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On the Spherical Quasi-Convexity of Quadratic Functions☆

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Abstract

In this paper the spherical quasi-convexity of quadratic functions on spherically convex sets is studied. Several conditions characterizing the spherical quasi-convexity of quadratic functions are presented. In particular, conditions implying spherical quasi-convexity of quadratic functions on the spherical positive orthant are given. Some examples are provided as applications of the obtained results.

Keywords: sphere, spherical quasi-convexity, quadratic functions, positive orthant.

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1. Introduction

In this paper we study the spherical quasi-convexity of quadratic functions on spherically convex sets, which is related to the problem of finding their minimizer. This problem of minimizing a quadratic function on the sphere has arisen to S. Z. Németh by trying to make certain fixed point theorems, surjectivity theorems, and existence theorems for complementarity problems and variational inequalities more explicit (see [1] and the related references therein). In particular, some existence theorems could be reduced to the optimization of a

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quadratic function on the intersection of the sphere and a cone. Indeed, consider a closed convex cone $K \subseteq \mathbb{R}^n$ with dual K^*. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping such that $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by $G(x) = \|x\|^2 F(x/\|x\|^2)$ and $G(0) = 0$ is differentiable at 0. Denote by $DG(0)$ the Jacobian matrix of G at 0. By [2, Corollary 8.1] and [3, Theorem18], if $\min_{\|u\|=1, u \in K} \langle DG(0)u, u\rangle > 0$, then the nonlinear complementarity problem defined by $K \ni x \perp F(x) \in K^*$ has a solution. Thus, we need to minimize a quadratic form on the intersection between a cone and the sphere. These sets are exactly the spherically convex sets; see [4]. Therefore, this leads to minimizing quadratic functions on spherically convex sets. In fact the optimization problem above reduces to the problem of calculating the scalar derivative, along cones introduced by S. Z. Németh (see [1] and the related references therein). Similar minimizations of quadratic functions on spherically convex sets are needed in the other settings (see [1] and the related references therein). Apart from the above, the motivation of this study is much wider. For instance, the quadratic constrained optimization problem on the sphere

$$\min\{\langle Qx, x\rangle \ : \ x \in C\}, \qquad C \subseteq \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \ : \ \|x\| = 1\}, \tag{1}$$

for a symmetric matrix Q, is a minimum eigenvalue problem in C, which includes the problem of finding the spectral norm of the matrix $-Q$ when $C = \mathbb{S}^{n-1}$ (see, e.g., [5]). It is important to highlight that the special case when C is the nonnegative orthant is of particular interest because the nonnegativity of the minimum value is equivalent to the copositivity of the matrix Q [6, Proposition 1.3] and to the nonnegativity of all Pareto eigenvalues of Q [6, Theorem 4.3]. As far as we are aware there are no methods for finding the Pareto spectra by using the intrinsic geometrical properties of the sphere, hence our study is expected to open new perspectives for detecting the copositivity of a symmetric matrix. More problems that deal with "spherical" constraints can be found in [7].

Optimization problems posed on the sphere have a specific underlying algebraic structure that could be exploited to greatly reduce the cost of obtaining the solutions; see [8, 9, 5, 10, 11, 12]. It is worth to point out that when a quadratic function is spherically quasi-convex, then a spherical strict local minimizer is equal to a spherical strict global minimizer. Therefore, it is natural to consider the problem of determining the spherically quasi-convex quadratic functions on spherically convex sets. The goal of the paper is to present necessary conditions and sufficient conditions for quadratic functions which are spherically quasi-convex on spherical convex sets. As a particular case, we exhibit several such results for the spherical positive orthant.

The paper can be considered as a first spherical analogue for the study of quasi-convexity of quadratic functions. Without the aim of completeness, we list here some of the main papers about the quasi-convexity of quadratic functions: [13, 14, 15, 16, 17]

The remainder of this paper is organized as follows. In Section 2, we recall some notations and basic results used throughout the paper. In Section 3 we present some general properties of spherically quasi-convex functions on spherically convex sets. In Section 4 we present some conditions characterizing quadratic spherically quasi-convex functions on a general spherically convex set. In Section 4.1 we present some properties of quadratic functions defined in the spherical positive orthant. We conclude this paper by making some final remarks in Section 5.

2. Basics results

In this section we present the notations and the auxiliary results used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$. Denote by $\mathbb{R}^n_+$ the nonnegative orthant and by $\mathbb{R}^n_{++}$ the positive orthant, that is,

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n)^\top : x_1 \geq 0, \dots, x_n \geq 0\}$$

and

$$\mathbb{R}^n_{++} = \{x = (x_1, \dots, x_n)^\top : x_1 > 0, \dots, x_n > 0\}.$$

Denote by e^i the i-th canonical unit vector in $\mathbb{R}^n$. A set $\mathcal{K}$ is called a *cone* if it is invariant under the multiplication with positive scalars and a *convex cone* if it is a cone which is also a convex set. The *dual cone* of a cone $\mathcal{K} \subset \mathbb{R}^n$ is the cone $\mathcal{K}^* := \{x \in \mathbb{R}^n : \langle x, y\rangle \geq 0, \ \forall\, y \in \mathcal{K}\}$. A cone $\mathcal{K} \subset \mathbb{R}^n$ is called *pointed* if $\mathcal{K} \cap \{-\mathcal{K}\} \subseteq \{0\}$, or equivalently, if $\mathcal{K}$ does not contain straight lines through the origin. Any pointed closed convex cone with nonempty interior will be called *proper cone*. The cone $\mathcal{K}$ is called *subdual* if $\mathcal{K} \subseteq \mathcal{K}^*$, *superdual* if $\mathcal{K}^* \subseteq \mathcal{K}$ and *self-dual* if $\mathcal{K}^* = \mathcal{K}$. The matrix I_n denotes the $n \times n$ identity matrix. Recall that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is *positive* if $a_{ij} > 0$ and *nonnegative* if $a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. A matrix $A \in \mathbb{R}^{n\times n}$ is *reducible* if there is permutation matrix $P \in \mathbb{R}^{n\times n}$ so that

$$P^T A P = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

$$B_{11} \in \mathbb{R}^{m\times m}, \ B_{22} \in \mathbb{R}^{(n-m)\times(n-m)}, \ B_{12} \in \mathbb{R}^{m\times(n-m)}, \quad m < n.$$

A matrix $A \in \mathbb{R}^{n\times n}$ is *irreducible* if it not reducible. In the following we state a version of *Perron-Frobenius theorem* for both positive matrices and nonnegative irreducible matrices, its proof can be found in [18, Theorem 8.2.11] and [18, Theorem 8.4.4], respectively.

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$ be either nonnegative and irreducible or positive. Then A has a dominant eigenvalue $\lambda_{max}(A) \in \mathbb{R}$ with associated eigenvector $v \in \mathbb{R}^n$ which satisfies the following properties:*

i) The eigenvalue $\lambda_{max}(A) > 0$ and its associated eigenvector $v \in \mathbb{R}^n_{++}$;

ii) The eigenvalue $\lambda_{max}(A) > 0$ has multiplicity one;

iii) Every other eigenvalue λ of A is less that $\lambda_{max}(A)$ in absolute value, i.e, $|\lambda| < \lambda_{max}(A)$;

iii) There are no other positive or non-negative eigenvectors of A except positive multiples of v.

Recall that $A \in \mathbb{R}^{n\times n}$ is *copositive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n_+$ and a *Z-matrix* is a matrix with nonpositive off-diagonal elements. Let $\mathcal{K} \subset \mathbb{R}^n$ be a pointed closed convex cone with nonempty interior, the *$\mathcal{K}$-Z-property* of a matrix $A \in \mathbb{R}^{n\times n}$ means that $\langle Ax, y\rangle \leq 0$ for all $(x, y) \in C(\mathcal{K})$, where $C(\mathcal{K}) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \ x \in \mathcal{K}, \ y \in \mathcal{K}^*, \langle x, y\rangle = 0\}$. If $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, then $\operatorname{diag}(x)$ will denote an $n \times n$ diagonal matrix with (i, i)-th entry equal to x_i, for $i = 1, \ldots, n$. Throughout the paper the *tangent hyperplane of the n-dimensional Euclidean sphere* $\mathbb{S}^{n-1}$ at a point $x \in \mathbb{S}^{n-1}$ is denoted by

$$T_x\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n \ : \ \langle x, v\rangle = 0\},$$

The *intrinsic distance on the sphere* between two arbitrary points $x, y \in \mathbb{S}^{n-1}$ is defined by

$$d(x, y) := \arccos\langle x, y\rangle. \tag{2}$$

It can be shown that $(\mathbb{S}^{n-1}, d)$ is a complete metric space, so that $d(x, y) \geq 0$ for all $x, y \in \mathbb{S}^{n-1}$, and $d(x, y) = 0$ if and only if $x = y$. It is also easy to check that $d(x, y) \leq \pi$ for all $x, y \in \mathbb{S}^{n-1}$, and $d(x, y) = \pi$ if and only if $x = -y$. The intersection curve of a plane though the origin of $\mathbb{R}^n$ with the sphere $\mathbb{S}^{n-1}$ is called a *geodesic*. If $x, y \in \mathbb{S}^{n-1}$ are such that $y \neq x$ and $y \neq -x$, then the unique *segment of minimal geodesic from to x to y* is

$$\gamma_{xy}(t) = \left(\cos(td(x, y)) - \frac{\langle x, y\rangle \sin(td(x, y))}{\sqrt{1 - \langle x, y\rangle^2}}\right) x + \frac{\sin(td(x, y))}{\sqrt{1 - \langle x, y\rangle^2}}\, y, \qquad t \in [0,\ 1]. \tag{3}$$

Let $x \in \mathbb{S}^{n-1}$ and $v \in T_x\mathbb{S}^{n-1}$ such that $\|v\| = 1$. The minimal segment of geodesic connecting x to $-x$, starting at x with velocity v at x is given by

$$\gamma_{x\{-x\}}(t) := \cos(t)\, x + \sin(t)\, v, \qquad t \in [0,\ \pi]. \tag{4}$$

Let $\Omega \subset \mathbb{S}^{n-1}$ be a spherically open set (i.e., a set open with respect to the induced topology in $\mathbb{S}^{n-1}$). The *gradient on the sphere* of a differentiable function $f : \Omega \to \mathbb{R}$ at a point $x \in \Omega$ is the vector defined by

$$\operatorname{grad} f(x) := \left[I_n - xx^T\right] Df(x) = Df(x) - \langle Df(x), x\rangle\, x, \tag{5}$$

where $Df(x) \in \mathbb{R}^n$ is the usual gradient of f at $x \in \Omega$. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set, $I \subset \mathbb{R}$ an open interval, $\Omega \subset \mathbb{S}^{n-1}$ a spherically open set and $\gamma : I \to \Omega$ a geodesic segment. If $f : \mathcal{D} \to \mathbb{R}$ is a differentiable function, then, since $\gamma'(t) \in T_{\gamma(t)}\mathbb{S}^{n-1}$ for all $t \in I$, the equality (5) implies

$$\frac{d}{dt} f(\gamma(t)) = \langle \operatorname{grad} f(\gamma(t)), \gamma'(t) \rangle = \langle Df(\gamma(t)), \gamma'(t) \rangle, \qquad \forall\, t \in I. \tag{6}$$

Definition 2. *The set $\mathcal{C} \subseteq \mathbb{S}^{n-1}$ is said to be* spherically convex *if for all x, $y \in \mathcal{C}$ all the minimal geodesic segments joining x to y are contained in $\mathcal{C}$.*

Example 3. *The set $S_+ = \{(x_1, \ldots, x_n) \in \mathbb{S}^{n-1} : x_1 \geq 0, \ldots, x_n \geq 0\}$ is a closed spherically convex set.*

We assume for convenience that *from now on all spherically convex sets are nonempty proper subsets of the sphere.* For each closet set $A \subset \mathbb{S}^{n-1}$, let $K_A \subset \mathbb{R}^n$ be the *cone spanned by A*, namely,

$$K_A := \{tx : x \in A,\ t \in [0, +\infty)\}. \tag{7}$$

Clearly, K_A is the smallest closed cone which contains A. In the next proposition we exhibit a relationship of spherically convex sets with the cones spanned by them; for the proof see [19].

Proposition 4. *The set $\mathcal{C}$ is spherically convex if and only if the cone $K_{\mathcal{C}}$ is convex and pointed.*

Let $\mathcal{C} \subset \mathbb{S}^{n-1}$ be a spherically convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is said to be spherically convex (respectively, strictly spherically convex) if for all minimal geodesic segment $\gamma : [0, 1] \to \mathcal{C}$, the composition $f \circ \gamma : [0, 1] \to \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense.

We end this section by stating some standard notations. We denote the *spherically open* and the *spherically closed ball* with radius $\delta > 0$ and center in $x \in \mathbb{S}^{n-1}$ by $B_\delta(x) := \{y \in \mathbb{S}^{n-1} : d(x, y) < \delta\}$ and $\bar{B}_\delta(x) := \{y \in \mathbb{S}^{n-1} : d(x, y) \leq \delta\}$, respectively. The *sub-level sets* of a function $f : \mathbb{R}^n \supseteq \mathcal{D} \to \mathbb{R}$ are denoted by

$$[f \leq c] := \{x \in \mathcal{D} : f(x) \leq c\}, \qquad c \in \mathbb{R}. \tag{8}$$

3. Spherically quasi-convex functions on spherically convex sets

In this section we study general properties of quasi-convex functions on the sphere. In particular, we present first order characterizations of differentiable quasi-convex functions on the sphere. Several results of this section have already appeared in [20], but here these results have more explicit statements and proofs. It is worth to remark that the quasi-convexity concept generalizes the convexity one, which was extensively studied in [4]. Let us start by defining this concept.

Definition 5. *Let $\mathcal{C} \subset \mathbb{S}^{n-1}$ be a spherically convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is said to be spherically quasi-convex (respectively, strictly spherically quasi-convex) if for all minimal geodesic segment $\gamma : [0,1] \to \mathcal{C}$, the composition $f \circ \gamma : [0,1] \to \mathbb{R}$ is quasi-convex (respectively, strictly quasi-convex) in the usual sense, i.e., $f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\}$ for all $t \in [0,1]$, (respectively, $f(\gamma(t)) < \max\{f(\gamma(0)), f(\gamma(1))\}$ for all $t \in [0,1]$).*

Naturally, from the above definition, it follows that spherically convex (respectively, strictly spherically convex) functions are spherically quasi-convex (respectively, strictly spherically quasi-convex), but the converse is not true; see [4].

Proposition 6. *Let $\mathcal{C} \subset \mathbb{S}^{n-1}$ be a spherically convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is spherically quasi-convex if and only if the sub-level sets $[f \leq c]$ are spherically convex for all $c \in \mathbb{R}$.*

Proof. Assume that f is spherically quasi-convex and $c \in \mathbb{R}$. Take $x, y \in [f \leq c]$ and $\gamma : [0,1] \to \mathbb{S}^{n-1}$ the minimal geodesic such that $\gamma(0) = x$ to $\gamma(1) = y$, see (3) and (4). Since f is a spherically quasi-convex function and $x, y \in [f \leq c]$ we have $f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\} \leq c$ for all $t \in [0,1]$, which implies that $\gamma(t) \in [f \leq c]$ for all $t \in [0,1]$. Hence we conclude that $[f \leq c]$ is a spherically convex set for all $c \in \mathbb{R}$. Conversely, we assume that $[f \leq c]$ is spherically convex for all $c \in \mathbb{R}$. Let $\gamma : [0,1] \to \mathcal{C}$ be a minimal geodesic segment. Since $[f \leq c]$ is a spherically convex set, we have $\gamma(t) \in [f \leq c]$ for all $t \in [0,1]$, which implies

$f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\}$ for all $t \in [0,1]$. Therefore, f is a spherically quasi-convex function and the proof is concluded. $\square$

Proposition 7. *Let $\mathcal{C} \subset \mathbb{S}^{n-1}$ be spherically convex and $f : \mathcal{C} \to \mathbb{R}$ be spherically quasi-convex. If $x^* \in \mathcal{C}$ is a strict local minimizer of f, then x^* is also a strict global minimizer of f in $\mathcal{C}$.*

Proof. If x^* is a strict local minimizer of f, then there exists a number $\delta > 0$ such that

$$f(x) > f(x^*), \qquad \forall\, x \in B_\delta(x^*) \setminus \{x^*\} = \{y \in \mathcal{C} \;:\; 0 < d(y, x^*) < \delta\}. \tag{9}$$

Assume by contradiction that x^* is not a strict global minimizer of f in $\mathcal{C}$. Thus, there exists $\bar{x} \in \mathcal{C}$ with $\bar{x} \neq x^*$ such that $f(\bar{x}) \leq f(x^*)$. Since C is spherically convex, we can take a minimal geodesic segment $\gamma : [0,1] \to \mathcal{C}$ joining x^* and $\bar{x}$, let's say, $\gamma(0) = x^*$ and $\gamma(1) = \bar{x}$. Considering that f is spherically quasi-convex we have $f(\gamma(t)) \leq \max\{f(x^*), f(\bar{x})\} = f(x^*)$ for all $t \in [0,1]$. On the other hand, for t sufficiently small we have $\gamma(t) \in B_\delta(x^*)$. Therefore, the last inequality contradicts (9) and the proof is concluded. $\square$

Proposition 8. *Let $\mathcal{C} \subset \mathbb{S}^{n-1}$ be a spherically convex set and $f : \mathcal{C} \to \mathbb{R}$ be a strictly spherically quasi-convex function. Then f has at most one local minimizer which is also a global minimizer of f.*

Proof. Assume by contradiction that f has two local minimizers $x^*, \bar{x} \in \mathcal{C}$ with $\bar{x} \neq x^*$. Thus, we can take a minimal geodesic segment $\gamma : [0,1] \to \mathcal{C}$ joining x^* and $\bar{x}$, let's say, $\gamma(0) = x^*$ and $\gamma(1) = \bar{x}$. Due to f being strictly spherically quasi-convex $f(\gamma(t)) < \max\{f(x^*), f(\bar{x})\}$ for all $t \in [0,1]$. Since we can take t sufficiently close to 0 or 1, the last inequality contradicts the assumption that $x^*, \bar{x}$ are two distinct local minimizers. Thus, f has at most one local minimizer. Since f is strictly quasi-convex, the local minimizer is strict. Therefore, Proposition 7 implies that the local minimizer is global and the proof is concluded. $\square$

Proposition 9. *Let $\mathcal{C} \subset \mathbb{S}^{n-1}$ be an open spherically convex set and $f : \mathcal{C} \to \mathbb{R}$ be a differentiable function. Then f is spherically quasi-convex if and only if*

$$f(x) \leq f(y) \Longrightarrow \langle Df(y), x\rangle - \langle x, y\rangle\langle Df(y), y\rangle \leq 0, \qquad \forall\, x, y \in \mathcal{C}. \tag{10}$$

Proof. Let $\gamma : I \to \mathcal{C}$ be a geodesic segment and consider the composition $f \circ \gamma : I \to \mathbb{R}$. The usual characterization of scalar quasi-convex functions implies that $f \circ \gamma$ is quasi-convex if and only if

$$f(\gamma(t_1)) \leq f(\gamma(t_2)) \Longrightarrow \frac{d}{dt}\left(f(\gamma(t_2))\right)(t_1 - t_2) \leq 0, \qquad \forall\, t_2, t_1 \in I. \tag{11}$$

On the other hand, for each $x, y \in \mathcal{C}$ with $y \neq x$ we have from (3) that γ_{xy} is the minimal geodesic segment from $x = \gamma_{xy}(0)$ to $y = \gamma_{xy}(1)$ and

$$\gamma'_{xy}(1) = \frac{\arccos\langle x, y\rangle}{\sqrt{1 - \langle x, y\rangle^2}}\left(yy^T - I_n\right)x \in T_y\mathbb{S}^{n-1}, \qquad y \neq -x.$$

Note that letting $x = \gamma(t_1)$ and $y = \gamma(t_2)$ we have that $\gamma_{xy}(t) = \gamma(t_1 + t(t_2 - t_1))$. Therefore, by using (6) we conclude that the condition in (11) is equivalent to (10) and the proof of the proposition follows. □

4. Spherically quasi-convex quadratic functions on spherically convex sets

In this section our aim is to present some conditions characterizing quadratic spherically quasi-convex functions on a general spherically convex set. For that we need some definitions: *From now on we assume that $\mathcal{K} \subset \mathbb{R}^n$ is a proper subdual cone, $\mathcal{C} = \mathbb{S}^{n-1} \cap \mathrm{int}(\mathcal{K})$ is an open spherically convex set and $A = A^T \in \mathbb{R}^{n\times n}$ with the associated quadratic function $q_A : \mathcal{C} \to \mathbb{R}$ defined by*

$$q_A(x) := \langle Ax, x\rangle. \tag{12}$$

We also need the *restriction on* $\mathrm{int}\,\mathcal{K}$ *of the Rayleigh quotient function* $\varphi_A : \mathrm{int}\,\mathcal{K} \to \mathbb{R}$ defined by

$$\varphi_A(x) := \frac{\langle Ax, x\rangle}{\|x\|^2}. \tag{13}$$

In the following propositions we present some equivalent characterizations of the convexity of q_A defined by (12) on spherically convex sets. Our first result is the following proposition.

Proposition 10. *Let q_A and φ_A be the functions defined in* (12) *and* (13), *respectively. The following statements are equivalent:*

(a) The quadratic function q_A is spherically quasi-convex;

(b) $\langle Ax, y\rangle \leq \langle x, y\rangle \max\{q_A(x),\ q_A(y)\}$ for all $x, y \in \mathbb{S}^{n-1} \cap \mathcal{K}$;

(c) $\dfrac{\langle Ax, y\rangle}{\langle x, y\rangle} \leq \max\{\varphi_A(x),\ \varphi_A(y)\}$ for all $x, y \in \mathcal{K}$ with $\langle x, y\rangle \neq 0$.

Proof. First of all, we assume that item (a) holds. Let $x, y \in \mathcal{C}$. Thus, either $q_A(x) \leq q_A(y)$ or $q_A(y) \leq q_A(x)$. Hence, by using Proposition 9 we conclude that either $\langle Ay, x\rangle \leq \langle x, y\rangle q_A(y)$ or $\langle Ax, y\rangle \leq \langle x, y\rangle q_A(x)$. Thus, since $A = A^T$ implies $\langle Ax, y\rangle = \langle Ay, x\rangle$, taking into account that $\mathcal{K}$ is a subdual cone and hence $\langle x, y\rangle \geq 0$, we have

$$\langle Ax, y\rangle \leq \max\{\langle x, y\rangle q_A(x), \langle x, y\rangle q_A(y)\} = \langle x, y\rangle \max\{q_A(x), q_A(y)\}, \ \forall\ x, y \in \mathcal{C}.$$

Therefore, by continuity we extend the above inequality to all $x, y \in \mathbb{S}^{n-1} \cap \mathcal{K}$ and, then item (b) holds. Conversely, we assume that item (b) holds. Let $x, y \in \mathcal{C}$ satisfying $q_A(x) \leq q_A(y)$. Then, by the inequality in item (b) and considering that $\mathcal{K}$ is a subdual cone, we have $\langle Ax, y\rangle \leq \langle x, y\rangle q_A(y)$. Hence, by using Proposition 9 we conclude that q_A is spherically quasi-convex and the proof of the equivalence between (a) and (b) is complete.

To establish the equivalence between (b) and (c), we assume first that item (b) holds. Let $x, y \in \mathcal{K}$ with $\langle x, y\rangle \neq 0$. Then, $x \neq 0$ and $y \neq 0$. Moreover, we have

$$u := \frac{x}{\|x\|} \in \mathbb{S}^{n-1} \cap \mathcal{K}, \qquad v := \frac{y}{\|y\|} \in \mathbb{S}^{n-1} \cap \mathcal{K}.$$

Hence, by using the inequality in item (b) with $x = u$ and $y = v$, we obtain the inequality in item (c). Conversely, suppose that (c) holds. Let $x, y \in \mathbb{S}^{n-1} \cap \mathcal{K}$. First assume that $\langle x, y\rangle \neq 0$. Since, $\|x\| = \|y\| = 1$, from the inequality in item (c) we conclude that

$$\frac{\langle Ax, y\rangle}{\langle x, y\rangle} \leq \max\{q_A(x),\ q_A(y)\}.$$

Due to $\mathcal{K}$ being a subdual cone, we have $\langle x, y\rangle \geq 0$, and hence the last inequality is equivalent to the inequality in item (b). Now, assume that $\langle x, y\rangle = 0$. Then, take two sequences $\{x^k\}, \{y^k\} \subset \mathcal{C}$ such that $\lim_{k\to+\infty} x^k = x$, $\lim_{k\to+\infty} y^k = y$ and $\langle x^k, y^k\rangle \neq 0$. Since $\mathcal{K}$ is a subdual cone, we have $\langle x^k, y^k\rangle > 0$ for all $k = 1, 2, \ldots$. Therefore, considering that $\|x^k\| = \|y^k\| = 1$ for all $k = 1, 2, \ldots$, we can apply again the inequality in item (c) to conclude

$$\langle Ax^k, y^k\rangle \leq \langle x^k, y^k\rangle \max\left\{q_A(x^k),\ q_A(y^k)\right\}, \qquad k = 1, 2, \ldots.$$

By tending with k to infinity, we conclude that the inequality in item (b) also holds for $\langle x, y\rangle = 0$ and the proof of the equivalence between (b) and (c) is complete. □

Corollary 11. *Assume that $\mathcal{K}$ is a self-dual cone. If the quadratic function q_A is spherically quasi-convex, then A has the $\mathcal{K}$-Z-property.*

Proof. Let $x, y \in \mathbb{R}^n \times \mathbb{R}^n$ such that $x \in \mathcal{K}$, $y \in \mathcal{K}^*$ and $\langle x, y\rangle = 0$. If $x = 0$ or $y = 0$ we have $\langle Ax, y\rangle = 0$. Thus, assume that $x \neq 0$ and $y \neq 0$. Considering that $\mathcal{K}$ is self-dual we have $x/\|x\|,\ y/\|y\| \in \mathbb{S}^{n-1} \cap \mathcal{K}$. Thus, since q_A is spherically quasi-convex and $\langle x/\|x\|,\ y/\|y\|\rangle = 0$, we obtain, from items (a) and (b) of Proposition 10, that $\langle Ax, y\rangle \leq 0$. Therefore, A has the $\mathcal{K}$-Z-property and the proof is concluded. □

Theorem 12. *The function q_A defined in* (12) *is spherically quasi-convex if and only if φ_A defined in* (13) *is quasi-convex.*

Proof. For $c \in \mathbb{R}$, let $[q_A \leq c] := \{y \in \mathcal{C} \ :\ q_A(x) \leq c\}$ and $[\varphi_A \leq c] := \{x \in \operatorname{int}(\mathcal{K}) \ :\ \varphi_A(x) \leq c\}$ be the sublevel sets of q_A and φ_A, respectively, where $c \in \mathbb{R}$. Let $\mathcal{K}_{[q_A \leq c]}$ be the cone spanned by $[q_A \leq c]$. Since $\mathcal{C} = \mathbb{S}^{n-1} \cap \operatorname{int}(\mathcal{K})$, we conclude that $x \in \operatorname{int}\mathcal{K}$ if and only if $x/\|x\| \in \mathcal{C}$. Hence, the definitions of $[q_A \leq c]$ and $[\varphi_A \leq c]$ imply that

$$\mathcal{K}_{[q_A \leq c]} = [\varphi_A \leq c]. \tag{14}$$

Now, we assume that q_A is spherically quasi-convex. Thus, from Poposition 6 we conclude that $[q_A \leq c]$ is spherically convex for all $c \in \mathbb{R}$. Hence, it follows from Proposition 4 that the cone $\mathcal{K}_{[q_A \le c]}$ is convex and pointed, which implies from (14) that $[\varphi_A \le c]$ is convex for all $c \in \mathbb{R}$. Therefore, φ_A is quasi-convex. Conversely, assume that φ_A is quasi-convex. Thus, $[\varphi_A \le c]$ is convex for all $c \in \mathbb{R}$. On the other hand, since $\mathcal{K}$ is a proper subdual cone, $\operatorname{int}\mathcal{K}$ is pointed. Thus, considering that $[\varphi_A \le c] \subset \operatorname{int}\mathcal{K}$ is a cone, it is also a pointed cone. Hence, from (14) it follows that $\mathcal{K}_{[q_A \le c]}$ is a pointed convex cone. Hence, Proposition 4 implies that $[q_A \le c]$ is spherically convex for all $c \in \mathbb{R}$. Therefore, by using Proposition 6, we conclude that q_A is spherically quasi-convex and the proof is completed. □

Corollary 13. *Assume that* $\{x \in \operatorname{int}(\mathcal{K}) : \ \langle A_c x, x\rangle < 0\} \neq \varnothing$, *where* $c \in \mathbb{R}$ *and* $A_c := A - cI_n$. *If* q_A *defined in* (12) *is spherically quasi-convex, then the cone*

$$\{x \in \mathcal{K} : \ \langle A_c x, x\rangle \le 0\}, \tag{15}$$

is convex.

Proof. Assume that q_A is spherically quasi-convex. Hence Theorem 12 implies that φ_A is quasi-convex and then $[\varphi_A \le c]$ is convex for all $c \in \mathbb{R}$. Since $\{x \in \operatorname{int}(\mathcal{K}) : \ \langle A_c x, x\rangle < 0\} \neq \varnothing$ we conclude that

$$\operatorname{closure}(\{x \in \operatorname{int}(\mathcal{K}) \ : \ \langle A_c x, x\rangle \le 0\}) = \{x \in \mathcal{K} \ : \ \langle A_c x, x\rangle \le 0\}.$$

where "closure" is the topological closure operator of a set. Thus, considering that $[\varphi_A \le c] = \{x \in \operatorname{int}(\mathcal{K}) \ : \ \langle A_c x, x\rangle \le 0\})$, we obtain that

$$\operatorname{closure}([\varphi_A \le c]) = \{x \in \mathcal{K} \ : \ \langle A_c x, x\rangle \le 0\},$$

Taking into account that $[\varphi_A \le c]$ is convex, the set $\operatorname{closure}([\varphi_A \le c])$ is also convex. Therefore, last equality implies that the set in (15) is convex. □

4.1. Spherically quasi-convex quadratic functions on the spherical positive orthant

In this section we present some properties of a quadratic function defined in the spherical positive orthant, which corresponds to $\mathcal{K} = \mathbb{R}^n_+$. We know that if A

has only one eigenvalue, then q_A is constant and, consequently, it is spherically quasi-convex. Therefore, *throughout this section we assume that A has at least two distinct eigenvalues. The domains $\mathcal{C}$ and $\mathrm{int}(\mathcal{K})$ of q_A and φ_A, respectively are given by*

$$\mathcal{C} := \mathbb{S}^{n-1} \cap \mathbb{R}^n_{++}, \qquad \mathrm{int}(\mathcal{K}) := \mathbb{R}^n_{++}, \tag{16}$$

We recall that q_A and φ_A are defined in (12) *and* (13)*, respectively.* Next we present a technical lemma which will be useful in the sequel.

Lemma 14. *Let $n \geq 2$ and $V = [v^1\ v^2\ v^3\ \cdots\ v^n] \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, $A = V^\top \Lambda V$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Assume that $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$. If $v^1 \in \mathbb{R}^n_+$ and $c \notin [\lambda_2, \lambda_n)$ then the sublevel set $[\varphi_A \leq c]$ is convex.*

Proof. By using that $VV^\top = I_n$ and $A = V^\top \Lambda V$ we obtain from the definition (13) that

$$[\varphi_A \leq c] = \left\{ x \in \mathbb{R}^n_{++} :\ \sum_{i=1}^{n} (\lambda_i - c)\langle v^i, x\rangle^2 \leq 0 \right\}. \tag{17}$$

We will show that $[\varphi_A \leq c]$ is convex for all $c \notin [\lambda_2, \lambda_n)$. If $c < \lambda_1$, then since $v^1, v^2, \ldots, v^n$ are linearly independent, we conclude from (17) that $[\varphi_A \leq c] = \{0\}$ and therefore it is convex. If $c = \lambda_1$, then from (17) we conclude that $[\varphi_A \leq c] = \mathcal{S} \cap \mathbb{R}^n_{++}$, where $\mathcal{S} := \{x \in \mathbb{R}^n\ : \langle v^2, x\rangle = 0,\ \ldots,\ \langle v^n, x\rangle = 0\}$, and hence $[\varphi_A \leq c]$ is convex. Assume that $\lambda_1 < c < \lambda_2$. By letting $y = V^\top x$, i.e., $y_i = \langle v^i, x\rangle$, for $i = 1, \ldots, n$, and since $v^1 \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n_{++}$, we have $y_1 > 0$ and from (17) we obtain that $[\varphi_A \leq c] = \mathcal{L} \cap V^\top \mathbb{R}^n_{++}$, where

$$\mathcal{L} := \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n :\ y_1 \geq \sqrt{\theta_2 y_2^2 + \ldots + \theta_n y_n^2} \right\},$$
$$\theta_i = \frac{\lambda_i - c}{c - \lambda_1}, \qquad i = 2, \ldots, n.$$

Since $\mathcal{L}$ and $V^\top \mathbb{R}^n_{++}$ are convex sets, we conclude that $[\varphi_A \leq c]$ is convex. If $c \geq \lambda_n$, then $[\varphi_A \leq c] = \mathbb{R}^n_{++}$ is convex, which concludes the proof. □

Lemma 15. *Let λ be an eigenvalue of A. If $\lambda I_n - A$ is copositive and $\lambda \leq c$, then*

$$[\varphi_A \leq c] = \mathbb{R}^n_{++}$$

and consequently it is a convex set.

Proof. Let $c \in \mathbb{R}$ and $[\varphi_A \leq c] = \{x \in \mathbb{R}^n_{++} : \langle Ax, x\rangle - c\|x\|^2 \leq 0\}$. Since $\lambda \leq c$ and $\lambda I_n - A$ is copositive, we have $\langle Ax, x\rangle - c\|x\|^2 \leq \langle Ax, x\rangle - \lambda\|x\|^2 = \langle (A - \lambda I_n)x, x\rangle \leq 0$ for all $x \in \mathbb{R}^n_{++}$, which implies that $[\varphi_A \leq c] = \mathbb{R}^n_{++}$. $\square$

The next theorem exhibits a series of implications and, in particular, conditions which imply that the quadratic function q_A is spherically quasi-convex.

Theorem 16. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ its eigenvalues. Consider the following statements:*

(i) q_A is spherically quasi-convex.

(ii) A is a Z-matrix.

(iii) $\lambda_2 I_n - A$ is copositive and there exists an eigenvector $v^1 \in \mathbb{R}^n_+$ of A corresponding to the eigenvalue λ_1 of A.

(iv) A is a Z-matrix and $\lambda_2 \geq a_{ii}$ for any $i \in \{1, 2, \ldots, n\}$.

(v) A is a Z-matrix, $\lambda_1 < \lambda_2$ and $\lambda_2 \geq a_{ii}$ for all $i \in \{1, 2, \ldots, n\}$.

(vi) A is an irreducible Z-matrix and $\lambda_2 \geq a_{ii}$ for all $i \in \{1, 2, \ldots, n\}$.

Then the following implications hold:

$$\begin{array}{ccccccc} & & \text{(v)} & & & & \\ & & \Downarrow & & & & \\ \text{(iv)} & \Leftarrow & \text{(iii)} & \Rightarrow & \text{(i)} & \Rightarrow & \text{(ii)} \\ & & \Uparrow & & & & \\ & & \text{(vi)} & & & & \end{array}$$

Proof.

(v)$\Rightarrow$(iii)$\Leftarrow$(vi): It is easy to verify that $\lambda_2 I_n - A$ is nonnegative and hence copositive. Moreover, Perron-Frobenius theorem applied to the matrix $\lambda_2 I_n - A$ implies that there exists an eigenvector $v^1 \in \mathbb{R}^n_+$ corresponding to the largest eigenvalue $\lambda_2 - \lambda_1$ of $\lambda_2 I_n - A$, which is also the eigenvector of A corresponding to λ_1.

(iii)⇒(i): If $c \leq \lambda_2$, then Lemma 14 implies that $[\varphi_A \leq c]$ is convex. If $c \geq \lambda_2$, then from Lemma 15 we have $[\varphi_A \leq c] = \mathbb{R}^n_{++}$, which is convex. Hence, $[\varphi_A \leq c]$ is convex for all $c \in \mathbb{R}$. Therefore, by using Theorem 12, we conclude that q_A is spherically quasi-convex function.

(i)⇒(ii): From Corollary 1, it follows that A has the $\mathbb{R}^n_+$-Z-property. It is easy to check that this is equivalent to A being a Z-matrix.

(iii)⇒(iv): Since (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii), it follows that A is a Z-matrix. Since $\lambda_2 I_n - A$ is copositive it follows that its diagonal elements are nonnegative. Hence, $\lambda_2 \geq a_{ii}$ for all $i \in \{1, 2, \ldots, n\}$.

□

Corollary 17. *Let $n \geq 2$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the eigenvalues of A. Assume that $-A$ is a positive matrix, $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$ and $0 < \lambda_2$. Then q_A is spherically quasi-convex.*

Proof. First note that the matrix $\lambda_2 I_n - A$ is a positive matrix and $\lambda_2 - \lambda_1 > 0$ is its largest eigenvalue. Thus, Theorem 1 implies that the eigenvalue $\lambda_2 - \lambda_1$ has the associated eigenvector $v^1 \in \mathbb{R}^n_{++}$. Since $(\lambda_2 I_n - A)v^1 = (\lambda_2 - \lambda_1)v^1$ we have $Av^1 = \lambda_1 v^1$. Hence v^1 is also an eigenvector of A associated to λ_1. Therefore, considering that A is a Z-matrix, $v^1 \in \mathbb{R}^n_+$, $\lambda_1 < \lambda_2$ and $\lambda_2 \geq a_{ii}$ for all $i \in \{1, 2 \ldots, n\}$, it follows from Theorem 16 (v)⇒(i) that q_A is spherically quasi-convex. □

In the following two examples we use Theorem 16 (iii)⇒(i) to present a class of quadratic quasi-convex functions defined in the spherical positive orthant.

Example 18. *Let $n \geq 3$ and $V = [v^1 \ v^2 \ v^3 \ \cdots \ v^n] \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, $A = V^\top \Lambda V$ and $\Lambda := \mathrm{diag}(\lambda, \mu, \ldots, \mu, \nu)$, where $\lambda, \mu, \nu \in \mathbb{R}$. Then q_A is a spherically quasi-convex, whenever*

$$v^1 - \sqrt{\frac{\nu - \mu}{\mu - \lambda}}|v^n| \in \mathbb{R}^n_+, \qquad \lambda < \mu < \nu, \tag{18}$$

where $|v^n| := (|v^n_1|, \ldots, |v^n_n|)$. Indeed, by using that $VV^\top = I_n$ and $A = V^\top \Lambda V$, *after some calculations we conclude that*

$$\langle Ax, x\rangle - \mu\|x\|^2 = (\mu - \lambda)\left[-\langle v^1, x\rangle^2 + \frac{\nu-\mu}{\mu-\lambda}\langle v^n, x\rangle^2\right]. \tag{19}$$

Thus, using the condition in (18) *and considering that* $x \in \mathbb{R}^n_{++}$*, we have*

$$-\langle v^1, x\rangle^2 + \frac{\nu-\mu}{\mu-\lambda}\langle v^n, x\rangle^2 \leq \frac{\nu-\mu}{\mu-\lambda}\left[-\langle |v^n|, x\rangle^2 + \langle v^n, x\rangle^2\right] \leq 0.$$

Hence, by combining the last inequality with (19)*, we conclude that* $\mu I_n - A$ *is copositive. Therefore, since* $v^1 \in \mathbb{R}^n_+$ *we can apply Theorem 16 (iii)⇒(i) with* $\lambda_2 = \mu$ *to conclude that* q_A *is a spherically quasi-convex function. For instance, taking* $\lambda < (\lambda+\nu)/2 < \mu < \nu$ *the vectors* $v^1 = (e^1 + e^n)/\sqrt{2}, v^2 = e^2, \ \ldots, \ v^{n-1} = e^{n-1}, v^n = (e^1 - e^n)/\sqrt{2}$*, satisfy* (18)*.*

Example 19. *Let* $n \geq 3$ *and* $V = [v^1 \ v^2 \ v^3 \ \cdots \ v^n] \in \mathbb{R}^{n\times n}$ *be an orthogonal matrix,* $A = V^\top \Lambda V$ *and* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$*. Then* q_A *is a spherically quasi-convex, whenever*

$$v^1 = (v^1_1, \ldots, v^1_n) \in \mathbb{R}^n_{++}, \qquad \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_2 + \frac{\alpha^2}{(n-2)}(\lambda_2 - \lambda_1), \tag{20}$$

where $\alpha := \min\{v^1_i \neq 0 : \ i = 1, \ldots, n\}$*. Indeed, by using that* $VV^\top = I_n$ *and the definition of the matrix* A*, we obtain that*

$$\langle Ax, x\rangle - \lambda_2\|x\|^2 = (\lambda_1 - \lambda_2)\langle v^1, x\rangle^2 + (\lambda_3 - \lambda_2)\langle v^3, x\rangle^2 + \cdots + (\lambda_n - \lambda_2)\langle v^n, x\rangle^2.$$

Since $\lambda_2 - \lambda_1 > 0$ *and* $0 \leq \lambda_j - \lambda_2 \leq \lambda_n - \lambda_2$ *for all* $j = 3, \ldots, n$*, the last equality becomes*

$$\langle Ax, x\rangle - \lambda_2\|x\|^2 \leq (\lambda_2 - \lambda_1)\left[-\langle v^1, x\rangle^2 + \frac{\lambda_n - \lambda_2}{\lambda_2 - \lambda_1}\left(\langle v^3, x\rangle^2 + \cdots + \langle v^n, x\rangle^2\right)\right]. \tag{21}$$

On the other hand, by using that $v^1_i \in \mathbb{R}_{++}$ *and* $v^1_i \geq \alpha$ *for all* $i = 1, \ldots, n$*, we obtain that*

$$\begin{aligned}\langle v^1, x\rangle^2 &= (v^1_1 x_1 + \cdots + v^1_n x_n)^2 \\ &\geq \alpha^2(x_1 + \cdots + x_n)^2 \geq \alpha^2(x_1^2 + \cdots + x_n^2) = \alpha^2\|x\|^2 \end{aligned} \tag{22}$$

for all $x \in \mathbb{R}^n_+$*. Moreover, taking into account that* $\|v^j\| = 1$ *for all* $j = 3, \ldots, n$*, it follows from the Cauchy-Schwarz inequality, that*

$$\langle v^3, x\rangle^2 + \cdots + \langle v^n, x\rangle^2 \leq \|v^3\|^2\|x\|^2 + \cdots + \|v^n\|^2\|x\|^2 \leq (n-2)\|x\|^2$$

for all $x \in \mathbb{R}^n_+$*. Thus, combining the last inequalities with* (21) *and* (22) *and considering that the last inequality in* (20) *is equivalent to* $-\alpha^2 + (n-2)(\lambda_n - \lambda_2)/(\lambda_2 - \lambda_1) \leq 0$*, we have*

$$\langle Ax, x\rangle - \lambda_2\|x\|^2 \leq (\lambda_2 - \lambda_1)\left[-\alpha^2 + (n-2)\frac{\lambda_n - \lambda_2}{\lambda_2 - \lambda_1}\right]\|x\|^2 \leq 0$$

for all $x \in \mathbb{R}^n_+$*. Hence, we conclude that* $\lambda_2 I_n - A$ *is copositive. Therefore, since* $v^1 \in \mathbb{R}^n_+$ *is the eigenvector of* A *corresponding to the eigenvalue* λ_1*, we apply Theorem 16 (iii)⇒(i), to conclude that* q_A *is spherically quasi-convex function. For instance,* $n \geq 3$*,* $A = V^\top \Lambda V$*,* $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ *and* $V = [v^1\ v^2\ v^3\ \cdots\ v^n] \in \mathbb{R}^{n\times n}$*, where* $\alpha = 1/\sqrt{n}$*,*

$$v^1 := \frac{1}{\sqrt{n}}\sum_{i=1}^{n} e^i, \quad v^j := \frac{1}{\sqrt{(n+1-j) + (n+1-j)^2}}\left[e^1 - (n+1-j)e^j + \sum_{i>j}^{n} e^i\right],$$

for $j = 2, \ldots, n$ *and* $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n < \lambda_2 + (1/[n(n-2)])(\lambda_2 - \lambda_1)$*, satisfy the orthogonality of* V *and the condition* (20)*.*

In the next theorem we establish the characterization for quasi-convex quadratic functions q_A on the spherical positive orthant where A is symmetric having only two distinct eigenvalues.

Theorem 20. *Let* $n \geq 3$ *and* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix with only two distinct eigenvalues, such that its smallest one has multiplicity one. Then,* q_A *is spherically quasi-convex if and only if there is an eigenvector of* A *corresponding to the smallest eigenvalue with all components nonnegative.*

Proof. Let $A := (a_{ij}) \in \mathbb{R}^{n\times n}$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A corresponding to an orthonormal set of eigenvectors $v^1, v^2, \ldots, v^n$, respectively. Then, we can assume without lose of generality that $\lambda_1 =: \lambda < \mu := \lambda_2 = \cdots = \lambda_n$. Thus, we have

$$A = V\Lambda V^T, \qquad V := [v^1\ v^2\ \ldots\ v^n] \in \mathbb{R}^{n\times n}, \qquad \Lambda := \mathrm{diag}(\lambda, \mu, \ldots, \mu) \in \mathbb{R}^{n\times n}. \tag{23}$$

First we assume that q_A is a spherically quasi-convex function. The matrix Λ can be equivalently written as follows

$$\Lambda = \mu I_n + (\lambda - \mu)D, \tag{24}$$

where $D := (d_{ij}) \in \mathbb{R}^{n\times n}$ has all entries 0 except the d_{11} entry which is 1. Then (24) and (23) imply

$$a_{ij} = (\lambda - \mu)v_i^1 v_j^1 \qquad i \neq j. \tag{25}$$

Since q_A is spherically quasi-convex and $e^i \in \mathcal{C} = \mathbb{S}^{n-1} \cap \mathbb{R}^n_{++}$ for all $i = 1, \ldots, n$, by using item (b) of Proposition 10 we conclude that $a_{ij} \leq 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Thus, since $\lambda < \mu$, we obtain form (25) that $0 \leq v_i^1 v_j^1$ for all $i \neq j$, which implies $v^1 \in \mathbb{R}^n_+$ or $-v^1 \in \mathbb{R}^n_+$. Therefore, there is an eigenvector corresponding to the smallest eigenvalue with all components nonnegative. Conversely, assume that $v^1 \in \mathbb{R}^n_+$. Then, applying Lemma 14 with $\lambda = \lambda_1 < \mu = \lambda_2 = \cdots = \lambda_n$ we conclude that $[\varphi_A \leq c]$ is convex for all $c \in \mathbb{R}$, and hence φ_A is quasi-convex. Therefore, by using Theorem 12, we conclude that q_A is spherically quasi-convex. □

In the following example we present a class of matrices satisfying the assumptions of Theorem 20.

Example 21. *Let $v \in \mathbb{R}^n_+$ and define the Householder matrix $H := I_n - 2vv^T/\|v\|^2$. The matrix H is nonsingular and symmetric. Moreover, $Hv = -v$ and letting $E := \{u \in \mathbb{R}^n \ : \ \langle v, u\rangle = 0\}$ we have $Hu = u$ for all $u \in E$. Since the dimension of E is $n-1$, we conclude that -1 and 1 are eigenvalues of H with multiplicities one and $n-1$, respectively. Furthermore, the eigenvector corresponding to the smallest eigenvalue of H has all components nonnegative. Therefore, Theorem 20 implies that $q_H(x) = \langle Hx, x\rangle$ is spherically quasi-convex.*

In order to give a *complete characterization of the spherical quasi-convexity of q_A for the case when A is diagonal*, in the following result we start with a necessary condition for q_A to be spherically quasi-convex on the spherical positive orthant.

Lemma 22. *Let $n \geq 3$, $\mathcal{C} = \mathbb{S}^{n-1} \cap \mathbb{R}^n_{++}$ and $A \in \mathbb{R}^{n \times n}$ be a nonsingular diagonal matrix. If q_A is spherically quasi-convex, then A has only two distinct eigenvalues, such that its smallest one has multiplicity one.*

Proof. The proof will be made by contradiction. First we assume that A has at least three distinct eigenvalues, among which exactly two are negative, or at least two distinct eigenvalues, among which exactly one is negative and has multiplicity greater than one, i.e.,

$$Ae^1 = -\lambda_1 e^1, \qquad Ae^2 = -\lambda_2 e^2, \quad Ae^3 = \lambda_3 e^3, \qquad \lambda_1, \lambda_2, \lambda_3 > 0 \tag{26}$$

with $-\lambda_1 < -\lambda_2 < 0 < \lambda_3$ or $-\lambda_1 = -\lambda_2 < 0 < \lambda_3$ and e^1, e^2, e^3 are canonical vectors of $\mathbb{R}^n$. Define the following two auxiliary vectors

$$v^1 := e^1 + t_1 e^3, \qquad v^2 := e^2 + t_2 e^3, \qquad t_i = \sqrt{\frac{\lambda_i}{\lambda_3}}, \qquad i = 1, 2. \tag{27}$$

Hence, (26) and (27) imply that $\langle Av^1, v^1 \rangle = 0$ and $\langle Av^2, v^2 \rangle = 0$ and since $v^1, v^2 \in \mathbb{R}^n_+$, we conclude that $v^1, v^2 \in \left\{ x \in \mathbb{R}^n_+ \; : \; \langle Ax, x \rangle \leq 0 \right\}$. However, using again (26) and (27) we obtain that

$$\langle A(v^1 + v^2), v^1 + v^2 \rangle = 2\sqrt{\lambda_1 \lambda_2} > 0,$$

and then $v^1 + v^2 \notin \left\{ x \in \mathbb{R}^n_+ \; : \; \langle Ax, x \rangle \leq 0 \right\}$. Thus, $\left\{ x \in \mathbb{R}^n_+ \; : \; \langle Ax, x \rangle \leq 0 \right\}$ is not convex. Finally, assume that A has at least three distinct eigenvalues or at least two distinct ones with the smallest one having multiplicity greater than one. Let λ, μ, ν be eigenvalues of A such that $\lambda < \mu < \nu$ or $\lambda = \mu < \nu$. Take a constant $c \in \mathbb{R}$ such that $\mu < c < \nu$. Letting $A_c := A - cI_n$ we conclude that $\lambda - c, \mu - c, \nu - c$ are eigenvalues of A_c and satisfy $\lambda - c < \mu - c < 0 < \nu - c$ or $\lambda - c = \mu - c < 0 < \nu - c$. Hence, by the first part of the proof, with A_c in the role of A, we conclude that $\left\{ x \in \mathbb{R}^n_+ \; : \; \langle A_c x, x \rangle \leq 0 \right\}$ is not convex. On the other hand, due to $e^i \in \mathbb{R}^n_+$ and $\langle Ae^i, e^i \rangle = \lambda - c < 0$, for some i, we obtain that $\left\{ x \in \mathbb{R}^n_{++} \; : \; \langle A_c x, x \rangle < 0 \right\} \neq \varnothing$. Thus, applying Corollary 13 with $\mathcal{K} = \mathbb{R}^n_+$ and taking into account that $\left\{ x \in \mathbb{R}^n_+ \; : \; \langle A_c x, x \rangle \leq 0 \right\}$ is not convex, we conclude that q_A is not spherically quasi-convex, which is absurd and the proof is complete. $\square$

To make the paper self-contained we state the result of [21, Theorem 1] explicitly here:

Theorem 23. *Let* $\mathcal{C} = \mathbb{S}^{n-1} \cap \mathbb{R}^n_{++}$ *and* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix. Then* q_A *is spherically convex if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $A = \lambda I_n$*. In this case* q_A *is a constant function.*

The next result gives a full characterization for q_A to be spherically quasi-convex quadratic function on the spherical positive orthant, where A *is a diagonal matrix*. The proof of this result is a combination of Theorem 20, Lemma 22 and Theorem 23. Before presenting the result we need the following definition: A function is called *merely spherically quasi-convex* if it is spherically quasi-convex, but it is not spherically convex.

Theorem 24. *Let* $n \geq 3$ *and* $A \in \mathbb{R}^{n\times n}$ *be a nonsingular diagonal matrix. Then* q_A *is merely spherically quasi-convex if and only if* A *has only two eigenvalues, such that its smallest one has multiplicity one and has a corresponding eigenvector with all components nonnegative.*

We end this section by showing that, if a symmetric matrix A has three eigenvectors in the nonnegative orthant associated to at least two distinct eigenvalues, then the associated quadratic function q_A cannot be spherically quasi-convex.

Lemma 25. *Let* $n \geq 3$ *and* $v^1, v^2, v^3 \in \mathbb{R}^n$ *be distinct eigenvectors of a symmetric matrix* A *associated to the eigenvalues* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$*, respectively, among which at least two are distinct. If* q_A *is spherically quasi-convex, then* $v^i \notin \mathbb{R}^n_+$ *for some* $i = 1, 2, 3$*.*

Proof. Assume by contradiction that $v^1, v^2, v^3 \in \mathbb{R}^n_+$. Without loss of generality we can also assume that $\|v^i\| = 1$, for $i = 1, 2, 3$. We have three possibilities: $\lambda_1 < \lambda_2 < \lambda_3$, $\lambda_1 = \lambda_2 < \lambda_3$ or $\lambda_1 < \lambda_2 = \lambda_3$. We start by analyzing the possibilities $\lambda_1 < \lambda_2 < \lambda_3$ or $\lambda_1 = \lambda_2 < \lambda_3$. First we assume that $\lambda_1 < \lambda_2 < 0 < \lambda_3$ or $\lambda_1 = \lambda_2 < 0 < \lambda_3$. Define the following vectors

$$w^1 := v^1 + t_1 v^3, \qquad w^2 := v^2 + t_2 v^3, \qquad t_1 := \sqrt{\frac{-\lambda_1}{\lambda_3}}, \qquad t_2 := \sqrt{\frac{-\lambda_2}{\lambda_3}}. \quad (28)$$

We have $\langle v^i, v^j \rangle = 0$ for all $i, j = 1, 2, 3$ with $i \neq j$, and since

$$Av^1 = \lambda_1 v^1, \qquad Av^2 = \lambda_2 v^2, \quad Av^3 = \lambda_3 v^3, \qquad v^1, v^2, v^3 \in \mathbb{R}^n_+, \tag{29}$$

we conclude from (28) that $\langle Aw^1, w^1 \rangle = 0$ and $\langle Aw^2, w^2 \rangle = 0$. Moreover, since $v^1, v^2, v^3 \in \mathbb{R}^n_+$ we conclude that $w^1, w^2 \in \{x \in \mathbb{R}^n_+ \,:\, \langle Ax, x \rangle \leq 0\}$. On the other hand, by using (29) and (28), we obtain that

$$\langle A(w^1 + w^2), w^1 + w^2 \rangle = 2\sqrt{\lambda_1 \lambda_2} > 0,$$

and then $w^1 + w^2 \notin \{x \in \mathbb{R}^n_+ \,:\, \langle Ax, x \rangle \leq 0\}$. Thus, $\{x \in \mathbb{R}^n_+ \,:\, \langle Ax, x \rangle \leq 0\}$ is not a convex cone. For the general case, take $c \in \mathbb{R}$ such that $\lambda_2 < c < \lambda_3$. Letting $A_c := A - cI_n$ we conclude that $\lambda_1 - c, \lambda_2 - c, \lambda_3 - c$ are eigenvalues of A_c and satisfying $\lambda_1 - c < \lambda_2 - c < 0 < \lambda_3 - c$ or $\lambda_1 - c = \lambda_2 - c < 0 < \lambda_3 - c$ with the three corresponding orthonormal eigenvectors $v^1, v^2, v^3 \in \mathbb{R}^n_+$. Hence, by the first part of the proof, with A_c in the role of A, we conclude that the cone $\{x \in \mathbb{R}^n_+ \,:\, \langle A_c x, x \rangle \leq 0\}$ is not convex. On the other hand, due to $v^1 \in \mathbb{R}^n_+$ and $\langle Av^1, v^1 \rangle = \lambda_1 - c < 0$, we have $\{x \in \mathbb{R}^n_{++} \,:\, \langle A_c x, x \rangle < 0\} \neq \varnothing$. Thus, applying Corollary 13 with $\mathcal{K} = \mathbb{R}^n_+$ and taking into account that $\{x \in \mathbb{R}^n_+ \,:\, \langle A_c x, x \rangle \leq 0\}$ is not convex, we conclude that q_A is not spherically quasi-convex, which is absurd. To analyze the possibility $\lambda_1 < \lambda_2 = \lambda_3$, first assume that $\lambda_1 < 0 < \lambda_2 = \lambda_3$ and define the vectors

$$w^1 := t_1 v^1 + v^3, \qquad w^2 := t_2 v^1 + v^3, \qquad t_1 = \sqrt{\frac{\lambda_2}{-\lambda_1}}, \qquad t_2 = \sqrt{\frac{\lambda_3}{-\lambda_1}},$$

and then proceed as above to obtain again a contradiction. Therefore, $v^i \notin \mathbb{R}^n_+$ for some $i = 1, 2, 3$ and the proof is complete. □

5. Final remarks

This paper is a continuation of [4, 19, 21], where intrinsic properties of the spherically convex sets and functions were studied. As far as we know this is the first study of spherically quasi-convex quadratic functions. As an even more challenging problem, we will work towards developing efficient algorithms for constrained optimization on spherically convex sets. Minimizing a quadratic function on the spherical nonnegative orthant is of particular interest because the nonnegativity of the minimum value is equivalent to the copositivity of the corresponding matrix [6, Proposition 1.3] and to the nonnegativity of its Pareto eigenvalues [6, Theorem 4.3]. Considering the intrinsic geometrical properties of the sphere will open interesting perspectives for detecting the copositivity of a matrix. We foresee further progress in these topics in the near future.

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[1] G. Isac, S. Z. Németh, Scalar and asymptotic scalar derivatives, Vol. 13 of Springer Optimization and Its Applications, Springer, New York, 2008, theory and applications.

[2] G. Isac, S. Z. Németh, Duality in nonlinear complementarity theory by using inversions and scalar derivatives, Math. Inequal. Appl. 9 (4) (2006) 781–795.

[3] S. Z. Németh, Scalar derivatives in Hilbert spaces, Positivity 10 (2) (2006) 299–314.

[4] O. P. Ferreira, A. N. Iusem, S. Z. Németh, Concepts and techniques of optimization on the sphere, TOP 22 (3) (2014) 1148–1170.

[5] S. T. Smith, Optimization techniques on Riemannian manifolds, in: Hamiltonian and gradient flows, algorithms and control, Vol. 3 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1994, pp. 113–136.

[6] J.-B. Hiriart-Urruty, A. Seeger, A variational approach to copositive matrices, SIAM Rev. 52 (4) (2010) 593–629.

[7] J. Malick, The spherical constraint in Boolean quadratic programs, J. Global Optim. 39 (4) (2007) 609–622.

[8] W. W. Hager, Minimizing a quadratic over a sphere, SIAM J. Optim. 12 (1) (2001) 188–208.

[9] W. W. Hager, S. Park, Global convergence of SSM for minimizing a quadratic over a sphere, Math. Comp. 74 (251) (2005) 1413–1423.

[10] A. M.-C. So, Deterministic approximation algorithms for sphere constrained homogeneous polynomial optimization problems, Math. Program. 129 (2, Ser. B) (2011) 357–382.

[11] L. Zhang, On the convergence of a modified algorithm for the spherical facility location problem, Oper. Res. Lett. 31 (2) (2003) 161–166.

[12] X. Zhang, C. Ling, L. Qi, The best rank-1 approximation of a symmetric tensor and related spherical optimization problems, SIAM J. Matrix Anal. Appl. 33 (3) (2012) 806–821.

[13] B. Martos, Subdefinite matrices and quadratic forms, SIAM J. Appl. Math. 17 (1969) 1215–1223.

[14] J. A. Ferland, Maximal domains of quasi-convexity and pseudo-convexity for quadratic functions, Math. Programming 3 (1972) 178–192.

[15] S. Schaible, Quasiconvex, pseudoconvex, and strictly pseudoconvex quadratic functions, J. Optim. Theory Appl. 35 (3) (1981) 303–338.

[16] S. Komlósi, Generalized convexity of a certain class of quadratic functions, Izv. Vyssh. Uchebn. Zaved. Mat. (9) (1984) 38–43.

[17] S. Karamardian, S. Schaible, J.-P. Crouzeix, Characterizations of generalized monotone maps, J. Optim. Theory Appl. 76 (3) (1993) 399–413.

[18] R. A. Horn, C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1985.

[19] O. P. Ferreira, A. N. Iusem, S. Z. Németh, Projections onto convex sets on the sphere, J. Global Optim. 57 (3) (2013) 663–676.

[20] S. Z. Németh, Five kinds of monotone vector fields, Pure Math. Appl. 9 (3-4) (1998) 417–428.

[21] O. P. Ferreira, S. Z. Németh, On the spherical convexity of quadratic functions, J. Glob. Optim. (2018). https://doi.org/10.1007/s10898-018-0710-6.