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ON THE FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS OF HALF-INTEGRAL WEIGHT OVER ARBITRARY ALGEBRAIC NUMBER FIELDS

By
Hisashi KOJIMA

Abstract. In Theorem 2.5 in previous paper [4], we determined the Fourier coefficients of the image of Shimura correspondence of modular forms \( f \) of half integral weight over arbitrary algebraic number fields in terms of those of \( f \). It seems that there is a gap in the proof. We give a correct proof of Theorem 2.5 in [4]. Moreover, we deduce useful formulas between the product of Fourier coefficients of \( f \) and the central value of quadratic twisted \( L \)-series associated with the image of Shimura correspondence of \( f \).

Introduction

Shimura [7] proved that the square of Fourier coefficients of a holomorphic Hilbert modular form of half-integral weight over a totally real number field gives essentially the critical value of the zeta function of the corresponding form of integral weight, which generalizes a previous result of Waldspurger [9] in the elliptic modular case. In [3] and [4], we extended Shimura [6] and [7] in the case of Hilbert modular forms of half-integral weight over arbitrary algebraic number fields. It seems that there is a gap in the proof of Theorem 2.5 in [4].

The purpose of this note is to deduce another useful formula between the product of Fourier coefficients of a modular form \( f \) of half-integral weight over an arbitrary algebraic number field and the central value of quadratic twisted \( L \)-series associated with the image of Shimura correspondence of \( f \). In the last section, we shall give a correct proof of Theorem 2.5 in [4].
§ 1. Fourier Coefficients of Modular Forms of Half-Integral Weight

Our notation follows closely that of [2], [4], [5] and [7]. Let \( x \in G_0 \) (resp. \( C_n' \)) be the element (resp. set) given in [4, pp. 29–30]. Take \( x \in G \cap U e^{-1} \), for \( U \) a sufficiently small open subgroup of \( C_n' \). Let \( f \) be an element of \( \mathcal{S}_{m+1/2} \), where \( \mathcal{S}_{m+1/2} \) is the space given in [4, p. 31] and [5, (2)]. Then define the inversion \( f^* \) of \( f \) by

\[
(1.1) \quad f^* = \psi(x) f \mid_{m+1/2}.
\]

Here \( \delta \) and \( f \mid_{m+1/2} \) are given in [4, p. 30] and [4, (1.16)]. We see that \( f^* \) belongs to \( \mathcal{S}_{m+1/2} \) (cf. [2, (4.19)]). Take a \( f \in \mathcal{S}_{m+1/2} \) (b, b'; \( \psi \)). Let \( \tau \) be an element of \( F^\times \) such that \( \tau > 0 \), \( \tau b = q^2 \tau \) with a fractional ideal \( q \) and a square free integral ideal \( I \). From [4, Lemma 1.2], we find an element \( h \in \mathcal{S}_{m+1/2} \) (b, b'; \( \phi \)) such that

\[
(1.2) \quad \mu_q(\xi, m) = \mu_f((r, q^{-1} m))
\]

for every \( \xi \in F^\times \) and fractional ideal \( m \) in \( F \), where \( \phi = \psi \), with the Hecke character \( \psi \), associated with the quadratic extension \( F(\sqrt{\xi})/F \). Let \( D \) be the set given in [4, (1.9)]. Define a function \( \Psi_{\tau, \xi}(f)(w) \) on \( D \) by

\[
(1.3) \quad \Psi_{\tau, \xi}(f)(w) = \int_{\mathfrak{C} \setminus D} \Theta_1(\xi, w; \eta_2) \Theta(-z)^{m+1/2} m \, d\chi
\]

for every \( w \in D \), where \( C = i^{m+2} \Gamma_{\tau} \Gamma_{\xi} \phi(\tau, \xi) N(\tau, \xi) \), \( \Gamma_{\tau} \) and \( \Theta_1(\xi, w; \eta_2) \) are given in [4, p. 39]. We deduced the following theorem [4, (2.33)].

**Theorem 0.1.** Let \( f \) be an element of \( \mathcal{S}_{m+1/2} \) (b, b'; \( \psi \)). Then

\[
(1.4) \quad \Psi_{\tau, \xi}(f)(w) = N(t, i, i) \sum_{m} \sum_{l \in \mathfrak{C} \setminus \mathfrak{C}} N(m) l^{m-1/2} \phi(l) \phi^*(l, i, i, m) \\times \mu_f((r, q^{-1} m)) e_1(f\mathfrak{C}(z)) e_1(lu) \prod_{i} c(\text{sgn}(l)) \\times \exp(-2\pi i \mathfrak{C}(z)vK_{2n}(4\pi l, llv))
\]

where \( m \) runs over all integral ideals, \( l \) runs over \( t, i, i, m \) under the condition \( (h_i^{-1} m, t, i, i) = 1 \), \( w = (z_1, \ldots, z_{i+1}, \ldots, z_{i+2}) \), \( z = (z, \ldots, z) \), \( \delta_i = u_i + i j u_i (1 \leq i \leq l) \), \( a = (u_{i+1}, \ldots, u_{i+2}) \), \( v = (u_{i+1}, \ldots, u_{i+2}) \), \( l^{m-1} = \prod_{i} (l)\phi(l) \phi(l) \phi(l) \phi(l) \phi(l) \), and \( |l| = \prod_{i=1}^n |l|^{l} \).
We shall give a correct proof of Theorem 0.1, that is, Theorem 2.5 in [4] in Section 2.

We showed the following in [4, pp. 47-48].

THEOREM 0.2. Let $f$ be an element of $S_{2m,\omega}(b, b'; \psi)$. Suppose that $f$ is a common eigenform of $T_v$ for each $v \in \mathfrak{h}$, i.e.,

$$(1.5) \quad f|T_v = \chi(v)N_v^{-1}f$$

for each $v \in \mathfrak{h}$.

Then there exists the normalized eigenform $g$ belonging to $S_{2m,\omega}(2^{-1}c, \psi^2)$ attached to $\chi$ such that

$$(1.6) \quad \mu_f(r, a^{-1}) g = (\gamma, \ldots, \gamma)$$

where $\gamma = (0, \ldots, 0, 4\omega_0 + 3, \ldots, 4\omega_r + 3)$ with $\omega = (0, \ldots, 0, \omega_{r+1}, \ldots, \omega_{r+s})$.

Let $g$ be the above element of $S_{2m,\omega}(2^{-1}c, \psi^2)$ in Theorem 0.2. Take the matrix $\pi = \begin{pmatrix} s & 1 \\ 0 & 1 \end{pmatrix}$ with $s \in F_f^\times$ such that $\omega = 2^{-1}c$. Define

$$(1.7) \quad (J_2, g)(p) = \psi(\det p)^{-1} g(p\pi)$$

for every $p \in \mathfrak{G}_A$.

Then $J_2 g$ belongs to $S_{2m,\omega}(2^{-1}c, \psi^{-2})$. We put $g^* = J_2^{-1} g = (g^*_1)$.

Here we assume the following condition.

$$(1.8) \quad \begin{array}{l}
(i) \quad \psi_p(x) = (\text{sgn } x_p)^{m_p} |x_p|^{2\mu_p} (x \in F^\times_p), \text{ where } (\text{sgn } x_p)^{m_p} = \\
\prod_{i=1}^{m_p} |x_p|^{|\sum_{j=1}^{\infty} |x_p|x_p|^{2\mu_p}} (x_0, \ldots, x_{r_1}) \in F^\times_0, \\
|\mu_p|^{2\mu_p} = \prod_{i=1}^{m_p} |\mu_p|^{2\mu_p} (x_0, \ldots, x_{r_1}) \in \mathbb{R}^{m_p} \text{ and } \sum_{i=1}^{m_p} \lambda_i + \sum_{i=1}^{m_p} \mu_{r_1+i} = 0.
\end{array}$$

(ii) If $v$ is a common prime of 2 and $r$, then $\varphi_e$ satisfies either

(a) $(\tau r)_e = h_e = 4r$ and $\varphi_e(1 + 4x) = \varphi_e(1 + 4x^2)$ for every $x \in \mathfrak{a}_e$, or

(b) $(\tau r)_e \neq h_e < 4r$.

(iii) If $f' \in S_{2m+(1/2)m_r,\omega}(b, b'; \psi)$ and $f'|T_{r} = N_{r}^{-1} \chi(v)x'$ for every $v \not\mid h^{-1}r$, then $f'$ is a constant times $f$.

We shall deduce the following theorem.

THEOREM 1. Let $f \in S_{m+(1/2)m_r,\omega}(b, b'; \psi)$ be an eigenform of all Hecke operators $T_v$ satisfying $f|T_v = N_v^{-1} \chi(v)f$. Suppose that $f$, $\tau$, $h$, $c$, $\psi$ and $\varphi$ satisfy the condition (1.8), and $g$ and $g^*$ are the elements in Theorem 0.2. Then
\[ \mu(\tau, q^{-1}b; f, \psi) \mu(\tau, q^{-1}b; f^*, \bar{\psi}) \langle g, g \rangle / \langle f, f \rangle \]
\[ = Q \sum_{\sigma \in \Gamma} \mu(\tau) \psi(\tau) N(\tau) \Gamma(\tau) D(0, g, \varphi, t^{-1}h^{-1}r), \]
where \( D(0, g, \varphi, t^{-1}h^{-1}r) \) is given in \([4, \text{ p. } 37]\), \( Q = 2^{(r/2)-(m)+3n-1} \pi^{-m} \).

Let \( \eta \) be an element in \([4, \text{ p. } 38]\). Put \( \hat{h}(\xi) = \langle \Theta(\xi, \rho; \eta, g(p)) \rangle \), where \( \Theta(\xi, \rho; \eta, g(p)) \) is the function given in \([4, (2.4)]\) and \( g \) is the function given in Theorem 0.2. By \([7, \text{ Proposition } 5.8]\) and \([2, \text{ Theorem } 5.2 \text{ and the arguments in p. } 440]\), we have
\[ \hat{h}(\xi) = A h(\xi) \]
with a constant \( A \) under the assumption (1.8), where \( h(\xi) \) is the function given in (1.2). Since \( \langle h, h \rangle = e^{m^{-1}/2} \varphi(\xi) N(\rho)^{-1} \psi(\xi)^{-1} \langle f, f \rangle \) and
\[ \langle \Theta(\xi, \rho; \eta, g(p)) \rangle = \int \Theta(\xi, \rho; \eta, g(p)) \rho(p) \frac{1}{\rho(\xi)} \varphi(\xi) \psi(\xi) N(\rho)^{-1} \psi(\xi)^{-1} \langle f, f \rangle, \]
we obtain
\[ \hat{h}(\xi) = A h(\xi) \]
with \( \Phi, h, C \) as in \([4, \text{ p. } 39]\). As shown at \([7, \text{ p. } 340]\), \( A h(\xi) = \langle \Theta(\xi, \rho; \eta, g(p)) \rangle \) implies that
\[ \hat{h}(\xi) = \langle \Theta(\xi, \rho; \sigma), g^*(p) \rangle = \sum \langle \Theta(\xi, w; \sigma, g^*(p)) \rangle, \]
where \( \sigma, g^* \) (resp. \( \Theta(\xi, \rho; \sigma) \)) is the symbol given in \([7, (6.2)]\) (resp. \([4, (2.4)]\)).

Given a function \( f \) on \( D \) and \( z = (\xi, \sigma) \) in \( G \), we put
\[ f_{\mid \sigma}(\sigma(\xi)) = (c_{\xi} + d_{\xi})^{-1} f(\sigma(\xi)), \]
where \( \sigma = (\xi_1, \ldots, \xi_n, \delta_{n+1}, \ldots, \delta_{n+k}) \), \( z = (\xi_1, \ldots, \xi_n) \) and \( \delta_{n+i} = z_{n+i} + j w_{n+i} \).

Let \( \Gamma = \Gamma[z, b] \) (cf. \([4, \text{ p. } 29]\)). We put
\[ f_{\mid \sigma}(\xi) = (c_{\xi} + d_{\xi})^{-1} f(\xi), \]
where \( \xi = (z_1, \ldots, z_n) \).

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\begin{equation}
E(\mathfrak{f}, \varphi; \Gamma) = \sum_{z \in R} \varphi_{\sigma}(d_z) \psi(\varphi(d_z) \Phi^{-1}) N(\mathfrak{f})^{2s} \zeta_{\varphi}(d_z, 2^m z + (i\delta - m)/2, w, 2^m z + iy)_{|m|z}
\end{equation}

\[ C(\mathfrak{f}, \varphi; \Gamma) = L_{2s}(2s, \varphi) E(3, \varphi; \Gamma) \]

Here \( R \) is a set of representatives for \( P \setminus \{ G \cap P \Delta]\{x, h\} \), for \( x \in R \), we define \( \mathfrak{f}_x \) by writing \( \alpha = p^x \) with \( p \in P \) and \( x \in D(x, h) \), and setting \( \mathfrak{f}_x = d_p \). We put

\begin{equation}
L_{2s}(s, \varphi) = \sum_{m=0}^{\infty} \varphi^*(m) N(m)^{-s}.
\end{equation}

We obtain the following proposition.

\textbf{Proposition 2.} Let \( \Gamma = \Gamma[2^{-1}]\{1, 2, 3\} \) and let \( \mathcal{A}(\varphi) \) be the function in [4, (4.1)]. Then

\begin{equation}
\int_{\Gamma \setminus \mathcal{A}} h(\mathfrak{f}) \mathcal{A}(\varphi) E(3, \varphi + 1/2, \Gamma) \zeta^{m+(1/2)} v_{1/2} w^2 d \mathfrak{f} = D_{\varphi}^{1/2} \mathcal{A}(\varphi) \mathcal{A}(\psi) (2\pi)^{-2m_2 - m_1 + (1/2)} e^{-2m_2 + (1/2)v} \times \Gamma(2m_2 - m_1 + (1/2)v)
\end{equation}

By (1.13) and Proposition 2, we see that \( \mathcal{A} \) times the integral in (1.17) is equal to

\begin{equation}
\sum_{\mathfrak{f}} \left( \int_{\Gamma \setminus \mathcal{A}} \mathcal{A}(\varphi) \Theta^2(\mathfrak{f}, w; \sigma_z) E(3, \varphi + 1/2, \Gamma) \zeta^{m+(1/2)} v_{1/2} w^2 d \mathfrak{f}, \mathcal{A}(\varphi) \right).
\end{equation}

By the same method as that of [7, pp. 543–544], we have the following equation (cf. [4, (4.19)])

\begin{equation}
A N(\eta r)^{-1} \mathcal{A}(\mathfrak{f})^{-1} \mathcal{A}(\varphi) \mathcal{A}(\psi) \zeta^{m+(1/2)} v_{1/2} \pi^{1/2} \pi^{1/2}
\end{equation}

By the same method as that of [7, pp. 543–544], we have the following equation (cf. [4, (4.19)])

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\end{equation}
\[ \times \Gamma'(2a+n, -i\mu + i\nu) \sum_{m} \mu_f, (\tau, q^{-1}\text{int}) N(m)^{-2\tau} \]

\[ = \sum_{\ell} \left( \sum_{\mathfrak{p} \in B} \psi^* (\mathfrak{p}) N(\mathfrak{p})^{2\ell + 1} S_{\mathfrak{p}} (w, \bar{s}, \theta_{\mathfrak{p}} (w)) \right), \]

where \( B \) is determined by \( G \cap P_a D [2^{-1} b^{-1} \mathfrak{n}, 2b] = \prod_{\mathfrak{p} \in B} P_{\mathfrak{p}} \Gamma \). The ideals \( \mathfrak{p} \) are as in (1.15), and run through a set of representatives for the ideal class group of \( F \). Here

\[ (1.20) \quad S_{\mathfrak{p}} (w, s) = \sum_{\zeta, b} \sigma_1 (\gamma \zeta) \mu_f (b) |\zeta, w|^{-m} |\zeta, w|/ |\eta(w)|^{-2m_{\mathfrak{p}} - n_{\mathfrak{p}} + mi} \]

\[ \times \left| \frac{\zeta + b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\eta(w)} \right|^{-2(m_{\mathfrak{p}} + n_{\mathfrak{p}} + mi)}, \]

where \([*, *]\) and \( \eta(*)\) are symbols given in [4, (2.3)] and the sum is over the pairs \( (\zeta, b) \in V \times \mathfrak{p} / \sigma^x \) such that \( \zeta \neq 0 \) and \( \det \zeta = -b^2 \) with \( V = \{ \zeta \in M_2 (F) | \text{tr} \zeta = 0 \} \). Furthermore, we have chosen \( \gamma \in F_x^* \) such that \( \gamma v = \mathfrak{p} \gamma \) and \( \gamma_s = 1 \) for \( v \in \mathfrak{p} \). By [7, 7.14a, 7.14b], we have the following.

**Proposition 3.** Let \( q \) range through a set of representatives for \( 2^{-1} t_x \text{rch}/t_x \text{rch} \) and let \( \Gamma^x = \Gamma [2^{-1} b^{-1}, t_x \text{rch}] \). Then there exist functions \( T_{\mathfrak{p}} (w, s) \) such that

\[ (1.21) \quad S_{\mathfrak{p}} (w, s) = (-1)^{[m]} 2^n \sum_q T_{\mathfrak{p}} (w, s) (q, 0, 1) \]

\[ \sum_{\mathfrak{p} \in B} \psi^* (\mathfrak{p}) N(\mathfrak{p})^{2\ell} T_{\mathfrak{p}} (w, s - 1/2) = N(2^{-1} t_x \text{rch})^{2\ell} C(w, s; \Gamma \gamma) E(w, s; \Gamma \gamma) \]

By Proposition 3, we find that the expression of (1.19) is equal to the value at \( s = t \) of

\[ (1.22) \quad (-1)^{[m]} 2^d \sum_{\lambda} \langle N(2^{-1} t_x \text{rch})^{4 + t + 1} C(\lambda, s + 1/2; \Gamma \gamma) E(\lambda, t + 1/2; \Gamma \gamma), \theta' (\lambda) \rangle. \]

The equality (1.22) becomes

\[ (1.23) \quad (-1)^{[m]} 2^d \text{vol} (\Gamma [2b^{-1}, 2^{-1} \text{rch}] \setminus D)^{-1} \sum_{\lambda} N(2^{-1} t_x \text{rch})^{4 + t + 1} \]

\[ \times \int_{\Gamma \setminus D} \frac{C(\lambda, s + 1/2; \Gamma \gamma) E(\lambda, t + 1/2; \Gamma \gamma) \theta' (\lambda) \gamma^{2m}}{d\lambda}. \]
The integral appeared in (1.23) is equal to

\[ \int_{\Gamma \backslash \mathcal{O}} \frac{C(z)E(z, i + 1/2; \Gamma')}{\Gamma'} \, d\gamma = \sum_{\varphi \neq \psi} \phi_\varphi(d_2) \phi_\psi^*(d_2 \mathcal{O}_z^{-1}) N(\mathcal{O}_z)^{2s+1} \times \int_{\Psi_\varphi} \gamma_\varphi^2(\delta) C_\varphi(z) \gamma^{n+2(\mu+3m-i\zeta)/2} w^{2(i+1/2)\mu+\mu} \, d\mu, \]

where \( \Psi_\varphi = \Gamma \cap \mathcal{O}_z^{-1} \backslash \mathcal{O} \), \( g_\varphi = g_\varphi \|_{2m} \mathfrak{a}^{-1} \) and \( C_\varphi(z) = C(z, i + 1/2; \Gamma') \|_{m} \mathfrak{a}^{-1} \).

By [7, Lemma 3.8], we have

\[ \int_{\Psi_\varphi} g_\varphi^2(\delta) C_\varphi^2(z) \gamma^{n+2(\mu+3m-i\zeta)/2} w^{2(i+1/2)\mu+\mu} \, d\mu = \frac{1}{\mu} \log(\delta^2 \gamma^{n+2(\mu+3m-i\zeta)/2} w^{2(i+1/2)\mu+\mu}) \gamma \frac{\mu}{\delta} \gamma^{n+2(\mu+3m-i\zeta)/2} w^{2(i+1/2)\mu+\mu} \, d\mu, \]

and

\[ \phi_\varphi(d_2) \phi_\psi^*(d_2 \mathcal{O}_z^{-1}) N(\mathcal{O}_z)^{-2s+1} \gamma^{n+2(\mu+3m-i\zeta)/2} w^{2(i+1/2)\mu+\mu} C_\varphi(z) \]

where \( \phi_\varphi(d_2) \phi_\psi^*(d_2 \mathcal{O}_z^{-1}) N(\mathcal{O}_z)^{-2s+1} \gamma^{n+2(\mu+3m-i\zeta)/2} w^{2(i+1/2)\mu+\mu} C_\varphi(z) \)

where \( \Phi_\varphi \) is given in [4, (1.36) and (1.37)] and [2, p. 409] for a fractional ideal \( \mathfrak{m} \) and a signature \( \sigma \in \{ \pm 1 \}^n \), and \( \zeta(y, w, bh; \delta u + (u + m + i\zeta)/2, \delta u + (u - m + i\zeta)/2, 2(s + 1/2) + i\mu) \) is given in [4, (3.21)].

We note the formula (cf. [1, p. 334]),

\[ \int_{\mathfrak{m}} y^s K_v(y) K_{-v}(y) \, dy = 2^{i-2} \frac{\Gamma(\frac{1-s^+ + s^+ + 1}{2}) \Gamma(\frac{1-s^- + s^- + 1}{2}) \Gamma(\frac{1-s^+ + s^- + 1}{2}) \Gamma(\frac{1-s^+ + s^- + 1}{2})}{\Gamma(i+1)} \]

By the same method as that of [4, p. 59], we see that the integral (1.24) is equal to

\[ \text{(1.26)} \int_{\mathfrak{m}} y^s K_v(y) K_{-v}(y) \, dy = 2^{i-2} \frac{\Gamma(\frac{1-s^+ + s^+ + 1}{2}) \Gamma(\frac{1-s^- + s^- + 1}{2}) \Gamma(\frac{1-s^+ + s^- + 1}{2}) \Gamma(\frac{1-s^+ + s^- + 1}{2})}{\Gamma(i+1)} \]
\begin{equation}
(1.27) \quad \bar{\gamma}(\phi) 2^{\gamma} \mathcal{N}(2x, b^{-1} d^{-1}) \sum_{x} \mathcal{N}(x) 2^{x+2i} \sum_{f, a} \mu(f) \bar{\phi}^{*}(f) \mathcal{N}(1)^{1-x} \mathcal{N}(\eta)^{2x} \times \sum_{a \in F, a \neq 0} c(a) \mathcal{N}(a) \mathcal{N}(b)^{-2a} \times \frac{\phi^{*}(ab^{-1}) \mathcal{N}(2\pi)^{1/2} \gamma^{*}(\mathcal{N}(\eta)^{1-x})}{\gamma^{*}(\mathcal{N}(\eta)^{1-x})} \times \Gamma^{*}((t - s)u_{r} + v + (1/2)u_{z})^{-1} \times \Gamma^{*}((t + s)u_{r} - v + (1/2)u_{z})^{-1} \times \Gamma^{*}((t + s)u_{r} - v + (1/2)u_{z}) M(s, t). \tag{1.28}
\end{equation}

where

\begin{equation}
M(s, t) = \int_{y > 0} \exp(-2\pi y) \xi(y, t; \mathcal{N}(\eta)^{1-x}) dy \tag{1.29}
\end{equation}

Here \(\xi(y, t; \mathcal{N}(\eta)^{1-x})\) is the function in \([7, p. 530]\). Therefore we find that the equality (1.23) is equal to

\begin{equation}
(1.28) \quad (-1)^{l+m} 2^{\mu} \text{vol}(\mathbb{H}^{l+m} \setminus \mathbb{D})^{-1} \mathcal{D}^{\mu} \mathcal{D}^{\nu} \mathcal{N}(\eta)^{1-x} \times \sum_{\mathcal{N}(\eta)^{1-x}} \mu(f) \bar{\phi}^{*}(f) \mathcal{N}(1)^{1-x} \times \mathcal{N}(\mathcal{N}(\eta)^{1-x}) \times \sum_{m, n} c(t^{-1} \mathcal{N}(\eta)^{1-x}) \mathcal{N}(\mathcal{N}(\eta)^{1-x}) \times 2^{2u_{z} - i\eta} \frac{(4\pi)^{-u_{z} + (2x - 2)b^{-1} d^{-1}}}{2^{2u_{z} - i\eta}} \times \Gamma^{*}(2\mathcal{N}(\eta)^{1-x} + (1/2)u_{z})^{-1} \times \Gamma^{*}((t - s)u_{r} + v + (1/2)u_{z})^{-1} \times \Gamma^{*}((t + s)u_{r} - v + (1/2)u_{z})^{-1} \times \Gamma^{*}((t + s)u_{r} - v + (1/2)u_{z}) M(s, t) \tag{1.28}
\end{equation}

where \(u = (1, \ldots, 1)\). Put \(Y_{l}(s, t) = \sum_{m, n} c(t^{-1} \mathcal{N}(\eta)^{1-x}) \mathcal{N}(\mathcal{N}(\eta)^{1-x}). \)
We note that
\[
\lim_{s \to \pm \infty} Y_i(s, s) = D(0, g^*, \varphi, t^{-1} \text{htc})
\]
and
\[
M(s, s) = i^{\lfloor m \rfloor} 2^{-\lfloor u \rfloor - \lfloor m \rfloor + 3 \lfloor u \rfloor} \Gamma'(m)(2\pi)^{\lfloor m \rfloor - \lfloor m \rfloor} (2\pi)^{-(1/2)u} \\
\times 2^{-(1/2)u} \zeta(2m) \psi_i(s + (m - i\lambda)/2) \Gamma''(s + (1 + m - i\lambda)/2)^{-1}
\]
(cf. [7, (4.18)])

Therefore, by (1.12), (1.19) and (1.28), we have
\[
(1.29) \quad i^{\lfloor m \rfloor} 2^{-\lfloor u \rfloor + 3 \lfloor u \rfloor} (1/\sqrt{2\pi})^{\lfloor m \rfloor} \theta_d(1/2) \tau_s^{\lfloor u \rfloor + (1/2)u} |\tau_s|^{-3} N(qz)^{-1} \langle g, g \rangle \\
\times \mu_f(\tau, q^{-1}) \text{vol}(\Gamma[2b^{-1}, 2^{-1} \text{htc}] \backslash D)^{-1} \langle f, f \rangle^{-1} N(qz^{-1}) \\
\times 2^{-n/2 - \lfloor u \rfloor} 2^{-3/2} \psi_d(\tau) |\tau_s|^{(1/2)u} \psi_i(\tau) \pi^{n/2} \tau^{2\lfloor u \rfloor} 2^2 \tau^{2\lfloor u \rfloor} \\
\times \sum_m \mu_f(\tau, q^{-1} bm) N(m)^{-1} \\
= (-1)^{\lfloor m \rfloor} 2^d \text{vol}(\Gamma[2b^{-1}, 2^{-1} \text{htc}] \backslash D)^{-1} h(2\tau)^{-1} N(qz)^{-1} Y_i(s, s)(2\pi)^{2} 2^{-2n} \tau(4\pi)^{-u} \\
\times \Gamma''(v + 1/2) \Gamma'(-v + 1/2) i^{\lfloor m \rfloor} 2^{-\lfloor m \rfloor + i\lambda} \Gamma'(m) \\
\times (2\pi)^{\lfloor m \rfloor - \lfloor m \rfloor} (2\pi)^{-(1/2)u} \zeta(2m) 2^{-n/2} \zeta(1/2 + \lfloor m \rfloor - i\lambda).
\]

Letting \( s \) tend to \( +\infty \) we deduce our Theorem 1.

§ 2. A Correct Proof of Theorem 0.1

We use the notation in [4] and [5]. The changes of [4] are as follows:

(1) [4, (2.15)] should read
\[
ev_r(z) = \prod_{i=1}^{r} e^{-2\Re(\zeta^{(r+1)} u_{n+1})}. \quad ev_r(z/2) = \prod_{i=1}^{r} e^{\Re(\zeta^{(1)} z_{n+1})}.
\]

(2) [4, (2.24)] should read
\[
\text{This proposition implies that}
\]
\[
\mathfrak{S}(y(\beta^{-1}(\delta)))^{-n}\mathcal{G}_{m-n}(\beta^{-1}(\delta), tu, t(\beta^2)) \phi_n(t(\beta^2)/2)
\]
\[\times \varepsilon_s(\sqrt{-1}(\tau^2) 3\mathfrak{S}(\beta(\beta^{-1}(\delta)))^{-1}/4) \exp(-\pi(|t|v)^2w(\beta(\beta^{-1}(\delta)))^{-1})
\]
\[\times \mathfrak{S}(\beta^{-1}(\delta))^{m+1/2}m^1n^1(j(\beta, \beta^{-1}(\delta)))^n\mathcal{W}(\beta^{-1}(\delta))
\]
\[= (y'/j(y^{-1}\beta^{-1}, \delta')(j(y^{-1}\beta^{-1}, \delta'))^{-n}J_{m-n}(y^{-1}\beta^{-1}, \delta')
\]
\[\times \mathcal{G}_{m-n}(\delta', tu)\phi_n(t(\beta^2)/2)(y^{-1}\beta^{-1}, \delta')^{-n}
\]
\[\times \varepsilon_s(\sqrt{-1}(\tau^2) 3\mathfrak{S}(\beta'')^{-1}/4) \exp(-\pi(|t|v)^2w(\beta'^{-1}(\delta''))^{-1})h(y^{-1}\beta^{-1}(\delta''))
\]
\[\times \mathfrak{S}(y^{-1}\beta^{-1}(\delta'))^{m+1/2}m^1n^1w(y^{-1}\beta^{-1}(\delta'))^2;
\]

(3) The line 11 in [4, p. 44]:
\[
\phi_{\tau}(t(\beta^2)/2)\phi_{\tau}(\alpha, t) = \phi_{\tau}(t(\beta^2)/2)
\]
should read
\[
-\phi_{\tau}(t(\beta^2)/2)\phi_{\tau}(\alpha, t) = \phi_{\tau}(t(\beta^2)/2)
\]

(4) [4, (2.25)] should read
\[
(y')^{-n}\phi_{\tau}(t(\beta^2)/2)\mathcal{G}_{m-n}(\delta', tu)J_{m}(\beta, \beta^{-1}(\delta')) t^n h(\beta^{-1}(\delta'))
\]
\[\times \varepsilon_s(\sqrt{-1}(\tau^2) 3\mathfrak{S}(\beta'')^{-1}/4) \exp(-\pi(|t|v)^2w(\beta'^{-1}(\delta''))^{-1})\mathfrak{S}(\beta'^{-1}(\delta'))^{m+1/2}m^1n^1.
\]

(5) The element \( \ell \) in [4, (2.33)] runs over \( t, t_0, n \) under the condition that \( \ell \).

We sketch a correct proof of Theorem 0.1. Let \( f \) be an element of \( \mathcal{S}_{m+1}^{(1/2)}(b, b'; \psi) \). Since \( f \) is holomorphic with respect to \( z_1, \ldots, z_n \), the function \( g_{\tau, \lambda}(w) \) in [4, (2.11)] is holomorphic with respect to \( z'_1, \ldots, z'_{n+2} \), where \( w = (z'_1, \ldots, z'_n, \lambda_{n+1}, \ldots, \lambda_{n+2}) \) (cf. [4, p. 408], [4, (2.14)] and [5, (2)]). To determine the Fourier coefficients of \( g_{\tau, \lambda}(w) \), it is sufficient to calculate \( g_{\tau, \lambda}(w) \) for \( z'_1 = i y'_1, \ldots, z'_n = i y'_n (y'_1 > 0, \ldots, y'_n > 0) \). We put \( h_1 = 0, \ldots, h_{n+1} = 0 \) in [4, (2.15) and (2.16)]. By [6, pp. 772–777], [6, pp. 783–785], [8, pp. 1015–1024], [8, Theorem 1.2] and [8, Proposition 1.3], we can prove the proposition 2.3 in [4] in the case of \( (h_1, \ldots, h_{n+1}) = (0, \ldots, 0) \). We note [5, (6), (7), (8) and (9)]. By the same method as that of [4], we deduce
On the Fourier coefficients of Hilbert modular forms

\[ \Psi_{r,i}(f)(w) = N(t_i/t) \sum \sum_{m, l | t_i, t^{-1}m} N(m)l^{-1} |l|^{-1} \varphi_{r,l}(f) \varphi^*(f/t) \mu_f(r, (r \cdot q)^{-1} m) \]

\[ \times e_{\overline{r}}(hu) \prod_{i=1}^{\ell} \left( \frac{c(\text{sgn}(|l_i|)) \exp(-2\pi \Im(z)) v K_2, (4\pi |l|/v)}{2} \right) \]

for \( w = (iy_1', \ldots, iy_{r_i+1}', \ldots, iy_{r_i+r_j}') \), where \( m \) runs over all integral ideals, \( l \) runs over \( t_i \tau^{-1} m \) under the condition \( (m^{-1} r/t_i, r \tau) = 1 \), \( \bar{a}_{r_i} = u_{r_i+1} + \bar{b}_{r_i+1} \), \( z = (iy_1', \ldots, iy_{r_i}') \), \( u = (u_{r_i+1}', \ldots, u_{r_i+r_j}') \), \( v = (v_{r_i+1}', \ldots, v_{r_i+r_j}') \), \( l^{m-1} = \prod_{i=1}^{t_i} (|l_i|)^{m-1} \) and \( |l| = \prod_{i=1}^{t_i} |l_i^{r_i+1}| \). Therefore we deduce Theorem 0.1.

References


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