |
| 787 | URBN | URBAN OUTFITTERS INC | 8-Feb-2010 | 19-Mar-2017 | 2273 | 5651 | Retail-Family Clothing Stores |
| 788 | BRK.B | BERKSHIRE HATHAWAY | 16-Feb-2010 | | 2265 | 9997 | |
| 789 | HP | HELMERICH & PAYNE | 1-Mar-2010 | | 2252 | 1381 | Drilling Oil & Gas Wells |
| 790 | DISCA | DISCOVERY INC | 1-Mar-2010 | | 2252 | 4841 | Cable & Other Pay Television Services |

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|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 791 | OKE | ONEOK INC | 15-Mar-2010 | | 2238 | 4923 | Natural Gas Transmission & Distribution |
| 792 | CERN | CERNER CORP | 30-Apr-2010 | | 2192 | 7373 | Services-Computer Integrated Systems Design |
| 793 | KMX | CARMAX INC | 28-Jun-2010 | | 2133 | 5500 | Retail-Auto Dealers & Gasoline Stations |
| 794 | QEP | QEP RESOURCES INC | 1-Jul-2010 | 30-Jun-2015 | 2130 | 1311 | Crude Petroleum & Natural Gas |
| 795 | CB | CHUBB LTD | 15-Jul-2010 | | 2116 | 6331 | Fire, Marine & Casualty Insurance |
| 796 | TYC | TYCO INTERNATIONAL PLC | 27-Aug-2010 | 5-Sep-2016 | 2073 | 3669 | Communications Equipment, NEC |
| 797 | IR | INGERSOLL-RAND PLC | 17-Nov-2010 | | 1991 | 3585 | Air-Cond & Warm Air Heatg Equip & Comm & Incl Refrig Equip |
| 798 | NFX | NEWFIELD EXPLORATION CO | 20-Dec-2010 | | 1958 | 1311 | Crude Petroleum & Natural Gas |
| 799 | FFIV | F5 NETWORKS INC | 20-Dec-2010 | | 1958 | 7373 | Services-Computer Integrated Systems Design |
| 800 | NFLX | NETFLIX INC | 20-Dec-2010 | | 1958 | 7841 | Services-Video Tape Rental |
| 801 | ATUS | ALTICE USA INC | 20-Dec-2010 | 23-Jun-2016 | 1958 | 4841 | Cable & Other Pay Television Services |
| 802 | MMI.3 | MOTOROLA MOBILITY HLDGS INC | 4-Jan-2011 | 24-May-2012 | 1943 | 3663 | Radio & TV Broadcasting & Communications Equipment |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|--------------------------------|------------------|------------------|------------|------------------|---|
| 803 | NE | NOBLE CORP PLC | 18-Jan-2011 | 19-Jul-2015 | 1929 | 1381 | Drilling Oil & Gas Wells |
| 804 | JOY | JOY GLOBAL INC | 28-Feb-2011 | 7-Oct-2015 | 1888 | 3532 | Mining Machinery & Equip (No Oil & Gas Field Mach & Equip) |
| 805 | COV | COVIDIEN PLC | 1-Mar-2011 | 26-Jan-2015 | 1887 | 3845 | Electromedical & Electrotherapeutic Apparatus |
| 806 | EW | EDWARDS LIFESCIENCES CORP | 1-Apr-2011 | | 1856 | 3842 | Orthopedic, Prosthetic & Surgical Appliances & Supplies |
| 807 | BLK | BLACKROCK INC | 4-Apr-2011 | | 1853 | 6282 | Investment Advice |
| 808 | CMG | CHIPOTLE MEXICAN GRILL INC | 28-Apr-2011 | | 1829 | 5812 | Retail-Eating Places |
| 809 | ANRZQ | ALPHA NATURAL RESOURCES INC | 2-Jun-2011 | 1-Oct-2012 | 1794 | 1220 | Bituminous Coal & Lignite Mining |
| 810 | MPC | MARATHON PETROLEUM CORP | 1-Jul-2011 | | 1765 | 2911 | Petroleum Refining |
| 811 | ACN | ACCENTURE PLC | 6-Jul-2011 | | 1760 | 8742 | Services-Management Consulting Services |
| 812 | MOS | MOSAIC CO | 26-Sep-2011 | | 1678 | 2870 | Agricultural Chemicals |
| 813 | TEL | TE CONNECTIVITY LTD | 17-Oct-2011 | | 1657 | 3678 | Electronic Connectors |
| 814 | XYL | XYLEM INC | 1-Nov-2011 | | 1642 | 3561 | Pumps & Pumping Equipment |
| 815 | CBE | COOPER INDUSTRIES PLC | 23-Nov-2011 | 2-Dec-2012 | 1620 | 3640 | Electric Lighting & Wiring Equipment |
| 816 | SO7 | SOUTHERN CO GAS | 13-Dec-2011 | 30-Jun-2016 | 1600 | 4924 | Natural Gas Distribution |
| 817 | PRGO | PERRIGO CO PLC | 19-Dec-2011 | | 1594 | 2834 | Pharmaceutical Preparations |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 818 | BWA | BORGWARNER INC | 19-Dec-2011 | | 1594 | 3714 | Motor Vehicle Parts & Accessories |
| 819 | DLTR | DOLLAR TREE INC | 19-Dec-2011 | | 1594 | 5331 | Retail-Variety Stores |
| 820 | TRIP | TRIPADVISOR INC | 21-Dec-2011 | | 1592 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 821 | WPX | WPX ENERGY INC | 3-Jan-2012 | 23-Mar-2014 | 1579 | 1311 | Crude Petroleum & Natural Gas |
| 822 | CCI | CROWN CASTLE INTL CORP | 14-Mar-2012 | | 1508 | 6798 | Real Estate Investment Trusts |
| 823 | FOSL | FOSSIL GROUP INC | 4-Apr-2012 | 4-Jan-2016 | 1487 | 3873 | Watches, Clocks, Clockwork Operated Devices/Parts |
| 824 | PSX | PHILLIPS 66 | 1-May-2012 | | 1460 | 2911 | Petroleum Refining |
| 825 | KMI | KINDER MORGAN INC | 25-May-2012 | | 1436 | 4923 | Natural Gas Transmission & Distribution |
| 826 | ALXN | ALEXION PHARMACEUTICALS INC | 25-May-2012 | | 1436 | 2836 | Biological Products, (No Diagnostic Substances) |
| 827 | LRCX | LAM RESEARCH CORP | 5-Jun-2012 | | 1425 | 3559 | Special Industry Machinery, NEC |
| 828 | MNST | MONSTER BEVERAGE CORP | 29-Jun-2012 | | 1401 | 2086 | Bottled & Canned Soft Drinks & Carbonated Waters |
| 829 | STX | SEAGATE TECHNOLOGY PLC | 2-Jul-2012 | | 1398 | 3572 | Computer Storage Devices |
| 830 | ESV | ENSCO PLC | 31-Jul-2012 | 29-Mar-2016 | 1369 | 1381 | Drilling Oil & Gas Wells |
| 831 | LYB | LYONDELLBASELL INDUSTRIES NV | 5-Sep-2012 | | 1333 | 2820 | Plastic Material, Synth Resin/Rubber, Cellulos (No Glass) |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|---|
| 832 | PNR | PENTAIR PLC | 1-Oct-2012 | | 1307 | 3561 | Pumps & Pumping Equipment |
| 833 | ADT.4 | ADT CORP | 1-Oct-2012 | 2-May-2016 | 1307 | 7380 | Services-Miscellaneous Business Services |
| 834 | KRFT | KRAFT FOODS GROUP INC | 2-Oct-2012 | 2-Jul-2015 | 1306 | 2000 | Food and Kindred Products |
| 835 | PETM | PETSMART INC | 5-Oct-2012 | 11-Mar-2015 | 1303 | 5990 | Retail-Retail Stores, NEC |
| 836 | DG | DOLLAR GENERAL CORP | 3-Dec-2012 | | 1244 | 5331 | Retail-Variety Stores |
| 837 | GRMN | GARMIN LTD | 12-Dec-2012 | | 1235 | 3812 | Search, Detection, Navigation, Guidance, Aeronautical Sys |
| 838 | APTIV | APTIV PLC | 24-Dec-2012 | | 1223 | 3714 | Motor Vehicle Parts & Accessories |
| 839 | ABBV | ABBVIE INC | 2-Jan-2013 | | 1214 | 2836 | Biological Products, (No Diagnostic Substances) |
| 840 | PVH | PVH CORP | 14-Feb-2013 | | 1171 | 2300 | Apparel & Other Finished Prods of Fabrics & Similar Matl |
| 841 | REGN | REGENERON PHARMACEUTICALS | 1-May-2013 | | 1095 | 2834 | Pharmaceutical Preparations |
| 842 | MAC | MACERICH CO | 9-May-2013 | | 1087 | 6798 | Real Estate Investment Trusts |
| 843 | KSU | KANSAS CITY SOUTHERN | 24-May-2013 | | 1072 | 4011 | Railroads, Line-Haul Operating |
| 844 | GM | GENERAL MOTORS CO | 7-Jun-2013 | | 1058 | 3711 | Motor Vehicles & Passenger Car Bodies |
| 845 | ZTS | ZOETIS INC | 24-Jun-2013 | | 1041 | 2834 | Pharmaceutical Preparations |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|----------------------------|------------------|------------------|------------|------------------|--|
| 846 | NWSA | NEWS CORP | 1-Jul-2013 | | 1034 | 2711 | Newspapers: Publishing or Publishing & Printing |
| 847 | NLSN | NIELSEN HOLDINGS PLC | 9-Jul-2013 | | 1026 | 8700 | Services-Engineering, Accounting, Research, Management |
| 848 | DAL | DELTA AIR LINES INC | 11-Sep-2013 | | 962 | 4512 | Air Transportation, Scheduled |
| 849 | AME | AMETEK INC | 23-Sep-2013 | | 950 | 3823 | Industrial Instruments For Measurement, Display, and Control |
| 850 | VRTX | VERTEX PHARMACEUTICALS INC | 23-Sep-2013 | | 950 | 2834 | Pharmaceutical Preparations |
| 851 | RIG | TRANSOCEAN LTD | 29-Oct-2013 | 25-Jul-2017 | 914 | 1381 | Drilling Oil & Gas Wells |
| 852 | KORS | MICHAEL KORS HOLDINGS LTD | 13-Nov-2013 | | 899 | 2300 | Apparel & Other Finished Prods of Fabrics & Similar Matl |
| 853 | ALLE | ALLEGION PLC | 2-Dec-2013 | | 880 | 3420 | Cutlery, Handtools & General Hardware |
| 854 | GGP | GGP INC | 10-Dec-2013 | | 872 | 6798 | Real Estate Investment Trusts |
| 855 | MHK | MOHAWK INDUSTRIES INC | 23-Dec-2013 | | 859 | 2273 | Carpets & Rugs |
| 856 | ADS | ALLIANCE DATA SYSTEMS CORP | 23-Dec-2013 | | 859 | 7374 | Services-Computer Processing & Data Preparation |
| 857 | FB | FACEBOOK INC | 23-Dec-2013 | | 859 | 7370 | Services-Computer Programming, Data Processing, Etc. |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|--|
| 858 | TSCO | TRACTOR SUPPLY CO | 24-Jan-2014 | | 827 | 5200 | Retail-Building Materials, Hardware, Garden Supply |
| 859 | GMCR | KEURIG GREEN MOUNTAIN INC | 24-Mar-2014 | 6-Mar-2016 | 768 | 2090 | Miscellaneous Food Preparations & Kindred Products |
| 860 | ESS | ESSEX PROPERTY TRUST | 2-Apr-2014 | | 759 | 6798 | Real Estate Investment Trusts |
| 861 | GOOGL | ALPHABET INC | 3-Apr-2014 | | 758 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 862 | UAA | UNDER ARMOUR INC | 1-May-2014 | | 730 | 2300 | Apparel & Other Finished Prods of Fabrics & Similar Matl |
| 863 | AVGO | BROADCOM INC | 8-May-2014 | | 723 | 3674 | Semiconductors & Related Devices |
| 864 | XEC | CIMAREX ENERGY CO | 23-Jun-2014 | | 677 | 1311 | Crude Petroleum & Natural Gas |
| 865 | AMG | AFFILIATED MANAGERS GRP INC | 1-Jul-2014 | | 669 | 6282 | Investment Advice |
| 866 | MLM | MARTIN MARIETTA MATERIALS | 2-Jul-2014 | | 668 | 1400 | Mining & Quarrying of Nonmetallic Minerals (No Fuels) |
| 867 | DISCA | DISCOVERY INC | 7-Aug-2014 | | 632 | 4841 | Cable & Other Pay Television Services |
| 868 | MNK | MALLINCKRODT PLC | 19-Aug-2014 | 25-Jul-2017 | 620 | 2834 | Pharmaceutical Preparations |
| 869 | UHS | UNIVERSAL HEALTH SVCS INC | 22-Sep-2014 | | 586 | 8062 | Services-General Medical & Surgical Hospitals, NEC |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|-----------------------------|------------------|------------------|------------|------------------|---|
| 870 | URI | UNITED RENTALS INC | 22-Sep-2014 | | 586 | 7350 | Services-Miscellaneous Equipment Rental & Leasing |
| 871 | LVL | LEVEL 3 COMMUNICATIONS INC | 5-Nov-2014 | 12-Oct-2017 | 542 | 4813 | Telephone Communications (No Radiotelephone) |
| 872 | RCL | ROYAL CARIBBEAN CRUISES LTD | 5-Dec-2014 | | 512 | 4400 | Water Transportation |
| 873 | HCA | HCA HEALTHCARE INC | 27-Jan-2015 | | 459 | 8062 | Services-General Medical & Surgical Hospitals, NEC |
| 874 | ENDP | ENDO INTERNATIONAL PLC | 27-Jan-2015 | 1-Mar-2017 | 459 | 2834 | Pharmaceutical Preparations |
| 875 | SWKS | SKYWORKS SOLUTIONS INC | 12-Mar-2015 | | 415 | 3674 | Semiconductors & Related Devices |
| 876 | HSIC | SCHEIN (HENRY) INC | 18-Mar-2015 | | 409 | 5047 | Wholesale-Medical, Dental & Hospital Equipment & Supplies |
| 877 | AAL | AMERICAN AIRLINES GROUP INC | 23-Mar-2015 | | 404 | 4512 | Air Transportation, Scheduled |
| 878 | SLG | SL GREEN REALTY CORP | 23-Mar-2015 | | 404 | 6798 | Real Estate Investment Trusts |
| 879 | EQIX | EQUINIX INC | 23-Mar-2015 | | 404 | 6798 | Real Estate Investment Trusts |
| 880 | HBI | HANESBRANDS INC | 23-Mar-2015 | | 404 | 2300 | Apparel & Other Finished Prods of Fabrics & Similar Matl |
| 881 | O | REALTY INCOME CORP | 7-Apr-2015 | | 389 | 6798 | Real Estate Investment Trusts |
| 882 | QRVO | QRVO INC | 12-Jun-2015 | | 323 | 3674 | Semiconductors & Related Devices |
| 883 | JBHT | HUNT (JB) TRANSPRT SVCS INC | 1-Jul-2015 | | 304 | 4213 | Trucking (No Local) |

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| Idx | Ticker | Company Name | Date First Added | Date Last Appear | Total Days | Company SIC Code | Industry |
|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 884 | BXLT | BAXALTA INC | 1-Jul-2015 | 2-Jun-2016 | 304 | 2836 | Biological Products, (No Diagnostic Substances) |
| 885 | WRK | WESTROCK CO | 2-Jul-2015 | | 303 | 2650 | Paperboard Containers & Boxes |
| 886 | CPGX | COLUMBIA PIPELINE GROUP INC | 2-Jul-2015 | 5-Jul-2016 | 303 | 4922 | Natural Gas Transmission |
| 887 | KHC | KRAFT HEINZ CO | 3-Jul-2015 | | 302 | 2030 | Canned, Frozen & Preserved Fruit, Veg & Food Specialties |
| 888 | AAP | ADVANCE AUTO PARTS INC | 9-Jul-2015 | | 296 | 5531 | Retail-Auto & Home Supply Stores |
| 889 | PYPL | PAYPAL HOLDINGS INC | 20-Jul-2015 | | 285 | 7374 | Services-Computer Processing & Data Preparation |
| 890 | SIG | SIGNET JEWELERS LTD | 29-Jul-2015 | 18-Mar-2018 | 276 | 5944 | Retail-Jewelry Stores |
| 891 | ATVI | ACTIVISION BLIZZARD INC | 31-Aug-2015 | | 243 | 7372 | Services-Prepackaged Software |
| 892 | UAL | UNITED CONTINENTAL HLDGS INC | 3-Sep-2015 | | 240 | 4512 | Air Transportation, Scheduled |
| 893 | FOXA | TWENTY-FIRST CENTURY FOX INC | 21-Sep-2015 | | 222 | 4888 | Newspapers: Publishing or Publishing & Printing |
| 894 | NWSA | NEWS CORP | 21-Sep-2015 | | 222 | 2711 | Cable & Other Pay Television Services |
| 895 | CMCSA | COMCAST CORP | 21-Sep-2015 | 13-Dec-2015 | 222 | 4841 | Insurance Agents, Brokers & Service |
| 896 | VRSK | VERISK ANALYTICS INC | 8-Oct-2015 | | 205 | 6411 | |

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|-----|--------|------------------------------|------------------|------------------|------------|------------------|--|
| 897 | HPE | HEWLETT PACKARD ENTERPRISE | 2-Nov-2015 | | 180 | 3571 | Electronic Computers |
| 898 | FCPT | FOUR CORNERS PROPERTY TR INC | 10-Nov-2015 | 10-Nov-2015 | 172 | 6798 | Real Estate Investment Trusts |
| 899 | SYF | SYNCHRONY FINANCIAL | 18-Nov-2015 | | 164 | 6141 | Personal Credit Institutions |
| 900 | ILMN | ILLUMINA INC | 19-Nov-2015 | | 163 | 3826 | Laboratory Analytical Instruments |
| 901 | CSRA | CSRA INC | 30-Nov-2015 | 3-Apr-2018 | 152 | 7370 | Services-Computer Programming, Data Processing, Etc. |
| 902 | CHD | CHURCH & DWIGHT INC | 29-Dec-2015 | | 123 | 2840 | Soap, Detergents, Cleaning Preparations, Perfumes, Cosmetics |
| 903 | WLTW | WILLIS TOWERS WATSON PLC | 5-Jan-2016 | | 116 | 6411 | Insurance Agents, Brokers & Service |
| 904 | EXR | EXTRA SPACE STORAGE INC | 19-Jan-2016 | | 102 | 6798 | Real Estate Investment Trusts |
| 905 | FRT | FEDERAL REALTY INVESTMENT TR | 1-Feb-2016 | | 89 | 6798 | Real Estate Investment Trusts |
| 906 | CFG | CITIZENS FINANCIAL GROUP INC | 1-Feb-2016 | | 89 | 6020 | Crude Petroleum & Natural Gas |
| 907 | CXO | CONCHO RESOURCES INC | 22-Feb-2016 | | 68 | 1311 | Water Supply |
| 908 | AWK | AMERICAN WATER WORKS CO INC | 4-Mar-2016 | | 57 | 4941 | |
| 909 | UDR | UDR INC | 7-Mar-2016 | | 54 | 6798 | Real Estate Investment Trusts |

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|-----|--------|---------------------|------------------|------------------|------------|------------------|--|
| 910 | HOLX | HOLOGIC INC | 30-Mar-2016 | | 31 | 3844 | X-Ray Apparatus & Tubes & Related Irradiation Apparatus |
| 911 | CNC | CENTENE CORP | 30-Mar-2016 | | 31 | 6324 | Hospital & Medical Service Plans |
| 912 | FL | FOOT LOCKER INC | 4-Apr-2016 | | 26 | 5661 | Retail-Shoe Stores |
| 913 | UAA | UNDER ARMOUR INC | 8-Apr-2016 | | 22 | 2300 | Apparel & Other Finished Prods of Fabrics & Similar Matl |
| 914 | ULTA | ULTA BEAUTY INC | 18-Apr-2016 | | 12 | 5990 | Retail-Retail Stores, NEC |
| 915 | GPN | GLOBAL PAYMENTS INC | 25-Apr-2016 | | 5 | 7374 | Services-Computer Processing & Data Preparation |