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Maximum asymmetry of copulas revisited

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Abstract: Motivated by the nice characterization of copulas A for which $d_\infty(A, A^t)$ is maximal as established independently by Nelsen [11] and Klement & Mesiar [7], we study maximum asymmetry with respect to the conditioning-based metric D_1 going back to Trutschnig [12]. Despite the fact that $D_1(A, A^t)$ is generally not straightforward to calculate, it is possible to provide both, a characterization and a handy representation of all copulas A maximizing $D_1(A, A^t)$. This representation is then used to prove the existence of copulas with full support maximizing $D_1(A, A^t)$. A comparison of D_1- and d_∞-asymmetry including some surprising examples rounds off the paper.

Keywords: Copula, exchangeability, symmetry, complete dependence, Markov kernel

MSC: 60E05, 62E10, 28A12, 62H05

1 Introduction

A pair (X, Y) of random variables is called exchangeable (or symmetric), if (X, Y) and (Y, X) have the same distribution. Obviously exchangeable random variables are necessarily identically distributed but not vice versa. Trying to quantify asymmetry of identically distributed (continuous) random variables, copulas naturally come into play. In fact, if X, Y have both distribution function F and F is continuous, by Sklar's theorem (X, Y) is exchangeable if, and only if, the copula A coincides with its transpose A^t (see Section 2). Independently of each other, Nelsen [11] as well as Klement & Mesiar [7] proved in 2006 that $d_\infty(A, A^t) \leq \frac{1}{3}$ with equality if, and only if, $A(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$ and $A(\frac{1}{3}, \frac{2}{3}) = 0$ or $A^t(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$ and $A^t(\frac{1}{3}, \frac{2}{3}) = 0$ holds (A^t as usual denoting the transpose, defined by $A^t(x, y) = A(y, x)$). As direct consequence, a copula A has maximal d_∞-asymmetry if, and only if,

$$\mu_A([0, \tfrac{1}{3}] \times [\tfrac{1}{3}, \tfrac{2}{3}]) = \mu_A([\tfrac{1}{3}, \tfrac{2}{3}] \times [\tfrac{2}{3}, 1]) = \mu_A([\tfrac{2}{3}, 1] \times [0, \tfrac{1}{3}]) = \tfrac{1}{3}$$

or

$$\mu_{A^t}([0, \tfrac{1}{3}] \times [\tfrac{1}{3}, \tfrac{2}{3}]) = \mu_{A^t}([\tfrac{1}{3}, \tfrac{2}{3}] \times [\tfrac{2}{3}, 1]) = \mu_{A^t}([\tfrac{2}{3}, 1] \times [0, \tfrac{1}{3}]) = \tfrac{1}{3}$$

is fulfilled, implying that the area of the support $supp(A)$ of each such copula A is at most $\frac{1}{3}$. For a discussion of the multivariate case we refer to [5].

In the current paper we replace d_∞ by the metric D_1 on the family of all two-dimensional copulas $\mathcal{C}$ as introduced in [12] and study analogous questions. Working with (mutually) completely dependent copulas we first show $\max_{A \in \mathcal{C}} D_1(A, A^t) = \frac{1}{2}$ and then provide a surprisingly simple and handy representation of all copulas A fulfilling $D_1(A, A^t) = \frac{1}{2}$. Based on this representation we then provide examples showing that, contrary to the case of maximal d_∞-asymmetry, copulas with maximal D_1-asymmetry can have full support,

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and study the interrelation of maximal d_∞- and D_1-asymmetry. In particular, we prove that for each A with maximal D_1-asymmetry we have $d_\infty(A, A^t) \in [0, \frac{1}{4}]$ and that for each A with maximal d_∞-asymmetry we have $D_1(A, A^t) = \frac{4}{9}$.

The rest of the paper is organized as follows: Section 2 gathers some preliminaries and notations that will be used throughout the paper. Section 3 first solves a minimization problem in the class of all completely dependent copulas and provides a characterization of all mutually completely dependent copulas with maximal D_1-asymmetry. The afore-mentioned general handy characterization/representation of all copulas with maximal D_1-asymmetry is presented in Section 4. Section 5 contains the comparison of the two notions quantifying asymmetry and the Appendix complements some tedious calculations concerning the minimization problem in Section 3.

2 Notation and Preliminaries

For every metric space (Ω, d) the Borel σ-field on Ω will be denoted by $\mathcal{B}(\Omega)$, δ_x will denote the Dirac measure (concentrated) at $x \in \Omega$. λ and λ_2 will denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}^2)$ respectively. For every probability measure ν on $\mathcal{B}(\Omega)$ the support of ν, i.e. the complement of the union of all open sets U fulfilling $\nu(U) = 0$, will be denoted by $supp(\nu)$.

In the sequel $\mathcal{C}$ will denote the family of all two-dimensional *copulas*, $\mathcal{P}_\mathcal{C}$ the family of all *doubly stochastic measures*, i.e. the family of all probability measures on $\mathcal{B}([0, 1]^2)$ whose marginals are uniformly distributed on $[0, 1]$. For every $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C, d_∞ will denote the uniform metric on $\mathcal{C}$, given by

$$d_\infty(A, B) = \sup_{(x,y)\in[0,1]^2} |A(x, y) - B(x, y)|.$$

As usual, M denotes the minimum copula, Π the product copula, W the lower Fréchet-Hoeffding bound, and A^t the transpose of $A \in \mathcal{C}$, defined by $A^t(x, y) = A(y, x)$. For every $A \in \mathcal{C}$ the diagonal $\delta_A : [0, 1] \to [0, 1]$ is defined by $\delta_A(x) = A(x, x)$. For further background on copulas we refer to [2] and [10].

To keep notation simple and consistent throughout the paper we will let A, B, C (and the corresponding versions with subindices) denote copulas, E, U, V, S will denote Borel sets, H will denote two-dimensional distribution functions, F and G univariate distribution functions.

Suppose that (Ω_1, d_1) and (Ω_2, d_2) are metric spaces. A *Markov kernel* from Ω_1 to $\mathcal{B}(\Omega_2)$ is a mapping $K : \Omega_1 \times \mathcal{B}(\Omega_2) \to [0, 1]$ such that $x \mapsto K(x, E)$ is measurable for every fixed $E \in \mathcal{B}(\Omega_2)$ and $E \mapsto K(x, E)$ is a probability measure for every fixed $x \in \Omega_1$. Given real-valued random variables X, Y on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is called a *regular conditional distribution of Y given X* if for every $E \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), E) = \mathbb{E}(\mathbf{1}_E \circ Y|X)(\omega) \tag{1}$$

holds $\mathbb{P}$-a.e. It is well known that for each pair (X, Y) of real-valued random variables a regular conditional distribution $K(\cdot, \cdot)$ of Y given X exists, that $K(\cdot, \cdot)$ is unique $\mathbb{P}^X$-a.s. (i.e. unique for $\mathbb{P}^X$-almost all $x \in \mathbb{R}$) and that $K(\cdot, \cdot)$ only depends on $\mathbb{P}^{X\otimes Y}$. Hence, given $C \in \mathcal{C}$ and $(X, Y) \sim C$, we will denote (a version of) the regular conditional distribution of Y given X by $K_C(\cdot, \cdot)$, directly view it as Markov kernel from $[0, 1]$ to $\mathcal{B}([0, 1])$, and refer to $K_C(\cdot, \cdot)$ simply as *regular conditional distribution of C* or as *Markov kernel of C*. Note that for every $C \in \mathcal{C}$, its regular conditional distribution $K_C(\cdot, \cdot)$, and every Borel set $E \in \mathcal{B}([0, 1]^2)$ we have the following *disintegration* (here $E_x := \{y \in [0, 1] : (x, y) \in E\}$ denotes the x-section of E for every $x \in [0, 1]$)

$$\int_{[0,1]} K_C(x, E_x)\, d\lambda(x) = \mu_C(E), \tag{2}$$

so in particular

$$\int_{[0,1]} K_C(x, U)\, d\lambda(x) = \lambda(U) \tag{3}$$

for every $U \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (3) induces a unique element $\mu \in \mathcal{P}_{\mathcal{C}}$ via (2). For more details and properties of conditional expectation, regular conditional distributions, and disintegration see [6] and [8]. For examples underlining the usefulness of Markov kernels in the copula setting we refer, for instance, to [3, 13, 14] as well as to [2] and the reference therein.

$\mathcal{T}$ will denote the family of all λ-preserving transformations on $[0,1]$, $\mathcal{T}_b$ the subclass of all bijections in $\mathcal{T}$. A copula C will be called *completely dependent (mutually completely dependent)* if there exists a transformation $h \in \mathcal{T}$ ($h \in \mathcal{T}_b$) such that $K(x,E) = \mathbf{1}_E(h(x)) = \delta_{h(x)}(E)$ is a Markov kernel of C. $\mathcal{C}_d$ will denote the family of all completely dependent copulas, A_h will denote the completely dependent copula induced by $h \in \mathcal{T}$. For alternative characterizations of (mutual) complete dependence we refer to [12] and the reference therein.

The metric D_1 on $\mathcal{C}$ is defined by

$$D_1(A,B) = \int_{[0,1]} \underbrace{\int_{[0,1]} \left| K_A\left(x,[0,y]\right) - K_B\left(x,[0,y]\right) \right| d\lambda(x)\, d\lambda(y).}_{=:\Phi_{A,B}(y)}$$

According to [12] (also see [2, 3]) the resulting metric space $(\mathcal{C}, D_1)$ is complete and separable, convergence with respect to D_1 implies convergence with respect to d_∞ (but not vice versa) and $\max_{A,B\in\mathcal{C}} D_1(A,B) = \frac{1}{2}$.

3 Maximal D_1-asymmetry of completely dependent copulas

As already mentioned in the introduction, according to [7, 11] $\max_{A\in\mathcal{C}} d_\infty(A,A^t) = \frac{1}{3}$ holds. We are now going to prove $\max_{A\in\mathcal{C}} D_1(A,A^t) = \frac{1}{2}$ and proceed in several steps: First we minimize the function $\alpha \mapsto D_1(\alpha A_1 + (1-\alpha)A_2, \Pi)$ for every pair A_1, A_2 of completely dependent copulas, concentrate on completely dependent copulas A_1, A_2 for which $D_1(A_1, A_2)$ is maximal, and then characterize all $h \in \mathcal{T}_b$ such that $D_1(A_h, A_h^t)$ is maximal.

Fix $A_1, A_2 \in \mathcal{C}_d$, let h_1, h_2 denote the corresponding λ-preserving transformations and consider $\alpha \in [0,1]$. Setting $C_\alpha = \alpha A_1 + (1-\alpha)A_2$ with $\alpha \in [0,1]$ we have

$$\begin{aligned} f_{h_1,h_2}(\alpha) := D_1(C_\alpha, \Pi) &= \int_{[0,1]} \int_{[0,1]} \left| K_{C_\alpha}\left(x,[0,y]\right) - K_\Pi\left(x,[0,y]\right) \right| d\lambda(x) d\lambda(y) \\ &= \int_{[0,1]} \int_{[0,1]} \left| \alpha \mathbf{1}_{[0,y]}(h_1(x)) + (1-\alpha)\mathbf{1}_{[0,y]}(h_2(x)) - y \right| d\lambda(x) d\lambda(y). \end{aligned}$$

A straightforward calculation (see Appendix) yields

$$\begin{aligned} f_{h_1,h_2}(\alpha) = \frac{1}{3}(1 - 3\alpha + 3\alpha^2) &+ 2\int_{[0,1]} C_{h_1,h_2}(y,y) d\lambda(y) - 2\int_{[0,1]} y C_{h_1,h_2}(y,y) d\lambda(y) \\ &- 2\alpha \int_{[0,\alpha]} C_{h_1,h_2}(y,y) d\lambda(y) + 2\int_{[0,\alpha]} y C_{h_1,h_2}(y,y) d\lambda(y) \\ &- 2(1-\alpha) \int_{[0,1-\alpha]} C_{h_1,h_2}(y,y) d\lambda(y) + 2\int_{[0,1-\alpha]} y C_{h_1,h_2}(y,y) d\lambda(y), \end{aligned}$$

where C_{h_1,h_2} denotes the copula defined by $C_{h_1,h_2}(x,y) = \lambda(h_1^{-1}([0,x]) \cap h_2^{-1}([0,y]))$ for all $x, y \in [0,1]$ (see [9]). Obviously $f_{h_1,h_2} \in C^1([0,1])$ and we get

$$\begin{aligned} f'_{h_1,h_2}(\alpha) &= 2\alpha - 1 - 2\left(\alpha C_{h_1,h_2}(\alpha,\alpha) + \int_{[0,\alpha]} C_{h_1,h_2}(y,y)d\lambda(y)\right) + 2\alpha C_{h_1,h_2}(\alpha,\alpha) \\ &\quad - 2\left(-(1-\alpha)C_{h_1,h_2}(1-\alpha,1-\alpha) - \int_{[0,1-\alpha]} C_{h_1,h_2}(y,y)d\lambda(y)\right) \\ &\quad - 2(1-\alpha)C_{h_1,h_2}(1-\alpha,1-\alpha) \\ &= 2\alpha - 1 - 2\int_{[0,\alpha]} C_{h_1,h_2}(y,y)d\lambda(y) + 2\int_{[0,1-\alpha]} C_{h_1,h_2}(y,y)d\lambda(y). \end{aligned}$$

Hence $f'_{h_1,h_2} \in C^1([0,1])$ and

$$f''_{h_1,h_2}(\alpha) = 2 - 2C_{h_1,h_2}(\alpha,\alpha) - 2C_{h_1,h_2}(1-\alpha,1-\alpha) \geq 2 - 2\alpha - 2(1-\alpha) = 0,$$

so f_{h_1,h_2} is convex on $[0,1]$. Considering $f'_{h_1,h_2}(\frac{1}{2}) = 0$ it follows that f_{h_1,h_2} attains its global minimum at the point $\frac{1}{2}$, and we get

$$\min_{\alpha\in[0,1]} D_1(C_\alpha, \Pi) = \frac{1}{12} + 2\int_{[0,1]} (1-y)C_{h_1,h_2}(y,y)d\lambda(y) - 2\int_{[0,\frac{1}{2}]} (1-2y)C_{h_1,h_2}(y,y)d\lambda(y). \tag{4}$$

Remark 3.1. The minimum can also be attained at a point $\alpha \neq \frac{1}{2}$. Consider $t \in (0, \frac{1}{2})$ and let $h_1, h_2 \in \mathcal{T}_b$ be defined by $h_1(x) = x$ and $h_2(x) = (t-x)\mathbf{1}_{[0,t)}(x) + x\mathbf{1}_{[t,1]}(x)$. Then $C_{h_1,h_2}(y,y) = y$ for all $y \in [t,1]$ and for every $\alpha \in (t, \frac{1}{2})$ we get

$$f'_{h_1,h_2}(\alpha) = 2\alpha - 1 + 2\int_{[\alpha,1-\alpha]} y\, d\lambda(y) = 2\alpha - 1 + (1-\alpha)^2 - \alpha^2 = 0.$$

Hence $D_1(C_\alpha, \Pi)$ is minimal for every $\alpha \in [t, \frac{1}{2}]$.

Example 3.2. Consider $C_\alpha = \alpha M + (1-\alpha)W$. Then $h_1(x) = x$ and $h_2(x) = 1 - x$ for all $x \in [0,1]$ and we get $C_{h_1,h_2}(y,y) = \lambda\left([0,y] \cap [1-y,1]\right) = W(y,y)$. So

$$\begin{aligned} \min_{\alpha\in[0,1]} D_1(C_\alpha, \Pi) &= D_1\left(\frac{M+W}{2}, \Pi\right) \\ &= \frac{1}{12} + 2\int_{[0,1]} (1-y)W(y,y)d\lambda(y) - 2\int_{[0,\frac{1}{2}]} (1-2y)W(y,y)d\lambda(y) \\ &= \frac{1}{12} + 2\int_{[\frac{1}{2},1]} (1-y)(2y-1)d\lambda(y) = \frac{1}{6}. \end{aligned}$$

The value $\frac{1}{6}$ attained by the copula $C_{\frac{1}{2}}$ in Example 3.2 is as close as we can possibly get to Π. In fact, for arbitrary $A_1, A_2 \in \mathcal{C}_d$ with corresponding transformations $h_1, h_2 \in \mathcal{T}$ we have

$$\begin{aligned} D_1\left(\frac{A_1 + A_2}{2}, \Pi\right) &= \frac{1}{12} + 2\int_{[0,1]} (1-y)C_{h_1,h_2}(y,y)d\lambda(y) - 2\int_{[0,\frac{1}{2}]} (1-2y)C_{h_1,h_2}(y,y)d\lambda(y) \\ &\geq \frac{1}{12} + 2\int_{[0,1]} (1-y)C_{h_1,h_2}(y,y)d\lambda(y) - 2\int_{[0,\frac{1}{2}]} (1-y)C_{h_1,h_2}(y,y)d\lambda(y) \\ &= \frac{1}{12} + 2\int_{[\frac{1}{2},1]} (1-y)C_{h_1,h_2}(y,y)d\lambda(y) \\ &\geq \frac{1}{12} + 2\int_{[\frac{1}{2},1]} (1-y)(2y-1)d\lambda(y) = \frac{1}{6}. \end{aligned} \tag{5}$$

with equality if and only if $C_{h_1,h_2}(y,y) = 0$ on $[0, \frac{1}{2}]$ and $C_{h_1,h_2}(y,y) = 2y - 1$ on $y \in [\frac{1}{2}, 1]$. The following theorem adds some equivalent conditions to this observation.

Theorem 3.3. *Suppose that $h_1, h_2 \in \mathcal{T}$. Then the following conditions are equivalent for the corresponding completely dependent copulas $A_1 := A_{h_1}, A_2 := A_{h_2}$.*

(a) $D_1(\frac{A_1+A_2}{2}, \Pi) = \frac{1}{6}$,
(b) $C_{h_1,h_2}(y,y) = \delta_W(y)$,
(c) $\|h_1 - h_2\|_1 = \int_{[0,1]} |h_1(x) - h_2(x)| d\lambda(x) = \frac{1}{2}$,
(d) $D_1(A_1, A_2) = \frac{1}{2}$ *(i.e. A_1 and A_2 have maximum D_1-distance).*

Proof. The equivalence of (a) and (b) has already been shown, the fact that (c) and (d) are equivalent was proved in [12]. It therefore suffices to show that (b) and (c) are equivalent, which can be done as follows: Using Proposition 15 in [12] (saying that the D_1-distance of two completely dependent copulas coincides with the L^1-distance of the corresponding λ-preserving transformations) we get

$$\begin{aligned} \|h_1 - h_2\|_1 &= \int_{[0,1]}\int_{[0,1]} \left(\mathbf{1}_{[0,y]}(h_1(x)) - \mathbf{1}_{[0,y]}(h_2(x))\right)^2 d\lambda(x)d\lambda(y) \\ &= \int_{[0,1]}\int_{[0,1]} \left(\mathbf{1}_{h_1^{-1}([0,y])}(x) - 2\cdot\mathbf{1}_{h_1^{-1}([0,y])\cap h_2^{-1}([0,y])}(x) + \mathbf{1}_{h_2^{-1}([0,y])}(x)\right) d\lambda(x)d\lambda(y) \\ &= 2\int_{[0,1]} y d\lambda(y) - 2\int_{[0,1]} C_{h_1,h_2}(y,y)d\lambda(y) \\ &= 1 - 2\int_{[0,1]} \delta_W(y)d\lambda(y) = \frac{1}{2}, \end{aligned} \tag{6}$$

so (b) implies (c). To complete the proof set $\delta_{12}(y) := C_{h_1,h_2}(y,y)$ and assume that $\delta_W(y_0) < \delta_{12}(y_0)$ holds for some $y_0 \in (0,1)$. Considering continuity of δ_{12} the fact that $\|h_1 - h_2\|_1 < \frac{1}{2}$ holds follows immediately. □

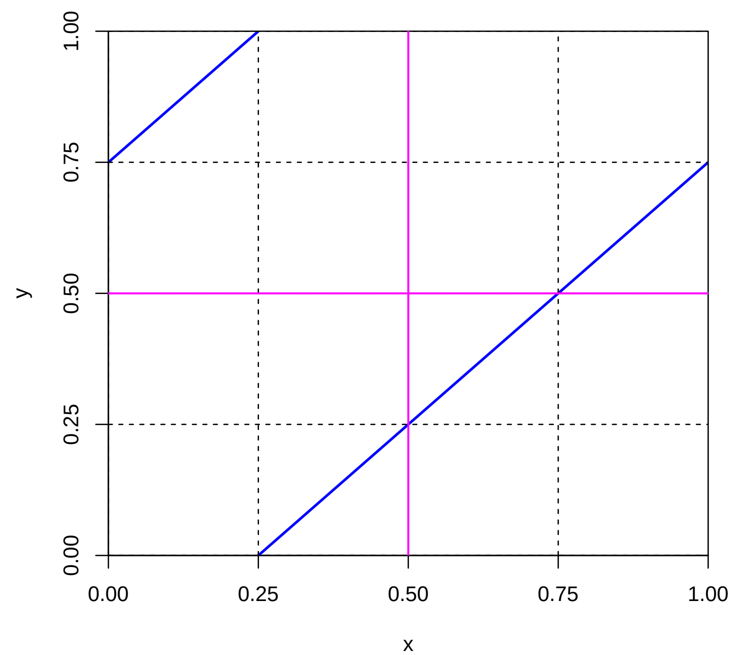

Figure 1: Completely dependent copula A_h with maximal D_1-asymmetry as considered in Example 3.4.

We now turn to the situation $h_1 = h \in \mathcal{T}_b$, $h_2 = h^{-1}$ and start with the following example.

Example 3.4. Let $h \in \mathcal{T}_b$ be defined by $h(x) = x + \frac{3}{4}$ for $x \in [0, \frac{1}{4}]$ and $h(x) = x - \frac{1}{4}$ for $x \in (\frac{1}{4}, 1]$ (see Figure 1). In this case we easily get $\|h - h^{-1}\|_1 = \frac{1}{2}$, implying $D_1(A_h, A_h^t) = D_1(A_h, A_{h^{-1}}) = \frac{1}{2}$, where the last equality follows from (see Lemma 10 in [12])

$$\begin{aligned} A_h^t(x,y) = A_h(y,x) &= \lambda\left([0,y] \cap h^{-1}([0,x])\right) = \lambda^{h^{-1}}\left([0,y] \cap h^{-1}([0,x])\right) \\ &= \lambda\left(h([0,y]) \cap [0,x]\right) = A_{h^{-1}}(x,y). \end{aligned}$$

Example 3.4 implies

$$\sup_{h \in \mathcal{T}_b} D_1(A_h, A_h^t) = \sup_{A \in \mathcal{C}} D_1(A, A^t) = \frac{1}{2}.$$

In other words, referring to the quantity $\kappa(A) := 2D_1(A, A^t)$ as D_1-*asymmetry* of $A \in \mathcal{C}$, the maximum D_1-asymmetry of a copula is 1 and there exists a mutually completely dependent copula A_h with $\kappa(A_h) = 1$. Notice that the definition of $\kappa : \mathcal{C} \to [0,1]$ implies that $\kappa(A) = 0$ if, and only if, $A = A^t$, i.e. if A is symmetric.

The subsequent theorem builds upon Theorem 3.3 and provides an easy characterization of all mutually completely dependent copulas A_h having maximal D_1-asymmetry.

Theorem 3.5. *Consider $h \in \mathcal{T}_b$. Then the following conditions are equivalent:*

(a) *A_h has maximal D_1-asymmetry (i.e. $\kappa(A_h) = 1$),*
(b) $\|h - h^{-1}\|_1 = \frac{1}{2}$,
(c) $\lambda\left(h^{-1}([0, \frac{1}{2}]) \cap h([0, \frac{1}{2}])\right) = 0$.

Proof. Condition (c) implies $C_{h,h^{-1}}(\frac{1}{2}, \frac{1}{2}) = 0$, from which $C_{h,h^{-1}}(y,y) = W(y,y)$ for all $y \in [0,1]$ follows immediately. The remaining implications are a direct consequence of Theorem 3.3. □

4 Maximal D_1-asymmetry of general copulas

We now turn to the general situation and provide necessary and sufficient conditions for a copula $A \in \mathcal{C}$ to have maximal D_1-asymmetry (Φ_{A,A^t} as at the end of Section 2).

Theorem 4.1. *For $A \in \mathcal{C}$ the following statements are equivalent:*

(a) A has maximum D_1-asymmetry,
(b) $\Phi_{A,A^t}(\frac{1}{2}) = 1$,
(c) There exists a Borel set $U \in \mathcal{B}([0,1])$ with the following properties:

$$\lambda(U \cap [0, \tfrac{1}{2}]) = \lambda(U \cap [\tfrac{1}{2}, 1]) = \tfrac{1}{4},\ \mu_A(U \times [0, \tfrac{1}{2}]) = \tfrac{1}{2},\ \mu_A([0, \tfrac{1}{2}] \times U) = 0. \tag{7}$$

Proof. The equivalence of the first and the second assertion is a direct consequence of the results in [12] (for all copulas A, B the function $\Phi_{A,B}$ is Lipschitz-continuous with Lipschitz constant 2 and bounded from above by the tent map $T(y) = \min\{2y, 2(1-y)\}$).

The fact that the second condition implies the third one can be proved as follows. Considering

$$1 = \Phi_{A,A^t}\left(\tfrac{1}{2}\right) = \int_{[0,1]} \underbrace{\left|K_A\left(x, \left[0, \tfrac{1}{2}\right]\right) - K_{A^t}\left(x, \left[0, \tfrac{1}{2}\right]\right)\right|}_{=:g(x)\in[0,1]} d\lambda(x)$$

it follows immediately that the set $\Lambda = \{x \in [0,1] : g(x) = 1\}$ fulfills $\lambda(\Lambda) = 1$. Setting $U := \{x \in [0,1] : K_A(x, [0, \frac{1}{2}]) = 1\}$, applying Scheffé's theorem ([1]) and disintegration we get

$$\begin{aligned} 1 = \Phi_{A,A^t}\left(\tfrac{1}{2}\right) &= \int_{[0,1]} \left|K_A\left(x, \left[0, \tfrac{1}{2}\right]\right) - K_{A^t}\left(x, \left[0, \tfrac{1}{2}\right]\right)\right| d\lambda(x) \\ &= 2\int_U \left(K_A\left(x, \left[0, \tfrac{1}{2}\right]\right) - K_{A^t}\left(x, \left[0, \tfrac{1}{2}\right]\right)\right) d\lambda(x) \\ &= 2\mu_A(U \times [0, \tfrac{1}{2}]) - 2\mu_{A^t}(U \times [0, \tfrac{1}{2}]). \end{aligned} \tag{8}$$

Additionally considering

$$\begin{aligned} \mu_A(U \times [0, \tfrac{1}{2}]) &= \int_{U\cap\Lambda} K_A\left(x, \left[0, \tfrac{1}{2}\right]\right) d\lambda(x) = \int_{U\cap\Lambda} 1\, d\lambda = \lambda(U \cap \Lambda) = \lambda(U), \\ \mu_A([0, \tfrac{1}{2}] \times U) &= \mu_{A^t}(U \times [0, \tfrac{1}{2}]) = \int_{U\cap\Lambda} K_{A^t}\left(x, \left[0, \tfrac{1}{2}\right]\right) d\lambda(x) = \int_{U\cap\Lambda} 0\, d\lambda = 0 \end{aligned}$$

the last two identities in eq. (7) and $\lambda(U) = \frac{1}{2}$ follow immediately and it remains to show that U fulfills $\lambda(U \cap (\frac{1}{2}, 1]) = \frac{1}{4}$. Eq. (7) implies $\mu_A((U \times [0, \frac{1}{2}]) \setminus ([0, \frac{1}{2}] \times U)) = \frac{1}{2}$. Hence, using $([0, \frac{1}{2}] \times U)^c = ((\frac{1}{2}, 1] \times U) \cup ([0,1] \times U^c)$ and the fact that μ_A is doubly stochastic we get

$$\begin{aligned} \tfrac{1}{2} &= \mu_A\left((U \times [0, \tfrac{1}{2}]) \setminus ([0, \tfrac{1}{2}] \times U)\right) \\ &= \mu_A\left((U \cap (\tfrac{1}{2}, 1]) \times (U \cap [0, \tfrac{1}{2}])\right) + \mu_A\left(U \times ([0, \tfrac{1}{2}] \cap U^c)\right) \\ &\le \min\{\lambda(U \cap (\tfrac{1}{2}, 1]), \lambda(U \cap [0, \tfrac{1}{2}])\} + \lambda([0, \tfrac{1}{2}] \cap U^c) \\ &= \min\{r, \tfrac{1}{2} - r\} + r =: \ell(r) \end{aligned}$$

where $r = \lambda(U \cap (\frac{1}{2}, 1])$. In case of $r < \frac{1}{4}$ we would get $\ell(r) = 2r < \frac{1}{2}$, so $r \ge \frac{1}{4}$ holds. Considering $\mu_A(U \times (\frac{1}{2}, 1]) = 0$, $\mu_A((\frac{1}{2}, 1] \times U) = \frac{1}{2}$ and proceeding analogously yields $r \le \frac{1}{4}$, from which we finally get $\lambda(U \cap (\frac{1}{2}, 1]) = \frac{1}{4}$.

In case assertion three is fulfilled, disintegration implies that both $K_A(x, [0, \frac{1}{2}]) = \mathbf{1}_U(x)$ and $K_{A^t}(x, [0, \frac{1}{2}]) = \mathbf{1}_{U^c}(x)$ hold λ-a.e., from which, considering

$$\Phi_{A,A^t}(\tfrac{1}{2}) = \int_{[0,1]} |K_A(x, [0, \tfrac{1}{2}]) - K_{A^t}(x, [0, \tfrac{1}{2}])| d\lambda(x) = \int_{[0,1]} |\mathbf{1}_U(x) - \mathbf{1}_{U^c}(x)| d\lambda(x) = 1$$

the second assertion follows immediately. □

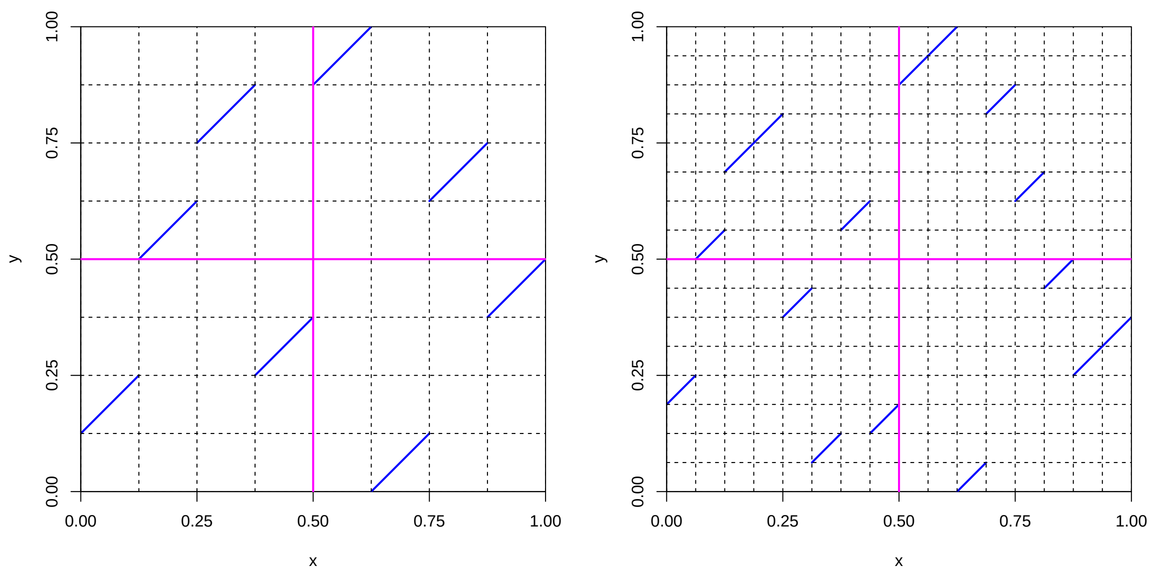

Figure 2: Shuffles with maximum D_1-asymmetry. For the shuffle in the left panel the set U is given by $U = [0, \frac{1}{8}] \cup [\frac{3}{8}, \frac{1}{2}] \cup [\frac{5}{8}, \frac{3}{4}] \cup [\frac{7}{8}, 1]$.

Figure 2 depicts some shuffles with maximum D_1-asymmetry. Interpreting these shuffles as checkmin copulas and replacing the minimum copula M by any other copula B, as direct consequence of Theorem 4.1 the resulting copulas have maximum D_1-asymmetry too. Assertion three in Theorem 4.1 can further be simplified, the following lemma holds:

Lemma 4.2. *Suppose that $\kappa(A) = 1$ and that U is a Borel set fulfilling eq. (7). Then there exists a measurable partition of $[0, 1]$ into sets U_1, V_1, U_2, V_2 of length $\frac{1}{4}$ fulfilling the following properties:*

1. $U_1 \subseteq [0, \frac{1}{2}]$, $V_1 = [0, \frac{1}{2}] \setminus U_1$, $U_2 \subseteq (\frac{1}{2}, 1]$, $V_2 = (\frac{1}{2}, 1] \setminus U_2$,
2. $\mu_A(U_1 \times V_1) = \mu_A(V_1 \times V_2) = \mu_A(U_2 \times U_1) = \mu_A(V_2 \times U_2) = \frac{1}{4}$.

Furthermore, setting $F_i(x) := 4\lambda(U_i \cap [0, x])$ and $G_i(x) := 4\lambda(V_i \cap [0, x])$ for $i \in \{1, 2\}$ and $x \in [0, 1]$, there exist copulas $C_1, C_2, C_3, C_4 \in \mathcal{C}$ such that the following identity holds for all $x, y \in [0, 1]$:

$$A(x, y) = \frac{1}{4}\left(C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) + C_4(G_2(x), F_2(y))\right) \tag{9}$$

Proof. Set $U_1 = U \cap [0, \frac{1}{2}]$, $U_2 = U \cap (\frac{1}{2}, 1]$, $V = [0, 1] \setminus U$, $V_1 = V \cap [0, \frac{1}{2}]$ and $V_2 = V \cap (\frac{1}{2}, 1]$. Then eq. (7) implies $\lambda(U_1) = \lambda(U_2) = \frac{1}{4} = \lambda(V_1) = \lambda(V_2)$ as well as

$$0 = \mu_A([0, \tfrac{1}{2}] \times U) = \mu_A(U_1 \times U_1) + \mu_A(U_1 \times U_2) + \mu_A(V_1 \times U_1) + \mu_A(V_1 \times U_2) \tag{10}$$

$$0 = \mu_A(V \times [0, \tfrac{1}{2}]) = \mu_A(V_1 \times U_1) + \mu_A(V_1 \times V_1) + \mu_A(V_2 \times U_1) + \mu_A(V_2 \times V_1). \tag{11}$$

Considering $\frac{1}{4} = \mu_A(U_1 \times [0, \frac{1}{2}]) = \mu_A(U_1 \times U_1) + \mu_A(U_1 \times V_1)$ and using eq. (10), $\mu_A(U_1 \times V_1) = \frac{1}{4}$ follows immediately. Additionally, $\mu_A(V_1 \times V_2) = \frac{1}{4}$ is a direct consequence of $\frac{1}{4} = \mu_A(V_1 \times (\frac{1}{2}, 1]) = \mu_A(V_1 \times U_2) + \mu_A(V_1 \times V_2)$ and eq. (10).

Using the fact that $K_A(x,[0,\frac{1}{2}])(x) = \mathbf{1}_U(x)$ and $K_A(x,(\frac{1}{2},1]) = \mathbf{1}_V(x)$ hold λ-a.e. and applying disintegration we get

$$\mu_A([0,\tfrac{1}{2}]^2) = \int_{[0,\frac{1}{2}]} K_A(x,[0,\tfrac{1}{2}])d\lambda(x) = \int_{U_1} K_A(x,[0,\tfrac{1}{2}])d\lambda(x) = \lambda(U_1) = \tfrac{1}{4} = \mu_A(U_1 \times V_1).$$

Having this, $\mu_A((\frac{1}{2},1]\times[0,\frac{1}{2}]) = \frac{1}{4}$ follows and, using eq. (11), we get

$$\tfrac{1}{4} = \mu_A((\tfrac{1}{2},1]\times[0,\tfrac{1}{2}]) = \mu_A(U_2 \times U_1) + \mu_A(U_2 \times V_1).$$

Since $\mu_A(U_2 \times V_1) > 0$ would imply

$$\tfrac{1}{4} = \lambda(V_1) \geq \mu_A(U \times V_1) = \mu_A(U_1 \times V_1) + \mu_A(U_2 \times V_1) > \mu_A(U_1 \times V_1) = \tfrac{1}{4}$$

$\mu_A(U_2 \times U_1) = \frac{1}{4}$ follows. To show $\mu_A(V_2 \times U_2) = \frac{1}{4}$ we proceed analogously, use $\frac{1}{4} = \mu_A(V_2 \times (\frac{1}{2},1])$ and the fact that $\mu_A(V_2 \times V_2) > 0$ would imply $\frac{1}{4} \geq \mu_A(V \times V_2) > \frac{1}{4}$, a contradiction.
The proof of eq. (9) is now a straightforward application of Sklar's Theorem and the fact that the sets $U_1 \times V_1, V_1 \times V_2, U_2 \times U_1, V_2 \times U_2$ are pairwise disjoint:

$$\begin{aligned} A(x,y) = \mu_A([0,x]\times[0,y]) &= \frac{1}{4}\Big(4\mu_A(U_1 \cap [0,x] \times V_1 \cap [0,y]) + 4\mu_A(V_1 \cap [0,x] \times V_2 \cap [0,y]) \\ &\quad + 4\mu_A(U_2 \cap [0,x] \times U_1 \cap [0,y]) + 4\mu_A(V_2 \cap [0,x] \times U_2 \cap [0,y])\Big) \\ &= \frac{1}{4}\left(C_1(F_1(x),G_1(y)) + C_2(G_1(x),G_2(y)) + C_3(F_2(x),F_1(y)) + C_4(G_2(x),F_2(y))\right) \end{aligned}$$

□

The reverse implication of Lemma 4.2 holds as well:

Lemma 4.3. *Suppose that $U_1, U_2 \in \mathcal{B}([0,1])$ fulfill $U_1 \subseteq [0,\frac{1}{2}]$, $U_2 \subseteq (\frac{1}{2},1]$, $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$, and let C_1, C_2, C_3, C_4 be arbitrary copulas. Set $V_1 = [0,\frac{1}{2}] \setminus U_1$, $V_2 = (\frac{1}{2},1] \setminus U_2$ as well as $F_i(x) := 4\lambda(U_i \cap [0,x])$ and $G_i(x) := 4\lambda(V_i \cap [0,x])$ for $i \in \{1,2\}$ and $x \in [0,1]$. Then the function $A : [0,1]^2 \to [0,1]$, defined by*

$$A(x,y) = \tfrac{1}{4}\left(C_1(F_1(x),G_1(y)) + C_2(G_1(x),G_2(y)) + C_3(F_2(x),F_1(y)) + C_4(G_2(x),F_2(y))\right), \tag{12}$$

is a copula with maximum D_1-asymmetry.

Proof. First of all notice that the distribution functions F_1, F_2, G_1, G_2 are absolutely continuous and let $\nu_{F_1}, \nu_{F_2}, \nu_{G_1}, \nu_{G_2}$ denote the corresponding probability measures on $\mathcal{B}([0,1])$. It is clear from eq. (12) and Sklar's theorem that A is two-dimensional distribution function. Since A also fulfills the boundary conditions of a copula, $A \in \mathcal{C}$ follows and it suffices to prove $\kappa(A) = 1$, which can be done as follows: Setting $A_1(x,y) := C_1(F_1(x),G_1(y))$ yields a continuous distribution function A_1, whose corresponding probability measure ϑ_{A_1} fulfills $\vartheta_{A_1}([0,1]^2) = 1$ as well as

$$\begin{aligned} \vartheta_{A_1}\left([\underline{x},\overline{x}]\times[\underline{y},\overline{y}]\right) &= A_1(\overline{x},\overline{y}) - A_1(\underline{x},\overline{y}) - A_1(\overline{x},\underline{y}) + A_1(\underline{x},\underline{y}) \\ &= C_1(F_1(\overline{x}),G_1(\overline{y})) - C_1(F_1(\underline{x}),G_1(\overline{y})) - C_1(F_1(\overline{x}),G_1(\underline{y})) \\ &\quad + C_1(F_1(\underline{x}),G_1(\underline{y})) \end{aligned}$$

for all $[\underline{x},\overline{x}]\times[\underline{y},\overline{y}] \subseteq [0,1]^2$. Specializing to $[\underline{y},\overline{y}] = [0,1]$ we get

$$\vartheta_{A_1}\left([\underline{x},\overline{x}]\times[0,1]\right) = C_1(F_1(\overline{x}),1) - C_1(F_1(\underline{x}),1) = F_1(\overline{x}) - F_1(\underline{x}) = \nu_{F_1}([\underline{x},\overline{x}])$$

for all $[\underline{x},\overline{x}] \subset [0,1]$, implying $\vartheta_{A_1}(U_1 \times [0,1]) = \nu_{F_1}(U_1) = 1$. Proceeding in the same manner yields $\vartheta_{A_1}([0,1] \times V_1) = \nu_{G_1}(V_1) = 1$, from which altogether we get $\vartheta_{A_1}(U_1 \times V_1) = 1$. Setting $A_2(x,y) :=$

$C_2(G_1(x), G_2(y))$, $A_3(x, y) := C_3(F_2(x), F_1(y))$, $A_4(x, y) := C_4(G_2(x), F_2(y))$, and proceeding in the same manner shows $\nu_{A_2}(V_1 \times V_2) = \nu_{A_3}(U_2 \times U_1) = \nu_{A_4}(V_2 \times U_2) = 1$. Using the fact that U_1, U_2, V_1, V_2 are pairwise disjoint yields

$$\mu_A(U_1 \times V_1) = \mu_A(V_1 \times V_2) = \mu_A(U_2 \times U_1) = \mu_A(V_2 \times U_2) = \tfrac{1}{4},$$

from which, setting $U = U_1 \cup U_2$ the assertion follows as direct consequence of Theorem 4.1. □

Summing up, we have proved the following handy characterization of all copulas with maximum D_1-asymmetry.

Theorem 4.4. *The following statements are equivalent for $A \in \mathcal{C}$:*

(a) A has maximum D_1-asymmetry.
(b) There exist sets $U_1, U_2 \in \mathcal{B}([0,1])$ with $U_1 \subseteq [0, \frac{1}{2}]$, $U_2 \subseteq (\frac{1}{2}, 1]$, $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$, and copulas $C_1, C_2, C_3, C_4 \in \mathcal{C}$ such that (with the notation of Lemma 4.3) the following identity holds:

$$\begin{aligned} A(x, y) = \tfrac{1}{4}\big(&C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) \\ &+ C_4(G_2(x), F_2(y))\big) \end{aligned}$$

Replacing the sets U_1, U_2 in Lemma 4.2 by Borel sets $U_1' \subseteq [0, \frac{1}{2}]$, $U_2' \subseteq (\frac{1}{2}, 1]$ with $\lambda(U_1 \Delta U_1') = \lambda(U_2 \Delta U_2') = 0$ obviously results in the same copula A (Δ denoting the symmetric difference). Considering the equivalence relation $\sim$ induced via $(U_1, U_2) \sim (U_1', U_2')$ if and only if $\lambda(U_1 \Delta U_1') = \lambda(U_2 \Delta U_2') = 0$ and letting $\mathcal{E}$ denote the induced equivalence classes in

$$\big\{(U_1, U_2) : U_1 \in \mathcal{B}([0, \tfrac{1}{2}]), U_2 \in \mathcal{B}((\tfrac{1}{2}, 1]), \lambda(U_1) = \lambda(U_2) = \tfrac{1}{4}\big\}$$

we get the following result ($\mathcal{C}^4 := \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}$):

Corollary 4.5. *There is a one-to-one-correspondence between $\mathcal{E} \times \mathcal{C}^4$ and $\{A \in \mathcal{C} : \kappa(A) = 1\}$.*

Lemma 4.3 has the following consequence:

Corollary 4.6. *Suppose that $U_1 \in \mathcal{B}([0, \frac{1}{2}])$, $U_2 \in \mathcal{B}((\frac{1}{2}, 1])$ fulfill $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$ and set $V_1 = [0, \frac{1}{2}] \setminus U_1$, $V_2 = (\frac{1}{2}, 1] \setminus U_2$ and $S = U_1 \times V_1 \cup V_1 \times V_2 \cup U_2 \times U_1 \cup V_2 \times U_2$. Then there exists an absolutely continuous copula A with $\kappa(A) = 1$ and $\mu_A(S) = 1$.*

Proof. Defining $f : [0, 1]^2 \to [0, \infty)$ as $f(x, y) = 4 \cdot \mathbf{1}_S(x, y)$ yields the probability density of a copula A with the desired properties. □

Remark 4.7. Given sets U_1, U_2 with the afore-mentioned properties, Theorem 4.4 allows not only to construct absolutely continuous copulas A with $\kappa(A) = 1$ and $\mu_A(S) = 1$ as mentioned in Corollary 4.6. In fact, since the four copulas C_1, C_2, C_3, C_4 may be chosen arbitrarily, copulas B with $\kappa(B) = 1$, $\mu_B(S) = 1$ and arbitrary singular mass $\mu_B^{sing}([0, 1]^2) \in [0, 1]$ may easily be constructed (see [14] for the definition of the singular component).

We conclude this section with an additional example illustrating that copulas with maximum D_1-asymmetry (contrary to copulas with maximal d_∞-asymmetry) may distribute mass on the full unit square.

Example 4.8. There exists an absolutely continuous copula A with $\kappa(A) = 1$ and full support $supp(A) = [0, 1]^2$. In fact, letting Ω denote the set constructed in the proof of Lemma 3.1 (by starting with the Smith-Volterra-Cantor set, pasting affine copies of the set in the holes of the set and proceeding in the same manner) in [4], setting $U_1 = \frac{1}{2}\,\Omega$ as well as $U_2 = U_1 + \frac{1}{2}$, and applying Corollary 4.6 yields a set S with $\lambda_2(S) = \frac{1}{4}$ and $\lambda_2(O \cap S) > 0$ for every open set $O \subset [0, 1]^2$. As a consequence, the corresponding (absolutely continuous) copula A fulfills $\mu_A(O) > 0$ for every open set $O \subset [0, 1]^2$, implying that A has full support.

5 The interrelation with maximal d_∞-asymmetry

In this section we study the interrelation between maximal D_1- and maximal d_∞-asymmetry. We start with an example of a sequence $(A_k)_{k\in\mathbb{N}}$ of copulas with maximum D_1-asymmetry that converges to Π with respect to d_∞. In other words: There are copulas with maximum D_1-asymmetry whose d_∞-asymmetry is arbitrarily close to zero.

Example 5.1. For $m \in \mathbb{N}$ let the copula A_m correspond to a uniform distribution on 4^m segments as depicted in Figure 3 for the cases $m = 1, 2, 3, 4$. More formally, if $x \in [\frac{2j}{2^{m+1}}, \frac{2j+1}{2^{m+1}}]$ for some $j \in \{0, 1, \dots, 2^{m-1} - 1\}$, then the Markov kernel $K_{A_m}(x, E)$ of A_m is given by

$$K_{A_m}(x, E) = \frac{1}{2^{m-1}} \sum_{i=0}^{2^{m-1}-1} \mathbf{1}_E \left(x - \tfrac{2j}{2^{m+1}} + \tfrac{2i+1}{2^{m+1}}\right) \tag{13}$$

and for all other cases the formulas are analogous - the probability measure $K_{A_m}(x, \cdot)$ is a uniform discrete distribution on 2^{m-1} points. Theorem 4.4 implies that $\kappa(A_m) = 1$ holds for every $m \in \mathbb{N}$. Moreover the construction of $(A_m)_{m\in\mathbb{N}}$ implies that for every point $(x, y) \in \{0, \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k}, 1\}^2$ with $k \in \mathbb{N}$ the sequence $(A_m(x, y))_{m\in\mathbb{N}}$ is eventually constant. In fact, for $(x, y) = (\frac{i}{2^k}, \frac{j}{2^k})$ we have $A_m(x, y) = \frac{ij}{4^k}$ for every $m \geq k + 1$. Since the set $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k$ with $\Lambda_k = \{0, \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k}, 1\}^2$ is dense in $[0, 1]^2$, the sequence $(A_m)_{m\in\mathbb{N}}$ converges on a dense set to Π, from which $\lim_{n\to\infty} d_\infty(A_m, \Pi) = 0$ follows.

In order to show that the d_∞-asymmetry of copulas A with $\kappa(A) = 1$ is bounded from above by $\frac{1}{4}$ we will use the following simple lemma:

Lemma 5.2. *Every copula $A \in \mathcal{C}$ with $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ fulfills $d_\infty(A, A^t) \leq \frac{1}{4}$.*

Proof. Considering $|A(x, y) - A^t(x, y)| = |A(y, x) - A^t(y, x)|$ it suffices to prove the result for points $(x, y) \in [0, 1]^2$ with $y \leq x$. Additionally, without loss of generality, we may assume that $A(x, y) \geq A^t(x, y)$ holds (otherwise A and A^t change place). Moreover, Lipschitz continuity of copulas implies $|A(x, y) - A^t(x, y)| \leq \frac{1}{4}$ for all $(x, y) \in [0, 1]^2 \setminus [\frac{1}{4}, \frac{3}{4}]^2$. In fact, for $(x, y) \in [0, 1] \times [0, \frac{1}{4}]$ we have $A(x, y), A^t(x, y) \in [0, \frac{1}{4}]$, from which $|A(x, y) - A^t(x, y)| \leq \frac{1}{4}$ follows immediately. For $(x, y) \in [\frac{3}{4}, 1] \times [\frac{1}{4}, 1]$ Lipschitz continuity of copulas yields $A(x, y), A^t(x, y) \in [y - \frac{1}{4}, y]$, implying $|A(x, y) - A^t(x, y)| \leq \frac{1}{4}$. The remaining case $(x, y) \in [0, 1] \times [\frac{3}{4}, 1]$ can be handled analogously. Additionally, considering that $A(x, y), A^t(x, y) \in [0, \frac{1}{4}]$ holds for all $(x, y) \in [0, \frac{1}{2}]^2$ (since $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$) the proof is complete if $A(x, y) - A^t(x, y) \leq \frac{1}{4}$ is proved for all $(x, y) \in \Omega_1 \cup \Omega_2$ where $\Omega_1 = [\frac{1}{2}, \frac{3}{4}] \times [\frac{1}{4}, \frac{1}{2}]$ and Ω_2 denotes the set of all points in the triangle with vertices $(\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})$.
For $(x, y) \in \Omega_1$ we obviously have

$$A(y, x) = y - \mu_A\big([0, y] \times [x, 1]\big) \geq y - \tfrac{1}{4},$$

implying

$$A(x, y) - A^t(x, y) = A(x, y) - A(y, x) \leq A(x, y) - y + \tfrac{1}{4} \leq \tfrac{1}{4}.$$

In case of $(x, y) \in \Omega_2$ (coordinate-wise) monotonicity yields $A(y, x) \geq A(\frac{1}{2}, x) \geq A(\frac{1}{2}, y)$. Considering

$$A(x, x) - A(x, y) - A(\tfrac{1}{2}, x) + A(\tfrac{1}{2}, y) \geq 0,$$

using Lipschitz continuity and coordinate-wise monotonicity we finally get

$$A(x, y) - A^t(x, y) = A(x, y) - A(y, x) \leq x - \tfrac{1}{2} + A(\tfrac{1}{2}, y) - A(\tfrac{1}{2}, y) = x - \tfrac{1}{2} \leq \tfrac{1}{4},$$

which completes the proof. □

Having this, the proof of the following theorem is straightforward.

Theorem 5.3. *Every copula A having maximum D_1-asymmetry satisfies $d_\infty(A, A^t) \leq \frac{1}{4}$ and the upper bound $\frac{1}{4}$ is best possible.*

Proof. Suppose that $A \in \mathcal{C}$ has maximum D_1-asymmetry. Theorem 4.4 implies $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$, so Lemma 5.2 yields $d_\infty(A, A^t) \leq \frac{1}{4}$. Since the completely dependent copula A_h considered in Example 3.4 obviously fulfills $A_h(\frac{1}{2}, \frac{1}{4}) = \frac{1}{4}$ as well as $A_h(\frac{1}{4}, \frac{1}{2}) = 0$, the proof is complete. □

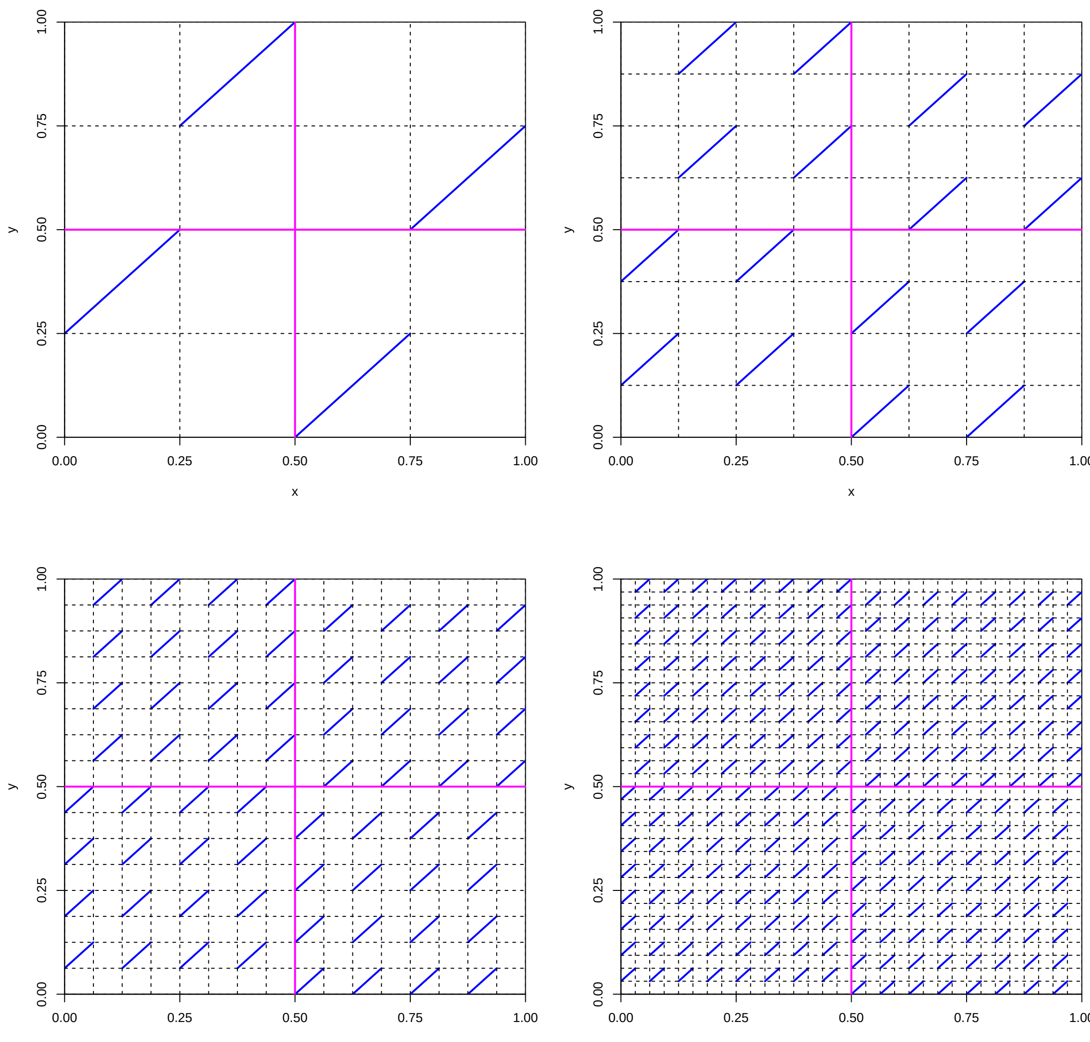

Figure 3: The copulas A_1, A_2, A_3, A_4 used in Example 5.1; each of them concentrates its mass uniformly on the blue segments.

Due to the simple form of copulas with maximum d_∞-asymmetry (see [7, 11]) it is possible to calculate the D_1-asymmetry for all these copulas.

Theorem 5.4. *All copulas $A \in \mathcal{C}$ with $d_\infty(A, A^t) = \frac{1}{3}$ fulfill $D_1(A, A^t) = \frac{4}{9}$.*

Proof. As mentioned in the introduction (also see [11]) $d_\infty(A, A^t)$ is maximal if, and only if, $A(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$ and $A(\frac{1}{3}, \frac{2}{3}) = 0$ or $A^t(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$ and $A^t(\frac{1}{3}, \frac{2}{3}) = 0$ holds. Without loss of generality we may consider the case $A(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$ and $A(\frac{2}{3}, \frac{1}{3}) = 0$. Using the fact that μ_A is doubly stochastic,

$$\mu_A([0, \tfrac{1}{3}] \times [\tfrac{1}{3}, \tfrac{2}{3}]) = \mu_A([\tfrac{1}{3}, \tfrac{2}{3}] \times [\tfrac{2}{3}, 1]) = \mu_A([\tfrac{2}{3}, 1] \times [0, \tfrac{1}{3}]) = \tfrac{1}{3}$$

follows, and we can find copulas $A_1, A_2, A_3 \in \mathcal{C}$ such that $\mu_A = \frac{1}{3}\mu_{A_1}^{f_{12}} + \frac{1}{3}\mu_{A_2}^{f_{23}} + \frac{1}{3}\mu_{A_3}^{f_{31}}$ holds. Thereby the function $f_{ij} : [0, 1]^2 \to [\frac{i-1}{3}, \frac{i}{3}] \times [\frac{j-1}{3}, \frac{j}{3}]$ is given by

$$f_{ij}(x, y) = (\tfrac{x+i-1}{3}, \tfrac{y+j-1}{3})$$

for each $(i, j) \in \{1, 2, 3\}^2$ and $\mu_A^{f_{ij}}$ denotes the push-forward of μ_A via f_{ij} for every $A \in \mathcal{C}$. In the same manner $\mu_{A^t} = \frac{1}{3}\mu_{A_1^t}^{f_{21}} + \frac{1}{3}\mu_{A_2^t}^{f_{32}} + \frac{1}{3}\mu_{A_3^t}^{f_{13}}$ follows.

For $x \in [0, \frac{1}{3}]$ we get

$$|K_A(x, [0, y]) - K_{A^t}(x, [0, y])| = \begin{cases} 0 & \text{if } y \in [0, \frac{1}{3}], \\ K_{A_1}(3x, [0, 3y - 1]) & \text{if } y \in (\frac{1}{3}, \frac{2}{3}], \\ 1 - K_{A_3^t}(3x, [0, 3y - 2]) & \text{if } y \in (\frac{2}{3}, 1], \end{cases}$$

for $x \in (\frac{1}{3}, \frac{2}{3}]$

$$|K_A(x, [0, y]) - K_{A^t}(x, [0, y])| = \begin{cases} K_{A_1^t}(3x - 1, [0, 3y]) & \text{if } y \in [0, \frac{1}{3}], \\ 1 & \text{if } y \in (\frac{1}{3}, \frac{2}{3}], \\ 1 - K_{A_2}(3x - 1, [0, 3y - 2]) & \text{if } y \in (\frac{2}{3}, 1], \end{cases}$$

follows, and for $x \in (\frac{2}{3}, 1]$ we have

$$|K_A(x, [0, y]) - K_{A^t}(x, [0, y])| = \begin{cases} K_{A_3}(3x - 2, [0, 3y]) & \text{if } y \in [0, \frac{1}{3}], \\ 1 - K_{A_2^t}(3x - 2, [0, 3y - 1]) & \text{if } y \in (\frac{1}{3}, \frac{2}{3}], \\ 0 & \text{if } y \in (\frac{2}{3}, 1]. \end{cases}$$

Using change of coordinates and disintegration we easily get

$$\begin{aligned} s_{1,2} &:= \int\limits_{[\frac{1}{3}, \frac{2}{3}]} \int\limits_{[0, \frac{1}{3}]} |K_A(x, [0, y]) - K_{A^t}(x, [0, y])| d\lambda(x) d\lambda(y) \\ &= \frac{1}{3} \int\limits_{[\frac{1}{3}, \frac{2}{3}]} \int\limits_{[0,1]} K_{A_1}(x, [0, 3y - 1]) d\lambda(x) d\lambda(y) = \frac{1}{3} \int\limits_{[\frac{1}{3}, \frac{2}{3}]} (3y - 1) d\lambda(y) = \frac{1}{18}. \end{aligned}$$

Proceeding analogously for

$$s_{i,j} := \int\limits_{[\frac{j}{3}, \frac{j+1}{3}]} \int\limits_{[\frac{i}{3}, \frac{i+1}{3}]} |K_A(x, [0, y]) - K_{A^t}(x, [0, y])| d\lambda(x) d\lambda(y)$$

with $(i, j) \in \{1, 2, 3\}^2$ yields

$$s_{1,1} = 0,\ s_{1,3} = \tfrac{1}{18},\ s_{2,1} = \tfrac{1}{18},\ s_{2,2} = \tfrac{1}{9},\ s_{2,3} = \tfrac{1}{18},\ s_{3,1} = \tfrac{1}{18},\ s_{3,2} = \tfrac{1}{18},\ s_{3,3} = 0,$$

from which we immediately get $D_1(A, A^t) = \frac{6}{18} + \frac{1}{9} = \frac{4}{9}$. □

Combining Theorem 5.3 and Theorem 5.4 shows that there exists no copula having both, maximum d_∞-asymmetry and maximum D_1-asymmetry.

Appendix

In the sequel we derive the formulas for f_{h_1,h_2} as mentioned at the beginning of Section 3. Doing so we consider four different cases and consider

$$K_{C_\alpha}(x,[0,y]) - y = \alpha \mathbf{1}_{[0,y]}(h_1(x)) + (1-\alpha)\mathbf{1}_{[0,y]}(h_2(x)) - y$$
$$= \begin{cases} 1-y & \text{if } x \in h_1^{-1}([0,y]) \cap h_2^{-1}([0,y]), \\ \alpha - y & \text{if } x \in h_1^{-1}([0,y]) \cap h_2^{-1}([y,1]), \\ 1-\alpha-y & \text{if } x \in h_1^{-1}([y,1]) \cap h_2^{-1}([0,y]), \\ -y & \text{if } x \in h_1^{-1}([y,1]) \cap h_2^{-1}([y,1]). \end{cases}$$

Case 1: $x \in h_1^{-1}([0,y]) \cap h_2^{-1}([0,y])$,

$$\int_{[0,1]}\int_{[0,1]} (1-y)\mathbf{1}_{h_1^{-1}([0,y])\cap h_2^{-1}([0,y])}(x)d\lambda(x)d\lambda(y)$$
$$= \int_{[0,1]} (1-y)\lambda(h_1^{-1}([0,y]) \cap h_2^{-1}([0,y]))d\lambda(y)$$
$$= \int_{[0,1]} (1-y)C_{h_1,h_2}(y,y)d\lambda(y)$$
$$= \int_{[0,1]} C_{h_1,h_2}(y,y)d\lambda(y) - \int_{[0,1]} yC_{h_1,h_2}(y,y)d\lambda(y).$$

Case 2: $x \in h_1^{-1}([0,y]) \cap h_2^{-1}((y,1])$,

$$\int_{[0,1]}\int_{[0,1]} |\alpha-y|\mathbf{1}_{h_1^{-1}([0,y])\cap h_2^{-1}((y,1])}(x)d\lambda(x)d\lambda(y)$$
$$= \int_{[0,1]} |\alpha-y|\lambda(h_1^{-1}([0,y]) \cap h_2^{-1}((y,1]))d\lambda(y)$$
$$= \int_{[0,1]} |\alpha-y|\left(y - C_{h_1,h_2}(y,y)\right)d\lambda(y)$$
$$= \int_{[0,\alpha]} (\alpha-y)\left(y - C_{h_1,h_2}(y,y)\right)d\lambda(y) + \int_{(\alpha,1]} (y-\alpha)\left(y - C_{h_1,h_2}(y,y)\right)d\lambda(y)$$
$$= \int_{[0,\alpha]} \left(\alpha y - y^2 - \alpha C_{h_1,h_2}(y,y) + yC_{h_1,h_2}(y,y)\right)d\lambda(y) + \int_{(\alpha,1]} \left(y^2 - \alpha y + \alpha C_{h_1,h_2}(y,y) - yC_{h_1,h_2}(y,y)\right)d\lambda(y)$$
$$= \frac{\alpha^3}{2} - \frac{\alpha^3}{3} - \alpha\int_{[0,\alpha]} C_{h_1,h_2}(y,y)d\lambda(y) + \int_{[0,\alpha]} yC_{h_1,h_2}(y,y)d\lambda(y)$$
$$+ \frac{1-\alpha^3}{3} - \frac{\alpha-\alpha^3}{2} + \alpha\int_{(\alpha,1]} C_{h_1,h_2}(y,y)d\lambda(y) - \int_{(\alpha,1]} yC_{h_1,h_2}(y,y)d\lambda(y)$$
$$= \frac{1}{6}\left(2 - 3\alpha + 2\alpha^3\right) + \alpha\int_{[0,1]} C_{h_1,h_2}(y,y)d\lambda(y) - 2\alpha\int_{[0,\alpha]} C_{h_1,h_2}(y,y)d\lambda(y)$$
$$- \int_{[0,1]} yC_{h_1,h_2}(y,y)d\lambda(y) + 2\int_{[0,\alpha]} yC_{h_1,h_2}(y,y)d\lambda(y).$$

Case 3: $x \in h_1^{-1}((y,1]) \cap h_2^{-1}([0,y])$,

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,1]} |1-\alpha-y| \mathbf{1}_{h_1^{-1}((y,1])\cap h_2^{-1}([0,y])}(x)d\lambda(x)d\lambda(y) \\
&= \int_{[0,1]} |1-\alpha-y|\lambda(h_1^{-1}((y,1]) \cap h_2^{-1}([0,y]))d\lambda(y) \\
&= \int_{[0,1]} |1-\alpha-y| \left(y - C_{h_1,h_2}(y,y)\right) d\lambda(y) \\
&= \int_{[0,1-\alpha]} (1-\alpha-y)\left(y - C_{h_1,h_2}(y,y)\right) d\lambda(y) + \int_{(1-\alpha,1]} (y-1+\alpha)\left(y - C_{h_1,h_2}(y,y)\right) d\lambda(y) \\
&= \int_{[0,1-\alpha]} \left((1-\alpha)y - y^2 - (1-\alpha)C_{h_1,h_2}(y,y) + yC_{h_1,h_2}(y,y)\right) d\lambda(y) \\
&\quad + \int_{(1-\alpha,1]} \left(y^2 - (1-\alpha)y + (1-\alpha)C_{h_1,h_2}(y,y) - yC_{h_1,h_2}(y,y)\right) d\lambda(y) \\
&= \frac{(1-\alpha)^3}{2} - \frac{(1-\alpha)^3}{3} - (1-\alpha)\int_{[0,1-\alpha]} C_{h_1,h_2}(y,y)d\lambda(y) + \int_{[0,1-\alpha]} yC_{h_1,h_2}(y,y)d\lambda(y) \\
&\quad + \frac{1-(1-\alpha)^3}{3} - \frac{(1-\alpha)-(1-\alpha)^3}{2} + (1-\alpha)\int_{(1-\alpha,1]} C_{h_1,h_2}(y,y)d\lambda(y) \\
&\quad - \int_{(1-\alpha,1]} yC_{h_1,h_2}(y,y)d\lambda(y) \\
&= \frac{1}{6}\left(1 - 3\alpha + 6\alpha^2 - 2\alpha^3\right) + (1-\alpha)\int_{[0,1]} C_{h_1,h_2}(y,y)d\lambda(y) - 2(1-\alpha)\int_{[0,1-\alpha]} C_{h_1,h_2}(y,y)d\lambda(y) \\
&\quad - \int_{[0,1]} yC_{h_1,h_2}(y,y)d\lambda(y) + 2\int_{[0,1-\alpha]} yC_{h_1,h_2}(y,y)d\lambda(y).
\end{aligned}$$

Case 4: $x \in h_1^{-1}((y,1]) \cap h_2^{-1}((y,1])$,

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,1]} y\mathbf{1}_{h_1^{-1}((y,1])\cap h_2^{-1}((y,1])}(x)d\lambda(x)d\lambda(y) \\
&= \int_{[0,1]} y\lambda(h_1^{-1}((y,1]) \cap h_2^{-1}((y,1]))d\lambda(y) \\
&= \int_{[0,1]} y\left(1 - 2y + C_{h_1,h_2}(y,y)\right) d\lambda(y) \\
&= \int_{[0,1]} \left(y - 2y^2 + yC_{h_1,h_2}(y,y)\right) d\lambda(y) \\
&= \frac{1}{2} - \frac{2}{3} + \int_{[0,1]} yC_{h_1,h_2}(y,y)d\lambda(y) \\
&= -\frac{1}{6} + \int_{[0,1]} yC_{h_1,h_2}(y,y)d\lambda(y).
\end{aligned}$$

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