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Review Article

On Ulam's Type Stability of the Linear Equation and Related Issues

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This is a survey paper concerning stability results for the linear functional equation in single variable. We discuss issues that have not been considered or have been treated only briefly in other surveys concerning stability of the equation. In this way, we complement those surveys.

1. Introduction

It is a commonly accepted conviction that the issue of stability of functional equations has been motivated by a problem raised by Ulam (cf. [1]) in 1940 in his talk at the University of Wisconsin. The problem can be stated as follows.

Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta, \quad x, y \in G_1, \tag{1}$$

then a homomorphism $T : G_1 \to G_2$ exists with

$$d(f(x), T(x)) < \varepsilon, \quad x, y \in G_1? \tag{2}$$

The first (partial) answer to it was published in 1941 by Hyers [2]. It reads as follows.

Let E and Y be Banach spaces and $\varepsilon > 0$. Then, for every $g : E \to Y$ with

$$\sup_{x,y\in E} \|g(x+y) - g(x) - g(y)\| \le \varepsilon, \tag{3}$$

there is a unique solution $f : E \to Y$ of the Cauchy equation

$$f(x+y) = f(x) + f(y) \tag{4}$$

such that

$$\sup_{x\in E} \|g(x) - f(x)\| \le \varepsilon. \tag{5}$$

Nowadays, we describe that result of Hyers simply saying that *Cauchy functional equation* (4) *is Hyers-Ulam stable (or has the Hyers-Ulam stability)*. Next, Hyers and Ulam published some further stability results for polynomial functions, isometries, and convex functions in [3–6].

For the last 50 years, that issue has been a very popular subject of investigations and we refer the reader to monographs and surveys [7–17] for further information, references, some discussions, and examples of recent results. Below, we present only one such example, which is an extension of the result of Hyers [2] and is composed of the outcomes from [18–21] (cf. [22, 23]; see also [24]).

Before we do this, let us yet recall that a function is called *additive* provided it is a solution of (4).

Theorem 1. *Let E_1 and E_2 be normed spaces, $c \ge 0$, $p \ne 1$ fixed real numbers. Assume also that $f : E_1 \to E_2$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \le c(\|x\|^p + \|y\|^p), \quad x, y \in E_1 \setminus \{0\}. \tag{6}$$

If $p \ge 0$ and E_2 is complete, then there is a unique additive function $T : E_1 \to E_2$ with

$$\|f(x) - T(x)\| \le \frac{c\|x\|^p}{|2^{p-1} - 1|}, \quad x \in E_1 \setminus \{0\}. \tag{7}$$

If $p < 0$, then f is additive.

In this paper, we focus on stability of a linear functional equation of the first order, in single variable and some related results; in this way, we complement to some extent the information provided in surveys [7–9, 25, 26]. Let us yet mention that the equation plays a significant role in the investigations of stability of the functional equations in several variables; for suitable examples, we refer the reader, for example, to [8, 27–31].

2. Preliminaries

In what follows, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote, as usually, the sets of positive integers, integers, rationals, reals, and complex numbers, respectively; moreover, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$.

Let us recall that the linear functional equation of the order $m \in \mathbb{N}$ has the form

$$\varphi(x) = \sum_{i=1}^{m} a_i(x)\varphi(\xi_i(x)) + F(x), \tag{8}$$

where S is a nonempty set, X is a linear space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and the functions $F : S \to X$, $\xi_i : S \to S$, and $a_i : S \to \mathbb{F}$ for $i = 1, \ldots, m$ are given. The unknown function is $\varphi : S \to X$. We refer the reader to [7–9, 25, 26] for surveys on stability results for that equation (with arbitrary m) and its generalizations. In this paper, we focus only on the case $m = 1$, when the equation takes the form

$$\varphi(x) = a_1(x)\varphi(\xi_1(x)) + F(x). \tag{9}$$

It is easily seen that the following functional equation

$$\Phi(x, \varphi(x), \varphi(\xi(x))) = \varphi(x), \tag{10}$$

with suitable functions ξ and Φ, is its natural generalization. Next, if ξ_1 is bijective, then we can rewrite (9) in the form

$$\varphi\left(\xi_1^{-1}(x)\right) = a_1\left(\xi_1^{-1}(x)\right)\varphi(x) + F\left(\xi_1^{-1}(x)\right), \tag{11}$$

and a natural generalization of it is the functional equation

$$\varphi(\eta(x)) = H(x, \varphi(x)) \tag{12}$$

with suitable functions η and H.

We discuss stability results for those three functional equations and some related issues that have not been treated at all or only briefly in [7–9, 25, 26].

The following general definition (cf. [25]) describes the main idea of the notion of stability that we use in this paper; for comments on various possible definitions of stability, we refer the reader to [16, 17, 32] (given two nonempty sets, A and B, by A^B we denote, as usual, the family of all functions mapping B into A).

Definition 2. Let $n \in \mathbb{N}$, S be a nonempty set, (X, d) a metric space, $\mathscr{C} \subset {\mathbb{R}_+}^{S^n}$ nonempty, $\mathscr{T}$ a function mapping $\mathscr{C}$ into ${\mathbb{R}_+}^S$, and $\mathscr{F}_1, \mathscr{F}_2$ functions mapping nonempty set $\mathscr{D} \subset X^S$ into X^{S^n}. We say that the equation

$$\mathscr{F}_1\varphi(x_1, \ldots, x_n) = \mathscr{F}_2\varphi(x_1, \ldots, x_n) \tag{13}$$

is $\mathscr{T}$-stable provided, for any $\varepsilon \in \mathscr{C}$ and $\varphi_0 \in \mathscr{D}$ with

$$d(\mathscr{F}_1\varphi_0(x_1, \ldots, x_n), \mathscr{F}_2\varphi_0(x_1, \ldots, x_n)) \le \varepsilon(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in S, \tag{14}$$

there is a solution $\varphi \in \mathscr{D}$ of (13) such that

$$d(\varphi(x), \varphi_0(x)) \le \mathscr{T}\varepsilon(x), \quad x \in S. \tag{15}$$

In the case where $\mathscr{C}$ consists of all constant functions from ${\mathbb{R}_+}^{S^n}$ and $\mathscr{T}(\mathscr{C})$ contains only constant functions, the $\mathscr{T}$-stability is usually called the Hyers-Ulam (or the Ulam-Hyers) stability.

3. Stability Results

In this section, we present various examples of stability results. We do not compare them, in general. The readers can easily do it themselves.

The first theorem is a well known example of the Hyers-Ulam stability result for a particular case of functional equation (10) (its probabilistic versions have been given by Miheţ in [33] and Miheţ and Zaharia in [34, 35]).

Theorem 3 (see [36, Theorem 2]). *Let S be a nonempty set, (X, d) a complete metric space, $\xi : S \to S$, $F : S \times X \to X$, $\lambda \in [0, 1)$, and*

$$d(F(t, u), F(t, v)) \le \lambda d(u, v), \quad t \in S, u, v \in X. \tag{16}$$

If $\varphi : S \to X$, $\delta > 0$ and

$$d(\varphi(t), F(t, \varphi(\xi(t)))) \le \delta, \quad t \in S, \tag{17}$$

then there is a unique solution $\psi : S \to X$ of the functional equation

$$\psi(t) = F(t, \psi(\xi(t))) \tag{18}$$

such that

$$d(\psi(t), \varphi(t)) \le \frac{\delta}{1 - \lambda}, \quad t \in S. \tag{19}$$

To formulate the next result (which is a generalization of Theorem 3), we recall that a mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it is nondecreasing and

$$\lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0, \infty). \tag{20}$$

Theorem 4 (see [37, Theorem 2.2]). *Let S be a nonempty set, (X, d) a complete metric space, and $\xi : S \to S$, $F : S \times X \to X$. Assume also that*

$$d(F(t, u), F(t, v)) \le \gamma(d(u, v)), \quad t \in S, u, v \in X, \tag{21}$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, and let $\varphi : S \to X$, $\delta > 0$ be such that (17) *holds. Then, there is a unique solution $\psi : S \to X$ of* (18) *such that*

$$\rho(\psi, \varphi) := \sup_{t \in S} d(\psi(t), \varphi(t)) < \infty. \tag{22}$$

Moreover,

$$\rho(\psi, \varphi) - \gamma(\rho(\psi, \varphi)) \le \delta. \tag{23}$$

Below, we present several other (less known) similar stability results for particular cases of (10), obtained in an analogous way as Theorems 3 and 4, that is, by the fixed point methods.

Theorem 5 (see [38, Theorem 2.1]). *Let S be a nonempty set, (X, d) a complete metric space, and functions $\xi : S \to S$, $F : S \times X \to X$, and $\alpha : S \to (0, \infty)$ fulfil*

$$\begin{aligned} &\alpha(\xi(t))\, d(F(t, u(\xi(t))), F(t, v(\xi(t)))) \\ &\quad \le \lambda \alpha(t)\, d(u(\xi(t)), v(\xi(t))) \end{aligned} \tag{24}$$

for any $t \in S$, $u, v \in X^S$, and a fixed $\lambda \in [0, 1)$. If $\varphi : S \to X$ satisfies the inequality

$$d(\varphi(t), F(t, \varphi(\xi(t)))) \le \alpha(t), \quad t \in S, \tag{25}$$

then there exists a solution $\psi : S \to X$ of (18) *such that*

$$d(\psi(t), \varphi(t)) \le \frac{\alpha(t)}{1 - \lambda}, \quad t \in S. \tag{26}$$

The subsequent theorem also concerns (18).

Theorem 6 (see [39, Theorem 2.2]). *Let S be a nonempty set, (X, d) a complete metric space, $\xi : S \to S$, $F : S \times X \to X$, and*

$$\begin{aligned} &d(F(t, u), F(t, v)) \\ &\quad \le \alpha_1(u, v)\, d(u, v) + \alpha_2(u, v)\, d(u, F(t, u)) \\ &\quad\quad + \alpha_3(u, v)\, d(v, F(t, v)) + \alpha_4(u, v)\, d(u, F(t, v)) \\ &\quad\quad + \alpha_5(u, v)\, d(v, F(t, u)), \quad t \in S, u, v \in X, \end{aligned} \tag{27}$$

where $\alpha_1, \ldots, \alpha_5 : X \times X \to \mathbb{R}_+$ fulfil the inequality

$$\sum_{i=1}^{5} \alpha_i(u, v) \le \lambda, \quad u, v \in X \tag{28}$$

for a fixed $\lambda \in [0, 1)$. If $\varphi : S \to X$, $\delta > 0$ and (17) *holds, then there is a unique function $\psi : S \to X$ satisfying* (18) *and such that*

$$d(\psi(t), \varphi(t)) \le \frac{(2 + \lambda)\delta}{2(1 - \lambda)}, \quad t \in S. \tag{29}$$

Recall that a mapping $\gamma : [0, \infty] \to [0, \infty]$ is called a *generalized strict comparison function* if it is nondecreasing, $\gamma(\infty) = \infty$, and

$$\begin{aligned} &\lim_{n \to \infty} \gamma^n(t) = 0, \quad t \in (0, \infty), \\ &\lim_{t \to \infty} (t - \gamma(t)) = \infty. \end{aligned} \tag{30}$$

The following is one more generalization of Theorem 3.

Theorem 7 (see [40, Theorem 3.1]). *Let S be a nonempty set, (X, d) a complete metric space, and $\xi : S \to S$, $F : S \times X \to X$. Assume also that*

$$d(F(t, u), F(t, v)) \le \gamma(d(u, v)), \quad t \in S, u, v \in X, \tag{31}$$

where $\gamma : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function, and let $\varphi : S \to X$, $\delta > 0$ be such that (17) *holds. Then, there is a unique function $\psi : S \to X$ satisfying* (18) *and*

$$d(\psi(t), \varphi(t)) \le \sup\{s \in (0, \infty) : s - \gamma(s) \le \delta\}, \quad t \in S. \tag{32}$$

The next result involves a generalization of condition (17) (with a constant replaced by a suitable function on the right hand side of the inequality).

Theorem 8 (see [41, Theorem 4.1]). *Let S be a nonempty set, (X, d) a complete metric space, $\xi : S \to S$, $F : S \times X \to X$, $g : S \to \mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and*

$$d(F(t, u), F(t, v)) \le |g(t)|\, d(u, v), \quad t \in S, u, v \in X. \tag{33}$$

Assume that $\varphi : S \to X$ satisfies

$$d(\varphi(t), F(t, \varphi(\xi(t)))) \le \delta(t), \quad t \in S \tag{34}$$

with a mapping $\delta : S \to \mathbb{R}_+$ for which there exists an $L \in [0, 1)$ such that

$$|g(t)|\, \delta(\xi(t)) \le L\delta(t), \quad t \in S. \tag{35}$$

Then, there is a unique solution $\psi : S \to X$ of (18) *such that*

$$d(\psi(t), \varphi(t)) \le \frac{\delta(t)}{1 - L}, \quad t \in S. \tag{36}$$

Let us mention here that an analogous result for the complete probabilistic metric spaces has been obtained in [42].

Another result on the stability of (18) comes from [43] (for some related results cf. [44]). To formulate it, we define, for given nonempty sets S, Z and functions $\xi : S \to S$, $F : S \times Z \to Z$, an operator $\mathscr{L}_\xi^F : Z^S \to Z^S$ by

$$\mathscr{L}_\xi^F(g)(t) := F(t, g(\xi(t))), \quad g \in Z^S, t \in S. \tag{37}$$

Theorem 9 (see [43, Corollary 2.1]). *Let S be a nonempty set, (X, d) a complete metric space, $F : S \times X \to X$, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, and*

$$d(F(t, u), F(t, v)) \le \Lambda(t, d(u, v)), \quad t \in S, u, v \in X. \tag{38}$$

Assume also that $\xi : S \to S$ and $\delta : S \to \mathbb{R}_+$ are such that

$$\sum_{n=0}^{\infty} (\mathscr{L}_\xi^\Lambda)^n(\delta)(t) =: \sigma(t) < \infty, \quad t \in S, \tag{39}$$

$\varphi : S \to X$ fulfils (34) *and, for every $t \in S$, $\Lambda_t := \Lambda(t, \cdot)$ is nondecreasing and $F(t, \cdot)$ is continuous. Then, the limit*

$$\psi(t) := \lim_{n \to \infty} (\mathscr{L}_\xi^F)^n(\varphi)(t) \tag{40}$$

exists for every $t \in S$,

$$d(\psi(t), \varphi(t)) \le \sigma(t), \quad t \in S \tag{41}$$

and ψ is a solution of (18)*. Moreover, if, for every $t \in S$, Λ_t is subadditive (i.e., $\Lambda_t(a + b) \le \Lambda_t(a) + \Lambda_t(b)$ for $a, b \in \mathbb{R}_+$) and $M \in \mathbb{N}$, then $\psi : S \to X$ is the unique solution of* (18) *with*

$$d(\psi(t), \varphi(t)) \le M\sigma(t), \quad t \in S. \tag{42}$$

Now, we present a result from [45].

Theorem 10 (see [45, Theorem 2.2]). *Let S be a nonempty set, (X, d) a complete metric space, $\xi : S \to S$, $F : X \times X \to X$, $\lambda, \mu \in \mathbb{R}_+$, and*

$$d(F(s,u), F(t,v)) \leq \mu d(s,t) + \lambda d(u,v), \quad s,t,u,v \in X. \tag{43}$$

Assume also that $\varphi : S \to X$, $\Phi : S \to \mathbb{R}_+$ are such that

$$\begin{gathered} d(\varphi(t), F(\varphi(t), \varphi(\xi(t)))) \leq \Phi(t), \quad t \in S, \\ \lambda\Phi(\xi(t)) + \mu\Phi(t) \leq L\Phi(t), \quad t \in S \end{gathered} \tag{44}$$

with an $L \in [0,1)$. Then, there is a unique solution $\psi : S \to X$ of the equation

$$\psi(t) = F(\psi(t), \psi(\xi(t))) \tag{45}$$

such that

$$d(\psi(t), \varphi(t)) \leq \frac{\Phi(t)}{1-L}, \quad t \in S. \tag{46}$$

The next two stability outcomes were obtained in [46].

Theorem 11 (see [46, Theorem 2]). *Let S be a nonempty set, (X, d) a complete metric space, $F : S \times X \times X \to X$, $\xi : S \to S$, $\alpha : S \to (0, \infty)$, and $L \in [0,1)$. Assume also that functions $\lambda, \mu : S \to \mathbb{R}_+$ satisfy the inequality*

$$\lambda(x)\alpha(x) + \mu(x)\alpha(\xi(x)) \leq L\alpha(x), \quad x \in S \tag{47}$$

and $F : S \times X \times X \to X$ is such that

$$\begin{aligned} &d(F(x, u(x), u(\xi(x))), F(x, v(x), v(\xi(x)))) \\ &\quad \leq \lambda(x) d(u(x), v(x)) + \mu(x) d(u(\xi(x)), v(\xi(x))), \\ &\qquad x \in S, u, v \in X^S. \end{aligned} \tag{48}$$

If $\varphi : S \to X$ fulfils

$$d(\varphi(x), F(x, \varphi(x), \varphi(\xi(x)))) \leq \alpha(x), \quad x \in S, \tag{49}$$

then there exists a unique solution $\psi : S \to X$ of the functional equation

$$\psi(x) = F(x, \psi(x), \psi(\xi(x))) \tag{50}$$

such that

$$d(\varphi(x), \psi(x)) \leq \frac{\alpha(x)}{1-L}, \quad x \in S. \tag{51}$$

Theorem 12 (see [46, Theorem 5]). *Let S be a nonempty set, X a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\xi : S \to S$, $a : S \to \mathbb{F}$, $h : S \to X$, $\alpha : S \to (0, \infty)$, and $L \in [0,1)$. Assume also that functions $\lambda, \mu : S \to \mathbb{R}_+$ satisfy the inequalities*

$$\begin{gathered} \mu(x)\alpha(\xi(x)) \leq (L - \lambda(x))\alpha(x), \quad x \in S, \\ (|a(x)| - \mu(x)) \|u(\xi(x)) - v(\xi(x))\| \\ \leq \lambda(x)\|u(x) - v(x)\|, \quad x \in S, u, v \in X^S. \end{gathered} \tag{52}$$

If $\varphi : S \to X$ fulfils

$$\|\varphi(x) - a(x)\varphi(\xi(x)) - h(x)\| \leq \alpha(x), \quad x \in S, \tag{53}$$

then there exists a unique solution $\psi : S \to X$ of the functional equation

$$\psi(x) = a(x)\psi(\xi(x)) + h(x) \tag{54}$$

such that

$$\|\varphi(x) - \psi(x)\| \leq \frac{\alpha(x)}{1-L}, \quad x \in S. \tag{55}$$

Moreover,

$$\begin{aligned} \psi(x) = h(x) &+ \lim_{n\to\infty}\left(\varphi(\xi^n(x))\prod_{i=0}^{n-1}a(\xi^i(x))\right. \\ &\left. + \sum_{j=0}^{n-2}h(\xi^{j+1}(x))\prod_{i=0}^{j}a(\xi^i(x))\right), \quad x \in S. \end{aligned} \tag{56}$$

The next theorem has been applied in [47] to prove stability of the Pexiderized linear functional equation

$$\psi(\xi(t)) = p(t)\varphi(t) + q(t). \tag{57}$$

Theorem 13 (see [47, Theorem 2.1]). *Let S be a nonempty set, X a Banach space over $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, $\xi : S \to S$, $p : S \to \mathbb{F} \setminus \{0\}$, $q : S \to X$, $\alpha : S \to \mathbb{R}_+$, $L \in (0,1)$, and*

$$\alpha(\xi(t)) \leq L|p(\xi(t))|\alpha(t), \quad t \in S. \tag{58}$$

If $\varphi : S \to X$ satisfies

$$\|\varphi(\xi(t)) - p(t)\varphi(t) - q(t)\| \leq \alpha(t), \quad t \in S, \tag{59}$$

then there is a unique function $\psi : S \to X$ such that

$$\psi(\xi(t)) = p(t)\psi(t) + q(t), \quad t \in S, \tag{60}$$

$$\|\varphi(t) - \psi(t)\| \leq \frac{\alpha(t)}{(1-L)|p(t)|}, \quad t \in S. \tag{61}$$

Theorem 14 (see [47, Theorem 2.5]). *Let S be a nonempty set, X a Banach space over $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, $\xi : S \to S$ a bijection, $p : S \to \mathbb{F} \setminus \{0\}$, $q : S \to X$, $\alpha : S \to \mathbb{R}_+$, $L \in (0,1)$, and*

$$|p(t)|\alpha(\xi^{-1}(t)) \leq L\alpha(t), \quad t \in S. \tag{62}$$

If $\varphi : S \to X$ satisfies (59)*, then there is a unique function $\psi : S \to X$ such that* (60) *holds and*

$$\|\varphi(t) - \psi(t)\| \leq \frac{1}{1-L}\alpha(\xi^{-1}(t)), \quad t \in S. \tag{63}$$

The authors have also proved in [47] a stability result for the system of homogeneous linear equations

$$\psi(\xi_i(t)) = p_i(t)\psi(t), \quad i \in I, \tag{64}$$

which gives a partial affirmative answer to a problem posed by G.L. Forti during the 13th International Conference on Functional Equations and Inequalities (Małe Ciche, Poland, September 13–19, 2009).

The below theorem has been used in [48] to prove a stability result for the following functional equation

$$\psi(F(x,y)) = H(\psi(x), \psi(y), x, y), \tag{65}$$

with suitable functions F and H.

Theorem 15 (see [48, Theorem 1]). *Let S be a nonempty set, (X,d) a complete metric space, $\xi : S \to S$, $F : X \times S \to X$, $\lambda \in \mathbb{R}_+$, and*

$$d(F(u,t), F(v,t)) \le \lambda d(u,v), \quad t \in S, u, v \in X. \tag{66}$$

If $\varphi : S \to X$, $\delta : S \to \mathbb{R}_+$ are such that

$$d(\varphi(t), F(\varphi(\xi(t)), t)) \le \delta(t), \quad t \in S, \tag{67}$$

and the series $\sum_{i=0}^{\infty} \lambda^i \delta(\xi^i(t))$ converges for every $t \in S$, then there is a unique solution $\psi : S \to X$ of the functional equation

$$\psi(t) = F(\psi(\xi(t)), t) \tag{68}$$

with

$$d(\psi(t), \varphi(t)) \le \sum_{i=0}^{\infty} \lambda^i \delta(\xi^i(t)), \quad t \in S. \tag{69}$$

Let us also mention that the probabilistic stability of the following particular cases of (10) and (18)

$$f(\varphi(g(t))) = \varphi(t) \tag{70}$$

was investigated in [49]. Further results on stability of this equation can be found, for instance, in [50–52].

The next result deals with linear equation (54) and is due to Trif [53]. We will show its application in the sequel, in the section concerning solutions of a simplified version of the linear equation.

Theorem 16 (see [53, Theorem 2.1]). *Let S be a nonempty set, $\xi : S \to S$, X a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $a : S \to \mathbb{F}$, $h : S \to X$, and $\epsilon : S \to \mathbb{R}_+$ such that*

$$\omega(x) := \sum_{j=0}^{\infty} \frac{\epsilon(\xi^j(x))}{\prod_{k=0}^{j} |a(\xi^k(x))|} < \infty, \quad x \in S. \tag{71}$$

If $\varphi : S \to X$ satisfies the inequality

$$\|\varphi(\xi(x)) - a(x)\varphi(x) - h(x)\| \le \epsilon(x), \quad x \in S, \tag{72}$$

then there exists a unique solution $\tilde{\varphi} : S \to X$ of (54) with

$$\|\varphi(x) - \tilde{\varphi}(x)\| \le \omega(x), \quad x \in S. \tag{73}$$

Moreover,

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \left(\frac{\varphi(\xi^n(x))}{\prod_{j=0}^{n-1} a(\xi^j(x))} - \sum_{k=0}^{n-1} \frac{h(\xi^k(x))}{\prod_{j=0}^{k} a(\xi^j(x))} \right), \quad x \in S. \tag{74}$$

Actually, condition (74) has not been included in the statement of [53, Theorem 2.1], but it can be easily derived from the proof of the theorem. For some investigations of condition (71), we refer the reader to [54].

We end this section with quite general stability results for difference equations that have been obtained in [55].

Theorem 17 (see [55, Theorem 1]). *Let X be an abelian group, d a complete, and invariant metric in X, $a_n : X \to X$ a continuous isomorphism for every $n \in \mathbb{N}_0$, $\{\varepsilon_n\}_{n\in\mathbb{N}_0} \subset (0,\infty)$, $\{\lambda_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_+$, and $\{x_n\}_{n\in\mathbb{N}_0}$, $\{b_n\}_{n\in\mathbb{N}_0} \subset X$. Suppose that*

$$d(x_{n+1}, a_n(x_n) + b_n) \le \varepsilon_n, \quad n \in \mathbb{N}_0,$$

$$L_0 := \liminf_{n\to\infty} \frac{\varepsilon_{n-1}\lambda_n}{\varepsilon_n} > 1, \tag{75}$$

$$d(a_n(x), a_n(y)) \ge \lambda_n d(x,y), \quad x, y \in X, n \in \mathbb{N}_0.$$

Then there exists a unique sequence $\{y_n\}_{n\in\mathbb{N}_0} \subset X$ such that

$$y_{n+1} = a_n(y_n) + b_n, \quad n \in \mathbb{N}_0, \tag{76}$$

$$d(x_n, y_n) \le M\varepsilon_{n-1}, \quad n \in \mathbb{N}, \tag{77}$$

with an $M \in \mathbb{R}_+$.

Remark 18 (see [55, Remark 3]). It follows from [56, Remark 2.3] that, in the case

$$\liminf_{n\to\infty} \frac{\varepsilon_{n-1}\lambda_n}{\varepsilon_n} = 1, \tag{78}$$

the conclusion of Theorem 17 is not generally true.

Theorem 19 (see [55, Theorem 2]). *Let (X,d) be a metric space, $\{x_n\}_{n\in\mathbb{N}_0} \subset X$, $\{a_n\}_{n\in\mathbb{N}_0} \subset X^X$, $\{\varepsilon_n\}_{n\in\mathbb{N}_0} \subset (0,\infty)$, and*

$$d(x_{n+1}, a_n(x_n)) \le \varepsilon_n, \quad n \in \mathbb{N}_0. \tag{79}$$

Suppose that there exists $\{\lambda_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_+$ with

$$L := \limsup_{n\to\infty} \frac{\varepsilon_{n-1}\lambda_n}{\varepsilon_n} < 1, \qquad d(a_n(x), a_n(y)) \le \lambda_n d(x,y), \quad x, y \in X, n \in \mathbb{N}_0. \tag{80}$$

Then there exist a sequence $\{y_n\}_{n\in\mathbb{N}_0} \subset X$ and an $M > 0$ such that

$$y_{n+1} = a_n(y_n), \quad n \in \mathbb{N}_0, \tag{81}$$

$$d(x_n, y_n) \le M\varepsilon_{n-1}, \quad n \in \mathbb{N}. \tag{82}$$

Remark 20 (see [55, Remark 3]). There is no uniqueness of the sequence $\{y_n\}_{n\in\mathbb{N}}$ in Theorem 19, which follows from [56, Remark 2.2].

If

$$\limsup_{n\to\infty}\frac{\varepsilon_{n-1}\lambda_n}{\varepsilon_n}=1, \tag{83}$$

then the conclusion of Theorem 19 is not generally true (cf. [56, Remark 2.3]).

We refer the reader to [57] (and the references therein) for further stability results for linear difference equations of higher orders.

4. Iterative Stability

Let $I=(0,d]$ for a $d>0$ and $\xi: I\to I, a, h: I\to\mathbb{R}$ given functions. Consider the linear nonhomogenous equation

$$\varphi(\xi(x))=a(x)\varphi(x)+h(x),\quad x\in I, \tag{84}$$

and its homogenous version

$$\varphi(\xi(x))=a(x)\varphi(x),\quad x\in I, \tag{85}$$

where $\varphi: I\to\mathbb{R}$ is unknown.

Brydak [58] (cf. [59, Definition 2]) introduced the notion of stability (later called *iterative stability*), which for (84) means that for every $\varepsilon>0$ there exists a $\delta>0$ such that if a continuous function $\psi: I\to\mathbb{R}$ satisfies the condition

$$\left|\psi\left(\xi^n(x)\right)-G_n(x)\psi(x)+G_n(x)\sum_{i=0}^{n-1}\frac{h\left(\xi^i(x)\right)}{G_{i+1}(x)}\right|<\delta, \quad x\in I,\ n\in\mathbb{N}, \tag{86}$$

then there exists a continuous solution φ of (84) such that

$$|\psi(x)-\varphi(x)|<\varepsilon,\quad x\in I, \tag{87}$$

where

$$G_n(x):=\prod_{i=0}^{n-1}a\left(\xi^i(x)\right),\quad x\in I. \tag{88}$$

In general, the following two hypotheses have been used in investigations of that stability.

(H1) ξ is a strictly increasing continuous function and $0<\xi(x)<x$ for $x\in I$.

(H2) a is a continuous function such that $a(x)\neq 0$ for $x\in I$.

It is known that if (H1) and (H2) hold, then continuous solutions of (84) and (85) defined on I depend on an arbitrary function (cf. [60, Theorem 2.1]). The crucial assumption here is that 0 does not belong to the domain of the solutions.

Let us yet introduce the following two assumptions.

(A) The limit $G(x):=\lim_{n\to\infty}G_n(x)$ exists, G is continuous in I and $G(x)\neq 0$ for $x\in I$.

(B) There exists an interval $J\subset I$ such that the sequence $\{G_n\}_{n\in\mathbb{N}}$ converges uniformly to the zero function on J.

Brydak [58] proved that if either (A) holds and

$$\gamma:=\inf_{x\in I}\lim_{n\to\infty}G_n(x)\neq 0, \tag{89}$$

or (B) holds, then (84) is iteratively stable (cf. also [7]). Turdza [61] considered the same problem in the case where $a: I\to\mathbb{F}$, $h,\varphi: I\to Y$, $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, and Y is a Banach space over $\mathbb{F}$. He proved that if (H1), (H2), and (A) hold and $\gamma\neq 0$, then (84) is iteratively stable (cf. [7] for suitable comments).

Choczewski et al. [59, Theorem 1] have also introduced the following definition of stability, which according to the comment following Definition 2 can be called the Hyers-Ulam stability.

Definition 21 (see [59, Definition 21]). Equation (84) is called stable in the class $C(I)$ consisting of the all functions continuous in the interval I, if there exists a $K>0$ such that for any $\epsilon>0$ and solution $\psi\in C(I)$ of the inequality

$$|\psi(\xi(x))-a(x)\psi(x)-h(x)|\leq\epsilon,\quad x\in I, \tag{90}$$

there exists a solution $\varphi\in C(I)$ of (84) with

$$|\psi(x)-\varphi(x)|\leq K\epsilon,\quad x\in I. \tag{91}$$

They showed (under hypotheses (H1) and (H2)) that if (85) is stable (iteratively stable, resp.) and has a continuous solution $\psi: I\to\mathbb{R}$, then so is (84); a very recent and more general result of this type will be presented at the end of this section.

For an ample and much more detailed discussion of the results concerning iterative stability, we refer the reader to survey paper [7]. Below, we present some outcomes obtained by Turdza in [62], which have not been included in [7].

The notion of iterative stability has been introduced in [62] for functional equation (12), that is, for the equation

$$\varphi(\eta(x))=H(x,\varphi(x)), \tag{92}$$

with suitable given functions η and H and the unknown function φ.

The author has used in his considerations the following hypotheses.

(H_1) The function η is continuous and strictly increasing in the interval $I=[\rho,b)$, $\eta(x)<x$ for $x\in I\setminus\{\rho\}$, and $\eta(\rho)=\rho$.

(H_2) The function $H(x,y)$ is defined in a set $\Omega\subset I\times E$ and takes values in E (E is a nonempty set), and for every fixed $x\in I$ the function $H(x,\cdot)$ is invertible in the set $\Omega_x:=\{y:(x,y)\in\Omega\}$ (provided $\Omega_x\neq\emptyset$).

(H_3) For any $x_0\in I$ and function φ_0, which is continuous in the interval $I_0=[\eta(x_0),x_0)$ and such that $\varphi_0(\eta(x_0))=H(x_0,\varphi_0(x_0))$, there exists exactly one function φ that is continuous in I and satisfies (92) and the condition $\varphi(x)=\varphi_0(x)$ for $x\in I_0$.

(H_4) For every $x_0 \in I$, there exists an $L > 0$ such that for any continuous solution ψ of the inequality

$$\left|\psi\left(\eta^n(x)\right) - H_n\left(x, \psi(x)\right)\right| \le \epsilon, \quad n \in \mathbb{N},\ x \in I, \tag{93}$$

where $H_0(x, y) = y$ and

$$H_{n+1}(x, y) = H\left(\eta^n(x), H_n(x, y)\right) \quad n \in \mathbb{N}_0, \tag{94}$$

and continuous solution φ of (92), fulfilling the condition

$$\left|\psi(x) - \varphi(x)\right| \le \epsilon, \quad x \in I_0, \tag{95}$$

the subsequent inequality is valid

$$\left|\psi(x) - \varphi(x)\right| \le L\epsilon, \quad x \in I. \tag{96}$$

Let I be a nontrivial interval and $C(I)$ denote the class of all functions defined and continuous in I. The next two definitions have been introduced in [62].

Definition 22 (see [62, Definition 1]). Equation (92) is iteratively stable in the interval I in the class $C(I)$, if there exists an $L > 0$ such that, for any $\epsilon > 0$ and solution $\psi \in C(I)$ of the system of inequalities (93), there exists a solution $\phi \in C(I)$ of (92) satisfying (96).

Definition 23 (see [62, Definition 2]). Equation (92) is stable in the interval I in the class $C(I)$, if there exists an $L > 0$ such that, for any $\epsilon > 0$ and solution $\psi \in C(I)$ of the inequality

$$\left|\psi(\eta(x)) - H(x, \psi(x))\right| \le \epsilon, \quad x \in I, \tag{97}$$

there exists a solution $\varphi \in C(I)$ of (92) satisfying (96).

Actually the term "iterative stable" has been used in [62] instead of "iteratively stable," but it seems that the latter one is more correct and consistent with [59, Definition 2].

The notions of stability described in Definitions 22 and 23 are closely related. Namely, we have the following.

Theorem 24 (see [62, Theorem 2]). *Let hypotheses* (H_1) *and* (H_2) *be valid,* (97) *hold, and*

$$\left|H(x, y_1) - H(x, y_2)\right| \le m(x)\left|y_1 - y_2\right|, \quad (x, y_1), (x, y_2) \in \Omega \tag{98}$$

for a function $m : I \to [0, 1)$ *such that*

$$\sup_{x \in [\rho, \rho+\delta]} m(x) < 1 \tag{99}$$

with a $\delta \in (0, b-\rho)$. *Then, Definitions 22 and 23 are equivalent.*

The subsequent two theorems concern iterative stability (the first one has actually been proved in [63]).

Theorem 25 (see [62, Theorem 1]). *Let hypotheses* (H_1) *and* (H_2) *be valid. If there exists an* $M > 0$ *with*

$$\left|H_n(x, y_1) - H_n(x, y_2)\right| \le M\left|y_1 - y_2\right|, \quad (x, y_1), (x, y_2) \in \Omega,\ n \in \mathbb{N}, \tag{100}$$

then for every $\rho_0 \in (0, b-\rho)$ (92) *is iteratively stable in* $[\rho, b-\rho_0]$.

Theorem 26 (see [62, Theorem 4]). *Let hypotheses* (H_1)–(H_4) *be valid with* $\rho = 0$. *Assume also that there exist* $M, \mu > 0$ *and continuous function* $m : I \to \mathbb{R}_+$ *such that*

$$\left|\xi(x) - x\right| \le Mx^{1+\mu} \tag{101}$$

in a neighbourhood of zero, (98) *holds, and*

$$m(x) \le 1 - x, \quad x \in [0, c) \tag{102}$$

with a $c \in (0, b)$. *Then, for every* $x_0 \in [0, b)$ (92) *is iteratively stable in the interval* $[0, x_0]$.

The next theorem corresponds to Theorem 26.

Theorem 27 (see [62, Theorem 3]). *Let hypotheses* (H_1) *and* (H_2) *be valid. If condition* (98) *is fulfilled with a function* $m : I \to [0, 1)$ *such that* (99) *holds, then for every* $\rho_0 \in (0, b-\rho)$ (92) *is stable in the interval* $[\rho, b-\rho_0]$ *in the class* $C(I)$.

A connection between the stability and the continuous dependance of (92) on a given function H has been investigated in [62, Theorems 5 and 6]. Below, we present those results.

Theorem 28 (see [62, Theorem 5]). *Let* (H_1) *be valid and functions* H_1, H_2 *satisfy hypothesis* (H_2) *with* $H = H_i$ *for* $i = 1, 2$. *Let* $\varphi_1, \varphi_2 : I \to \mathbb{R}$ *be such that*

$$\varphi_i(\xi(x)) = H_i\left(x, \varphi_i(x)\right), \quad x \in I,\ i = 1, 2. \tag{103}$$

Assume also that there is an $m : I \to [0, 1)$ *such that* (99) *holds and*

$$\left|H_2(x, y_1) - H_2(x, y_2)\right| \le m(x)\left|y_1 - y_2\right|, \quad (x, y_1), (x, y_2) \in \Omega. \tag{104}$$

If $x_0 \in I$, $\epsilon > 0$, *and*

$$\begin{gathered}\left|H_1(x, y) - H_2(x, y)\right| \le \epsilon, \quad (x, y) \in \Omega, \\ \left|\varphi_1(x) - \varphi_2(x)\right| \le \epsilon, \quad x \in \left[\eta(x_0), x_0\right],\end{gathered} \tag{105}$$

then there exists an $S(x_0) > 0$ *such that*

$$\left|\varphi_1(x) - \varphi_2(x)\right| \le S(x_0)\epsilon, \quad x \in I,\ x \le x_0. \tag{106}$$

Theorem 29 (see [62, Theorem 6]). *Assume that* (H_1) *and* (H_2) *are valid and functions* H_n *satisfy hypothesis* (H_2) *with* $H = H_n$ *for* $n \in \mathbb{N}$. *Let the sequence* $\{H_n\}_{n\in\mathbb{N}}$ *converge to* H *uniformly on* Ω. *If the equations*

$$\varphi_n(\xi(x)) = H_n\left(x, \varphi_n(x)\right), \quad x \in I,\ n \in \mathbb{N} \tag{107}$$

are stable in I *with constants* K_n *and*

$$L := \sup_{n\in\mathbb{N}} K_n < \infty, \tag{108}$$

then for every solution φ *of* (92) *there exists a sequence* $\{\varphi_n\}_{n\in\mathbb{N}}$ *of solutions of* (107), *which converges to* φ *uniformly on* I.

Now, we show how some considerations concerning the iterative stability can be expressed in terms of difference equations; we will only deal with (85). Let us assume that $\xi(0) := 0$. Then, hypothesis *(H1)* implies that 0 is an attractive fixed point of ξ. Indeed, for every $x_0 \in I$, the sequence $\{x_n\}_{n\in\mathbb{N}}$, where

$$x_n := \xi^n(x_0), \tag{109}$$

tends to 0, since $\xi(x) < x$ for all $x \in I$. Moreover, $\xi([0, x_0]) \subsetneq [0, x_0]$ for every $x_0 \in I$.

Let φ be a solution of (85). For a fixed $x_0 \in I$, put

$$y_n := \varphi(\xi^n(x_0)), \quad n \in \mathbb{N}_0. \tag{110}$$

Then, by (85), we have

$$y_n = a(x_{n-1})\, y_{n-1}, \quad n \in \mathbb{N}. \tag{111}$$

On the other hand, by (85) and (88),

$$y_n = G_n(x_0)\, y_0, \quad n \in \mathbb{N}. \tag{112}$$

Let $\psi : I \to \mathbb{R}$ be a function such that

$$\left|\psi(\xi(x)) - a(x)\,\psi(x)\right| < \delta, \quad x \in I. \tag{113}$$

Fix an $x_o \in I$ and put

$$z_n := \psi(\xi^n(x_0)), \quad n \in \mathbb{N}. \tag{114}$$

Then, we get

$$\left|z_n - a(x_{n-1})\, z_{n-1}\right| < \delta, \quad n \in \mathbb{N}. \tag{115}$$

Next, condition (86) with $h(x) \equiv 0$ yields

$$\left|z_n - G_n(x_0)\, z_0\right| < \delta, \quad n \in \mathbb{N}. \tag{116}$$

Hence, by (112), we obtain

$$\begin{aligned}\left|z_n - y_n\right| &= \left|z_n - G_n(x_0)\, z_0 + G_n(x_0)\, z_0 - y_n\right| \\ &\le \left|z_n - G_n(x_0)\, z_0\right| + \left|G_n(x_0)\right|\left|z_0 - y_0\right|, \quad n \in \mathbb{N},\end{aligned} \tag{117}$$

and consequently, by (116),

$$\left|z_n - y_n\right| \le \delta + \left|G_n(x_0)\right|\left|z_0 - y_0\right|, \quad n \in \mathbb{N}. \tag{118}$$

So, in the particular case where $z_0 = y_0$, that is, $\varphi(x_0) = \psi(x_0)$, we have

$$\left|z_n - y_n\right| \le \delta, \quad n \in \mathbb{N}. \tag{119}$$

Thus, we have shown that, in particular, if there is a $d \in I$ with $\varphi(x_0) = \psi(x_0)$ for $x_0 \in (\xi(d), d]$, then (87) holds with $\varepsilon = \delta$.

We end this section with a very simple, but useful (we hope) observation, which is a simplified version of [64, Theorem 1]; it corresponds to the already mentioned [59, Theorem 1] and, in view of Theorem 24, it concerns relation between iterative stabilities of some special cases of (84) and (85). Using it, we can also deduce easily from Theorem 17 some stability results for (76) in the special case when all a_n are additive.

Let S be a nonempty set, X a normed space, $\mathscr{C} \subset {\mathbb{R}_+}^S$ nonempty, $\mathscr{T}$ a function mapping $\mathscr{C}$ into ${\mathbb{R}_+}^S$, and $\mathscr{F}$ a function mapping a nonempty set $\mathscr{U} \subset X^S$ into X^S and such that

$$\mathscr{F}(\psi_1 + \psi_2)(x) = \mathscr{F}\psi_1(x) + \mathscr{F}\psi_2(x), \quad \psi_1, \psi_2 \in \mathscr{U},\ x \in S, \tag{120}$$

where for simplicity we write $\mathscr{F}\psi_i := \mathscr{F}(\psi_i)$ for $i = 1, 2$ and $(\psi_1 + \psi_2)(x) := \psi_1(x) + \psi_2(x)$ for $x \in S$. Assume also that $\mathscr{U}$ is a subgroup of X^S; that is,

$$\psi_1 - \psi_2 \in \mathscr{U}, \quad \psi_1, \psi_2 \in \mathscr{U}. \tag{121}$$

Now, we are in a position to present the following theorem (cf. Definition 2).

Theorem 30. *Let $\mu : S \to X$. Suppose that the equation*

$$\mathscr{F} f(x) = \mu(x), \quad x \in S \tag{122}$$

admits a solution $\psi_0 \in \mathscr{U}$. Then, the equation

$$\mathscr{F} f(x) = 0, \quad x \in S \tag{123}$$

is $\mathscr{T}$-stable if and only if so is (122).

Proof. Since the proof is very elementary and short, we present it here for the convenience of the readers.

Assume first that (122) is $\mathscr{T}$-stable. Let $\delta \in \mathscr{C}$ and $\phi \in \mathscr{U}$ satisfy the condition

$$\left\|\mathscr{F}\phi(x)\right\| \le \delta(x), \quad x \in S. \tag{124}$$

Write $\phi_0 := \phi + \psi_0$. Then, $\phi_0 \in \mathscr{U}$ and

$$\left\|\mathscr{F}\phi_0(x) - \mu(x)\right\| = \left\|\mathscr{F}\phi(x)\right\| \le \delta(x), \quad x \in S. \tag{125}$$

Hence, there exists a solution $\eta_0 \in \mathscr{U}$ of (122) such that

$$\left\|\phi_0(x) - \eta_0(x)\right\| \le \mathscr{T}\delta(x), \quad x \in S. \tag{126}$$

Clearly, $\eta := \eta_0 - \psi_0 \in \mathscr{U}$ is a solution of (123) and

$$\left\|\phi(x) - \eta(x)\right\| = \left\|\phi_0(x) - \eta_0(x)\right\| \le \mathscr{T}\delta(x), \quad x \in S. \tag{127}$$

The proof of the necessary condition is analogous. But, again for the convenience of the readers, we present it below. So, assume that (123) is $\mathscr{T}$-stable. Let $\delta \in \mathscr{C}$ and $\phi_0 \in \mathscr{U}$ satisfy

$$\left\|\mathscr{F}\phi_0(x) - \mu(x)\right\| \le \delta(x), \quad x \in S. \tag{128}$$

Write $\phi := \phi_0 - \psi_0$. Then,

$$\left\|\mathscr{F}\phi(x)\right\| \le \delta(x), \quad x \in S. \tag{129}$$

Hence, there exists a solution $\eta \in \mathscr{U}$ of (123) such that

$$\left\|\phi(x) - \eta(x)\right\| \le \mathscr{T}\delta(x), \quad x \in S. \tag{130}$$

Clearly, $\eta_0 := \eta + \psi_0 \in \mathscr{U}$ is a solution of (122) and

$$\left\|\phi_0(x) - \eta_0(x)\right\| = \left\|\phi(x) - \eta(x)\right\| \le \mathscr{T}\delta(x), \quad x \in S. \tag{131}$$

□

Remark 31. It is easily seen that the assumption that (122) admits a solution $\psi_0 \in \mathcal{U}$ is very important in the proof of Theorem 30; an analogous hypothesis is also applied in [59, Theorem 1].

In the next section, we present some remarks on the issue of the existence of solutions of (122), resulting from some stability outcomes obtained for the equation.

5. A Description of Solutions

Let, as before, S be a nonempty set, $\xi : S \to S$, X a Banach space, and $h : S \to X$. In this section, we show how to derive from Theorem 16, in a very easy way, a description of solutions of the equation

$$\varphi(\xi(x)) = \varphi(x) + h(x), \quad x \in S \tag{132}$$

under assumption (139). Note that (132) is a particular case of (84) (with $a(x) \equiv 1$).

First, let us rewrite Theorem 16 in a simplified form with $a(x) \equiv 1$.

Corollary 32. *Let $\epsilon : S \to \mathbb{R}_+$ be such that*

$$\omega(x) := \sum_{j=0}^{\infty} \epsilon\left(\xi^j(x)\right) < \infty, \quad x \in S. \tag{133}$$

If $\varphi : S \to X$ satisfies

$$\left\|\varphi(\xi(x)) - \varphi(x) - h(x)\right\| \le \epsilon(x), \quad x \in S, \tag{134}$$

then there exists a unique solution $\widetilde{\varphi} : S \to X$ of (132) *with*

$$\left\|\varphi(x) - \widetilde{\varphi}(x)\right\| \le \omega(x), \quad x \in S. \tag{135}$$

Moreover,

$$\widetilde{\varphi}(x) = \lim_{n\to\infty}\left(\varphi\left(\xi^n(x)\right) - \sum_{k=0}^{n-1} h\left(\xi^k(x)\right)\right), \quad x \in S. \tag{136}$$

Let us next introduce some notions.

We say that a function $\varphi : S \to X$ is ξ-*invariant* provided $\varphi \circ \xi = \varphi$. Define an equivalence relation $\mathcal{R}(\xi) \subset S^2$ by

$$\mathcal{R}(\xi) := \left\{(x, y) \in S^2 : \xi^m(x) = \xi^k(y) \text{ with some } k, m \in \mathbb{N}_0\right\} \tag{137}$$

and write

$$[x]_\xi := \left\{y \in S : (x, y) \in \mathcal{R}(\xi)\right\}, \quad x \in S. \tag{138}$$

It is easily seen that a function $\varphi : S \to X$ is ξ-invariant if and only if φ is constant on $[x]_\xi$ for every $x \in S$.

Now, we are ready to present the following description of solutions of functional equation (132).

Corollary 33. *Let $\varphi : S \to X$ be ξ-invariant. Suppose that*

$$\omega(x) := \sum_{j=0}^{\infty} \left\|h\left(\xi^j(x)\right)\right\| < \infty, \quad x \in S. \tag{139}$$

Then, there exists a unique solution $\widetilde{\varphi} : S \to X$ of (132) *such that* (135) *holds. Moreover,*

$$\widetilde{\varphi}(x) = \varphi(x) - \lim_{n\to\infty} \sum_{k=0}^{n} h\left(\xi^k(x)\right), \quad x \in S. \tag{140}$$

Proof. Observe that (133) holds with

$$\epsilon(x) := \|h(x)\|, \quad x \in S, \tag{141}$$

and φ fulfils (134). Thus, it is enough to use Corollary 32. □

6. Stability of Intervals and Regions

In this section, we assume that *(H1)* and the following hypothesis (instead of *(H2)*) are valid:

(H3) $a : I \to (0, \infty)$ is a continuous function.

Then, $G_n(x) > 0$ for $x \in I$, where G_n is given by (88). For each $x_* \in I$, put $I_* := [\xi(x_*), x_*]$. Let $\varphi_0 : I_* \to \mathbb{R}$ be a continuous function such that

$$\varphi_0(\xi(x_*)) = a(x_*)\varphi_0(x_*). \tag{142}$$

Then, there exists a unique continuous function $\varphi_* : I \to \mathbb{R}$ satisfying (85) such that $\varphi_*(x) = \varphi_0(x)$ for $x \in I_*$ (see [60, Theorem 2.1]).

Czerni [65, 66] has considered stability and uniform stability of real intervals for (85). First, we present the results concerning the case where the studied intervals do not depend on x. Next, we proceed to the stability of regions, that is, to the case where the interval changes continuously with x.

For simplicity, let us restrict our attention to the case where the studied intervals have the form $[c, \infty)$ for some $c > 0$. The interval $[c, \infty)$ is called a stable interval of (85) if for every $\varepsilon > 0$ and every $x_* \in I$ there exists a $\delta > 0$ such that if a continuous function $\varphi_0 : I_* \to \mathbb{R}$ satisfies the condition

$$\varphi_0(x) > c - \delta, \quad x \in \left[\xi(x_*), x_*\right], \tag{143}$$

then for its extension $\varphi_* : I \to \mathbb{R}$ fulfilling equation (85) the condition

$$\varphi_*(x) > c - \varepsilon, \quad x \in \left[0, x_*\right] \tag{144}$$

holds (see [65, Definition 3]).

Theorem 34 (see [65, Theorem 4]). *Let $\xi : I \to I$, $a : I \to \mathbb{R}$ satisfy (H1) and (H3), respectively. Then, $[c, \infty)$, where $c > 0$, is a stable interval of* (85) *if and only if*

$$a(x) \ge 1, \quad x \in I. \tag{145}$$

To explain the above theorem, we suppose that $a(x_0) < 1$ for some $x_0 \in I$. By the continuity of a, we can assume without loss of generality that $x_0 \in (0, d)$. Then, there exists an $x_* \in I$ such that $x_0 \in (\xi(x_*), x_*)$. Let $\varepsilon := c - a(x_0)\, c$. Then, for each $\delta > 0$, if $\varphi_0(x_0) \in (c - \delta, c)$ for a continuous function $\varphi_0 : I_* \to \mathbb{R}$, then by (85)

$$\varphi_*(\xi(x_0)) = a(x_0)\varphi_0(x_0) = \frac{c - \varepsilon}{c}\varphi_0(x_0) < c - \varepsilon. \tag{146}$$

Therefore, the interval $[c, \infty)$ cannot be stable. In other words, if $a(x_0) < 1$ for some $x_0 \in I$, then we can take such I_* and $\varphi_0 : I_* \to (c-\delta, \infty)$ that $\varphi_*(\xi(x_0)) < c - \varepsilon$.

The condition that $a(x) \geq 1$ for $x \in I$ implies that for each solution φ of (85) and each $x_0 \in I$ we have

$$\varphi(x_n) = G_n(x_0)\varphi(x_0) \geq \varphi(x_0), \quad n \in \mathbb{N}, \tag{147}$$

where x_n is given by (109). Hence, if $\varphi(x_0) \geq c - \delta$ for an $x_0 \in I$ and a $\delta > 0$, then $\varphi(x_n) \geq c - \delta$ for all $n \in \mathbb{N}$. Consequently, for any $\varepsilon > 0$, we can take any $\delta \leq \varepsilon$ in (143) to obtain (144). Moreover, such a δ does not depend on the choice of x_*. Furthermore, by (147), we obtain that $[c, \infty)$ is an invariant set.

The condition $a(x) \geq 1$ for $x \in I$ in Theorem 34 can be slightly weakened. In the case where $a(x) \geq 1$ for all x from a vicinity of 0, we can replace, in Theorem 34, I with the interval $(0, s]$, where s is arbitrarily taken from this vicinity.

A different situation is if we consider the problem of interval stability for some particular $x_* \in I$. It may happen that, in the case where condition (145) does not hold, we can still find for all $\varepsilon > 0$ and for some $x_* \in I$ a $\delta > 0$ such that (143) implies (144) for every φ_0 satisfying (142). More precisely, for a fixed $x_* \in I$, to obtain the stability of $[c, \infty)$, we need to assume that $G_n(x) \geq 1$ for $x \in I_*$. Then, by (147),

$$\varphi(x_n) \geq \varphi(x_0) > c - \delta \geq c - \varepsilon \tag{148}$$

if $\delta \leq \varepsilon$. We will say that the interval $[c, \infty)$ is stable with respect to the set A if for all $x_* \in A$ and $\varepsilon > 0$ there exists such a $\delta > 0$ that (143) implies (144) for every φ_0 satisfying (142).

Theorem 34 can be now restated in the following form.

Theorem 35. *Let $\xi : I \to I$, $a : I \to \mathbb{R}$ satisfy (H1) and (H3), respectively. Then, $[c, \infty)$, where $c > 0$, is a stable interval of* (85) *with respect to an $x_* \in I$ if and only if*

$$G_n(x) \geq 1, \quad x \in [\xi(x_*), x_*]. \tag{149}$$

Czerni [65] has also considered the stability of regions of the form

$$[\psi, \infty) := \{(x, y) \in \mathbb{R}^2 : 0 < x \leq d, \psi(x) \leq y\}, \tag{150}$$

where $\psi : I \to \mathbb{R}$ is a continuous function which satisfies the inequality

$$\psi(\xi(x)) \leq a(x)\psi(x), \quad x \in I. \tag{151}$$

The constant interval $[c, \infty)$ is now replaced by interval $[\psi(x), \infty)$ varying continuously with x. Let us note that if $a(x) \geq 1$ for $x \in I$, then for any $c > 0$ the constant function given by $\psi(x) = c$ satisfies inequality (151).

Using the assumption that the function ψ fulfills inequality (151), Czerni proved the following theorem.

Theorem 36 (see [66, Theorem 2.2]). *Let $\xi : I \to I$, $a : I \to \mathbb{R}$ satisfy (H1) and (H3), respectively, and $x_* \in I$. Assume that $\psi : I \to \mathbb{R}$ is a continuous solution of inequality* (151). *Then, if there exists a $k \in \mathbb{N}$ such that the region $[\psi, \infty)$ is stable with respect to $\xi^k(x_*)$, then $[\psi, \infty)$ is stable with respect to each $x_0 \in [\xi^k(x_*), x_*]$.*

The above theorem has been used in the proof of the next result about stability of the region $[\psi, \infty)$.

Theorem 37 (see [66, Theorem 3.2]). *Let $\xi : I \to I$, $a : I \to \mathbb{R}$ satisfy (H1) and (H3), respectively, and $x_* \in I$. Assume that for a continuous solution $\psi : I \to \mathbb{R}$ of inequality* (151) *there exists a $k \in \mathbb{N}$ such that*

$$\psi(\xi(x)) < a(x)\psi(x), \quad x \in [\xi^{k+1}(x_*), \xi^k(x_*)]. \tag{152}$$

Then, the region $[\psi, \infty)$ is stable with respect to each $x_0 \in [\xi^k(x_*), x_*]$.

Let us note that, the assumption on function ψ is the counterpart of condition (149) in the case of stability of $[c, \infty)$. In the proof of the above theorem, it is showed that inequality (152) implies that the region $[\psi, \infty)$ is stable with respect to $\xi^k(x_*)$. Indeed, the compactness of $I_k := [\xi^{k+1}(x_*), \xi^k(x_*)]$ gives that there exists a positive minimum of $\psi(x) - (\psi(\xi(x))/a(x))$ over I_k. Taking any $\delta > 0$ smaller than this minimum (and, of course, smaller than a given ε), we obtain the stability of $[\psi, \infty)$.

We say that solutions of (85) depend continuously on initial conditions if, for each solution $\varphi : I \to \mathbb{R}$ of (85), each $x_* \in I$ and, for an arbitrary sequence $(\varphi_{0,n})_{n\in\mathbb{N}}$ converging uniformly to $\varphi|_{I_*}$, where each element $\varphi_{0,n} : I_* \to \mathbb{R}$ of the sequence is a continuous function satisfying (142), the sequence $\varphi_n : I \to \mathbb{R}$ of the extensions of the functions $\varphi_{0,n}$ fulfilling equation (85) tends uniformly to φ on the interval $(0, x_*]$ (see [66, Definition 1.2]).

In the case where solutions of (85) depend continuously on initial conditions, we have the following result.

Theorem 38 (see [66, Theorem 3.3]). *Let $\xi : I \to I$, $a : I \to \mathbb{R}$ satisfy (H1) and (H3), respectively. Assume that solutions of* (85) *depend continuously on initial conditions. If $\psi : I \to \mathbb{R}$ is a continuous solution of inequality* (151), *then the region $[\psi, \infty)$ is stable with respect to each $x_* \in (0, d)$.*

However, without assuming continuous dependency on initial conditions the following theorem holds.

Theorem 39 (see [67, Theorem 2.1]). *Let $\xi : I \to I$, $a : I \to \mathbb{R}$ satisfy (H1) and (H3), respectively, and $x_* \in I$. Assume that $\psi : I \to \mathbb{R}$ is a continuous solution of inequality* (151). *Then, the following conditions are equivalent.*

(i) *The region $[\psi, \infty)$ is not stable with respect to x_*.*

(ii) *There exists a sequence $(x_{0,k})_{k\in\mathbb{N}}$ of elements of I_* and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that*

$$\lim_{k\to\infty} G_{n_k}(x_{0,k}) = +\infty, \qquad \lim_{k\to\infty} \left(\psi(x_{0,k}) - \psi_{n_k}(x_{0,k})\right) = 0, \tag{153}$$

where ψ_{n_k} is given by

$$\psi_{n_k}(x) := \frac{\psi(\xi^{n_k}(x))}{G_{n_k}(x)}, \quad x \in I_*. \tag{154}$$

The results concerning the interval stability of (85) presented above and similar results for (12) (see [67, 68]) have been motivated by Shanholt's paper [69] concerning the stability of sets for difference equations. To compare these results with stability results in the theory of difference equation, see, for example, [70–72].

7. Nonstability

It seems to be difficult to give a suitable (but simple) definition of nonstability of functional equations; some examples of such definitions can be found in [54, 57, 73–76]. Probably, it should refer to Definition 2 and therefore also to the operator $\mathcal{T}$. Thus, we should speak of $\mathcal{T}$-nonstability. Below, we present an example of such a nonstability result for a linear difference equation (as before X stands for a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$).

Theorem 40 (see [74, Theorem 1]). *Assume that $\{\overline{a}_n\}_{n\in\mathbb{N}_0}$ is a sequence in $\mathbb{F} \setminus \{0\}$, $\{b_n\}_{n\in\mathbb{N}_0}$ is a sequence in X, and $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ is a sequence of nonnegative real numbers such that*

$$\lim_{n\to\infty} \frac{\varepsilon_n |\overline{a}_{n+1}|}{\varepsilon_{n+1}} = 1. \tag{155}$$

Then, there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying

$$\|x_{n+1} - \overline{a}_n x_n - b_n\| \le \varepsilon_n, \quad n \in \mathbb{N}_0 \tag{156}$$

and such that, for every sequence $\{y_n\}_{n\in\mathbb{N}_0}$ in X, given by

$$y_{n+1} = \overline{a}_n y_n + b_n, \quad n \in \mathbb{N}_0, \tag{157}$$

we have

$$\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\varepsilon_{n-1}} = \infty. \tag{158}$$

Clearly, Theorem 40 shows that (under assumption (155)) difference equation (157) is not $\mathcal{T}$-stable, for instance, for every operator $\mathcal{T} : (0,\infty)^{\mathbb{N}_0} \to (0,\infty)^{\mathbb{N}_0}$ such that

$$\mathcal{T}\left(\{\varepsilon_n\}_{n\in\mathbb{N}_0}\right) = \{\gamma_n \overline{\varepsilon}_n\}_{n\in\mathbb{N}_0}, \quad \{\varepsilon_n\}_{n\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0} \tag{159}$$

with a bounded sequence $\{\gamma_n\}_{n\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, where $\overline{\varepsilon}_n = \varepsilon_{n-1}$ for $n \in \mathbb{N}$ and $\overline{\varepsilon}_0 > 0$ can be completely arbitrary.

There arises a natural question if we can replace condition (155) by one of the following two conditions:

$$\liminf_{n\to\infty} \frac{\varepsilon_n |\overline{a}_{n+1}|}{\varepsilon_{n+1}} = 1, \qquad \limsup_{n\to\infty} \frac{\varepsilon_n |\overline{a}_{n+1}|}{\varepsilon_{n+1}} = 1. \tag{160}$$

It follows from [74, Examples 1–4] that this is not possible.

For further examples of similar nonstability results (also for other equations), we refer the reader to [54, 57, 73–76].

8. Multivalued Solutions

The issue of stability of functional equations in one variable has been investigated also for multivalued functions, and for suitable results we refer the reader to [77–80].

In this part of the paper, we present only one example of such results (on selections of set-valued maps satisfying linear inclusions), which is closely connected to the issue of stability of the corresponding functional equations.

Let S be a nonempty set and (Y, d) be a metric space. We will denote by $n(Y)$ the family of all nonempty subsets of Y. The real number

$$\delta(A) := \sup \{d(x, y) : x, y \in A\} \tag{161}$$

is said to be the *diameter* of a nonempty set $A \subset Y$. Given $F : S \to n(Y)$, we write cl F for the multifunction defined by

$$(\mathrm{cl}\, F)(x) := \mathrm{cl}\, F(x), \quad x \in S. \tag{162}$$

Each $f : S \to Y$ with

$$f(x) \in F(x), \quad x \in S, \tag{163}$$

is said to be a *selection* of the multifunction F.

The following result has been obtained in [77].

Theorem 41 (see [77, Theorem 2]). *Let $F : S \to n(Y)$, $\Psi : Y \to Y$, $\xi : S \to S$, $\lambda \in (0,\infty)$, and*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x, y), \quad x, y \in Y, \qquad \lim_{n\to\infty} \lambda^n \delta(F(\xi^n(x))) = 0, \quad x \in S. \tag{164}$$

(1) *If Y is complete and*

$$\Psi(F(\xi(x))) \subset F(x), \quad x \in S, \tag{165}$$

then, for each $x \in S$, the limit

$$\lim_{n\to\infty} \mathrm{cl}\left(\Psi^n \circ F \circ \xi^n\right)(x) =: f(x) \tag{166}$$

exists and f is a unique selection of the multifunction cl F such that

$$\Psi \circ f \circ \xi = f. \tag{167}$$

(2) *If*

$$F(x) \subset \Psi(F(\xi(x))), \quad x \in S, \tag{168}$$

then F is a single-valued function and

$$\Psi \circ F \circ \xi = F. \tag{169}$$

For a survey on further similar results, we refer the reader to [81].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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