A NEW HDG METHOD FOR DIRICHLET BOUNDARY CONTROL OF CONVECTION DIFFUSION PDES II: LOW REGULARITY*

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Abstract. In the first part of this work, we analyzed an unconstrained Dirichlet boundary control problem for an elliptic convection diffusion PDE and proposed a new hybridizable discontinuous Galerkin (HDG) method to approximate the solution. For the case of a 2D convex polygonal domain, we also proved an optimal superlinear convergence rate for the control under certain assumptions on the domain and on the target state. In this work, we revisit the convergence analysis without these assumptions; in this case, the solution can have low regularity and we use a different analysis approach. We again prove an optimal convergence rate for the control, and present numerical results to illustrate the convergence theory.

Key words. Dirichlet boundary control, hybridizable discontinuous Galerkin method (HDG), error analysis, low regularity, convection diffusion

AMS subject classifications. 49J20, 65M60

1. Introduction. In Part I of this work [26], we considered the following unconstrained Dirichlet boundary control problem: Minimize the cost functional

$$J(u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)}, \quad \gamma > 0, \tag{1}$$

subject to the elliptic convection diffusion equation

$$\begin{aligned} -\varepsilon\Delta y + \boldsymbol{\beta}\cdot\nabla y &= f \quad \text{in } \Omega,\\ y &= u \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where ε is a positive constant, $f \in L^2(\Omega)$, the vector field $\boldsymbol{\beta}$ satisfies

$$\nabla\cdot\boldsymbol{\beta} \le 0, \tag{3}$$

and $\Omega \subset \mathbb{R}^d$ $(d \ge 2)$ is a Lipschitz polyhedral domain with boundary $\Gamma = \partial\Omega$.

Many researchers have considered the numerical approximation of optimal control problem for convection diffusion equations [4, 24, 6, 20, 7, 40] and also optimal Dirichlet boundary control problems for the Poisson equation and other PDEs

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[9, 19, 30, 39, 34, 14, 5, 8, 25, 37, 11, 21, 1, 23, 22, 3]. However, the authors are unaware of any theoretical and numerical works in the literature concerning the above problem. Progress on this problem is an important step towards the analysis and approximation of Dirichlet boundary control problems for fluid flows.

Only in the last ten years have researchers developed thorough well-posedness, regularity, and finite element error estimation results for Dirichlet boundary control problems for the Poisson equation. Casas and Raymond in [9] obtained an optimal finite element error estimate of order $h^{1-1/s}$ even for some semilinear elliptic PDEs, where $s \geq 2$ depends on the largest angle of the boundary polygon. May et al. in [30] investigated the Poisson problem without constraints and improved the convergence rates obtained by Casas and Raymond for the state and the dual state. Deckelnick et al. in [19] considered domains in 2D and 3D with smooth boundary and proved an error estimate of order $h\sqrt{\ln h}$ for the control based on a variational discretization, and also obtained a superlinear convergence rate in the 2D case for special meshes. Apel et al. in [1] recently considered polygonal domains and obtained a superlinear convergence rate for the control for special meshes or higher order elements. In addition to standard finite element methods, Gong and Yan also obtained $O(h^{1-1/s})$ error estimates for a mixed finite element method [21].

Formally, the optimal control $u \in L^2(\Gamma)$ and the optimal state $y \in L^2(\Omega)$ minimizing the cost functional satisfy the optimality system

$$-\varepsilon\Delta y + \boldsymbol{\beta}\cdot\nabla y = f \quad \text{in } \Omega, \tag{4a}$$
$$y = u \quad \text{on } \partial\Omega, \tag{4b}$$
$$-\varepsilon\Delta z - \nabla\cdot(\boldsymbol{\beta} z) = y - y_d \quad \text{in } \Omega, \tag{4c}$$
$$z = 0 \quad \text{on } \partial\Omega, \tag{4d}$$
$$\varepsilon\nabla z\cdot\boldsymbol{n} - \gamma u = 0 \quad \text{on } \partial\Omega. \tag{4e}$$

In Part I, we showed in the 2D case that the optimal control is indeed determined by a weaker formulation of the above optimality system and we proved a regularity result for the solution.

We also introduced a new hybridzable discontinuous Galerkin (HDG) method to approximate the solution of the optimality system, and obtained an optimal superlinear convergence rate for the control when $\varepsilon = 1$. However, there are two main restrictions for our convergence results in Part I. First, we assumed the largest interior angle ω of the convex polygonal domain belongs to $[\pi/3, 2\pi/3)$. Second, we assumed the desired state y_d is in $H^s(\Omega)$ for some $s > 1/2$. When one of these conditions is not satisfied, the problem can have low regularity, i.e., $\boldsymbol{q} = -\nabla y \in H^{r_{\boldsymbol{q}}}(\Omega)$ for some $r_{\boldsymbol{q}} < 1/2$. In this case, $\boldsymbol{q}$ does not have a well-defined boundary trace and the analysis technique used in Part I is not applicable. Hence a different proof technique is required for this case. We briefly review the regularity theory and the new HDG algorithm in Section 2.

In this work, we use techniques from [28, 29] to remove the restrictions on the largest interior angle ω of the convex domain Ω and the desired state y_d in the diffusion dominated case. Specifically, in Section 3 we obtain optimal convergence rates for the control when $\varepsilon = 1$, $\omega \in [\pi/3, \pi)$, and $y_d \in H^s(\Omega)$ for some $s \geq 0$. We illustrate the low regularity convergence theory with numerical results in Section 4. Furthermore, we also present numerical results for convection dominated problems with $\varepsilon \ll 1$ to demonstrate the performance of the HDG method in this difficult case.

2. Background: Regularity and HDG Formulation. To begin, we briefly review the regularity results for the optimal control problem and the new HDG method from Part I. We set $\varepsilon = 1$ throughout this section.

2.1. Optimal Control Problem: Regularity. As in Part I, we use the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω, and let $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ denote the Sobolev norm and seminorm. We let $H^m(\Omega)$ denote the Sobolev space when $p = 2$ with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. Also, set $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$ and $H(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^2, \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\}$. We denote the L^2-inner products on $L^2(\Omega)$ and $L^2(\Gamma)$ by

$$(v, w)_\Omega = \int_\Omega vw \quad \forall v, w \in L^2(\Omega),$$
$$\langle v, w\rangle_\Gamma = \int_\Gamma vw \quad \forall v, w \in L^2(\Gamma).$$

For the analysis of the optimal control problem, we considered the following scenario in Part I. Suppose Ω is a convex polygonal domain, and let ω denote its largest interior angle. We have $1 < \pi/\omega \le 3$. We assume $\boldsymbol{\beta}$ satisfies

$$\boldsymbol{\beta} \in [L^\infty(\Omega)]^d, \quad \nabla\cdot\boldsymbol{\beta} \in L^\infty(\Omega), \quad \nabla\cdot\boldsymbol{\beta} \le 0, \quad \nabla\nabla\cdot\boldsymbol{\beta} \in [L^2(\Omega)]^d. \tag{5}$$

The mixed weak form of the formal optimality system (4a)-(4e) is

$$(\boldsymbol{q}, \boldsymbol{r})_\Omega - (y, \nabla\cdot\boldsymbol{r})_\Omega + \langle u, \boldsymbol{r}\cdot\boldsymbol{n}\rangle_\Gamma = 0, \tag{6a}$$
$$(\nabla\cdot(\boldsymbol{q} + \boldsymbol{\beta} y), w)_\Omega - (y\nabla\cdot\boldsymbol{\beta}, w)_\Omega = (f, w)_\Omega, \tag{6b}$$
$$(\boldsymbol{p}, \boldsymbol{r})_\Omega - (z, \nabla\cdot\boldsymbol{r})_\Omega = 0, \tag{6c}$$
$$(\nabla\cdot(\boldsymbol{p} - \boldsymbol{\beta} z), w)_\Omega = (y - y_d, w)_\Omega, \tag{6d}$$
$$\langle \gamma u + \boldsymbol{p}\cdot\boldsymbol{n}, \mu\rangle_\Gamma = 0, \tag{6e}$$

for all $(\boldsymbol{r}, w, \mu) \in H(\mathrm{div}, \Omega)\times L^2(\Omega)\times L^2(\Gamma)$. Also, we assume $f = 0$ for the theoretical result below; nonzero forcing can be treated by a simple change of variables as in [2, pg. 3623].

We proved the following well-posedness and regularity theorem in Part I [26].

THEOREM 2.1. *If $y_d \in H^{t^*}(\Omega)$ for some $0 \le t^* < 1$, then the optimal control problem has a unique solution $u \in L^2(\Gamma)$ and u is uniquely determined by the optimality system (6a)-(6e). Moreover, for any $s > 0$ satisfying $s \le \frac{1}{2} + t^*$ and $s < \min\{\frac{3}{2}, \frac{\pi}{\omega} - \frac{1}{2}\}$, we have $u \in H^s(\Gamma)$ and*

$$(\boldsymbol{q}, \boldsymbol{p}, y, z) \in [H^{s-\frac{1}{2}}(\Omega)]^d \cap H(\mathrm{div}, \Omega) \times [H^{s+\frac{1}{2}}(\Omega)]^d \times H^{s+\frac{1}{2}}(\Omega) \times H^{s+\frac{3}{2}}(\Omega).$$

Theorem 2.1 implies the regularity of the solution of the optimality system (6a)-(6e) depends on the desired state y_d and the domain Ω. As is known, solutions to Dirichlet boundary control problems can have low regularity; this causes difficulty for numerical analysis.

In Part I [26], for the numerical analysis of the new HDG method we assumed Ω is convex, $y_d \in H^{t^*}(\Omega)$ for some $t^* \in (1/2, 1)$, and $\pi/3 < \omega < 2\pi/3$. These assumptions give high regularity for the optimal control, i.e., $u \in H^{r_u}(\Gamma)$ for some $r_u \in (1, 3/2)$. Furthermore, the assumptions give $\boldsymbol{q} \in H^{r_{\boldsymbol{q}}}(\Omega)$ with $r_{\boldsymbol{q}} > 1/2$, which guarantees $\boldsymbol{q}$ has a well-defined trace on the boundary Γ. We used this property in the HDG convergence analysis.

In this paper we again assume Ω is convex, but we remove the restrictions on the desired state and the largest interior angle for the numerical analysis; i.e., we only require $t^* \in [0,1)$ and $\pi/3 \leq \omega < \pi$. In this case, the regularity of the optimal control can be low, i.e., $u \in H^{r_u}(\Gamma)$ for some $r_u \in [1/2, 1)$, and $\boldsymbol{q}$ is no longer guaranteed to have a well-defined L^2 boundary trace; however, the optimality system (6a)-(6e) can be understood in a standard weak sense.

2.2. The HDG Formulation. For the HDG method, we assume Ω is a polyhedral domain with $d \geq 2$. We use the same notation from Part I [26] to describe the method. For more information about HDG methods, see, e.g., [15, 16, 31, 32, 33, 10, 13, 18, 38, 17, 12, 35, 36].

Let $\mathcal{T}_h$ be a collection of disjoint elements that partition Ω, and let $\partial\mathcal{T}_h$ be the set $\{x \in \partial K : K \in \mathcal{T}_h\}$. For the analysis, we assume $\mathcal{T}_h$ is a conforming triangulation of Ω. Denote the elements of $\mathcal{T}_h$ by K and the faces of K by e. Denote $\mathcal{E}_h$ the set of all faces, $\mathcal{E}_h^\partial$ the set of faces such that $e \subset \Gamma$, and $\mathcal{E}_h^o = \mathcal{E}_h \setminus \mathcal{E}_h^\partial$. The mesh dependent inner products are denoted by

$$(w, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, v)_K, \qquad \langle \zeta, \rho \rangle_{\partial\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K}.$$

Let $\mathcal{P}^k(D)$ denote the set of polynomials of degree at most k on a domain D. As in Part I, we use the discontinuous finite element spaces

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in [L^2(\Omega)]^d : \boldsymbol{v}|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h\}, \tag{7}$$

$$W_h := \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h\}, \tag{8}$$

$$M_h := \{\mu \in L^2(\partial\mathcal{T}_h) : \mu|_e \in \mathcal{P}^{k+1}(e), \forall e \in \mathcal{E}_h\} \tag{9}$$

for the flux variables, scalar variables, and boundary trace variables, respectively. Note that the polynomial degree for the scalar and boundary trace variables is one order higher than the polynomial degree for the flux variables. We discussed this unusual choice for M_h in Part I.

Define $M_h(o)$ and $M_h(\partial)$ in the same way as M_h, but with $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ replacing $\mathcal{E}_h$, respectively. For any functions $w \in W_h$ and $\boldsymbol{r} \in \boldsymbol{V}_h$, we use ∇w and $\nabla \cdot \boldsymbol{r}$ to denote the gradient of w and the divergence of $\boldsymbol{r}$ taken piecewise on each element $K \in \mathcal{T}_h$.

To approximate the solution of the mixed weak form (6a)-(6e) of the optimality system, the HDG formulation considered here is modified from Part I to avoid the estimation of $\boldsymbol{q}$ on the boundary. In the 2D case, recall from Subsection 2.1 that $\boldsymbol{q}$ is not guaranteed to have a well-defined L^2 boundary trace since we consider a solution of the optimal control problem with low regularity.

The HDG method seeks approximate fluxes $\boldsymbol{q}_h, \boldsymbol{p}_h \in \boldsymbol{V}_h$, states $y_h, z_h \in W_h$, interior element boundary traces $\widehat{y}_h^o, \widehat{z}_h^o \in M_h(o)$, and boundary control $u_h \in M_h(\partial)$ satisfying

$$(\boldsymbol{q}_h, \boldsymbol{r}_1)_{\mathcal{T}_h} - (y_h, \nabla \cdot \boldsymbol{r}_1)_{\mathcal{T}_h} + \langle \widehat{y}_h^o, \boldsymbol{r}_1 \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h \setminus \mathcal{E}_h^\partial} + \langle u_h, \boldsymbol{r}_1 \cdot \boldsymbol{n} \rangle_{\mathcal{E}_h^\partial} = 0, \tag{10a}$$

$$\begin{aligned}(\nabla \cdot \boldsymbol{q}_h, w_1)_{\mathcal{T}_h} - (\boldsymbol{\beta} y_h, \nabla w_1)_{\mathcal{T}_h} - (\nabla \cdot \boldsymbol{\beta} y_h, w_1)_{\mathcal{T}_h} \\ + \langle (h^{-1} + \tau_1) y_h, w_1 \rangle_{\partial\mathcal{T}_h} + \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - \tau_1 - h^{-1}) \widehat{y}_h^o, w_1 \rangle_{\partial\mathcal{T}_h \setminus \mathcal{E}_h^\partial} \\ + \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - \tau_1 - h^{-1}) u_h, w_1 \rangle_{\mathcal{E}_h^\partial} = (f, w_1)_{\mathcal{T}_h},\end{aligned} \tag{10b}$$

for all $(\boldsymbol{r}_1, w_1) \in \boldsymbol{V}_h \times W_h$,

$$(\boldsymbol{p}_h, \boldsymbol{r}_2)_{\mathcal{T}_h} - (z_h, \nabla\cdot \boldsymbol{r}_2)_{\mathcal{T}_h} + \langle \widehat{z}_h^o, \boldsymbol{r}_2\cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0, \tag{10c}$$

$$\begin{aligned}&(\nabla\cdot \boldsymbol{p}_h, w_2)_{\mathcal{T}_h} - (y_h, w_2)_{\mathcal{T}_h} + (\boldsymbol{\beta} z_h, \nabla w_2)_{\mathcal{T}_h}\\ &+\langle (h^{-1}+\tau_2) z_h, w_2\rangle_{\partial\mathcal{T}_h} - \langle (h^{-1}+\tau_2+\boldsymbol{\beta}\cdot\boldsymbol{n})\widehat{z}_h^o, w_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = -(y_d, w_2)_{\mathcal{T}_h},\end{aligned} \tag{10d}$$

for all $(\boldsymbol{r}_2, w_2) \in \boldsymbol{V}_h \times W_h$,

$$\begin{aligned}&\langle \boldsymbol{q}_h\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial} + \langle (h^{-1}+\tau_1) y_h, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\ &+\langle (\boldsymbol{\beta}\cdot\boldsymbol{n} - \tau_1 - h^{-1})\widehat{y}_h^o, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0,\end{aligned} \tag{10e}$$

for all $\mu_1 \in M_h(o)$,

$$\begin{aligned}&\langle \boldsymbol{p}_h\cdot\boldsymbol{n}, \mu_2\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial} + \langle (h^{-1}+\tau_2) z_h, \mu_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\ &-\langle (\boldsymbol{\beta}\cdot\boldsymbol{n} + \tau_2 + h^{-1})\widehat{z}_h^o, \mu_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0,\end{aligned} \tag{10f}$$

for all $\mu_2 \in M_h(o)$, and the optimality condition

$$\langle \boldsymbol{p}_h\cdot\boldsymbol{n}, \mu_3\rangle_{\mathcal{E}_h^\partial} + \gamma\langle u_h, \mu_3\rangle_{\mathcal{E}_h^\partial} + \langle (h^{-1}+\tau_2) z_h, \mu_3\rangle_{\mathcal{E}_h^\partial} = 0, \tag{10g}$$

for all $\mu_3 \in M_h(\partial)$.

Here, τ_1 and τ_2 are stabilization functions defined on $\partial\mathcal{T}_h$ that satisfy the same conditions as in Part I:
(A1) τ_2 is piecewise constant on $\partial\mathcal{T}_h$.
(A2) $\tau_1 = \tau_2 + \boldsymbol{\beta}\cdot\boldsymbol{n}$.
(A3) For any $K \in \mathcal{T}_h$, $\min\,(\tau_2 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})|_{\partial K} > 0$.
Conditions **(A2)** and **(A3)** imply

$$\min\,(\tau_1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})|_{\partial K} > 0 \quad \text{for any } K \in \mathcal{T}_h. \tag{11}$$

This completes the formulation of the HDG method.

Notice that formulation (10) is slightly different from formulation (3.4) in Part I; specifically, equations (b) and (d) are modified. A straightforward computation shows that both are equivalent; see Part I, Section 3.2. Formulation (10) above allows us to achieve error estimates in the low regularity case considered here.

3. Error Analysis. Next, we perform a convergence analysis of the above HDG method in the diffusion dominated case. Therefore, we set $\varepsilon = 1$ in this section.

3.1. Assumptions and Main Result. As in Part I, we assume throughout that Ω is a bounded convex polyhedral domain and $\boldsymbol{\beta}$ satisfies

$$\boldsymbol{\beta} \in [C(\overline{\Omega})]^d, \quad \nabla\cdot\boldsymbol{\beta} \in L^\infty(\Omega), \quad \nabla\cdot\boldsymbol{\beta} \le 0, \quad \nabla\nabla\cdot\boldsymbol{\beta} \in [L^2(\Omega)]^d. \tag{12}$$

We assume the solution of the optimality system (6a)-(6e) is unique and has the following regularity properties:

$$y \in H^{r_y}(\Omega), \quad z \in H^{r_z}(\Omega), \quad \boldsymbol{q} \in [H^{r_{\boldsymbol{q}}}(\Omega)]^d \cap H(\text{div}, \Omega), \quad \boldsymbol{p} \in H^{r_{\boldsymbol{p}}}(\Omega), \tag{13a}$$

$$r_y \ge 1, \quad r_z \ge 2, \quad r_{\boldsymbol{q}} \ge 0, \quad r_{\boldsymbol{p}} \ge 1. \tag{13b}$$

In the 2D case, Theorem 2.1 guarantees this condition is satisfied.

As mentioned in Subsection 2.1, the regularity of $\boldsymbol{q}$ can be low and therefore $\boldsymbol{q}$ may not have a L^2 boundary trace. The $H(\mathrm{div},\Omega)$ regularity of $\boldsymbol{q}$ is critically important for the numerical analysis.

We also require the family of meshes $\{\mathcal{T}_h\}$ is a conforming quasi-uniform triangulation of Ω. This assumption on the meshes is stronger than in Part I; there we assumed $\{\mathcal{T}_h\}$ is a conforming quasi-uniform polyhedral mesh. Therefore, the analysis in Part I allows for a more general family of meshes; however, the analysis here allows us to treat the low regularity case.

We now state our main convergence result.

THEOREM 3.1. *Let*

$$\begin{aligned} s_{\boldsymbol{q}} &= \min\{r_{\boldsymbol{q}}, k+1\}, \qquad & s_y &= \min\{r_y, k+2\},\\ s_{\boldsymbol{p}} &= \min\{r_{\boldsymbol{p}}, k+1\}, \qquad & s_z &= \min\{r_z, k+2\}. \end{aligned} \tag{14}$$

If the above assumptions hold and $s_{\boldsymbol{q}} \in [0,1]$, then

$$\begin{aligned} \|u-u_h\|_{\varepsilon_h^\partial} &\lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac32}\|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac12}\|y\|_{s_y,\Omega},\\ \|y-y_h\|_{\mathcal{T}_h} &\lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac32}\|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac12}\|y\|_{s_y,\Omega},\\ \|\boldsymbol{p}-\boldsymbol{p}_h\|_{\mathcal{T}_h} &\lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac32}\|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac12}\|y\|_{s_y,\Omega},\\ \|z-z_h\|_{\mathcal{T}_h} &\lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac32}\|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac12}\|y\|_{s_y,\Omega}. \end{aligned}$$

If in addition the inequalities in (13b) are strict and $k \geq 1$, then

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-1}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-2}\|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-1}\|y\|_{s_y,\Omega}.$$

Remark 3.2. Note that we assume $s_{\boldsymbol{q}} \in [0,1]$. This is not a restriction since the case $s_{\boldsymbol{q}} > 1$ is treated in Part I on a more general family of meshes.

Specializing to the 2D case gives the following result:

COROLLARY 3.3. *Suppose $d=2$, $f=0$, $s_{\boldsymbol{q}} \in [0,1]$, and $y_d \in H^{t^*}(\Omega)$ for some $t^* \in [0,1)$. Let $\pi/3 \leq \omega < \pi$ be the largest interior angle of Γ, and let $r>0$ satisfy*

$$r \leq r_d := \frac12 + t^* \in [1/2, 3/2), \quad \text{and} \quad r < r_\Omega := \min\left\{\frac32, \frac{\pi}{\omega} - \frac12\right\} \in (1/2, 3/2].$$

If $k=1$, then

$$\begin{aligned} \|u-u_h\|_{\varepsilon_h^\partial} &\lesssim h^r(\|\boldsymbol{p}\|_{H^{r+1/2}(\Omega)} + \|z\|_{H^{r+3/2}(\Omega)} + \|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)} + \|y\|_{H^{r+1/2}(\Omega)}),\\ \|y-y_h\|_{\mathcal{T}_h} &\lesssim h^r(\|\boldsymbol{p}\|_{H^{r+1/2}(\Omega)} + \|z\|_{H^{r+3/2}(\Omega)} + \|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)} + \|y\|_{H^{r+1/2}(\Omega)}),\\ \|\boldsymbol{p}-\boldsymbol{p}_h\|_{\mathcal{T}_h} &\lesssim h^r(\|\boldsymbol{p}\|_{H^{r+1/2}(\Omega)} + \|z\|_{H^{r+3/2}(\Omega)} + \|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)} + \|y\|_{H^{r}+1/2(\Omega)}),\\ \|z-z_h\|_{\mathcal{T}_h} &\lesssim h^r(\|\boldsymbol{p}\|_{H^{r+1/2}(\Omega)} + \|z\|_{H^{r+3/2}(\Omega)} + \|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)} + \|y\|_{H^{r+1/2}(\Omega)}). \end{aligned}$$

If in addition $r > 1/2$, then

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\mathcal{T}_h} \lesssim h^{r-1/2}(\|\boldsymbol{p}\|_{H^{r+1/2}(\Omega)} + \|z\|_{H^{r+3/2}(\Omega)} + \|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)} + \|y\|_{H^{r+1/2}(\Omega)}).$$

Furthermore, if $k = 0$ then

$$
\begin{aligned}
\|u-u_h\|_{\varepsilon_h^\partial} &\lesssim h^{1/2}(\|\boldsymbol{p}\|_{H^1(\Omega)}+\|z\|_{H^2(\Omega)}+\|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)}+\|y\|_{H^{r+1/2}(\Omega)}),\\
\|y-y_h\|_{\mathcal{T}_h} &\lesssim h^{1/2}(\|\boldsymbol{p}\|_{H^1(\Omega)}+\|z\|_{H^2(\Omega)}+\|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)}+\|y\|_{H^{r+1/2}(\Omega)}),\\
\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\mathcal{T}_h} &\lesssim h^{1/2}(\|\boldsymbol{p}\|_{H^1(\Omega)}+\|z\|_{H^2(\Omega)}+\|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)}+\|y\|_{H^{r+1/2}(\Omega)}),\\
\|z-z_h\|_{\mathcal{T}_h} &\lesssim h^{1/2}(\|\boldsymbol{p}\|_{H^1(\Omega)}+\|z\|_{H^2(\Omega)}+\|\boldsymbol{q}\|_{H^{r-1/2}(\Omega)}+\|y\|_{H^{r+1/2}(\Omega)}).
\end{aligned}
$$

As in Part I, when $k = 1$ the convergence rates are optimal for the control and the flux $\boldsymbol{q}$ and suboptimal for the other variables. When $k = 0$ the convergence rates for all variables are suboptimal with one exception: If $y_d \in L^2(\Omega)$ only so that $t^* = 0$, then $u \in H^{1/2}(\Gamma)$ only and the convergence rate for the control is optimal. Also, if r_d or r_Ω is near $1/2$, then the convergence rate is nearly optimal for the control in the $k = 0$ case.

For standard finite elements for Dirichlet boundary control of the Poisson equation, May et al. used a duality argument to obtain improved convergence rates for the state and dual state [30]. We attempted to improve the above error estimates for the state, dual state, and fluxes using similar ideas, but we were unsuccessful. It appears entirely new proof techniques may be required to improve the convergence rates for these variables.

3.2. Preliminary material I. We split the preliminary material required for the proof into two parts. First, we give a brief overview of material closely related to the preliminary material in Part I: L^2 projections, HDG operators $\mathcal{B}_1$ and $\mathcal{B}_2$, and the well-posedness of the HDG equations.

As in Part I, we use the standard L^2 projections $\boldsymbol{\Pi} : [L^2(\Omega)]^d \to \boldsymbol{V}_h$, $\Pi : L^2(\Omega) \to W_h$, and $P_M : L^2(\partial\mathcal{T}_h) \to M_h$, which satisfy

$$
\begin{aligned}
(\boldsymbol{\Pi q}, \boldsymbol{r})_K &= (\boldsymbol{q}, \boldsymbol{r})_K, \qquad \forall \boldsymbol{r} \in [\mathcal{P}_k(K)]^d,\\
(\Pi y, w)_K &= (y, w)_K, \qquad \forall w \in \mathcal{P}_{k+1}(K),\\
\langle P_M m, \mu\rangle_e &= \langle m, \mu\rangle_e, \qquad \forall \mu \in \mathcal{P}_{k+1}(e).
\end{aligned} \tag{15}
$$

We have the following bounds:

$$\|\boldsymbol{q}-\boldsymbol{\Pi q}\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{q}}}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega}, \qquad \|y-\Pi y\|_{\mathcal{T}_h} \lesssim h^{s_y}\|y\|_{s_y,\Omega}, \tag{16a}$$

$$\|y-\Pi y\|_{\partial\mathcal{T}_h} \lesssim h^{s_y-\frac{1}{2}}\|y\|_{s_y,\Omega}, \qquad \|y-P_M y\|_{\partial\mathcal{T}_h} \lesssim h^{s_y-\frac{1}{2}}\|y\|_{s_y,\Omega}, \tag{16b}$$

$$\|w\|_{\partial\mathcal{T}_h} \lesssim h^{-\frac{1}{2}}\|w\|_{\mathcal{T}_h}, \ \forall w \in W_h, \tag{16c}$$

and similar projection error bounds for $\boldsymbol{p}$ and z.

In this paper, we do not use the same HDG formulation for the analysis that we used in Part I. We define the HDG operators $\mathcal{B}_1$ and $\mathcal{B}_2$ by

$$
\begin{aligned}
&\mathcal{B}_1(\boldsymbol{q}_h, y_h, \widehat{y}_h^o; \boldsymbol{r}_1, w_1, \mu_1)\\
&\quad = (\boldsymbol{q}_h, \boldsymbol{r}_1)_{\mathcal{T}_h} - (y_h, \nabla\cdot\boldsymbol{r}_1)_{\mathcal{T}_h} + \langle \widehat{y}_h^o, \boldsymbol{r}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + (\nabla\cdot\boldsymbol{q}_h, w_1)_{\mathcal{T}_h}\\
&\qquad - (\boldsymbol{\beta} y_h, \nabla w_1)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\beta} y_h, w_1)_{\mathcal{T}_h} + \langle h^{-1} y_h + \tau_1 y_h, w_1\rangle_{\partial\mathcal{T}_h}\\
&\qquad + \langle(\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1)\widehat{y}_h^o, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad - \langle \boldsymbol{q}_h\cdot\boldsymbol{n} + \boldsymbol{\beta}\cdot\boldsymbol{n}\widehat{y}_h^o + h^{-1}(y_h - \widehat{y}_h^o) + \tau_1(y_h - \widehat{y}_h^o), \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial},
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
&\mathcal{B}_2(\boldsymbol{p}_h, z_h, \widehat{z}_h^o; \boldsymbol{r}_2, w_2, \mu_2) \\
&\quad = (\boldsymbol{p}_h, \boldsymbol{r}_2)_{\mathcal{T}_h} - (z_h, \nabla\cdot \boldsymbol{r}_2)_{\mathcal{T}_h} + \langle \widehat{z}_h^o, \boldsymbol{r}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + (\nabla\cdot\boldsymbol{p}_h, w_2)_{\mathcal{T}_h} \\
&\qquad + (\boldsymbol{\beta} z_h, \nabla w_2)_{\mathcal{T}_h} + \langle h^{-1} z_h + \tau_2 z_h, w_2\rangle_{\partial\mathcal{T}_h} \\
&\qquad - \langle(\boldsymbol{\beta}\cdot\boldsymbol{n} + h^{-1} + \tau_2)\widehat{z}_h^o, w_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} \\
&\qquad - \langle \boldsymbol{p}_h\cdot\boldsymbol{n} - \boldsymbol{\beta}\cdot\boldsymbol{n}\widehat{z}_h^o + h^{-1}(z_h - \widehat{z}_h^o) + \tau_2(z_h - \widehat{z}_h^o), \mu_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
\tag{18}
$$

We emphasize that this is an equivalent definition to the one given in Part I that is more appropriate to obtain error estimates in the low regularity case.

We rewrite the HDG formulation of the optimality system (10) in terms of the HDG operators $\mathcal{B}_1$ and $\mathcal{B}_2$: find $(\boldsymbol{q}_h, \boldsymbol{p}_h, y_h, z_h, \widehat{y}_h^o, \widehat{z}_h^o, u_h) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times M_h(o) \times M_h(o) \times M_h(\partial)$ satisfying

$$
\begin{aligned}
\mathcal{B}_1(\boldsymbol{q}_h, y_h, \widehat{y}_h^o; \boldsymbol{r}_1, w_1, \mu_1) &= (f, w_1)_{\mathcal{T}_h} - \langle u_h, \boldsymbol{r}_1\cdot\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial} \\
&\quad - \langle(\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1)u_h, w_1\rangle_{\mathcal{E}_h^\partial},
\end{aligned}
\tag{19a}
$$

$$
\mathcal{B}_2(\boldsymbol{p}_h, z_h, \widehat{z}_h^o; \boldsymbol{r}_2, w_2, \mu_2) = (y_h - y_d, w_2)_{\mathcal{T}_h}, \tag{19b}
$$

$$
\gamma^{-1}\langle \boldsymbol{p}_h\cdot\boldsymbol{n} + h^{-1}z_h + \tau_2 z_h, \mu_3\rangle_{\mathcal{E}_h^\partial} = -\langle u_h, \mu_3\rangle_{\mathcal{E}_h^\partial}, \tag{19c}
$$

for all $(\boldsymbol{r}_1, \boldsymbol{r}_2, w_1, w_2, \mu_1, \mu_2, \mu_3) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times M_h(o) \times M_h(o) \times M_h(\partial)$.

For the convenience of the reader, we recall three results proven in Part I.

LEMMA 3.4 ([26]). *For any $(\boldsymbol{v}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$, we have*

$$
\begin{aligned}
&\mathcal{B}_1(\boldsymbol{v}_h, w_h, \mu_h; \boldsymbol{v}_h, w_h, \mu_h) \\
&\quad = (\boldsymbol{v}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle(h^{-1} + \tau_1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})(w_h - \mu_h), w_h - \mu_h\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} \\
&\qquad - \frac{1}{2}(\nabla\cdot\boldsymbol{\beta} w_h, w_h)_{\mathcal{T}_h} + \langle(h^{-1} + \tau_1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})w_h, w_h\rangle_{\mathcal{E}_h^\partial}, \\
&\mathcal{B}_2(\boldsymbol{v}_h, w_h, \mu_h; \boldsymbol{v}_h, w_h, \mu_h) \\
&\quad = (\boldsymbol{v}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle(h^{-1} + \tau_2 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})(w_h - \mu_h), w_h - \mu_h\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} \\
&\qquad - \frac{1}{2}(\nabla\cdot\boldsymbol{\beta} w_h, w_h)_{\mathcal{T}_h} + \langle(h^{-1} + \tau_2 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})w_h, w_h\rangle_{\mathcal{E}_h^\partial}.
\end{aligned}
$$

LEMMA 3.5 ([26]). *If **(A2)** holds, then*

$$
\mathcal{B}_1(\boldsymbol{q}_h, y_h, \widehat{y}_h^o; \boldsymbol{p}_h, -z_h, -\widehat{z}_h^o) + \mathcal{B}_2(\boldsymbol{p}_h, z_h, \widehat{z}_h^o; -\boldsymbol{q}_h, y_h, \widehat{y}_h^o) = 0.
$$

PROPOSITION 3.6 ([26]). *If **(A2)** holds, there exists a unique solution of the HDG equations* (19).

3.3. Preliminary material II. Next, we discuss preliminary material that is directly related to the low regularity case considered in this paper: the interpolation operators $\mathcal{I}_h^0$, $\mathcal{I}_h^1$, $\mathcal{I}_h$ and their properties.

Recall we assume the primary flux $\boldsymbol{q}$ only satisfies $\boldsymbol{q} \in [H^{r_{\boldsymbol{q}}}(\Omega)]^d \cap H(\text{div}, \Omega)$, where $r_{\boldsymbol{q}} \geq 0$. Therefore, the quantity $\|\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}\boldsymbol{q}\cdot\boldsymbol{n}\|_{\partial\mathcal{T}_h}$ is not well defined and the HDG analysis technique used in Part I is not applicable. We use analysis techniques from [28, 29] to avoid using the L^2 boundary trace of $\boldsymbol{q}$. Let us introduce some notation first.

Define the H^1-conforming piecewise linear finite element space W_h^c by

$$W_h^c := \{w_h^c \in H_0^1(\Omega) : w_h^c|_K \in \mathcal{P}_1(K),\ \forall K \in \mathcal{T}_h\}.$$

For any $K \in \mathcal{T}_h$, let $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{d+1}$ denote the standard barycentric coordinate functions defined on the simplex K. Define

$$\mathbb{S}(K) := S_1(K) + S_2(K) + \cdots + S_{d+1}(K), \tag{20}$$

where

$$S_i(K) := \Big(\prod_{j\neq i} \lambda_j\Big)\mathrm{span}\Big\{\prod_j \lambda_j^{\alpha_j} : \sum_j \alpha_j = k,\ \alpha_i = 0\Big\},\quad i = 1, 2, \ldots, d+1.$$

Now we define the interpolations operators $\mathcal{I}_h^0$, $\mathcal{I}_h^1$, $\mathcal{I}_h$. First, define $m_K : L^2(\partial K) \to \mathbb{R}$ by

$$m_K(\mu) := \frac{1}{d+1} \sum_{e\in\partial K} \frac{1}{|e|} \int_e \mu, \tag{21}$$

where $|e|$ denotes the $d-1$ dimensional Hausdorff measure of e. Next, the interpolation operator $\mathcal{I}_h^0 : L^2(\varepsilon_h) \to W_h^c$ is defined as follows:

$$\mathcal{I}_h^0\mu(a) = \begin{cases} \dfrac{1}{\#\omega_a} \displaystyle\sum_{K\in\omega_a} m_K(\mu) & \text{if } a \text{ is an interior node of } \mathcal{T}_h, \\ 0 & \text{if } a \text{ is a boundary node of } \mathcal{T}_h, \end{cases}$$

where $\omega_a := \{K \in \mathcal{T}_h : a \text{ is a vertex of } K\}$ and $\#\omega_a$ denotes the number of elements in ω_a.

Next, the interpolation operator $\mathcal{I}_h^1$ on $L^2(\Omega) \times L^2(\varepsilon_h)$ is defined elementwise as follows: for each K,

$$\mathcal{I}_h^1(w, \mu)|_K := \mathcal{I}_K^1(w, \mu) = w_1 + w_2,$$

where $(w_1, w_2) \in \mathbb{S}(K) \times (\prod_j \lambda_j)\mathcal{P}_k(K)$ is uniquely determined by

$$\begin{aligned} \langle w_1, m\rangle_e &= \langle \mu, m\rangle_e, \\ (w_2, n)_K &= (w - w_1, n)_K, \end{aligned}$$

for all $(m, n) \in \mathcal{P}_k(e) \times \mathcal{P}_k(K)$ and $e \in \partial K$.

Finally, for $(w, \mu) \in L^2(\Omega) \times L^2(\varepsilon_h)$, we define the third interpolation operator $\mathcal{I}_h$ by

$$\mathcal{I}_h(w, \mu) := \mathcal{I}_h^0\mu + \mathcal{I}_h^1(w - \mathcal{I}_h^0\mu, \mu - \mathcal{I}_h^0\mu).$$

It is straightforward to verify that $\mathcal{I}_h$ and $\mathcal{I}_h^1$ have the following properties; see [28, 29].

Lemma 3.7. *For any $(w, \mu) \in L^2(\Omega) \times L^2(\varepsilon_h)$ and $K \in \mathcal{T}_h$, we have*

$$(\mathcal{I}_h(w, \mu), n)_K = (w, n)_K, \tag{22a}$$

$$\langle \mathcal{I}_h(w, \mu), m\rangle_{\partial K} = \langle \mu, m\rangle_{\partial K}, \tag{22b}$$

for all $(m, n) \in \mathcal{P}_k(e) \times \mathcal{P}_k(K)$ and $e \in \partial K$, and

$$\|\mathcal{I}_h^1(w, \mu)\|_K \lesssim \|w\|_K + h^{\frac{1}{2}}\|\mu\|_{\partial K}. \tag{23}$$

Moreover, if $\mu|_\Gamma = 0$, we have

$$\mathcal{I}_h(w, \mu) \in H_0^1(\Omega). \tag{24}$$

In the next three lemmas, we assume $(\boldsymbol{v}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$ satisfy

$$(\boldsymbol{v}_h, \boldsymbol{r})_{\mathcal{T}_h} - (w_h, \nabla\cdot \boldsymbol{r})_{\mathcal{T}_h} + \langle \mu_h, \boldsymbol{r}\cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{25}$$

for all $\boldsymbol{r} \in \boldsymbol{V}_h$.

We begin with a key inequality; see Part I [26, Lemma 4.7] and also [35].

Lemma 3.8. *If $(\boldsymbol{v}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$ satisfy (25), then*

$$\|\nabla w_h\|_{\mathcal{T}_h} \lesssim \|\boldsymbol{v}_h\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|w_h - \mu_h\|_{\partial\mathcal{T}_h}. \tag{26}$$

The next two results are similar to Lemma 3.4 and Lemma 3.6 in [28]. Here, we have a different space M_h (with polynomials of degree $k+1$ instead of k) and we do not have a variable diffusion coefficient. However, the proofs of the next two results are very similar to the proofs in [28] and are omitted.

Lemma 3.9. *If $(\boldsymbol{v}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$ satisfy (25), then*

$$\begin{aligned} h^{-1}\sum_{K\in\mathcal{T}_h}\|w_h - m_K(\mu_h)\|_K + h^{-\frac{1}{2}}\sum_{K\in\mathcal{T}_h}\|\mu_h - m_K(\mu_h)\|_{\partial K} \\ \lesssim \|\boldsymbol{v}_h\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|w_h - \mu_h\|_{\partial\mathcal{T}_h}. \end{aligned} \tag{27}$$

Lemma 3.10. *If $(\boldsymbol{v}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$ satisfy (25), then*

$$\|\nabla\mathcal{I}_h(w_h,\mu_h)\|_{\mathcal{T}_h} \lesssim \|\boldsymbol{v}_h\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|w_h - \mu_h\|_{\partial\mathcal{T}_h}, \tag{28a}$$

$$h^{-1}\|w_h - \mathcal{I}_h(w_h,\mu_h)\|_{\mathcal{T}_h} \lesssim \|\boldsymbol{v}_h\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|w_h - \mu_h\|_{\partial\mathcal{T}_h}. \tag{28b}$$

3.4. Proof of Main Result. Now we move to the proof of the error estimates. We follow the strategy of Part I [26] and split the proof into seven steps. In the first five steps we use the rewriting of operators $\mathcal{B}_1$ and $\mathcal{B}_2$ in an explicit way and the proofs are different from the corresponding ones of Part I. Steps 6 and 7 use the properties of $\mathcal{B}_1$ and $\mathcal{B}_2$ recalled in Lemma 3.4 and Lemma 3.5 and are very similar to Steps 6 and 7 in the high regularity case in Part I. We include these proofs here to make this paper self-contained.

We first bound the error between the solution of the mixed form (6a)-(6d) of the optimality system and the solution

$$(\boldsymbol{q}_h(u), \boldsymbol{p}_h(u), y_h(u), z_h(u), \widehat{y}_h^o(u), \widehat{z}_h^o(u)) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times M_h(o) \times M_h(o)$$

of the auxiliary problem

$$\begin{aligned} \mathcal{B}_1(\boldsymbol{q}_h(u), y_h(u), \widehat{y}_h^o(u); \boldsymbol{r}_1, w_1, \mu_1) &= (f, w_1)_{\mathcal{T}_h} - \langle P_M u, \boldsymbol{r}_1\cdot\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial} \\ &\quad - \langle (\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1) P_M u, w_1\rangle_{\mathcal{E}_h^\partial}, \end{aligned} \tag{29a}$$

$$\mathcal{B}_2(\boldsymbol{p}_h(u), z_h(u), \widehat{z}_h^o(u); \boldsymbol{r}_2, w_2, \mu_2) = (y_h(u) - y_d, w_2)_{\mathcal{T}_h}, \tag{29b}$$

for all $(\boldsymbol{r}_1, \boldsymbol{r}_2, w_1, w_2, \mu_1, \mu_2) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times M_h(o) \times M_h(o)$. As in Part I, we use the notation

$$\begin{aligned} \delta^{\boldsymbol{q}} &= \boldsymbol{q} - \boldsymbol{\Pi}\boldsymbol{q}, & \varepsilon_h^{\boldsymbol{q}} &= \boldsymbol{\Pi}\boldsymbol{q} - \boldsymbol{q}_h(u), \\ \delta^y &= y - \Pi y, & \varepsilon_h^y &= \Pi y - y_h(u), \\ \delta^{\widehat{y}} &= y - P_M y, & \varepsilon_h^{\widehat{y}} &= P_M y - \widehat{y}_h(u), \\ \widehat{\boldsymbol{\delta}}_1 &= \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}} + (h^{-1} + \tau_1)(\delta^y - \delta^{\widehat{y}}), \end{aligned} \tag{30}$$

where $\widehat{y}_h(u) = \widehat{y}_h^o(u)$ on $\mathcal{E}_h^o$ and $\widehat{y}_h(u) = P_M u$ on $\mathcal{E}_h^\partial$. This definition gives $\varepsilon_h^{\widehat{y}} = 0$ on $\mathcal{E}_h^\partial$.

3.4.1. Step 1: The error equation for part 1 of the auxiliary problem (29a).

LEMMA 3.11. *For all* $(\boldsymbol{r}_1, w_1, \mu_1) \in \boldsymbol{V}_h \times W_h \times M_h(o)$, *we have*

$$
\begin{aligned}
&\mathcal{B}_1(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^{y}, \varepsilon_h^{\widehat{y}}, \boldsymbol{r}_1, w_1, \mu_1)\\
&\quad = -(\nabla\cdot\delta^{\boldsymbol{q}}, w_1)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + (\boldsymbol{\beta}\delta^y, \nabla w_1)_{\mathcal{T}_h}\\
&\qquad + (\nabla\cdot\boldsymbol{\beta}\delta^y, w_1)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, w_1\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\boldsymbol{\delta}}_1, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned} \tag{31}
$$

Proof. Using the definition of $\mathcal{B}_1$ in (17) gives

$$
\begin{aligned}
&\mathcal{B}_1(\boldsymbol{\Pi q}, \Pi y, P_M y, \boldsymbol{r}_1, w_1, \mu_1)\\
&\quad = (\boldsymbol{\Pi q}, \boldsymbol{r_1})_{\mathcal{T}_h} - (\Pi y, \nabla\cdot\boldsymbol{r_1})_{\mathcal{T}_h} + \langle P_M y, \boldsymbol{r_1}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad + (\nabla\cdot\boldsymbol{\Pi q}, w_1)_{\mathcal{T}_h} - (\boldsymbol{\beta}\Pi y, \nabla w_1)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\beta}\Pi y, w_1)_{\mathcal{T}_h}\\
&\qquad + \langle (h^{-1}+\tau_1)\Pi y, w_1\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1)P_M y, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n} + \boldsymbol{\beta}\cdot\boldsymbol{n}P_M y + (h^{-1}+\tau_1)(\Pi y - P_M y), \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
$$

Using properties of the L^2 projections (15) gives

$$
\begin{aligned}
&\mathcal{B}_1(\boldsymbol{\Pi q}, \Pi y, P_M y, \boldsymbol{r}_1, w_1, \mu_1)\\
&\quad = (\boldsymbol{q}, \boldsymbol{r_1})_{\mathcal{T}_h} - (y, \nabla\cdot\boldsymbol{r_1})_{\mathcal{T}_h} + \langle y, \boldsymbol{r_1}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad + (\nabla\cdot\boldsymbol{q}, w_1)_{\mathcal{T}_h} - (\nabla\cdot\delta^{\boldsymbol{q}}, w_1)_{\mathcal{T}_h} - (\boldsymbol{\beta}y, \nabla w_1)_{\mathcal{T}_h} + (\boldsymbol{\beta}\delta^y, \nabla w_1)_{\mathcal{T}_h}\\
&\qquad - (\nabla\cdot\boldsymbol{\beta}y, w_1)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, w_1)_{\mathcal{T}_h} + \langle (h^{-1}+\tau_1)\Pi y, w_1\rangle_{\partial\mathcal{T}_h}\\
&\qquad + \langle \boldsymbol{\beta}\cdot\boldsymbol{n}y, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}}, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - \langle (h^{-1}+\tau_1)P_M y, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}y, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad + \langle (h^{-1}+\tau_1)(\delta^y - \delta^{\widehat{y}}), \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
$$

The exact state y and flux $\boldsymbol{q}$ satisfy

$$
\begin{aligned}
(\boldsymbol{q}, \boldsymbol{r}_1)_{\mathcal{T}_h} - (y, \nabla\cdot\boldsymbol{r}_1)_{\mathcal{T}_h} + \langle y, \boldsymbol{r}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} &= -\langle u, \boldsymbol{r}_1\cdot\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial},\\
(\nabla\cdot\boldsymbol{q}, w_1)_{\mathcal{T}_h} - (\boldsymbol{\beta}y, \nabla w_1)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\beta}y, w_1)_{\mathcal{T}_h}&\\
+ \langle \boldsymbol{\beta}\cdot\boldsymbol{n}y, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} &= -\langle \boldsymbol{\beta}\cdot\boldsymbol{n}u, w_1\rangle_{\mathcal{E}_h^\partial} + (f, w_1)_{\mathcal{T}_h},
\end{aligned}
$$

for all $(\boldsymbol{r}_1, w_1) \in \boldsymbol{V}_h \times W_h$. This gives

$$
\begin{aligned}
&\mathcal{B}_1(\boldsymbol{\Pi q}, \Pi y, P_M y, \boldsymbol{r}_1, w_1, \mu_1)\\
&\quad = -\langle u, \boldsymbol{r}_1\cdot\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}u, w_1\rangle_{\mathcal{E}_h^\partial} + (f, w_1)_{\mathcal{T}_h} - (\nabla\cdot\delta^{\boldsymbol{q}}, w_1)_{\mathcal{T}_h} + (\boldsymbol{\beta}\delta^y, \nabla w_1)_{\mathcal{T}_h}\\
&\qquad + (\nabla\cdot\boldsymbol{\beta}\delta^y, w_1)_{\mathcal{T}_h} + \langle (h^{-1}+\tau_1)\Pi y, w_1\rangle_{\partial\mathcal{T}_h}\\
&\qquad - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}}, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - \langle (h^{-1}+\tau_1)P_M y, w_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad + \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + \langle (h^{-1}+\tau_1)(\delta^y - \delta^{\widehat{y}}), \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
$$

Here we used $\langle \boldsymbol{\beta}\cdot\boldsymbol{n}y, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0$, which holds since μ_1 is a single-valued function on the interior edges. Subtracting part 1 of the auxiliary problem (29a) from the above equality gives the result:

$$
\begin{aligned}
&\mathcal{B}_1(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}}, \boldsymbol{r}_1, w_1, \mu_1) \\
&\quad = -(\nabla\cdot\delta^{\boldsymbol{q}}, w_1)_{\mathcal{T}_h} + (\boldsymbol{\beta}\delta^y, \nabla w_1)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, w_1)_{\mathcal{T}_h} \\
&\qquad + \langle (h^{-1}+\tau_1)\Pi y, w_1\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}}, w_1\rangle_{\partial\mathcal{T}_h} - \langle (h^{-1}+\tau_1)P_M y, w_1\rangle_{\partial\mathcal{T}_h} \\
&\qquad - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial} + \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial} + \langle (h^{-1}+\tau_1)(\delta^y - \delta^{\widehat{y}}), \mu_1\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial} \\
&\quad = -(\nabla\cdot\delta^{\boldsymbol{q}}, w_1)_{\mathcal{T}_h} + (\boldsymbol{\beta}\delta^y, \nabla w_1)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, w_1)_{\mathcal{T}_h} \\
&\qquad - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial} - \langle \widehat{\boldsymbol{\delta}}_1, w_1\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\boldsymbol{\delta}}_1, \mu_1\rangle_{\partial\mathcal{T}_h\backslash\varepsilon_h^\partial}.
\end{aligned}
$$

∎

3.4.2. Step 2: Estimate for ε_h^q.

LEMMA 3.12. *We have*

$$
\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^y - \varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{q}}}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y-1}\|y\|_{s^y,\Omega}. \tag{32}
$$

Proof. Take $(\boldsymbol{v}_h, w_h, \mu_h) = (\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}})$. Since $\varepsilon_h^{\widehat{y}} = 0$ on ε_h^∂, the energy identity for $\mathcal{B}_1$ in Lemma 3.4 gives

$$
\begin{aligned}
&\mathcal{B}_1(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}}, \varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}}) \\
&\quad = (\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^{\boldsymbol{q}})_{\mathcal{T}_h} + \|(h^{-1}+\tau_1-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})^{\frac{1}{2}}(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2}\|(-\nabla\cdot\boldsymbol{\beta})^{\frac{1}{2}}\varepsilon_h^y\|_{\mathcal{T}_h}^2.
\end{aligned}
$$

Take $(\boldsymbol{r}_1, w_1, \mu_1) = (\boldsymbol{\varepsilon}_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}})$ in the error equation (31) in Lemma 3.11 to obtain

$$
\begin{aligned}
&(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^{\boldsymbol{q}})_{\mathcal{T}_h} + \|(h^{-1}+\tau_1-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})^{\frac{1}{2}}(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2}\|(-\nabla\cdot\boldsymbol{\beta})^{\frac{1}{2}}\varepsilon_h^y\|_{\mathcal{T}_h}^2 \\
&\qquad = -(\nabla\cdot\delta^{\boldsymbol{q}}, \varepsilon_h^y)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&\qquad\quad + (\boldsymbol{\beta}\delta^y, \nabla\varepsilon_h^y)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, \varepsilon_h^y)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&\qquad =: T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{33}
$$

We rewrite the term T_1 using the interpolation operator $\mathcal{I}_h$:

$$
\begin{aligned}
T_1 &= -(\nabla\cdot\delta^{\boldsymbol{q}}, \varepsilon_h^y)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&= -(\nabla\cdot\boldsymbol{q}, \varepsilon_h^y)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\Pi q}, \varepsilon_h^y)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&= -(\nabla\cdot\boldsymbol{q}, \varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{q}, \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} \\
&\quad + (\nabla\cdot\boldsymbol{\Pi q}, \varepsilon_h^y) - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&= -(\nabla\cdot\boldsymbol{q}, \varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} + (\boldsymbol{q}, \nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} \\
&\quad + (\nabla\cdot\boldsymbol{\Pi q}, \varepsilon_h^y) - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&= -(\nabla\cdot\boldsymbol{q}, \varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} + (\delta^{\boldsymbol{q}}, \nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} \\
&\quad + (\boldsymbol{\Pi q}, \nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\Pi q}, \varepsilon_h^y)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \\
&= -(\nabla\cdot\boldsymbol{q}, \varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} + (\delta^{\boldsymbol{q}}, \nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h}.
\end{aligned}
$$

The last step holds since

$$
\begin{aligned}
(\boldsymbol{\Pi q}, \nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} &= \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}})\rangle_{\partial\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\Pi q}, \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}))_{\mathcal{T}_h} \\
&= \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\Pi q}, \varepsilon_h^y)_{\mathcal{T}_h}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
T_1 &\le \|\nabla\cdot \boldsymbol{q}\|_{\mathcal{T}_h}\|\varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}})\|_{\mathcal{T}_h} + \|\delta^{\boldsymbol{q}}\|_{\mathcal{T}_h}\|\nabla\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}})\|_{\mathcal{T}_h}\\
&\lesssim h(\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac12}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h}) + h^{s_{\boldsymbol{q}}}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega}(\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac12}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h})\\
&\lesssim h^{s_{\boldsymbol{q}}}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega}(\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac12}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h}).
\end{aligned}
$$

Note that we used $s_{\boldsymbol{q}} \in [0,1]$.

Noticing that Lemma 3.8 implies

$$
\|\nabla\varepsilon_h^y\|_{\mathcal{T}_h} \lesssim \|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac12}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h}, \tag{34}
$$

and using Young's inequality, we have for T_2, T_3 and T_4:

$$
\begin{aligned}
T_2 &= (\boldsymbol{\beta}\delta^y,\nabla\varepsilon_h^y)_{\mathcal{T}_h} \le C\|\delta^y\|_{\mathcal{T}_h}^2 + \frac14\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h}^2 + \frac{1}{4h}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h}^2,\\
T_3 &= (\nabla\cdot\boldsymbol{\beta}\delta^y,\varepsilon_h^y)_{\mathcal{T}_h} \le C\|\delta^y\|_{\mathcal{T}_h}^2 + \frac12\|(-\nabla\cdot\boldsymbol{\beta})^{\frac12}\varepsilon_h^y\|_{\mathcal{T}_h}^2,\\
T_4 &= -\langle\widehat{\boldsymbol{\delta}}_1,\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} \le 4h\|\widehat{\boldsymbol{\delta}}_1\|_{\partial\mathcal{T}_h}^2 + \frac{1}{4h}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h}^2.
\end{aligned}
$$

Summing the estimates for $\{T_i\}_{i=1}^4$ and taking into account (30) and (16), we obtain the desired estimate. ∎

Remark 3.13. In Part I [26], we defined $\widehat{\boldsymbol{\delta}}_1 = \delta^{\boldsymbol{q}}\cdot\boldsymbol{n} + \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}} + (h^{-1}+\tau_1)(\delta^y-\delta^{\widehat{y}})$. It is not meaningful to estimate $\|\widehat{\boldsymbol{\delta}}_1\|_{\partial\mathcal{T}_h}$ if we only assume $r_{\boldsymbol{q}}\ge 0$. In this paper, we have $\widehat{\boldsymbol{\delta}}_1 = \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}} + (h^{-1}+\tau_1)(\delta^y-\delta^{\widehat{y}})$, and we can estimate $\|\widehat{\boldsymbol{\delta}}_1\|_{\partial\mathcal{T}_h}$.

3.4.3. Step 3: Estimate for ε_h^y by a duality argument. Next, for any Θ in $L^2(\Omega)$ we consider the dual problem

$$
\begin{aligned}
\boldsymbol{\Phi} - \nabla\Psi &= 0 \quad &&\text{in } \Omega,\\
\nabla\cdot\boldsymbol{\Phi} + \nabla\cdot(\boldsymbol{\beta}\Psi) &= \Theta \quad &&\text{in } \Omega,\\
\Psi &= 0 \quad &&\text{on } \partial\Omega.
\end{aligned} \tag{35}
$$

Since the domain Ω is convex, we have the regularity estimate

$$
\|\boldsymbol{\Phi}\|_{1,\Omega} + \|\Psi\|_{2,\Omega} \le C_{\text{reg}}\|\Theta\|_{\Omega}. \tag{36}
$$

We use the following notation in the next proof for the estimate of ε_h^y:

$$
\delta^{\boldsymbol{\Phi}} = \boldsymbol{\Phi} - \boldsymbol{\Pi}\boldsymbol{\Phi},\quad \delta^{\Psi} = \Psi - \Pi\Psi,\quad \delta^{\widehat{\Psi}} = \Psi - P_M\Psi. \tag{37}
$$

Lemma 3.14. *We have*

$$
\|\varepsilon_h^y\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega}.
$$

Proof. We take $\Theta = -\varepsilon_h^y$ in the dual problem (35) and $(\boldsymbol{r}_1, w_1, \mu_1) = (\boldsymbol{\Pi\Phi}, \Pi\Psi, P_M\Psi)$ in the error equation (31) in Lemma 3.11. Since $\Psi = 0$ on $\mathcal{E}_h^\partial$, we have

$$
\begin{aligned}
&\mathcal{B}_1(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}}; \boldsymbol{\Pi\Phi}, \Pi\Psi, P_M\Psi)\\
&\quad= (\varepsilon_h^{\boldsymbol{q}}, \boldsymbol{\Pi\Phi})_{\mathcal{T}_h} - (\varepsilon_h^y, \nabla\cdot\boldsymbol{\Pi\Phi})_{\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Pi\Phi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\qquad + (\nabla\cdot\varepsilon_h^{\boldsymbol{q}}, \Pi\Psi)_{\mathcal{T}_h} - (\boldsymbol{\beta}\varepsilon_h^y, \nabla\Pi\Psi)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\beta}\varepsilon_h^y, \Pi\Psi)_{\mathcal{T}_h} + \langle (h^{-1}+\tau_1)\varepsilon_h^y, \Pi\Psi\rangle_{\partial\mathcal{T}_h}\\
&\qquad + \langle (\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1)\varepsilon_h^{\widehat{y}}, \Pi\Psi\rangle_{\partial\mathcal{T}_h}\\
&\qquad - \langle \varepsilon_h^{\boldsymbol{q}}\cdot\boldsymbol{n} + \boldsymbol{\beta}\cdot\boldsymbol{n}\varepsilon_h^{\widehat{y}} + (h^{-1}+\tau_1)(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), P_M\Psi\rangle_{\partial\mathcal{T}_h}\\
&\quad= (\varepsilon_h^{\boldsymbol{q}}, \boldsymbol{\Phi})_{\mathcal{T}_h} - (\varepsilon_h^y, \nabla\cdot\boldsymbol{\Phi})_{\mathcal{T}_h} + (\varepsilon_h^y, \nabla\cdot\delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Pi\Phi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (\varepsilon_h^{\boldsymbol{q}}, \nabla\Psi)_{\mathcal{T}_h}\\
&\qquad + \langle \varepsilon_h^{\boldsymbol{q}}\cdot\boldsymbol{n}, \Psi\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{\beta}\varepsilon_h^y, \nabla\Psi)_{\mathcal{T}_h} + (\boldsymbol{\beta}\varepsilon_h^y, \nabla\delta^\Psi)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\beta}\varepsilon_h^y, \Psi)_{\mathcal{T}_h}\\
&\qquad + (\nabla\cdot\boldsymbol{\beta}\varepsilon_h^y, \delta^\Psi)_{\mathcal{T}_h} - \langle \varepsilon_h^{\boldsymbol{q}}\cdot\boldsymbol{n}, P_M\Psi\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\varepsilon_h^{\widehat{y}}, \delta^\Psi\rangle_{\partial\mathcal{T}_h}\\
&\qquad - \langle (h^{-1}+\tau_1)(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \delta^\Psi - \delta^{\widehat{\Psi}}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Here we used $\langle \boldsymbol{\beta}\cdot\boldsymbol{n}\varepsilon_h^{\widehat{y}}, \Psi\rangle_{\partial\mathcal{T}_h} = 0$ and $\langle \boldsymbol{\beta}\cdot\boldsymbol{n}\varepsilon_h^{\widehat{y}}, P_M\Psi\rangle_{\partial\mathcal{T}_h} = 0$, which both hold since $\varepsilon_h^{\widehat{y}}$ is a single-valued function on interior edges and $\varepsilon_h^{\widehat{y}} = 0$ on $\mathcal{E}_h^\partial$.

By the same argument as in the proof of Lemma 3.12 for the term T_1, we have

$$
\begin{aligned}
&(\varepsilon_h^y, \nabla\cdot\delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Pi\Phi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\
&\qquad\qquad = (\varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}), \nabla\cdot\boldsymbol{\Phi})_{\mathcal{T}_h} - (\nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}), \delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h}.
\end{aligned}
$$

Next, integration by parts gives

$$
(\boldsymbol{\beta}\varepsilon_h^y, \nabla\delta^\Psi)_{\mathcal{T}_h} = \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\varepsilon_h^y, \delta^\Psi\rangle_{\partial\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\beta}\varepsilon_h^y, \delta^\Psi)_{\mathcal{T}_h} - (\boldsymbol{\beta}\cdot\nabla\varepsilon_h^y, \delta^\Psi)_{\mathcal{T}_h}.
$$

This implies

$$
\begin{aligned}
&\mathcal{B}_1(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}}; \boldsymbol{\Pi\Phi}, \Pi\Psi, P_M\Psi)\\
&\quad= \|\varepsilon_h^y\|_{\mathcal{T}_h}^2 + \langle \boldsymbol{\beta}\cdot\boldsymbol{n}(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \delta^\Psi\rangle_{\partial\mathcal{T}_h} - (\nabla\varepsilon_h^y, \boldsymbol{\beta}\delta^\Psi)_{\mathcal{T}_h}\\
&\qquad + (\varepsilon_h^y - \mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}), \nabla\cdot\delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h} - (\nabla\mathcal{I}_h(\varepsilon_h^y, \varepsilon_h^{\widehat{y}}), \delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h}\\
&\qquad - \langle h^{-1}(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}) + \tau_1(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \delta^\Psi - \delta^{\widehat{\Psi}}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Also, since $\Psi = 0$ on $\mathcal{E}_h^\partial$, the error equation (31) in Lemma 3.11 gives

$$
\begin{aligned}
&\mathcal{B}_1(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^y, \varepsilon_h^{\widehat{y}}; \boldsymbol{\Pi\Phi}, \Pi\Psi, P_M\Psi)\\
&\quad= -(\nabla\cdot\delta^{\boldsymbol{q}}, \Pi\Psi)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, P_M\Psi\rangle_{\mathcal{T}_h}\\
&\qquad + (\boldsymbol{\beta}\delta^y, \nabla\Pi\Psi)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, \Pi\Psi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, \Pi\Psi - P_M\Psi\rangle_{\partial\mathcal{T}_h}\\
&\quad= -(\nabla\cdot\boldsymbol{q}, \Pi\Psi)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\Pi q}, \Psi)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \Psi\rangle_{\mathcal{T}_h}\\
&\qquad + (\boldsymbol{\beta}\delta^y, \nabla\Pi\Psi)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, \Pi\Psi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, \Pi\Psi - P_M\Psi\rangle_{\partial\mathcal{T}_h},\\
&\quad= (\nabla\cdot\boldsymbol{q}, \delta^\Psi)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{q}, \Psi)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\Pi q}, \Psi)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi q}\cdot\boldsymbol{n}, \Psi\rangle_{\mathcal{T}_h}\\
&\qquad + (\boldsymbol{\beta}\delta^y, \nabla\Pi\Psi)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, \Pi\Psi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, \Pi\Psi - P_M\Psi\rangle_{\partial\mathcal{T}_h},\\
&\quad= (\nabla\cdot\boldsymbol{q}, \delta^\Psi)_{\mathcal{T}_h} + (\boldsymbol{q}, \nabla\Psi)_{\mathcal{T}_h} - (\boldsymbol{\Pi q}, \nabla\Psi)_{\mathcal{T}_h}\\
&\qquad + (\boldsymbol{\beta}\delta^y, \nabla\Pi\Psi)_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\beta}\delta^y, \Pi\Psi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, \Pi\Psi - P_M\Psi\rangle_{\partial\mathcal{T}_h},\\
&\quad= (\nabla\cdot\boldsymbol{q}, \delta^\Psi) + (\delta^{\boldsymbol{q}}, \nabla\delta^\Psi)_{\mathcal{T}_h} + (\boldsymbol{\beta}\delta^y, \nabla\Pi\Psi)_{\mathcal{T}_h}\\
&\qquad + (\nabla\cdot\boldsymbol{\beta}\delta^y, \Pi\Psi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_1, \Pi\Psi - P_M\Psi\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

The two equalities above give

$$\begin{aligned}\|\varepsilon_h^y\|_{\mathcal{T}_h}^2 &= -\langle \boldsymbol{\beta}\cdot\boldsymbol{n}(\varepsilon_h^y-\varepsilon_h^{\widehat{y}}),\delta^{\Psi}\rangle_{\partial\mathcal{T}_h} + (\nabla\varepsilon_h^y,\boldsymbol{\beta}\delta^{\Psi})_{\mathcal{T}_h} + (\boldsymbol{\beta}\delta^y,\nabla\Pi\Psi)_{\mathcal{T}_h}\\ &\quad + (\nabla\cdot\boldsymbol{\beta}\delta^y,\Pi\Psi)_{\mathcal{T}_h} + \langle (h^{-1}+\tau_1)(\varepsilon_h^y-\varepsilon_h^{\widehat{y}}) + \widehat{\boldsymbol{\delta}}_1, \delta^{\Psi}-\delta^{\widehat{\Psi}}\rangle_{\partial\mathcal{T}_h}\\ &\quad - (\varepsilon_h^y-\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}}),\nabla\cdot\delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h} + (\nabla\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}}),\delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h}\\ &\quad + (\nabla\cdot\boldsymbol{q},\delta^{\Psi}) + (\delta^{\boldsymbol{q}},\nabla\delta^{\Psi})_{\mathcal{T}_h}\\ &=: \sum_{i=1}^{9} R_i.\end{aligned}$$

Bounds for R_1 to R_5 have been obtained in Part I [26]; we have

$$\sum_{i=1}^{5} R_i \lesssim (h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega})\|\varepsilon_h^y\|_{\mathcal{T}_h}.$$

For the terms R_6 and R_7, Lemma 3.10 and Lemma 3.12 give

$$\begin{aligned}R_6 &= -(\varepsilon_h^y-\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}}),\nabla\cdot\boldsymbol{\Phi})_{\mathcal{T}_h}\\ &\le \|\varepsilon_h^y-\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}})\|_{\mathcal{T}_h}\|\nabla\cdot\boldsymbol{\Phi}\|_{\mathcal{T}_h}\\ &\lesssim h(\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac12}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h})\|\nabla\cdot\boldsymbol{\Phi}\|_{\mathcal{T}_h}\\ &\lesssim (h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega})\|\varepsilon_h^y\|_{\mathcal{T}_h},\\ R_7 &= (\nabla\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}}),\delta^{\boldsymbol{\Phi}})_{\mathcal{T}_h}\\ &\le \|\nabla\mathcal{I}_h(\varepsilon_h^y,\varepsilon_h^{\widehat{y}})\|_{\mathcal{T}_h}\|\delta^{\boldsymbol{\Phi}}\|_{\mathcal{T}_h}\\ &\lesssim (\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac12}\|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\partial\mathcal{T}_h})\|\delta^{\boldsymbol{\Phi}}\|_{\mathcal{T}_h}\\ &\lesssim (h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega})\|\varepsilon_h^y\|_{\mathcal{T}_h}.\end{aligned}$$

For R_8, we have

$$\begin{aligned}R_8 &\le \|\nabla\cdot\boldsymbol{q}\|_{\mathcal{T}_h}\|\delta^{\Psi}\|_{\mathcal{T}_h} \lesssim h^2\|\Psi\|_{2,\Omega}\\ &\lesssim h^2\|\varepsilon_h^y\|_{\mathcal{T}_h}.\end{aligned}$$

Applying the triangle inequality for R_9 gives

$$R_9 \le \|\delta^{\boldsymbol{q}}\|_{\mathcal{T}_h}\|\nabla\delta^{\Psi}\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega}\|\varepsilon_h^y\|_{\mathcal{T}_h}.$$

Using $s_{\boldsymbol{q}}\in[0,1]$ and summing the estimates for R_1 to R_9 completes the proof. ∎

The triangle inequality gives optimal convergence rates for $\|\boldsymbol{q}-\boldsymbol{q}_h(u)\|_{\mathcal{T}_h}$ and $\|y-y_h(u)\|_{\mathcal{T}_h}$:

LEMMA 3.15.

$$\|\boldsymbol{q}-\boldsymbol{q}_h(u)\|_{\mathcal{T}_h} \le \|\delta^{\boldsymbol{q}}\|_{\mathcal{T}_h} + \|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{q}}}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y-1}\|y\|_{s^y,\Omega}, \tag{38a}$$

$$\|y-y_h(u)\|_{\mathcal{T}_h} \le \|\delta^{y}\|_{\mathcal{T}_h} + \|\varepsilon_h^{y}\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega}. \tag{38b}$$

3.4.4. Step 4: The error equation for part 2 of the auxiliary problem (29b). Next, we estimate the error between the exact state z and flux $\boldsymbol{p}$ satisfying the mixed form (6a)-(6d) of the optimality system and the solutions $z_h(u)$ and $\boldsymbol{p}_h(u)$ of the auxiliary problem. Define

$$
\begin{aligned}
&\delta^{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{\Pi}\boldsymbol{p}, && \varepsilon_h^{\boldsymbol{p}} = \boldsymbol{\Pi}\boldsymbol{p} - \boldsymbol{p}_h(u),\\
&\delta^{z} = z - \Pi z, && \varepsilon_h^{z} = \Pi z - z_h(u),\\
&\delta^{\widehat{z}} = z - P_M z, && \varepsilon_h^{\widehat{z}} = P_M z - \widehat{z}_h(u),\\
&\widehat{\boldsymbol{\delta}}_2 = -\boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{z}} + (h^{-1}+\tau_2)(\delta^z - \delta^{\widehat{z}}),
\end{aligned}
\tag{39}
$$

where $\widehat{z}_h(u) = \widehat{z}_h^o(u)$ on $\mathcal{E}_h^o$ and $\widehat{z}_h(u) = 0$ on $\mathcal{E}_h^\partial$. This gives $\varepsilon_h^{\widehat{z}} = 0$ on $\mathcal{E}_h^\partial$.

LEMMA 3.16. *For all $(\boldsymbol{r}_2, w_2, \mu_2) \in \boldsymbol{V}_h \times W_h \times M_h(o)$, we have*

$$
\begin{aligned}
&\mathcal{B}_2(\varepsilon_h^{\boldsymbol{p}}, \varepsilon_h^z, \varepsilon_h^{\widehat{z}}, \boldsymbol{r}_2, w_2, \mu_2)\\
&\qquad = -(\nabla\cdot\delta^{\boldsymbol{p}}, w_2)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}\boldsymbol{p}\cdot\boldsymbol{n}, \mu_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - (\boldsymbol{\beta}\delta^z, \nabla w_2)_{\mathcal{T}_h}\\
&\qquad\quad + (y - y_h(u), w_2)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_2, w_2\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\boldsymbol{\delta}}_2, \mu_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
\tag{40}
$$

The proof is similar to the proof of Lemma 3.11 and is omitted.

3.4.5. Step 5: Estimate for ε_h^p. We use the following discrete Poincaré inequality from [26] to estimate $\varepsilon_h^{\boldsymbol{p}}$.

LEMMA 3.17. *We have*

$$
\|\varepsilon_h^z\|_{\mathcal{T}_h} \le C(\|\nabla\varepsilon_h^z\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}).
\tag{41}
$$

LEMMA 3.18. *We have*

$$
\begin{aligned}
&\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}\\
&\qquad \lesssim h^{s_{\boldsymbol{p}}}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega} + h^{s_z-1}\|z\|_{s^z,\Omega} + h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega},
\end{aligned}
\tag{42a}
$$

$$
\|\varepsilon_h^z\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega} + h^{s_z-1}\|z\|_{s^z,\Omega} + h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega}.
\tag{42b}
$$

Proof. Take $(\boldsymbol{v}_h, w_h, \mu_h) = (\varepsilon_h^{\boldsymbol{p}}, \varepsilon_h^z, \varepsilon_h^{\widehat{z}})$. Since $\varepsilon_h^{\widehat{z}} = 0$ on ε_h^∂, the energy identity for $\mathcal{B}_2$ in Lemma 3.4 gives

$$
\begin{aligned}
&\mathcal{B}_2(\varepsilon_h^{\boldsymbol{p}}, \varepsilon_h^z, \varepsilon_h^{\widehat{z}}, \varepsilon_h^{\boldsymbol{p}}, \varepsilon_h^z, \varepsilon_h^{\widehat{z}})\\
&\quad = (\varepsilon_h^{\boldsymbol{p}}, \varepsilon_h^{\boldsymbol{p}})_{\mathcal{T}_h} + \|(h^{-1} + \tau_2 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})^{\frac{1}{2}}(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2}\|(-\nabla\cdot\boldsymbol{\beta})^{\frac{1}{2}}\varepsilon_h^z\|_{\mathcal{T}_h}^2.
\end{aligned}
$$

Take $(\boldsymbol{r}_2, w_2, \mu_2) = (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^z, \varepsilon_h^{\widehat{z}})$ in the error equation (40) in Lemma 3.16 to obtain

$$
\begin{aligned}
&(\varepsilon_h^{\boldsymbol{p}}, \varepsilon_h^{\boldsymbol{p}})_{\mathcal{T}_h} + \|(h^{-1} + \tau_2 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})^{\frac{1}{2}}(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2}\|(-\nabla\cdot\boldsymbol{\beta})^{\frac{1}{2}}\varepsilon_h^z\|_{\mathcal{T}_h}^2\\
&\qquad = -(\nabla\cdot\delta^{\boldsymbol{p}}, \varepsilon_h^z)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}\boldsymbol{p}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{z}}\rangle_{\partial\mathcal{T}_h}\\
&\qquad\quad - (\boldsymbol{\beta}\delta^z, \nabla\varepsilon_h^z)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\delta}}_2, \varepsilon_h^z - \varepsilon_h^{\widehat{z}}\rangle_{\partial\mathcal{T}_h} + (y - y_h(u), \varepsilon_h^z)_{\mathcal{T}_h}\\
&\qquad =: T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

Next, use Lemma 3.8 to get

$$
\|\nabla\varepsilon_h^z\|_{\mathcal{T}_h} \lesssim \|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}.
\tag{43}
$$

By the same argument as in the proof of Lemma 3.12, apply (43) and Young's inequality to obtain

$$
\begin{aligned}
T_1 &= -(\nabla\cdot\delta^{\boldsymbol{p}}, \varepsilon_h^z)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi p}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{z}}\rangle_{\partial\mathcal{T}_h}\\
&= -(\nabla\cdot\boldsymbol{p}, \varepsilon_h^z - \mathcal{I}_h(\varepsilon_h^z,\varepsilon_h^{\widehat{z}}))_{\mathcal{T}_h} + (\delta^{\boldsymbol{p}}, \nabla\mathcal{I}_h(\varepsilon_h^z,\varepsilon_h^{\widehat{z}}))_{\mathcal{T}_h}\\
&= -(\nabla\cdot\delta^{\boldsymbol{p}}, \varepsilon_h^z - \mathcal{I}_h(\varepsilon_h^z,\varepsilon_h^{\widehat{z}}))_{\mathcal{T}_h} + (\delta^{\boldsymbol{p}}, \nabla\mathcal{I}_h(\varepsilon_h^z,\varepsilon_h^{\widehat{z}}))_{\mathcal{T}_h}\\
&\le h\|\nabla\cdot\delta^{\boldsymbol{p}}\|_{\mathcal{T}_h} h^{-1}\|\varepsilon_h^z - \mathcal{I}_h(\varepsilon_h^z,\varepsilon_h^{\widehat{z}})\|_{\mathcal{T}_h} + \|\delta^{\boldsymbol{p}}\|_{\mathcal{T}_h}\|\nabla\mathcal{I}_h(\varepsilon_h^z,\varepsilon_h^{\widehat{z}})\|_{\mathcal{T}_h}\\
&\le Ch^2\|\nabla\cdot\delta^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + C\|\delta^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + \frac{1}{8}\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + \frac{1}{8h}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}^2,\\
T_2 &= -(\boldsymbol{\beta}\delta^z, \nabla\varepsilon_h^z)_{\mathcal{T}_h} \le C\|\delta^z\|_{\mathcal{T}_h}^2 + \frac{1}{8}\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + \frac{1}{8h}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}^2,\\
T_3 &= -\langle\widehat{\boldsymbol{\delta}}_2, \varepsilon_h^z - \varepsilon_h^{\widehat{z}}\rangle_{\partial\mathcal{T}_h} \le 8h\|\widehat{\boldsymbol{\delta}}_2\|_{\partial\mathcal{T}_h}^2 + \frac{1}{8}\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + \frac{1}{8h}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}^2.
\end{aligned}
$$

For the term T_4, we have

$$
\begin{aligned}
T_4 &= (y - y_h(u), \varepsilon_h^z)_{\mathcal{T}_h} \le \|y - y_h(u)\|_{\mathcal{T}_h}\|\varepsilon_h^z\|_{\mathcal{T}_h}\\
&\le C\|y - y_h(u)\|_{\mathcal{T}_h}(\|\nabla\varepsilon_h^z\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h})\\
&\le C\|y - y_h(u)\|_{\mathcal{T}_h}(\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h})\\
&\le C\|y - y_h(u)\|_{\mathcal{T}_h}^2 + \frac{1}{8}\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + \frac{1}{8h}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}^2.
\end{aligned}
$$

Summing T_1 to T_4 and using (16), Lemma 3.15, and (39) gives (42a); then (41), (42a), and (43) together imply (42b). ∎

The triangle inequality gives optimal convergence rates for $\|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_{\mathcal{T}_h}$ and $\|z - z_h(u)\|_{\mathcal{T}_h}$:

LEMMA 3.19.

$$
\|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega} + h^{s_z - 1}\|z\|_{s^z,\Omega} + h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega}, \tag{44a}
$$

$$
\|z - z_h(u)\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega} + h^{s_z - 1}\|z\|_{s^z,\Omega} + h^{s_{\boldsymbol{q}}+1}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega} + h^{s_y}\|y\|_{s^y,\Omega}. \tag{44b}
$$

3.4.6. Step 6: Estimates for $\|u - u_h\|_{\mathcal{E}_h^\partial}$ and $\|y - y_h\|_{\mathcal{T}_h}$. To obtain the main result, we estimate the error between the solution of the auxiliary problem and the HDG discretized optimality system (19). Define

$$
\begin{aligned}
\zeta_{\boldsymbol{q}} &= \boldsymbol{q}_h(u) - \boldsymbol{q}_h, \quad \zeta_y = y_h(u) - y_h, \quad \zeta_{\widehat{y}} = \widehat{y}_h(u) - \widehat{y}_h,\\
\zeta_{\boldsymbol{p}} &= \boldsymbol{p}_h(u) - \boldsymbol{p}_h, \quad \zeta_z = z_h(u) - z_h, \quad \zeta_{\widehat{z}} = \widehat{z}_h(u) - \widehat{z}_h,
\end{aligned}
$$

where $\widehat{y}_h = \widehat{y}_h^o$ on $\mathcal{E}_h^o$, $\widehat{y}_h = u_h$ on $\mathcal{E}_h^\partial$, $\widehat{z}_h = \widehat{z}_h^o$ on $\mathcal{E}_h^o$, and $\widehat{z}_h = 0$ on $\mathcal{E}_h^\partial$. This gives $\zeta_{\widehat{z}} = 0$ on $\mathcal{E}_h^\partial$.

Subtracting the two problems gives the error equations

$$
\mathscr{B}_1(\zeta_{\boldsymbol{q}}, \zeta_y, \zeta_{\widehat{y}}; \boldsymbol{r}_1, w_1, \mu_1) = -\langle P_M u - u_h, \boldsymbol{r}_1\cdot\boldsymbol{n} + (\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1)w_1\rangle_{\mathcal{E}_h^\partial}, \tag{45a}
$$

$$
\mathscr{B}_2(\zeta_{\boldsymbol{p}}, \zeta_z, \zeta_{\widehat{z}}; \boldsymbol{r}_2, w_2, \mu_2) = (\zeta_y, w_2)_{\mathcal{T}_h}. \tag{45b}
$$

LEMMA 3.20. *If **(A1)** and **(A2)** hold, then*

$$
\begin{aligned}
\gamma\|u - u_h\|_{\mathcal{E}_h^\partial}^2 + \|\zeta_y\|_{\mathcal{T}_h}^2 &= \langle\gamma u + \boldsymbol{p}_h(u)\cdot\boldsymbol{n} + h^{-1}z_h(u) + \tau_2 z_h(u), u - u_h\rangle_{\mathcal{E}_h^\partial}\\
&\quad - \langle\gamma u_h + \boldsymbol{p}_h\cdot\boldsymbol{n} + h^{-1}z_h + \tau_2 z_h, u - u_h\rangle_{\mathcal{E}_h^\partial}.
\end{aligned}
$$

Proof. We have

$$\langle \gamma u+\boldsymbol{p}_h(u)\cdot\boldsymbol{n}+h^{-1}z_h(u)+\tau_2 z_h(u), u-u_h\rangle_{\mathcal{E}_h^\partial}-\langle \gamma u_h+\boldsymbol{p}_h\cdot\boldsymbol{n}+h^{-1}z_h+\tau_2 z_h, u-u_h\rangle_{\mathcal{E}_h^\partial}$$
$$= \gamma\, \|u-u_h\|^2_{\mathcal{E}_h^\partial} + \langle \zeta_{\boldsymbol{p}}\cdot\boldsymbol{n}+h^{-1}\zeta_z+\tau_2\zeta_z, u-u_h\rangle_{\mathcal{E}_h^\partial}.$$

Next, Lemma 3.5 gives

$$\mathcal{B}_1(\zeta_{\boldsymbol{q}},\zeta_y,\zeta_{\widehat{y}};\zeta_{\boldsymbol{p}},-\zeta_z,-\zeta_{\widehat{z}})+\mathcal{B}_2(\zeta_{\boldsymbol{p}},\zeta_z,\zeta_{\widehat{z}};-\zeta_{\boldsymbol{q}},\zeta_y,\zeta_{\widehat{y}})=0.$$

Also, since τ_2 is piecewise constant on $\partial\mathcal{T}_h$ we have

$$\begin{aligned}
&\mathcal{B}_1(\zeta_{\boldsymbol{q}},\zeta_y,\zeta_{\widehat{y}};\zeta_{\boldsymbol{p}},-\zeta_z,-\zeta_{\widehat{z}})+\mathcal{B}_2(\zeta_{\boldsymbol{p}},\zeta_z,\zeta_{\widehat{z}};-\zeta_{\boldsymbol{q}},\zeta_y,\zeta_{\widehat{y}})\\
&\qquad= (\zeta_y,\zeta_y)_{\mathcal{T}_h} - \langle P_M u-u_h, \zeta_{\boldsymbol{p}}\cdot\boldsymbol{n}+(h^{-1}+\tau_1-\boldsymbol{\beta}\cdot\boldsymbol{n})\zeta_z\rangle_{\mathcal{E}_h^\partial}\\
&\qquad= (\zeta_y,\zeta_y)_{\mathcal{T}_h} - \langle P_M u-u_h, \zeta_{\boldsymbol{p}}\cdot\boldsymbol{n}+h^{-1}\zeta_z+\tau_2\zeta_z\rangle_{\mathcal{E}_h^\partial}\\
&\qquad= (\zeta_y,\zeta_y)_{\mathcal{T}_h} - \langle u-u_h, \zeta_{\boldsymbol{p}}\cdot\boldsymbol{n}+h^{-1}\zeta_z+\tau_2\zeta_z\rangle_{\mathcal{E}_h^\partial}.
\end{aligned}$$

The above equalities yield

$$(\zeta_y,\zeta_y)_{\mathcal{T}_h} = \langle u-u_h, \zeta_{\boldsymbol{p}}\cdot\boldsymbol{n}+h^{-1}\zeta_z+\tau_2\zeta_z\rangle_{\mathcal{E}_h^\partial}. \qquad \square$$

THEOREM 3.21. *We have*

$$\|u-u_h\|_{\varepsilon_h^\partial} \lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega}+h^{s_z-\frac32}\|z\|_{s_z,\Omega}+h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega}+h^{s_y-\frac12}\|y\|_{s_y,\Omega},$$
$$\|y-y_h\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega}+h^{s_z-\frac32}\|z\|_{s_z,\Omega}+h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega}+h^{s_y-\frac12}\|y\|_{s_y,\Omega}.$$

Proof. The optimality conditions yield $\gamma u+\boldsymbol{p}\cdot\boldsymbol{n}=0$ and $\gamma u_h+\boldsymbol{p}_h\cdot\boldsymbol{n}+h^{-1}z_h+\tau_2 z_h=0$ on $\mathcal{E}_h^\partial$. Therefore, the above lemma gives

$$\begin{aligned}
\gamma\,\|u-u_h\|^2_{\varepsilon_h^\partial}+\|\zeta_y\|^2_{\mathcal{T}_h} &= \langle \gamma u+\boldsymbol{p}_h(u)\cdot\boldsymbol{n}+h^{-1}z_h(u)+\tau_2 z_h(u), u-u_h\rangle_{\mathcal{E}_h^\partial}\\
&= \langle (\boldsymbol{p}_h(u)-\boldsymbol{p})\cdot\boldsymbol{n}+h^{-1}z_h(u)+\tau_2 z_h(u), u-u_h\rangle_{\mathcal{E}_h^\partial}.
\end{aligned}$$

Since $\widehat{z}_h(u)=z=0$ on $\mathcal{E}_h^\partial$, we have

$$\begin{aligned}
\|\boldsymbol{p}_h(u)-\boldsymbol{p}\|_{\partial\mathcal{T}_h} &\le \|\boldsymbol{p}_h(u)-\boldsymbol{\Pi}\boldsymbol{p}\|_{\partial\mathcal{T}_h}+\|\boldsymbol{\Pi}\boldsymbol{p}-\boldsymbol{p}\|_{\partial\mathcal{T}_h}\\
&\lesssim h^{-\frac12}\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}+h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega},\\
\|z_h(u)\|_{\varepsilon_h^\partial} &= \|z_h(u)-\Pi z+\Pi z-z+P_M z-\widehat{z}_h(u)\|_{\mathcal{E}_h^\partial}\\
&\le \|\varepsilon_h^z-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}+\|\Pi z-z\|_{\partial\mathcal{T}_h}.
\end{aligned}$$

This implies

$$\begin{aligned}
\|u-u_h\|_{\varepsilon_h^\partial}+\|\zeta_y\|_{\mathcal{T}_h} \lesssim{}& h^{-\frac12}\|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}+h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega}\\
&+h^{-1}\|\varepsilon_h^z-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}+h^{-\frac32}\|\delta^z\|_{\mathcal{T}_h}.
\end{aligned}$$

Lemma 3.18 and approximation properties of the L^2 projection give

$$\begin{aligned}
&\|u-u_h\|_{\mathcal{E}_h^\partial}+\|\zeta_y\|_{\mathcal{T}_h}\\
&\qquad\lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega}+h^{s_z-\frac32}\|z\|_{s^z,\Omega}+h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega}+h^{s_y-\frac12}\|y\|_{s^y,\Omega}.
\end{aligned}$$

The triangle inequality and Lemma 3.15 yield

$$\|y-y_h\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-\frac12}\|\boldsymbol{p}\|_{s^{\boldsymbol{p}},\Omega}+h^{s_z-\frac32}\|z\|_{s^z,\Omega}+h^{s_{\boldsymbol{q}}+\frac12}\|\boldsymbol{q}\|_{s^{\boldsymbol{q}},\Omega}+h^{s_y-\frac12}\|y\|_{s^y,\Omega}. \qquad \square$$

3.4.7. Step 7: Estimates for $\|p - p_h\|_{\mathcal{T}_h}$, $\|z - z_h\|_{\mathcal{T}_h}$, and $\|q - q_h\|_{\mathcal{T}_h}$.

LEMMA 3.22. *We have*

$$\|\zeta_{\boldsymbol{p}}\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-\frac{1}{2}} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac{3}{2}} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac{1}{2}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac{1}{2}} \|y\|_{s_y,\Omega},$$

$$\|\zeta_z\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-\frac{1}{2}} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac{3}{2}} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac{1}{2}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac{1}{2}} \|y\|_{s_y,\Omega}.$$

Proof. By the energy identity for $\mathcal{B}_2$ in Lemma 3.4, the second error equation (45b), and since $\zeta_{\widehat{z}} = 0$ on ε_h^{∂}, we have

$$\begin{aligned}
&\mathcal{B}_2(\zeta_{\boldsymbol{p}}, \zeta_z, \zeta_{\widehat{z}}; \zeta_{\boldsymbol{p}}, \zeta_z, \zeta_{\widehat{z}}) \\
&\quad= (\zeta_{\boldsymbol{p}}, \zeta_{\boldsymbol{p}})_{\mathcal{T}_h} + \langle (h^{-1} + \tau_2 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})(\zeta_z - \zeta_{\widehat{z}}), \zeta_z - \zeta_{\widehat{z}} \rangle_{\partial\mathcal{T}_h} - \frac{1}{2}(\nabla\cdot\boldsymbol{\beta}\zeta_z, \zeta_z)_{\mathcal{T}_h} \\
&\quad= (\zeta_y, \zeta_z)_{\mathcal{T}_h} \\
&\quad\le \|\zeta_y\|_{\mathcal{T}_h} \|\zeta_z\|_{\mathcal{T}_h} \\
&\quad\lesssim \|\zeta_y\|_{\mathcal{T}_h} (\|\nabla\zeta_z\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\zeta_z - \zeta_{\widehat{z}}\|_{\partial\mathcal{T}_h}) \\
&\quad\lesssim \|\zeta_y\|_{\mathcal{T}_h} (\|\zeta_{\boldsymbol{p}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\zeta_z - \zeta_{\widehat{z}}\|_{\partial\mathcal{T}_h}).
\end{aligned}$$

Here, for the last two inequalities we used the discrete Poincaré inequality in Lemma 3.17 and Lemma 3.8. This gives

$$\begin{aligned}
&\|\zeta_{\boldsymbol{p}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\zeta_z - \zeta_{\widehat{z}}\|_{\partial\mathcal{T}_h} \\
&\qquad\lesssim h^{s_{\boldsymbol{p}}-\frac{1}{2}} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac{3}{2}} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac{1}{2}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac{1}{2}} \|y\|_{s_y,\Omega}.
\end{aligned}$$

Using the discrete Poincaré inequality and Lemma 3.8 again yields

$$\begin{aligned}
\|\zeta_z\|_{\mathcal{T}_h} &\lesssim \|\nabla\zeta_z\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\zeta_z - \zeta_{\widehat{z}}\|_{\partial\mathcal{T}_h} \\
&\lesssim h^{s_{\boldsymbol{p}}-\frac{1}{2}} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac{3}{2}} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac{1}{2}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac{1}{2}} \|y\|_{s_y,\Omega}. \qquad \square
\end{aligned}$$

To obtain a positive convergence rate for $\boldsymbol{q}$, we need

$$r_y > 1, \quad r_z > 2, \quad r_{\boldsymbol{q}} > 0, \quad r_{\boldsymbol{p}} > 1. \tag{46}$$

LEMMA 3.23. *If **(A1)**, (46), and $k \ge 1$ hold, then*

$$\|\zeta_{\boldsymbol{q}}\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-1} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-2} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-1} \|y\|_{s_y,\Omega}.$$

Proof. By the energy identity in Lemma 3.4, the first error equation (45a), and since τ_2 is piecewise constant on $\partial\mathcal{T}_h$, we have

$$\begin{aligned}
&\mathcal{B}_1(\zeta_{\boldsymbol{q}}, \zeta_y, \zeta_{\widehat{y}}; \zeta_{\boldsymbol{q}}, \zeta_y, \zeta_{\widehat{y}}) \\
&\quad= (\zeta_{\boldsymbol{q}}, \zeta_{\boldsymbol{q}})_{\mathcal{T}_h} + \langle (h^{-1} + \tau_1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})(\zeta_y - \zeta_{\widehat{y}}), \zeta_y - \zeta_{\widehat{y}} \rangle_{\partial\mathcal{T}_h\setminus\mathcal{E}_h^{\partial}} \\
&\qquad - \frac{1}{2}(\nabla\cdot\boldsymbol{\beta}\zeta_y, \zeta_y)_{\mathcal{T}_h} + \langle (h^{-1} + \tau_1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n})\zeta_y, \zeta_y \rangle_{\mathcal{E}_h^{\partial}} \\
&\quad= -\langle P_M u - u_h, \zeta_{\boldsymbol{q}}\cdot\boldsymbol{n} + (\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_1)\zeta_y \rangle_{\mathcal{E}_h^{\partial}} \\
&\quad= -\langle P_M u - u_h, \zeta_{\boldsymbol{q}}\cdot\boldsymbol{n} - (h^{-1} + \tau_2)\zeta_y \rangle_{\mathcal{E}_h^{\partial}} \\
&\quad= -\langle u - u_h, \zeta_{\boldsymbol{q}}\cdot\boldsymbol{n} - (h^{-1} + \tau_2)\zeta_y \rangle_{\mathcal{E}_h^{\partial}} \\
&\quad\lesssim \|u - u_h\|_{\mathcal{E}_h^{\partial}} (\|\zeta_{\boldsymbol{q}}\|_{\mathcal{E}_h^{\partial}} + h^{-1} \|\zeta_y\|_{\mathcal{E}_h^{\partial}}) \\
&\quad\lesssim h^{-\frac{1}{2}} \|u - u_h\|_{\mathcal{E}_h^{\partial}} (\|\zeta_{\boldsymbol{q}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\zeta_y\|_{\mathcal{E}_h^{\partial}}).
\end{aligned}$$

This gives

$$\|\zeta_{\boldsymbol{q}}\|_{\mathcal{T}_h} \lesssim h^{-\frac{1}{2}} \|u-u_h\|_{\mathcal{E}_h^\partial} \lesssim h^{s_{\boldsymbol{p}}-1} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-2} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-1} \|y\|_{s_y,\Omega}. \qquad \square$$

The above lemma, the triangle inequality, Lemma 3.15, and Lemma 3.19 complete the proof of the main result:

THEOREM 3.24. *We have*

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-\frac{1}{2}} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac{3}{2}} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac{1}{2}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac{1}{2}} \|y\|_{s_y,\Omega},$$

$$\|z-z_h\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-\frac{1}{2}} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-\frac{3}{2}} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}+\frac{1}{2}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-\frac{1}{2}} \|y\|_{s_y,\Omega}.$$

If in addition (46) *is satisfied and* $k \geq 1$*, then*

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\boldsymbol{p}}-1} \|\boldsymbol{p}\|_{s_{\boldsymbol{p}},\Omega} + h^{s_z-2} \|z\|_{s_z,\Omega} + h^{s_{\boldsymbol{q}}} \|\boldsymbol{q}\|_{s_{\boldsymbol{q}},\Omega} + h^{s_y-1} \|y\|_{s_y,\Omega}.$$

4. Numerical Experiments. In this section, we report numerical experiments to illustrate our theoretical results. Furthermore, although we derived the a priori error estimates for diffusion dominated problems ($\varepsilon = 1$), we also present numerical results for convection dominated problems ($\varepsilon \ll 1$) to show the performance of the HDG method for this difficult case. For all computations, we take $\gamma = 1$, $\tau_2 = 1$, and $\tau_1 = \tau_2 + \boldsymbol{\beta} \cdot \boldsymbol{n}$ so that **(A1)-(A3)** are satisfied.

4.1. Smooth test. We begin with an example on a square domain $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$. The state, dual state, and convection coefficient are chosen as

$$y = -\varepsilon^{1/2}\pi(\sin(\pi x_1) + \sin(\pi x_2)), \;\; z = \varepsilon^{-1/2} \sin(\pi x_1)\sin(\pi x_2),$$
$$\boldsymbol{\beta} = -[x_1^2 \sin(x_2), \cos(x_1)e^{x_2}],$$

and the source term f and the desired state y_d are generated using the optimality system (4) with the above data. Since the solution is smooth, we do not use this test to illustrate the low regularity theory; instead, we use this test to study the performance of the HDG method when the problem becomes convection dominated.

We show the numerical results for $k = 1$ and $\varepsilon = 1$ in Table 1 and for $k = 1$ and $\varepsilon = 10^{-7}$ in Table 2. In the convection dominated case, the HDG method converges for all variables with at least a linear rate. In the diffusion dominated case, convergence rates are higher for all variables except the control. This example demonstrates that the error analysis will be different for the convection dominated case, as expected.

4.2. Non-smooth test. Next, we present numerical results for a 2D example problem similar to examples from [9, 21], where the case $\boldsymbol{\beta} = \boldsymbol{0}$ is considered. We consider a square domain $\Omega = [0,1/8] \times [0,1/8] \subset \mathbb{R}^2$, and choose $f = 0$ and $\gamma = 1$.

In the first test, we consider variable convection and choose the data as

$$\varepsilon = 1, \quad y_d = (x^2+y^2)^{-1/3}, \quad \text{and} \quad \boldsymbol{\beta} = -[x_1^2 \sin(x_2), \cos(x_1)e^{x_2}].$$

The largest interior angle is $\omega = \pi/2$, and therefore $r_\Omega = 3/2$. Also, we have $y_d \in H^{1/3-\eta}(\Omega)$ for any $\eta > 0$, and therefore $r_d = 5/6 - \eta$ for any $\eta > 0$. For this example, the value of r_d restricts the guaranteed regularity of the solution.

We do not have an exact solution for this problem; therefore, we generate numerical convergence rates by computing errors between approximate solutions computed

$h/\sqrt{2}$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\Omega}$	1.52E-01	5.05E-02	1.69E-02	5.75E-03	1.99E-03
order	-	1.59	1.58	1.55	1.53
$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\Omega}$	2.82E-02	7.33E-03	1.87E-03	4.74E-04	1.19E-04
order	-	1.94	1.97	1.98	1.99
$\|y-y_h\|_{0,\Omega}$	2.27E-02	3.40E-03	4.87E-04	7.06E-05	1.07E-05
order	-	2.74	2.80	2.79	2.73
$\|z-z_h\|_{0,\Omega}$	8.62E-03	1.21E-03	1.61E-04	2.09E-05	2.65E-06
order	-	2.83	2.91	2.95	2.98
$\|u-u_h\|_{0,\Gamma}$	9.84E-02	2.62E-02	6.70E-03	1.69E-03	4.24E-04
order	-	1.91	1.97	1.99	2.00

Table 1

Smooth test with $k = 1$ and $\varepsilon = 1$: Errors for the control u, state y, adjoint state z, and the fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

$h/\sqrt{2}$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\Omega}$	3.59E-05	1.94E-05	1.01E-05	5.19E-06	2.62E-06
order	-	0.90	0.93	0.97	0.99
$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\Omega}$	4.22E-05	1.84E-05	8.76E-06	4.28E-06	2.12E-06
order	-	1.20	1.07	1.03	1.02
$\|y-y_h\|_{0,\Omega}$	1.50E+01	3.85E+00	9.74E-01	2.45E-01	6.14E-02
order	-	1.96	1.98	1.99	2.00
$\|z-z_h\|_{0,\Omega}$	1.57E+01	3.55E+00	8.58E-01	2.11E-01	5.25E-02
order	-	2.14	2.05	2.02	2.01
$\|u-u_h\|_{0,\Gamma}$	3.77E+01	9.24E+00	2.28E+00	5.67E-01	1.41E-01
order	-	2.03	2.02	2.00	2.00

Table 2

Smooth test with $k = 1$ and $\varepsilon = 10^{-7}$: Errors for the control u, state y, adjoint state z, and the fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

on different meshes. Specifically, we compare approximate solutions computed on various meshes with the approximate solution on a fine mesh with 524288 elements, i.e., $h = 2^{-12}\sqrt{2}$.

When $k = 1$, the guaranteed theoretical convergence rates are given by Corollary 3.3 in Section 3:

$$\|y-y_h\|_{0,\Omega} = O(h^{5/6-\eta}), \qquad \|z-z_h\|_{0,\Omega} = O(h^{5/6-\eta}),$$
$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\Omega} = O(h^{1/3-\eta}), \qquad \|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\Omega} = O(h^{5/6-\eta}),$$

and

$$\|u-u_h\|_{0,\Gamma} = O(h^{5/6-\eta}).$$

Table 3 shows numerical results for this case. As in Part I, the numerically observed convergence rates match the theory for the control u and the primary flux $\boldsymbol{q}$, but are higher than the theoretical rates for the other variables. As mentioned in Part I, similar convergence behavior has been observed in other works [27, 30, 34, 21].

$h/\sqrt{2}$	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\Omega}$	1.49E-01	1.02E-01	7.51E-02	5.61E-02	4.31E-02
order	-	0.52	0.44	0.40	0.38
$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\Omega}$	2.62E-03	9.64E-04	3.55E-04	1.37E-04	5.21E-05
order	-	1.46	1.43	1.41	1.38
$\|y-y_h\|_{0,\Omega}$	1.02E-03	3.31E-04	1.23E-04	4.61E-05	1.82E-05
order	-	1.58	1.45	1.38	1.36
$\|z-z_h\|_{0,\Omega}$	5.94E-05	1.22E-05	2.44E-06	4.88E-07	9.62E-08
order	-	2.30	2.32	2.31	2.35
$\|u-u_h\|_{0,\Gamma}$	1.33E-02	6.39E-03	3.34E-03	1.82e-03	1.01E-03
order	-	1.02	0.94	0.85	0.85

TABLE 3

Non-smooth test with constant convection, $\varepsilon = 1$, and $k = 1$: Errors for the control u, state y, adjoint state z, and the fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

$h/\sqrt{2}$	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\Omega}$	2.22E-01	1.69E-01	1.22E-01	8.92E-02	6.56E-02
order	-	0.39	0.47	0.46	0.44
$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\Omega}$	8.60E-03	5.10E-03	2.75E-03	1.43E-03	7.31E-04
order	-	0.75	0.90	0.94	0.97
$\|y-y_h\|_{0,\Omega}$	2.96E-03	1.33E-03	4.91E-04	1.82E-04	6.97E-05
order	-	1.15	1.44	1.43	1.39
$\|z-z_h\|_{0,\Omega}$	3.82E-04	1.08E-04	2.89E-05	7.48E-06	1.90E-06
order	-	1.82	1.91	1.95	1.97
$\|u-u_h\|_{0,\Gamma}$	2.83E-02	1.79E-02	1.07E-02	6.14E-03	3.47E-03
order	-	0.66	0.75	0.80	0.82

TABLE 4

Non-smooth test with constant convection, $\varepsilon = 1$, and $k = 0$: Errors for the control u, state y, adjoint state z, and the fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

Next, for $k = 0$, Corollary 3.3 gives the suboptimal convergence rates

$$\|y-y_h\|_{0,\Omega} = O(h^{1/2-\eta}), \quad \|z-z_h\|_{0,\Omega} = O(h^{1/2-\eta}), \quad \|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\Omega} = O(h^{1/2-\eta}),$$

and

$$\|u-u_h\|_{0,\Gamma} = O(h^{1/2-\eta}).$$

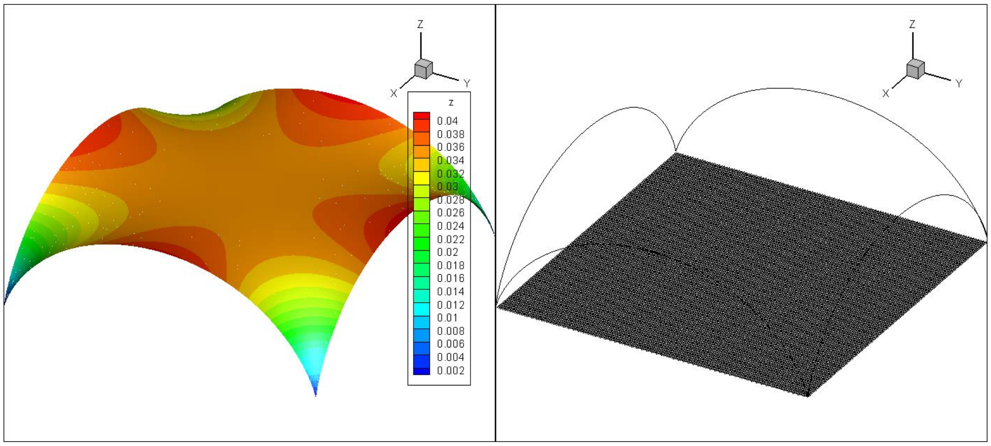
As in Part I, we observe much larger numerical convergence rates for all variables. Improving the analysis for the $k = 0$ case is again an interesting topic we leave to be considered elsewhere.

Numerical experiments for the same problem with constant convection coefficient $\boldsymbol{\beta} = [1, 1]$ gave similar results for both $k = 1$ and $k = 0$ (not shown).

In the second test, we consider variable convection and we choose the problem data

$$y_d = 1, \quad \text{and} \quad \boldsymbol{\beta} = -[x_1^2 \sin(x_2), \cos(x_1)e^{x_2}].$$

We compute the approximate solution using $k = 1$ for both $\varepsilon = 1$ and $\varepsilon = 10^{-6}$ to see the effect of strong convection on the solution.



(a) The computed state y (b) The computed boundary control u

FIG. 1. *Test with variable convection,* $y_d = 1$*,* $\varepsilon = 1$*, and* $k = 1$*.*

When $\varepsilon = 1$, since y_d is smooth we know from the high regularity convergence theory in Part I that

$$\|u - u_h\|_{0,\Gamma} = O(h^{3/2-\eta}).$$

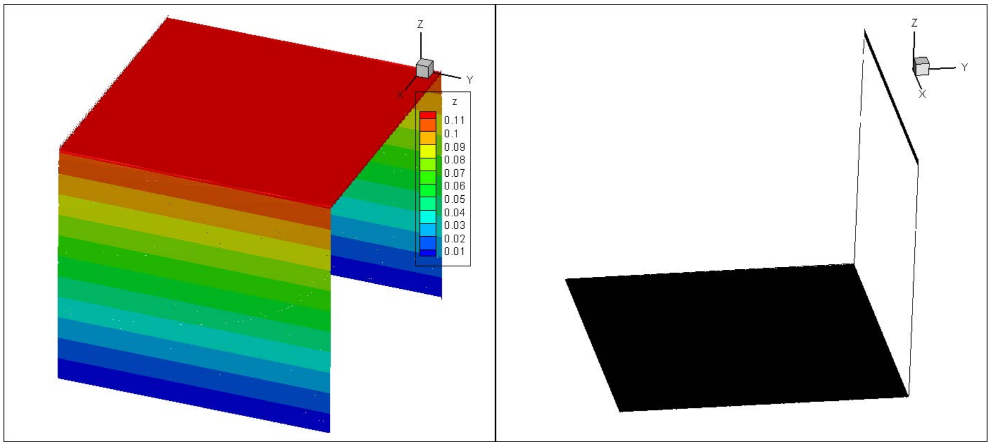
We observed this convergence rate in numerical experiments (not shown). The approximate state y and the approximate optimal boundary control u are shown in Figure 1.

Next, we demonstrate the performance of the HDG method in the convection dominated case with variable convection. We do not attempt to compute convergence rates here; instead for illustration we plot the state y and the boundary control u in Figure 2. We note that the computed state y is entirely different compared to the solution of the diffusion dominated problem. Also, the HDG method is able to capture the very sharp boundary layers in the solution with almost no oscillatory behavior.

5. Conclusion. In Part I of this work, we considered a Dirichlet boundary control problem for an elliptic convection diffusion equation and approximated the solution using a new HDG method. We also proved optimal convergence rates for the control under a high regularity assumption. In this paper, we removed the restrictions on the domain Ω and the desired state y_d from Part I and considered a low regularity scenario. We used very different HDG analysis techniques to prove optimal convergence rates for the control. We also presented numerical results for a convection dominated problem; the HDG method was able to capture sharp layers in the solution. A thorough investigation of the performance of HDG methods for the convection dominated case is underway.

As far as we are aware, this paper and Part I are the only existing analysis and numerical analysis explorations of this convection diffusion Dirichlet control problem. We leave many topics to be considered in future work, such as improving the HDG convergence analysis for the Dirichlet boundary control problem considered here and also applying HDG methods to Dirichlet control problems for fluids.

REFERENCES



(a) The computed state y (b) The computed boundary control u

FIG. 2. *Test with variable convection, $y_d = 1$, $\varepsilon = 10^{-6}$, and $k = 1$.*

[1] T. Apel, M. Mateos, J. Pfefferer, and A. Rösch, *Error estimates for Dirichlet control problems in polygonal domains*. http //arxiv.org/pdf/1704.08843v1, https://arxiv.org/abs/1704.08843v1.

[2] T. Apel, M. Mateos, J. Pfefferer, and A. Rösch, *On the regularity of the solutions of Dirichlet optimal control problems in polygonal domains*, SIAM J. Control Optim., 53 (2015), pp. 3620–3641, https://doi.org/10.1137/140994186, http://dx.doi.org/10.1137/140994186.

[3] N. Arada and J.-P. Raymond, *Dirichlet boundary control of semilinear parabolic equations. I. Problems with no state constraints*, Appl. Math. Optim., 45 (2002), pp. 125–143, https://doi.org/10.1007/s00245-001-0035-5, http://dx.doi.org/10.1007/s00245-001-0035-5.

[4] R. Becker and B. Vexler, *Optimal control of the convection-diffusion equation using stabilized finite element methods*, Numer. Math., 106 (2007), pp. 349–367, https://doi.org/10.1007/s00211-007-0067-0, http://dx.doi.org/10.1007/s00211-007-0067-0.

[5] F. Ben Belgacem, H. El Fekih, and H. Metoui, *Singular perturbation for the Dirichlet boundary control of elliptic problems*, M2AN Math. Model. Numer. Anal., 37 (2003), pp. 883–850, https://doi.org/10.1051/m2an:2003057, http://dx.doi.org/10.1051/m2an:2003057.

[6] M. Braack, *Optimal control in fluid mechanics by finite elements with symmetric stabilization*, SIAM J. Control Optim., 48 (2009), pp. 672–687, https://doi.org/10.1137/060653494, http://dx.doi.org/10.1137/060653494.

[7] E. Burman and P. Hansbo, *Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems*, Comput. Methods Appl. Mech. Engrg., 193 (2004), pp. 1437–1453, https://doi.org/10.1016/j.cma.2003.12.032, http://dx.doi.org/10.1016/j.cma.2003.12.032.

[8] E. Casas, M. Mateos, and J.-P. Raymond, *Penalization of Dirichlet optimal control problems*, ESAIM Control Optim. Calc. Var., 15 (2009), pp. 782–809, https://doi.org/10.1051/cocv:2008049, http://dx.doi.org/10.1051/cocv:2008049.

[9] E. Casas and J.-P. Raymond, *Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations*, SIAM J. Control Optim., 45 (2006), pp. 1586–1611, https://doi.org/10.1137/050626600, http://dx.doi.org/10.1137/050626600.

[10] A. Cesmelioglu, B. Cockburn, and W. Qiu, *Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier-Stokes equations*, Math. Comp., 86 (2017), pp. 1643–1670, https://doi.org/10.1090/mcom/3195, http://dx.doi.org/10.1090/mcom/3195.

[11] L. Chang, W. Gong, and N. Yan, *Weak boundary penalization for Dirichlet boundary control problems governed by elliptic equations*, J. Math. Anal. Appl., 453 (2017), pp. 529–557, https://doi.org/10.1016/j.jmaa.2017.04.016, http://dx.doi.org/10.1016/j.jmaa.2017.04.016.

[12] H. Chen, W. Qiu, K. Shi, and M. Solano, *A superconvergent HDG method for the Maxwell equations*, J. Sci. Comput., 70 (2017), pp. 1010–1029, https://doi.org/10.1007/s10915-016-0272-z.

[13] Y. Chen, B. Cockburn, and B. Dong, *Superconvergent HDG methods for linear, stationary, third-order equations in one-space dimension*, Math. Comp., 85 (2016), pp. 2715–2742, https://doi.org/10.1090/mcom/3091, http://dx.doi.org/10.1090/mcom/3091.

[14] S. Chowdhury, T. Gudi, and A. K. Nandakumaran, *Error bounds for a Dirichlet boundary control problem based on energy spaces*, Math. Comp., 86 (2017), pp. 1103–1126, https://doi.org/10.1090/mcom/3125, http://dx.doi.org/10.1090/mcom/3125.

[15] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems*, SIAM J. Numer. Anal., 47 (2009), pp. 1319–1365, https://doi.org/10.1137/070706616, http://dx.doi.org/10.1137/070706616.

[16] B. Cockburn, J. Gopalakrishnan, N. C. Nguyen, J. Peraire, and F.-J. Sayas, *Analysis of HDG methods for Stokes flow*, Math. Comp., 80 (2011), pp. 723–760, https://doi.org/10.1090/S0025-5718-2010-02410-X, http://dx.doi.org/10.1090/S0025-5718-2010-02410-X.

[17] B. Cockburn and K. Mustapha, *A hybridizable discontinuous Galerkin method for fractional diffusion problems*, Numer. Math., 130 (2015), pp. 293–314, https://doi.org/10.1007/s00211-014-0661-x, http://dx.doi.org/10.1007/s00211-014-0661-x.

[18] B. Cockburn and J. Shen, *A hybridizable discontinuous Galerkin method for the p-Laplacian*, SIAM J. Sci. Comput., 38 (2016), pp. A545–A566, https://doi.org/10.1137/15M1008014, http://dx.doi.org/10.1137/15M1008014.

[19] K. Deckelnick, A. Günther, and M. Hinze, *Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains*, SIAM J. Control Optim., 48 (2009), pp. 2798–2819, https://doi.org/10.1137/080735369, http://dx.doi.org/10.1137/080735369.

[20] L. Dede' and A. Quarteroni, *Optimal control and numerical adaptivity for advection-diffusion equations*, M2AN Math. Model. Numer. Anal., 39 (2005), pp. 1019–1040, https://doi.org/10.1051/m2an:2005044, http://dx.doi.org/10.1051/m2an:2005044.

[21] W. Gong and N. Yan, *Mixed finite element method for Dirichlet boundary control problem governed by elliptic PDEs*, SIAM J. Control Optim., 49 (2011), pp. 984–1014, https://doi.org/10.1137/100795632, http://dx.doi.org/10.1137/100795632.

[22] M. D. Gunzburger, L. Hou, and T. P. Svobodny, *Boundary velocity control of incompressible flow with an application to viscous drag reduction*, SIAM J. Control Optim., 30 (1992), pp. 167–181, https://doi.org/10.1137/0330011, http://dx.doi.org/10.1137/0330011.

[23] M. D. Gunzburger, L. S. Hou, and T. P. Svobodny, *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls*, RAIRO Modél. Math. Anal. Numér., 25 (1991), pp. 711–748, https://doi.org/10.1051/m2an/1991250607111, http://dx.doi.org/10.1051/m2an/1991250607111.

[24] M. Heinkenschloss and D. Leykekhman, *Local error estimates for SUPG solutions of advection-dominated elliptic linear-quadratic optimal control problems*, SIAM J. Numer. Anal., 47 (2010), pp. 4607–4638, https://doi.org/10.1137/090759902, http://dx.doi.org/10.1137/090759902.

[25] L. S. Hou and S. S. Ravindran, *A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier-Stokes equations*, SIAM J. Control Optim., 36 (1998), pp. 1795–1814, https://doi.org/10.1137/S0363012996304870, http://dx.doi.org/10.1137/S0363012996304870.

[26] W. Hu, M. Mateos, J. R. Singler, and Y. Zhang, *A new HDG method for Dirichlet boundary control of convection diffusion PDEs I: High regularity*. Submitted.

[27] W. Hu, J. Shen, J. R. Singler, Y. Zhang, and X. Zheng, *Hybridizable discontinuous Galerkin method for Dirichlet boundary control of elliptic PDEs*. submitted.

[28] B. Li and X. Xie, *Analysis of a family of HDG methods for second order elliptic problems*, J. Comput. Appl. Math., 307 (2016), pp. 37–51, https://doi.org/10.1016/j.cam.2016.04.027, http://dx.doi.org/10.1016/j.cam.2016.04.027.

[29] B. Li and X. Xie, *BPX preconditioner for nonstandard finite element methods for diffusion problems*, SIAM J. Numer. Anal., 54 (2016), pp. 1147–1168, https://doi.org/10.1137/140999268, http://dx.doi.org/10.1137/140999268.

[30] S. May, R. Rannacher, and B. Vexler, *Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems*, SIAM J. Control Optim., 51 (2013), pp. 2585–2611, https://doi.org/10.1137/080735734, http://dx.doi.org/10.1137/080735734.

[31] N. C. Nguyen, J. Peraire, and B. Cockburn, *An implicit high-order hybridizable discontinuous Galerkin method for linear convection-diffusion equations*, J. Comput. Phys.,

228 (2009), pp. 3232–3254, https://doi.org/10.1016/j.jcp.2009.01.030, http://dx.doi.org/10.1016/j.jcp.2009.01.030.

[32] N. C. Nguyen, J. Peraire, and B. Cockburn, *An implicit high-order hybridizable discontinuous Galerkin method for nonlinear convection-diffusion equations*, J. Comput. Phys., 228 (2009), pp. 8841–8855, https://doi.org/10.1016/j.jcp.2009.08.030, http://dx.doi.org/10.1016/j.jcp.2009.08.030.

[33] N. C. Nguyen, J. Peraire, and B. Cockburn, *A hybridizable discontinuous Galerkin method for Stokes flow*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 582–597, https://doi.org/10.1016/j.cma.2009.10.007, http://dx.doi.org/10.1016/j.cma.2009.10.007.

[34] G. Of, T. X. Phan, and O. Steinbach, *An energy space finite element approach for elliptic Dirichlet boundary control problems*, Numer. Math., 129 (2015), pp. 723–748, https://doi.org/10.1007/s00211-014-0653-x, http://dx.doi.org/10.1007/s00211-014-0653-x.

[35] W. Qiu and K. Shi, *An HDG method for convection diffusion equation*, J. Sci. Comput., 66 (2016), pp. 346–357, https://doi.org/10.1007/s10915-015-0024-5, http://dx.doi.org/10.1007/s10915-015-0024-5.

[36] W. Qiu and K. Shi, *A superconvergent HDG method for the incompressible Navier-Stokes equations on general polyhedral meshes*, IMA J. Numer. Anal., 36 (2016), pp. 1943–1967, https://doi.org/10.1093/imanum/drv067.

[37] S. S. Ravindran, *Finite element approximation of Dirichlet control using boundary penalty method for unsteady Navier–Stokes equations*, ESAIM: Mathematical Modelling and Numerical Analysis, 51 (2017), pp. 825–849.

[38] M. Stanglmeier, N. C. Nguyen, J. Peraire, and B. Cockburn, *An explicit hybridizable discontinuous Galerkin method for the acoustic wave equation*, Comput. Methods Appl. Mech. Engrg., 300 (2016), pp. 748–769, https://doi.org/10.1016/j.cma.2015.12.003, http://dx.doi.org/10.1016/j.cma.2015.12.003.

[39] B. Vexler, *Finite element approximation of elliptic Dirichlet optimal control problems*, Numer. Funct. Anal. Optim., 28 (2007), pp. 957–973, https://doi.org/10.1080/01630560701493305, http://dx.doi.org/10.1080/01630560701493305.

[40] N. Yan and Z. Zhou, *A RT mixed FEM/DG scheme for optimal control governed by convection diffusion equations*, J. Sci. Comput., 41 (2009), pp. 273–299, https://doi.org/10.1007/s10915-009-9297-x, http://dx.doi.org/10.1007/s10915-009-9297-x.