Large scale stochastic inventory routing problems with split delivery and service level constraints

Yugang Yu · Chengbin Chu · Haoxun Chen · Feng Chu

Abstract A stochastic inventory routing problem (SIRP) is typically the combination of stochastic inventory control problems and NP-hard vehicle routing problems, which determines delivery volumes to the customers that the depot serves in each period, and vehicle routes to deliver the volumes. This paper aims to solve a large scale multi-period SIRP with split delivery (SIRPSD) where a customer's delivery in each period can be split and satisfied by multiple vehicle routes if necessary. This paper considers SIRPSD under the multi-criteria of the total inventory and transportation costs, and the service levels of customers. The total inventory and transportation cost is considered as the objective of the problem to minimize, while the service levels of the warehouses and the customers are satisfied by some imposed constraints and can be adjusted according to practical requests. In order to tackle the SIRPSD with notorious computational complexity, we first propose an approximate model, which significantly reduces the number of decision variables compared to its corresponding exact model. We then develop a hybrid approach that combines the linearization of nonlinear constraints, the decomposition of the model into sub-models with Lagrangian relaxation, and a partial linearization approach for a sub model. A near optimal solution of the model found by the approach is used to construct a near optimal solution of the SIRPSD. Randomly

Y. Yu (✉)
Rotterdam School of Management, Erasmus University Rotterdam, Rotterdam, Netherlands
e-mail: yyugang@rsm.nl

Y. Yu
School of Management, Lanzhou University, Lanzhou, China

C. Chu
Laboratoire Génie Industriel, Ecole Centrale Paris, Paris, France

H. Chen
Industrial Systems Optimization Laboratory, Charles Delaunay Institute (FRE CNRS 2848), University of Technology of Troyes, 10010 Troyes, France

F. Chu
Laboratoire IBISC EA 4526, Université d'Evry Val d'Essonne, 40 rue du Pelvoux CE 1455, 91025 Evry Cedex, Evry, France

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generated instances of the problem with up to 200 customers and 5 periods and about 400 thousands decision variables where half of them are integer are examined by numerical experiments. Our approach can obtain high quality near optimal solutions within a reasonable amount of computation time on an ordinary PC.

Abbreviations

Indices

$i, j = 0, 1, \ldots, N$	Index of customer or depot, where $i, j = 1, \ldots, N$ are customer indexes, and 0 is the depot index,
$t = 1, \ldots, T$	Period index,

Parameters

C	Vehicle capacity in volume,
c_{ij}	Variable shipping cost per unit of product along arc (i, j) where $c_{ij} = c_{ji}$ and triangle inequality holds ($c_{ij} + c_{jk} \geq c_{ik}$),
c_{i0}^{b}	Traveling cost of an empty vehicle from customer i back directly to the depot,
f_t	Fixed vehicle cost per tour in period t,
h_{it}	Holding cost per unit product for customer i in period t,
I_{i0}	Initial inventory level at the beginning of period 1,
I_{it}	Inventory level of customer i at the end of period t,
$I_{it}^{+} = \max(0, I_{it})$	On-hand inventory of customer i at the end of period t, which excludes the stockout of $I_{it} < 0$,
V_i	The inventory capacity for customer i's warehouse,
α_{it}	Service level for customer i's demand in period t (probability that customer i's demand is satisfied in period t),
β_{it}	The service level of customer i's warehouse in period t (probability that customer i's warehouse is not overfilled in period t),
ζ_{it}	Stochastic demand of customer i in period t,
$\zeta_{i,(1,t)}$	$= \sum_{s=1}^{t} \zeta_{is}$, cumulative stochastic demand from period 1 to t,
$F_{i,(1,t)}(.)$	Accumulative probability distribution function of stochastic demand $\zeta_{i,(1,t)}$,

Decision variables

d_{it}	Delivery volume to customer i in period t,
q_{ijt}	Demand quantity transported on directed arc (i, j) in period t,
x_{ijt}	Number of the times that customer j is visited directly after customer i in period t.

1 Introduction

The inventory routing problem (IRP) is frequently found in a vendor managed inventory (VMI) system with one central vendor (depot) and multiple geographically dispersed customers. The depot operates vehicles with limited capacity for distributing products to its customers. The IRP aims to determine the delivery volume for every customer and

a set of feasible vehicle routes delivering the volumes in each period so that a system-wide total inventory and transportation cost is minimized. Such problems are common in VMI systems that have been adopted in many firms like P&G, Dell, HP, Barilla, Wal-Mart (Yu et al., 2009a, 2009b) and Air Products for gas distribution (Adelman 2004; Bell et al. 1983).

This paper focuses on a multiple-period stochastic inventory routing problem with split delivery (SIRPSD) where the depot has a fleet of homogeneous capacitated vehicles, and customers' demands are stochastic in each period (e.g. every day). The primary objective is the minimization of the total inventory and transportation cost. However, due to the stochastic nature of customer demands, the other criteria about service levels have to be satisfied by limiting the stockout probability at each customer and the probability of overfilling the stock of each customer. To consider these criteria simultaneously, various methods may be used; such as integrating multiple criteria in one objective function by using data development analysis (DEA) and analytic hierarchy process (AHP) (Koksalan and Tuncer 2009; Li and Ma 2008; Peng et al. 2008; Tchangani 2009). Alternatively, one can choose one criterion as the primary objective function to minimize, while the other secondary criteria are left as model constraints and analyzed later by sensitivity analysis. Motivated by the previous work of Chen (2007) on production planning, this paper selects the alternative method. Moreover, this research takes into consideration split delivery that allows the delivery volume to a customer in a period to be split and served by multiple vehicle routes. This is common in practice.

SIRP is a class of notoriously difficult problems and related literature can be classified in three categories: those who study a single period problem, those who study an infinite-period problem, and those who study a finite multi-period problem. A single period problem is firstly studied in Federgruen and Zipkin (1984) which considers the corresponding inventory control problem as a newsvendor problem and the corresponding routing problem as a TSP. Dror and Ball (1987) develop a heuristic technique to reduce a long-run problem to a single period problem. The infinite-period SIRP is mainly stemmed from Kleywegt et al. (2002, 2004) and Adelman (2004). They formulate the problem as a Markov decision problem (MDP) over an infinite horizon where dynamic programming can be applied to solve the problem. To make the MDP solvable, they assume that the state of the system can be observed at the beginning of each period so that an action can be taken, and that customer demands are observed after any action has been made. The state of the system is described through the inventory levels of all the customers. They aim to determine a policy for the MDP to minimize the expected total inventory and transportation cost plus possible revenue gained for each delivery over an infinite horizon. Further extensions of those researches can be found in Hvattum et al. (2009), and Lejeune and Ruszczynski (2007).

For multi-period SIRP, Trudeau and Dror (1992) and Dror and Trudeau (1996) consider stochastic demands over a rolling horizon. Both papers solve a slightly different model with a specific application to the distribution of oil and gas. In these models, a product has to be delivered from one depot to many customers whose demand is different in each period. Trudeau and Dror (1992) develop heuristics to solve their problems by minimizing the long-run average transportation costs, and Dror and Trudeau (1996) focus on maximizing operational efficiency (average number of units delivered in one hour of operation) and minimizing the average number of stockout in each period. Similar literature can be found in Jaillet et al. (2002) and Schwarz et al. (2006). The main differences between the above cited papers and our paper lie in:

(1) We consider split delivery in a stochastic inventory routing problem. That is, one customer's delivery volume can be satisfied by multiple vehicle routes. Split delivery is

extensively studied in pure VRP literature. Recently, Archetti et al. (2006, 2008) have proposed optimization-based heuristics and meta-heuristic approaches able to find very good feasible solutions of large SDVRP instances (up to 200 customers). A column generation approach improving the best lower and upper bounds for the instances is proposed by Belenguer et al. (2000). A method to find good solutions for instances with large average demand has been proposed by Jin et al. (2007). Chen et al. (2007) develop a heuristic algorithm combining a mixed integer program and a record-to-record travel algorithm that can produce very high-quality solutions for 21 new instances with 8 to 288 customers. However, split delivery is rarely considered in IRP, especially in SIRP.

(2) Besides considering the total cost, we consider the criteria of service levels for customer demands and warehouses, which are rarely treated in SIRP related literature.

(3) We consider stochastic demands of any probability distribution with the help of model simplification and a nonlinear objective function. The demands of each customer in multiple periods can be correlated.

This paper is a stochastic version of our previous work Yu et al. (2008) on a deterministic IRP, but they have distinct differences in modeling and solution approaches: (1) the objective function is nonlinear in this paper but linear in Yu et al. (2008). (2) the constraints of the two models are different; some constraints on the service levels are considered in this paper and they are nonlinear. Therefore, a new approach has to be developed to deal with those stochastic and nonlinear components. We borrow some ideas about how to construct a near optimal solution of the SIRPSD from its model's solution from our previous paper, but the construction approach has to be adapted to the new features of the SIRPSD and its model.

The contributions of the paper include: (1) study a new SIRPSD where the service levels of customers' demands and warehouses are considered, (2) propose a hybrid approach to find near-optimal solutions of the SIRPSD for large instances (i.e., with 200 customers). In order to efficiently solve such kind of large instances of the problem, we introduced the following approaches.

Firstly, we propose an approximate stochastic IRP model instead of an exact stochastic model. The approximate model allows us not to dedicate decision variables to individual vehicles since the vehicles are homogeneous. This can significantly reduce the number of decision variables. For example, if the vehicle fleet size is 20, our modeling only requires 1/20 vehicle related decision variables compared with an exact model where decision variables are dedicated to individual vehicles. Although the solution of such an approximate model might not be a feasible solution of the studied SIRPSD, the infeasibility can be effectively repaired without affecting solution quality. Moreover the optimal solution of the approximate model provides a lower bound of the optimal cost of our studied SIRPSD.

Secondly, we transform the approximate stochastic model into a simplified deterministic model which is easier to solve. Meanwhile some constraints are eliminated and the domains of some decision variables are reduced without losing optimal solutions (see Sect. 2.2).

Thirdly, we develop a Lagrangian relaxation approach to decompose the model into sub-models, which are an inventory problem and a vehicle routing problem, respectively. The inventory problem is nonlinear and is solved by a partial linearization approach. The routing problem is further decomposed into many smaller subproblems which can be quickly solved.

Finally, assignment problems are introduced to construct a feasible solution of the SIRPSD from its model's solution. Some local search improvements are also proposed to improve the quality of the found feasible solution of the SIRPSD.

Besides, our approach can provide a tight lower bound of the optimal cost of the studied SIRPSD for evaluating the quality of a feasible solution of the SIRPSD. The introduced Lagrangian relaxation approach can provide a lower bound of the approximate model, and then

a lower bound of the studied SIRPSD. The quality of the feasible solution of the SIRPSD can therefore be evaluated by the gap between the cost of the found solution of the SIRPSD and the lower bound given by the Lagrangian relaxation approach.

The rest of this paper is organized as follows. In Sect. 2, the approximate model is proposed and simplified. The near optimal solution of the approximate model is found in Sect. 3, based on which a near optimal solution of the studied SIRPSD is found in Sect. 4 and 5. In Sect. 6, the problem with a special probability distribution of customer demand is analyzed and the performance of our proposed approach is evaluated. Section 7 concludes the paper.

2 Approximate generic model and its simplification

The studied multi-period SIRPSD consists of a central depot, multiple customers, and a fleet of vehicles, where

(1) Each customer's demand is stochastic in each period, and the customers require the depot to satisfy their demands with a certain service level by limiting the possibility of stockout within a given value. The demand in each period is stochastic and obeys a given probability distribution. The exact demand of each customer is therefore unknown before its realization. Unsatisfied customer demand in each period can be backlogged but be limited with service level constraints.

(2) The depot is responsible for delivering a product to its customers to satisfy their requirements on demand and service level by a fleet of homogeneous and capacitated vehicles. Note that assuming homogeneous vehicles is realistic in practice and common in the literature (Fumero and Vercellis 1999; Hvattum et al. 2009; Yu et al. 2008). The fleet size of vehicles and the number of tours of a vehicle performed in each period are not constrained to minimize the total cost, but the decision results of the proposed model of the problem.

(3) A multi-period horizon is considered. Periodically, the depot has to make a plan for a given number of future periods about when and how much every customer should be replenished. Because the demands of each customer are stochastic but its delivery volumes for the given periods over the time horizon have to be determined at the beginning of the first period (which is also the current period), the customer's warehouse may be overfilled in the next period if the demand in the current period is low but the delivery volume for the next period is high. We therefore have to consider the service levels of the warehouses in order to model the limited overfilling possibility.

(4) Split delivery is allowed. In practice, if a customer's demand is large, the delivery volume of the customer is most likely to be served by multiple vehicle routes.

(5) The objective is to minimize the total inventory and transportation cost over a given time horizon subject to given service level constraints. The inventory cost depends on the inventory level of each customer at the end of each period. The transportation cost includes not only fixed costs such as vehicle depreciation, and drivers' rewards, but also variable costs that depend on transported quantity and traveled distance of a vehicle. This transportation cost structure, adopted by Fumero and Vercellis (1999), can not only model purely distance proportional cost components (such as fuel costs) in classical VRP but also model the transportation cost in the third party logistics where the transportation cost charged is usually proportional to the shipped volume.

The related notations are given as follows.

2.1 Approximate model

The approximate model for the SIRPSD (denoted by **P**) can therefore be formulated as:
    Model **P**:

$$Z = \min \sum_{t=1}^{T} E\left(\sum_{i=1}^{N} h_{it} I_{it}^{+}\right) + \sum_{t=1}^{T} \sum_{i=0, i\neq j}^{N} \sum_{j=0}^{N} c_{ij} q_{ijt} + \sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t} + \sum_{t=1}^{T} \sum_{i=1}^{N} c_{i0}^{b} x_{i0t} \quad (1)$$

Subject to

$$I_{it} = I_{i,0} + \sum_{s=1}^{t} d_{is} - \sum_{s=1}^{t} \zeta_{is}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T \quad (2)$$

$$\mathrm{Prob}(I_{it} \geq 0) \geq \alpha_{it}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T \quad (3)$$

$$\mathrm{Prob}(I_{i,t-1} + d_{it} \leq V_i) \geq \beta_{it}, \quad i = 1, \ldots, N, \quad t = 2, \ldots, T \quad (4)$$

$$I_{i,0} + d_{i1} \leq V_i, \quad i = 1, \ldots, N \quad (5)$$

$$\sum_{j=0, j\neq i}^{N} x_{ijt} = \sum_{j=0, j\neq i}^{N} x_{jit}, \quad i = 0, \ldots, N, \ t = 1, \ldots, T \quad (6)$$

$$\sum_{j=0, j\neq i}^{N} q_{jit} - \sum_{j=0, j\neq i}^{N} q_{ijt} = d_{it}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T \quad (7)$$

$$\sum_{i=1}^{N} q_{0it} = \sum_{i=1}^{N} d_{it}, \quad t = 1, \ldots, T \quad (8)$$

$$q_{ijt} \leq C \cdot x_{ijt}, \quad i = 0, \ldots, N, \ j = 1, \ldots, N, \ i \neq j, \ t = 1, \ldots, T \quad (9)$$

$$d_{it} \geq 0, \quad i = 1, \ldots, N,$$

$$q_{ijt} \geq 0, \quad i = 0, \ldots, N, \ j = 0, \ldots, N, \ j \neq i, \ t = 1, \ldots, T \quad (10)$$

$$x_{ijt} \geq 0 \quad \text{and integer} \quad i, j = 0, \ldots, N, \quad i \neq j, \ t = 1, \ldots, T \quad (11)$$

Equation (1) gives the total cost including the expected inventory cost for all customers
($\sum_{t=1}^{T} E(\sum_{i=1}^{N} h_{it} I_{it}^{+})$), the variable transportation cost ($\sum_{t=1}^{T} \sum_{j=0, j\neq i}^{N} \sum_{i=0}^{N} c_{ij} q_{ijt}$), the
fixed transportation cost ($\sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t}$) and the transportation cost of empty vehicles
back to the depot from customer i, ($\sum_{t=1}^{T} \sum_{i=1}^{N} c_{i0}^{b} x_{i0t}$). In $\sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t}$, the number of
the fixed costs (f_t) counted in $\sum_{i=1}^{N} f_t x_{i0t}$ equals the number of tours ($\sum_{i=1}^{N} x_{i0t}$) in period t.
Equation (1) is a stochastic version of the cost structure in Yu et al. (2008) and Fumero
and Vercellis (1999). Constraints (2) are the inventory balance constraints for individual
customers. Constraints (3) ensure that the probability for customer i's demand satisfied in
period t is no less than α_{it} for period $t = 1, \ldots, T$, which represents the service levels of the
depot to satisfy customer demand in each period. Constraints (4) describe the service levels
related to the customers' warehouses and guarantee that the probability of customer i's
warehouse capacity being able to accommodate its maximum inventory level is no less than
β_{it} at period $t = 2, \ldots, T$. Constraints (5) ensure that every customer's warehouse inventory
capacity should be no less than its maximum inventory level in period 1. Constraints (6)
ensure that the number of vehicles leaving from a customer or the depot is equal to that

of arriving vehicles. Constraints (7) are the product flow conservation equations, ensuring flow balance at each customer and eliminating possible subtours. Constraints (8) assure the total volume shipped from the depot equals the total delivery volume of all the customers in each period. Constraints (9) model the vehicle capacity and logical relationship between q_{ijt} and x_{ijt}.

Model **P** defines some necessary conditions of a feasible solution of the SIRPSD and therefore its optimal solution provides a lower bound of the optimal cost of the SIRPSD. However, in the decision variables q_{ijt} and x_{ijt}, no subscripts are dedicated to individual vehicles, and feasible q_{ijt} and x_{ijt} of Model **P** may be infeasible for the SIRPSD (Yu et al. 2007). q_{ijt} and x_{ijt} have to be split and assigned to individual vehicles in order to make them feasible for the studied SIRPSD. In the following, we firstly simplify the model (see Sect. 2.2), and find its near optimal solutions (see Sect. 3). How to construct a near optimal solution of the studied SIRPSD will be given later in Sect. 4.

2.2 Model Simplification

Model **P** can be simplified from three aspects (1) transforming the stochastic terms in Model **P** (i.e., (1), (2), (3) and (4)) into deterministic ones, (2) adding some valid constraints to reduce the feasible domains of the decision variables, (3) simplifying some decision variables.

Transformation of the stochastic terms   The stochastic terms are in (1), (2), (3) and (4). For the objective function (1), by substituting (2) into the objective function (1), $E(\sum_{i=1}^{N} h_{it} I_{it}^{+})$ can be reformulated as $\sum_{i=1}^{N} h_{it} E(I_{it}^{+}) = \sum_{i=1}^{N} h_{it} \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x) \, dF_{i,(1,t)}(x)$. Considering $F_{i,(1,t)}(0) = 0$ in practice, we have $\sum_{i=1}^{N} h_{it} E(I_{it}^{+}) = \sum_{i=1}^{N} h_{it} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x) F_{i,(1,t)}(x)|_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} + \sum_{i=1}^{N} h_{it} \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} F_{i,(1,t)}(x) dx = \sum_{i=1}^{N} h_{it} \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} F_{i,(1,t)}(x) dx$.

For constraints (3), substituting (2) into constraints (3), we have $\text{Prob}(I_{it} \geq 0) = \text{Prob}(I_{i,0} + \sum_{s=1}^{t} d_{is} \geq \sum_{s=1}^{t} \zeta_{is}) = \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} dF_{i,(1,t)}(x) = F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is}) - F_{i,(1,t)}(0) = F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is})$ because $F_{i,(1,t)}(0) = 0$ for all practical purposes. Consequently, constraint (3) can be reformulated as $F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is}) \geq \alpha_{it}$, or equivalently as

$$\sum_{s=1}^{t} d_{is} \geq F_{i,(1,t)}^{-1}(\alpha_{it}) - I_{i,0}, \quad i = 1, \dots, N, \ t = 1, \dots, T \qquad (3')$$

For constraints (4), substituting (2) into constraints (4), $\text{Prob}(I_{i,t-1} + d_{it} \leq V_i) = \text{Prob}(I_{i,0} + \sum_{s=1}^{t-1} d_{is} - \sum_{s=1}^{t-1} \zeta_{is} + d_{it} \leq V_i) = \text{Prob}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i \leq \sum_{s=1}^{t-1} \zeta_{is}) = \int_{I_{i,0}+\sum_{s=1}^{t} d_{is}-V_i}^{+\infty} dF_{i,(1,t-1)}(x) = 1 - F_{i,(1,t-1)}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i)$. Thus constraints (4) can be formulated as $F_{i,(1,t-1)}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i) \leq 1 - \beta_{it}$, or equivalently as:

$$\sum_{s=1}^{t} d_{is} \leq V_i + F_{i,(1,t-1)}^{-1}(1 - \beta_{it}) - I_{i,0}, \quad i = 1, \dots, N, \ t = 2, \dots, T \qquad (4')$$

As the result of the above transformation, constraints (2) are removed simultaneously.

Once $F_{i,(1,t-1)}^{-1}(\cdot)$ is known, constraints (3) and (4) can then be replaced with the linear constraints of (3') and (4').

Addition of valid constraints Without proof, the optimal solution of Model **P** must satisfy constraints (12) below:

$$q_{i0t} = 0, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T \tag{12}$$

The constraints imply that each vehicle must be empty when it returns to the depot.

Simplification of decision variables With Theorem 1 below, x_{ijt} for $i, j = 1, \ldots, N$ as integer can be simplified as binary variables, $x_{ijt} \in \{0, 1\}$.

Theorem 1 *If Model **P** is feasible, and c_{ij} $i, j = 1, \ldots, N$ satisfy the triangle inequality, then the model has an optimal solution where no two routes with the same direction have more than one common customer, i.e., $x_{ijt} \in \{0, 1\}$ for $i, j = 1, \ldots, N$.*

Here Theorem 1, taken from Yu et al. (2008), is given directly without proof. Theorem 1 is proved by Dror and Trudeau (1990) in case of VRP with split delivery.

Therefore constraints (11) can be replaced with

$$x_{ijt} \in \{0, 1\}, \quad j \neq i, \quad x_{i0t}, x_{0jt} \quad \text{integer } i, j = 1, \ldots, N \tag{11$'$}$$

According to the above analysis, Model **P** can be simplified as the following equivalent model, (denoted by **P$'$**).

Model **P$'$**:

$$\min Z = \sum_{t=1}^{T} \sum_{i=1}^{N} h_{it} \int_{0}^{I_{i,0} + \sum_{s=1}^{t} d_{is}} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} - x \right) dF_{i,(1,t)}(x) + \sum_{t=1}^{T} \sum_{j=1, j \neq i}^{N} \sum_{i=0}^{N} c_{ij} q_{ijt}$$

$$+ \sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t} + \sum_{t=1}^{T} \sum_{i=1}^{N} c_{i0}^{b} x_{i0t} \tag{1$'$}$$

Subject to constraints (5)–(10), (3$'$), (4$'$), (11$'$), and (12).

3 Solution methodology of Model P$'$

Model **P$'$** is obviously NP-hard as its simplified single period problem without considering inventory is SDVRP that is NP-hard. This motivates us to seek for approximate approaches to solve the problem and Lagrangian relaxation (LR) approach is selected since it can decompose our model into easily solvable sub-problems.

In this section the Lagrangian relaxation (LR) approach to find a near optimal solution of Model **P$'$** will be presented. The solution will then be used to construct a feasible near optimal solution of the SIRPSD using a heuristic approach. The optimal dual value obtained by the Lagrangian relaxation approach provides a lower bound of the optimal cost of the studied SIRPSD for evaluating the quality of the feasible solution of SIRPSD.

3.1 Lagrangian relaxation

In Model **P$'$**, the constraints that complicate the resolution of this problem are constraints (9) which couple q_{ijt} and x_{ijt}. They are relaxed by introducing non-negative Lagrange multipliers $\lambda = (\lambda_{ijt})_{(N+1) \times N \times T}$ with a penalty term added to the objective function (1$'$). The corresponding Lagrangian relaxed problem (denoted by **RP**) can be formulated as:

Model **RP**:

$$Z_\lambda(d, q, x) = \sum_{t=1}^{T} \sum_{i=1}^{N} h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} - x \right) dF_{i,(1,t)}(x)$$

$$+ \sum_{t=1}^{T} \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} c_{ij} q_{ijt} + \sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t} + \sum_{t=1}^{T} \sum_{i=1}^{N} c_{i0}^b x_{i0t}$$

$$+ \sum_{t=1}^{T} \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} \lambda_{ijt}(q_{ijt} - C \cdot x_{ijt})$$

$$= \sum_{t=1}^{T} \sum_{i=1}^{N} h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} - x \right) dF_{i,(1,t)}(x)$$

$$+ \sum_{t=1}^{T} \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} (\lambda_{ijt} + c_{ij}) q_{ijt} + \sum_{t=1}^{T} \sum_{i=1}^{N} (c_{i0}^b + f_t) x_{i0t}$$

$$- C \sum_{t=1}^{T} \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} \lambda_{ijt} x_{ijt} \tag{13}$$

subject to $\lambda \geq 0$, (3′), (4′), (5)–(8), (10), (11′), and (12).

The problem can therefore be decomposed into the following two independent subproblems while global minimization is reserved.

The inventory subproblem (denoted by **INV**) determining the d, q values can be formulated as:

$$Z_\lambda^1(d, q) = \min \sum_{t=1}^{T} \sum_{i=1}^{N} h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} - x \right) dF_{i,(1,t)}(x)$$

$$+ \sum_{t=1}^{T} \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} (\lambda_{ijt} + c_{ij}) q_{ijt} \tag{14}$$

subject to constraints (3′), (4′), (5), (7), (8), (10) and (12).

The routing subproblem (denoted by **ROU**) determining the x values can be formulated as:

$$Z_\lambda^2(x) = \min \sum_{t=1}^{T} \sum_{i=1}^{N} (c_{i0}^b + f_t) x_{i0t} - C \sum_{t=1}^{T} \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} \lambda_{ijt} x_{ijt} \tag{15}$$

subject to (6) and (11′).

For **ROU**, it can be further decomposed into T independent subproblems, one for each period, given by:

$$Z_\lambda(x_{(t)}) = \min \sum_{i=1}^{N} (c_{i0}^b + f_t) x_{i0t} - C \sum_{j=1,j\neq i}^{N} \sum_{i=0}^{N} \lambda_{ijt} x_{ijt} \tag{16}$$

subject to constraints $(11')$ and

$$\sum_{j=0, j\neq i}^{N} x_{ijt} = \sum_{j=0, j\neq i}^{N} x_{jit}, \quad i = 0, \ldots, N \tag{17}$$

Denote $D(\lambda)$ be the function of the optimal objective value of **RP** for any given Lagrange multipliers $\{\lambda_{ijt}\}_{(N+1)\times N\times T}$. The Lagrangian dual problem (denoted by **DP**) is

Model **DP**:

$$\max_{\lambda} D(\lambda) \tag{18}$$

where $\max D(\lambda) = \max\{Z_{\lambda}(d, q, x)|$ s.t. $\lambda \geq 0$, $(3')$, $(4')$, (5)–(8), (10), $(11')$, and $(12)\}$.

For each given $\{\lambda_{ijt}\}_{(N+1)\times N\times T} > 0$, we have:

$Z_{\lambda}(d, q, x) = Z_{\lambda}^1(d, q) + Z_{\lambda}^2(x) = Z_{\lambda}^1(d, q) + \sum_{t\in T} Z_{\lambda}^{2t}(x_{(t)})$ and $Z_{\lambda}(d, q, x) \leq Z^*$, where Z^* is the optimal value of model **P'**.

3.2 Partial linearization for subproblem INV

The subproblem **INV** is a nonlinear programming problem and can not be easily solved by using a commercial software such as Lingo. Fortunately, the objective function of **INV** is obviously a convex and increasing function of d_{it} $t = 1, \ldots, T$, and all the constraints of **INV** are linear. It can be solved by using a partial linearization method proposed by Patriksson (1993) as follows.

Defining $G_{it}(d_{i1}, \ldots, d_{it}) = h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x) F'_{i,(1,t)}(x) dx$, we have

$$\sum_{t=1}^{T} \sum_{i=1}^{N} h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} - x \right) dF_{i,(1,t)}(x) = \sum_{t=1}^{T} \sum_{i=1}^{N} G_{it}(d_{i1}, \ldots, d_{it})$$

The gradient of $G_{it}(d_{i1}, \ldots, d_{it})$ with respect to d_{iv} $v = 1, \ldots, t$ is

$$\frac{\partial G_{it}(d_{i1}, \ldots, d_{iv}, \ldots, d_{it})}{\partial d_{iv}} = \frac{\partial h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x) F'_{i,(1,t)}(x) dx}{\partial d_{iv}}$$

$$= \frac{\partial h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is}) F'_{i,(1,t)}(x) dx}{\partial d_{iv}}$$

$$- \frac{\partial h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} x F'_{i,(1,t)}(x) dx}{\partial d_{iv}}$$

$$= \left[h_{it} \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} \frac{\partial (I_{i,0} + \sum_{s=1}^{t} d_{is})}{\partial d_{iv}} F'_{i,(1,t)}(x) dx \right.$$

$$\left. + h_{it} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} \right) \int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} F''_{i,(1,t)}(x) dx \right]$$

$$- h_{it} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} \right) F'_{i,(1,t)} \left(I_{i,0} + \sum_{s=1}^{t} d_{is} \right)$$

$$= h_{it} \int_0^{I_{i,0}+\sum_{s=1}^t d_{is}} F'_{i,(1,t)}(x) dx$$

$$= h_{it} F_{i,(1,t)}\left(I_{i,0} + \sum_{s=1}^t d_{is}\right), \quad v = 1, \ldots, t$$

At any point d_{it}^k, $t = 1, \ldots, T$, $i = 1, \ldots, N$, the linearization of $\sum_{t=1}^T \sum_{i=1}^N h_{it} \int_0^{I_{i,0}+\sum_{s=1}^t d_{is}}$
$(I_{i,0} + \sum_{s=1}^t d_{is} - x) dF_{i,(1,t)}(x)$ is thus $\sum_{t=1}^T \sum_{i=1}^N h_{it} \int_0^{I_{i,0}+\sum_{s=1}^t d_{is}^k} (I_{i,0} + \sum_{s=1}^t d_{is}^k - x) dF_{i,(1,t)}(x) + \sum_{t=1}^T \sum_{i=1}^N \sum_{s=1}^t (d_{is} - d_{is}^k) h_{it} F_{i,(1,t)} (I_{i,0} + \sum_{s=1}^t d_{is}^k))$.

The linearization of $Z_\lambda^1(d, q)$ at the point, denoted by $\bar{Z}^1((d, q), (d^k, q^k))$, is thus

$$\bar{Z}^1((d, q), (d^k, q^k)) = \sum_{t=1}^T \sum_{i=1}^N h_{it} \int_0^{I_{i,0}+\sum_{s=1}^t d_{is}^k} \left(I_{i,0} + \sum_{s=1}^t d_{is}^k - x\right) dF_{i,(1,t)}(x)$$

$$+ \sum_{t=1}^T \sum_{i=1}^N \sum_{s=1}^t (d_{is} - d_{is}^k) h_{it} F_{i,(1,t)}\left(I_{i,0} + \sum_{s=1}^t d_{is}^k\right)$$

$$+ \sum_{t=1}^T \sum_{j=1, j\neq i}^N \sum_{i=0}^N (\lambda_{ijt} + c_{ij}) q_{ijt}. \tag{19}$$

The partial linearization method solves a linear programming problem and performs a line search at each iteration. For our problem, at each iteration k, the method solves the following linear programming problem (denoted by IP^k):

IP^k:

$$\min \bar{Z}^1((d, q), (d^k, q^k)) \tag{20}$$

subject to constraints $(3')$, $(4')$, (5), (7), (8), (10) and (12). and performs a line search to minimize $Z_\lambda^1(d, q)$ for subproblem **INV**, i.e.,

$$\min_\rho \{Z_\lambda^1(d, q) | (d, q) = \rho(d^k, q^k) + (1 - \rho)(\bar{d}^k, \bar{q}^k), 0 \le \rho \le 1\}, \tag{21}$$

where $(\bar{d}^k, \bar{q}^k)$ is an optimal solution of IP^k. The starting point (d^{k+1}, q^{k+1}) of the iteration $k + 1$ is taken as the solution of the line search at the iteration k.

The iterative procedure continues until (d^k, q^k) also solves IP^k. That is, $\bar{Z}^1((d, q), (d^k, q^k)) = \bar{Z}^1((d, q), (\bar{d}^k, \bar{q}^k))$. Initially at $k = 0$, (d^k, q^k) is taken as a feasible solution of subproblem **INV**.

3.3 Minimum cost flow for ROU subproblem

The constraint matrix of subproblem **ROU** is totally unimodular and the right-hand side are integers, so every basic feasible solution is integral (see Wolsey 1998). In other words, solving the problem as a linear program using the simplex method always yields an integral solution. We can then relax $x_{ijt} \in \{0, 1\}$ to $0 \le x_{ijt} \le 1$, which transforms subproblem **ROU** into a minimum cost flow (MCF) problem. MCF can be solved by the scaling out-of-kilter algorithm (Ciupala 2005) in the polynomial time of $O(m(m + n \log n))$ where m, and $\underline{n}$ are the number of arcs $(N + 1) * (N + 1)$, and the number of nodes $(N + 1)$, respectively.

3.4 Subgradient method for the dual problem

The Lagrangian relaxation approach maximizes the dual objective (18) by using subgradient method. We use adaptive step sizing strategy to set the step size of the method in each iteration. The algorithm steps for the Lagrangian relaxation (Yu et al. 2005) is given as follows.

Step 0. Give an initial value $\lambda^0 = 0$, $\theta^0 = 1$ and $k = 0$
Step 1. Calculate subproblem **INV** and subproblem **ROU**.
Step 2. Calculate step size s^k in iteration k by

$$s^k = \beta(L^* - \tilde{L}^k)/\|g^k\|^2$$

where β is a parameter with $0 < \beta < 1$, $\tilde{L}^k$ is current lower bound, $\tilde{L}^{[k]}$ is the best dual obtained prior to iteration k, L^* is estimated by $(1 + \frac{\omega}{\theta^\rho})\tilde{L}^{[k]}$, where $\omega \in [0.1, 1.0]$, $\rho \in [1.1, 1.5]$, $\theta^{k+1} = \max(1, \theta^k - 1)$ (θ^k is value of θ in iteration k if $L^k > \tilde{L}^{[k]}$, otherwise $\theta^{k+1} = \theta^k + 1$), and $\|g^k\|^2 = \sum_{t=1}^{T}\sum_{j=1, j\neq i}^{N}\sum_{i=0}^{N}(q_{ijt}^k - C \cdot x_{ijt}^k)^2$.
Step 3. Calculate $\lambda_{ijt}^{k+1} = \max\{\lambda_{ijt}^k + s^k(q_{ijt}^k - C \cdot x_{ijt}^k), 0\}$.
Step 4. Check stopping criteria. The criteria may be given by

(1) $\sum_{ijt} |\lambda_{ijt}^{k+1} - \lambda_{ijt}^k| \leq \varepsilon_1$ or $\|\lambda^{k+1} - \lambda^k\| \leq \varepsilon_2$, or
(2) a given maximal iteration time reached; where ε_1 and ε_2 are given little positive numbers.

If the criterion is met, stop and output all required results. Otherwise, set $k = k + 1$ and go to Step 1.

The solution of the Lagrangian relaxation problem not only provides a lower bound of Model **P′**, then the SIRPSD, but also can be used to construct a near optimal feasible solution of Model **P′**.

3.5 Feasible solution construction for the Model **P′**

Based on d, q obtained by solving the Lagrangian relaxed problem, similar to Yu et al. (2008), a feasible solution Model **P′** can be constructed by solving the following problem, denoted by **FP**.

Model **FP**:

$$Z_\lambda(x) = \min \sum_{t=1}^{T}\sum_{i=1}^{N}(c_{i0}^b + f_t)x_{i0t} \tag{22}$$

Subject to

$$\sum_{j=0, j\neq i}^{N} x_{ijt} = \sum_{j=0, j\neq i}^{N} x_{jit}, \quad i = 0, \ldots, N, \ t = 1, \ldots, T \tag{23}$$

$$\left\lceil \frac{q_{ijt}}{C} \right\rceil \leq x_{ijt}, \quad i = 0, \ldots, N, \ j = 1, \ldots, N, \ i \neq j, \ t = 1, \ldots, T \tag{24}$$

$$x_{ijt} \in \{0, 1\}, j \neq i, \quad x_{i0t}, x_{0jt} \quad \text{integer } i, j = 1, \ldots, N, \ t = 1, \ldots, T \tag{25}$$

where q_{ijt} $i = 0, \ldots, N, \ j = 1, \ldots, N, \ i \neq j$ is obtained from the solution of the relaxed problem solved by the subgradient method in Sect. 3.4. The problem FP can be reformulated

as a minimal cost flow problem again by relaxing x to:

$$\left\lceil \frac{q_{ijt}}{C} \right\rceil \leq x_{ijt} \leq 1, \quad i = 1, \ldots, N, \ j = 1, \ldots, N, \ i \neq j, \ t = 1, \ldots, T \quad (26)$$

$$x_{0jt} \geq \left\lceil \frac{q_{0jt}}{C} \right\rceil, \quad j = 1, \ldots, N, \ t = 1, \ldots, T \quad (27)$$

The problem can be decomposed into T sub-problems, one for each period.

In order to obtain a good feasible solution of Model $\mathbf{P}'$, the feasible solution is constructed based on every solution obtained in every iteration of the subgradient method in Sect. 3.4. The best one (with the smallest total cost measured by (1′)) is selected as the final one.

4 Feasible solution construction and improvement for the SIRPSD

This section provides an approach to repair an obtained feasible solution of Model $\mathbf{P}'$ in Sect. 3.5 to a feasible solution of the studied SIRPSD, and improve it with some local search improvements.

4.1 Repair a solution of P′ to a feasible solution of SIRPSD

In Sect. 3.5, a feasible solution, d, q, x of Model $\mathbf{P}'$, is obtained, but this solution is not implementable since q has not been allocated to individual vehicles for constructing feasible routes. Moreover, the solution may define an infeasible solution to the original problem SIRPSD (Yu et al. 2007). An example will be given later in Fig. 3. Moreover, we do not know when (in which period) the infeasibility will occur to which customers. In this case, in order to obtain a feasible solution of the SIRPSD, we have to check q_{ijt} and x_{ijt} of every customer for feasibility of its related routes (Step 3 below), and fix each infeasible route to a feasible one (Step 4 below). To ease the trace of every individual customer, the sequence of customers and delivery volumes (q_{ijt}) to check have to be defined first (Steps 1&2 below). Moreover, because the customer nodes are numbered following arc directions in the corresponding transportation graph (a directed graph), when we check node i, it is already known that all inflow-outflow matches of its preceding nodes have been feasible and the adjustment of its inflows is then not required.

Because the method is the same for every period, for simplification we omit the subscript t of the corresponding variables and parameters in the following discussion. Similar to the solution repair approach in (Yu et al. 2008), the method has the following steps.

Step 1. *Build a directed transportation graph with specifically numbered nodes*. With x values in the feasible solution of Model $\mathbf{P}'$, a *directed transportation graph*, as exemplified in Fig. 1(a), can be defined in each period t where two customer nodes (or a customer node and the depot node) i and j are connected x_{ij} times by directed arcs (i, j) if $x_{ij} \geq 1$. The depot is split into two virtual ones: an outgoing depot (numbered as 0) and an incoming depot (numbered as 0′). The directed arcs associated with $\{x_{ji} | x_{ji} \geq 1, j = 0, \ldots, N\}$ are called *incoming* arcs of customer node i. The directed arcs associated with $\{x_{ij} | x_{ij} \geq 1, j = 0, \ldots, N\}$ are called *outgoing* arcs of customer node i. The customer nodes are virtually numbered here by applying the rule: Starting from the depot node 0, an unnumbered customer node i can be numbered as next node if and only if all the nodes connected to i with a direct arc have been numbered. For example, in Fig. 1(a), before
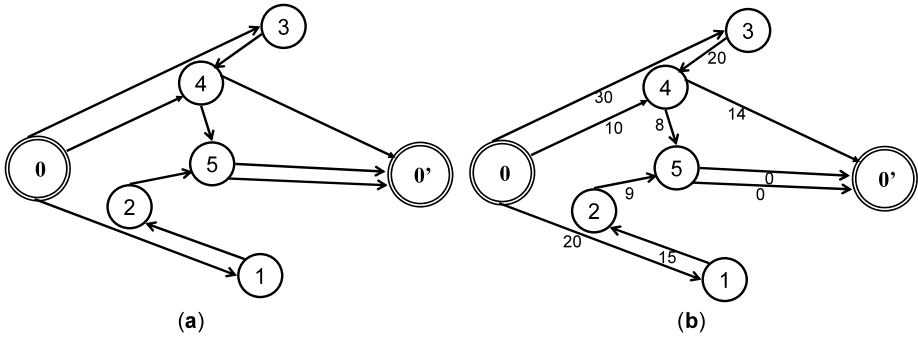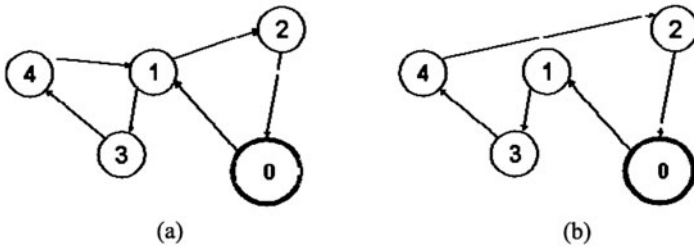
Fig. 1 A directed transportation graph

Fig. 2 An example of a sub-cycle

starting the numbering, only the two depot nodes are numbered. Customer nodes 1 and 3 are the only two nodes that satisfy the rule, and one of them can be numbered as 1. One of these two can be selected and numbered as 1 arbitrarily. We select the first one. Then, customer nodes 2 and 3 become the next ones satisfying the rule and we can select one of them to be numbered as 2. This process continues until all customer nodes are numbered. The numbering will give a sequence to assign arc flows to vehicle routes.

Although constraints (7) ensure that there is no subtour in the graph, the nodes in the direct graph may not be successfully numbered by simply applying the above rule. A typical situtation, illustrated in Fig. 2(a), may happen in the numbering. Customer nodes 1, 3, 4 form a sub-cycle (it is not a subtour) where customer 1 is revisited but constraints (7) are satisfied. In this case, before numbering the customers, such a sub-cycle has to be eliminated. Our elimination approach is illustrated by transforming Figs. 2(a) to 2(b). Let q_{ij}^a, q_{ij}^b denote the quantities transported on an arc from node i to j in Figs. 2(a) and 2(b), respectively. The transformation of Fig. 2(a) to 2(b) can be made by setting $q_{42}^b = q_{12}^a, q_{34}^b = q_{34}^a - (q_{41}^a - q_{12}^a), q_{13}^b = q_{13}^a - (q_{41}^a - q_{12}^a)$ and provides the total cost reduction of $(q_{41}^a c_{41} + q_{12}^a c_{12} - q_{42}^b c_{42}) + (q_{41}^a - q_{12}^a)(c_{13} + c_{43})$.

Step 2 . *Weight all directed arcs in the transportation graph.* In the feasible solution of Model **P**′, each $x_{ij} \geq 1$ corresponds to q_{ij}. We weight all directed arcs with q_{ij} to obtain a weighted graph shown in Fig. 1(b). For customer node i, $\{q_{ji}|x_{ji} \geq 1, j = 0, \ldots, N\}$ form its *inflows*, and $\{q_{ij}|x_{ij} \geq 1, j = 0, \ldots, N\}$ form its *outflows*.

Step 3. *Assign q on directed arcs to individual vehicle routes.* Starting from numbered customer node 1, we check nodes $2, \ldots, n$ successively. For every customer node checked, each of its inflows has to be matched with one outflow. If the outflow is no larger than the

inflow, the match is feasible to the construction of a vehicle route. If all inflow-outflow matches can be found and are feasible in the directed transportation graph, a set of feasible routes can be naturally traced (constructed) from a feasible solution d, q, x of Model $\mathbf{P}'$. The solution of Model $\mathbf{P}'$ is then a feasible solution of the SIRPSD. In step 3, an assignment problem is solved to match inflows with outflows of each customer node for further constructing vehicle routes. The objective function of the assignment problem is to find feasible matches as many as possible (by introducing the cost coefficient C_{ln}^A below) and make possible adjustment of infeasible solutions as easy as possible (by introducing the cost coefficient C_{ln}^B below). We here define the assignment problem (denoted as **AP**) formally for node i and the details can refer to our earlier paper (Yu et al. 2008).

Model **AP**:

$$\min \sum_{l=1}^{L} \sum_{n=1}^{L} (C_{ln}^A + C_{ln}^B) u_{ln}. \tag{28}$$

Subject to:

$$\sum_{l} u_{ln} = 1, \quad n = 1, \ldots, L \tag{29}$$

$$\sum_{n} u_{ln} = 1, \quad l = 1, \ldots, L \tag{30}$$

$$u_{ln} \in \{0, 1\}, \quad l, n = 1, \ldots, L \tag{31}$$

where L is the number of inflows and outflows of node i. The decision variables are denoted as $u_{lm} = 1$ if incoming arc l is matched with outgoing arc m, and $u_{lm} = 0$ otherwise. $q_+^l(i)$ is the l-th inflow (the flow of the l-th incoming arc) of node i that corresponds to $\{q_{ji}|x_{ji} = 1, \ j = 0, 1, \ldots, N, j \neq i\}$. $q_-^l(i)$ is the l-th outflow (the flow of the l-th outgoing arc) of node i that corresponds to $\{q_{ij}|x_{ij} = 1, j = 0, 1, \ldots, N, j \neq i\}$. $\bar{q}_-^{\min}(i)$ is the minimum outflow of node i, that is, $\bar{q}_-^{\min}(i) = \min_l\{q_-^l(i)|l = 1, \ldots, L\}$. If outflow m of node i is no larger than inflow l, which leads to a *feasible* match, we set $C_{lm}^A = 0$. Otherwise, we set $C_{lm}^A = M$ if $q_+^l(i) < q_-^m(i)$ to penalize an *infeasible* match where M is a very large positive number. $C_{lm}^B = \bar{q}_-^{\min}(k) - q_+^l(i)$ if $q_+^l(i) < q_-^m(i)$ and $\bar{q}_-^{\min}(k) - q_+^l(i) > 0$ where k is the immediate successor of node i and the end of the m-th outgoing arc of node i, $C_{lm}^B = 0$ otherwise.

Note that, because x_{0i} may be an integer larger than 1 (corresponding to multiple arcs), for well defining the assignment problem, q_{0i} has to be split into multiple quantities which are assigned to the multiple arcs, one quantity for each arc. This can be realized by setting each inflow of node i equal to a corresponding outflow of node i. The remaining unassigned quality can be split and assigned arbitrarily to these inflows without violating vehicle capacity.

If the minimum objective value of the **AP** for each customer node $i, i = 1, \ldots, N$ is zero, the inflows and the outflows of all customers can be matched to trace a set of feasible routes. As a result, all customers are assigned to feasible routes, and a feasible solution of IRPSD is thus obtained. Otherwise, $\sum_{l=1}^{L} \sum_{n=1}^{L} (C_{ln}^A + C_{ln}^B) u_{ln} > 0$ for some nodes in Model **AP** and go to the next step.

Step 4. *Adjust each infeasible inflow-outflow match into a feasible one*. If the minimum objective value of model AP for a customer is not equal to zero, some matches are infeasible and have to be adjusted. As a typical situation shown in Fig. 3(a), the dotted inflow (10) of node 3 from node 1 can not be feasibly matched to either of the two outflows of node 3.

Note: the numbers on each direct arc (i,j) and node i are q_{ij} and d_i respectively

(a)   (b)
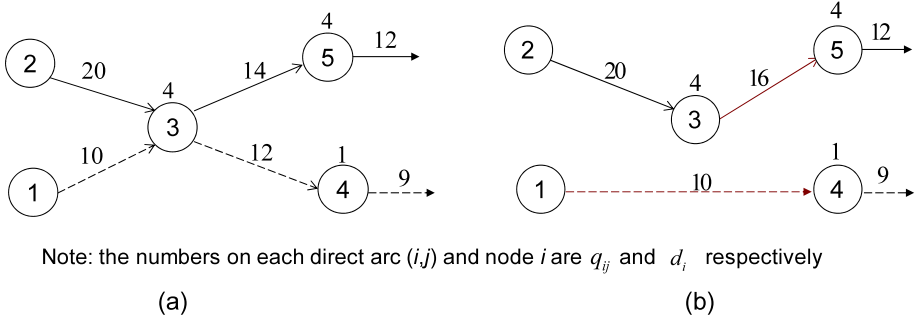
Fig. 3 An example for value adjustment

In this case, the minimum objective value of the corresponding assignment problem must be larger than 0 and the adjustment of the values x, q is inevitable. In this case, the values x, q in Fig. 3(a) are adjusted to those in Fig. 3(b).

The procedure of the matching and value adjustment for all customer nodes in every period is performed one by one following the given node sequence and a feasible solution of the studied SIRPSD can then obtained if all customer nodes have been examined.

4.2 Local search improvement of the feasible solution of the SIRPSD

A feasible solution of the SIRPSD can be improved further by local search. This subsection presents several local search heuristics to reduce the total cost of the SIRPSD as follows.

4.2.1 Relocation of customer delivery between two routes

Multiple vehicle routes may serve a common customer, such as in Fig. 4(a) where customer 1 is a common customer of routes 1 and 2. A relocation method shown in Fig. 4(b)–(c) can be used to reduce the routing cost; compared to Fig. 3(a), Fig. 3(b) relocates 10 units of product from route 2 to 1 such that route 2 can visit customer 3 directly from the depot (0) by avoiding a detour via customer 1. Similarly, we get Fig. 3(c). The variable transportation costs are reduced by Figs. 3(b) and 3(c) can then be calculated as $(80c_{01} + 50c_{01} + 70c_{13}) - (70c_{03} + 60c_{01}) = (80 \times 2 + 50 \times 2 + 70 \times 2) - (70 \times 3 + 60 \times 2) = 70$ and $(80c_{01} + 50c_{01} + 40c_{12}) - (90c_{01} + 40c_{02}) = (80 \times 2 + 50 \times 2 + 40 \times 2) - (90 \times 2 + 40 \times 2) = 80$ for those two relocations respectively. Because $80 > 70$, the relocation in Fig. 3(c) is selected to save 80 units of the total cost.

As an special case, it is possible that multiple routes visit a common customer but one of them may drop-off zero unit of the product. For example, in Fig. 5(a), route 1 visits customer 3 without delivering any unit of the product. In this case, by deleting the common customer from route 1, the variable transportation cost can be reduced by $90(c_{03} + c_{34} - c_{04}) = 90(2 + 2 - 3) = 90$ according to the triangle inequalities for c_{ij}.

The procedure to implement the above local search improvement is as follows. *Step 1*, at every customer node, check whether the customer is served by multiple vehicle routes. If no, go to the next customer node. Otherwise, go to step 2. *Step 2*: implement the local search improvement (illustrated in Figs. 3 or 4) if it can bring a cost saving. After the local search, go to step 1 for checking the next customer node.
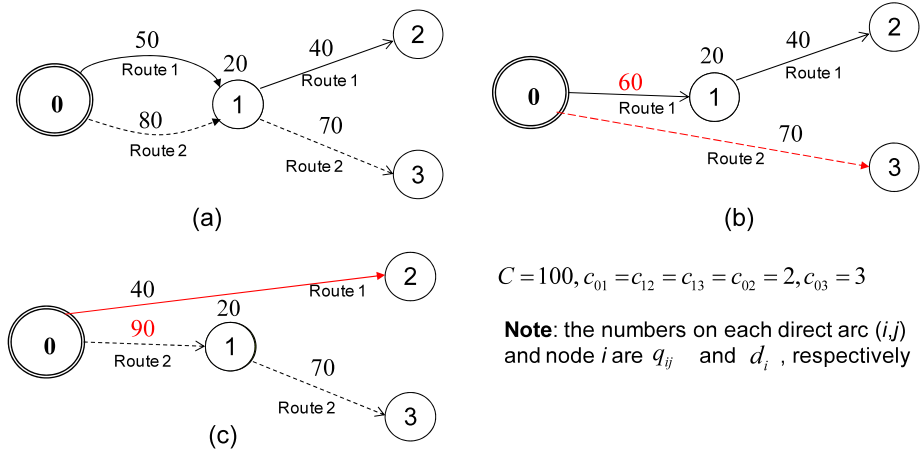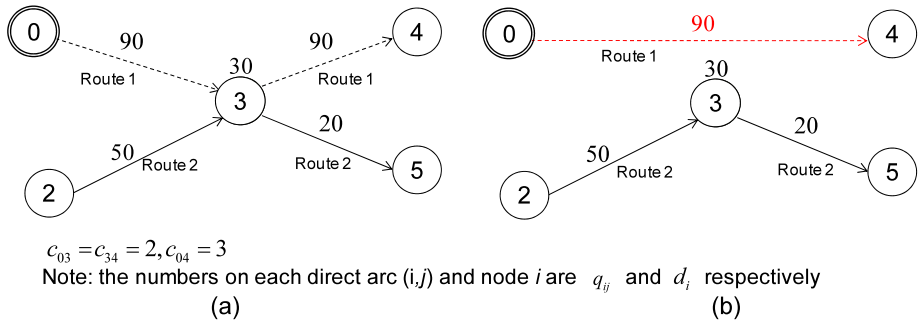
Fig. 4 Example 1 for local search improvement

$C = 100, c_{01} = c_{12} = c_{13} = c_{02} = 2, c_{03} = 3$

Note: the numbers on each direct arc (i,j) and node i are q_{ij} and d_i, respectively

$c_{03} = c_{34} = 2, c_{04} = 3$
Note: the numbers on each direct arc (i,j) and node i are q_{ij} and d_i respectively

Fig. 5 Example 2 for local search improvement

4.2.2 Reduce the number of routes

The number of routes can be reduced by merging two vehicles not fully loaded to make all routes as fully loaded as possible if the merging reduces the total cost. With the reduction of route number, it is possible to increase the total vehicle utilization, and reduce the fixed cost $\sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t}$. In this local search, every pair of routes is checked, and two vehicle routes are merged if the merged route is feasible and the total transportation cost is reduced. Figures 6(b)–6(c) illustrates two options of merging routes in Fig. 6(a). In Fig. 6(b), the merging can reduce the transportation cost by $f_t + 50c_{02} + c_{40}^b - 50(c_{01} + c_{14} + c_{42}) = 100 + 50 + 3 - 50(1 + 1 + 1) = 3$. In Fig. 6(c), the merging can reduce the transportation cost by $f_t + 30c_{01} + c_{30}^b - 30(c_{02} + c_{23} + c_{31}) = 100 + 30 + 3 - 30(1 + 1 + 1) = 43$. Because $43 > 3$, the route merging in Fig. 6(b) is selected.

The algorithm to reduce the number routes works as follow. *Step 1*: pick out all q_{0i}, $i = 1, \ldots, N$ that are positive. *Step 2*: check each $q_{0i} > 0$ sequentially and find all pairs of routes that can be merged. *Step 3*: find the largest cost reductions from all the merging options with the methods in Fig. 6 and merge the two routes. *Step 4*: similarly, check all the rest pairs of routes sequentially for possible merging that can reduce the transportion cost.
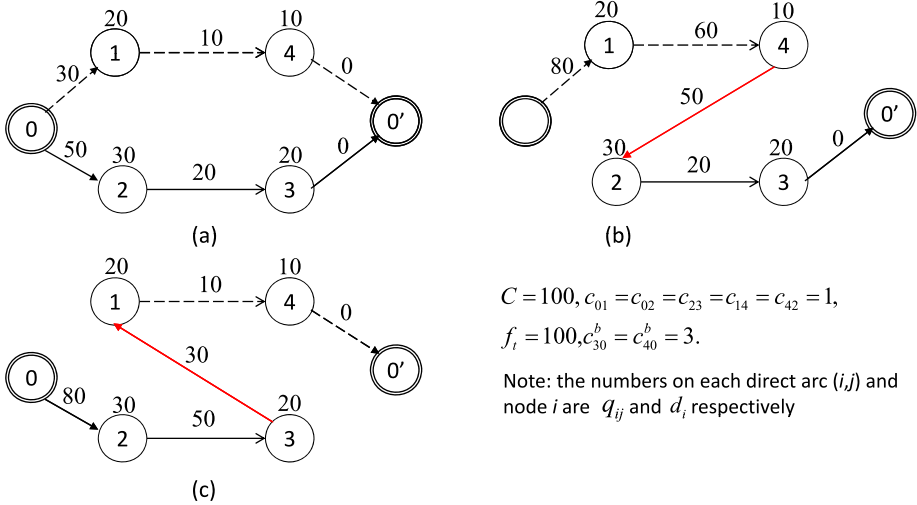
Fig. 6 Example 3 for local search improvement

$C = 100, c_{01} = c_{02} = c_{23} = c_{14} = c_{42} = 1,$
$f_t = 100, c_{30}^b = c_{40}^b = 3.$

Note: the numbers on each direct arc (i,j) and node i are q_{ij} and d_i respectively

5 Normal distribution for demands and solution evaluation

In Model **P** or **P′**, the stochastic demand in each period is in a generic form that is applicable to different stochastic demand distributions, such as normal distribution, Weibull Distribution, Log Normal Distribution. In order to apply our proposed model and solution approaches to numerical examples, we have to specify the distributions of the customers' stochastic demands.

For a given distribution of ζ_{it} in our Model **P**, we need to obtain $F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is})$ values for (1′), constraints (3′) and (4′), and update these constraints. We here show the update process for the normal distribution of ζ_{it}. The normal distribution the most commonly used both in theory and practice.

5.1 Normal distribution for stochastic demands

Supposing that ζ_{it} is a random variable subject to a normal distribution with mean u_i and standard deviation σ_i. That is,

$$\zeta_{it} \sim N(u_i, \sigma_i^2) \tag{32}$$

We have a probability density function: $f_{it}(\zeta_{it}) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-\frac{(\zeta_{it} - u_i)^2}{2\sigma_i^2}}$ $-\infty \le \zeta_{it} \le +\infty$.

The accumulative customer demand $\zeta_{i,(1,t)} = \sum_{s=1}^{t} \zeta_{is}$ obeys

$$\zeta_{i,(1,t)} \sim N(tu_i, t\sigma_i^2) \tag{33}$$

Defining $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$, we have

$$F_{i,(1,t)}(r) = P(\zeta_{i,(1,t)} \le r) = P\left(\frac{\zeta_{i,(1,t)} - tu_i}{\sqrt{t}\sigma_i} \le \frac{r - tu_i}{\sqrt{t}\sigma_i}\right) = \Phi\left(\frac{r - tu_i}{\sqrt{t}\sigma_i}\right) \tag{34}$$

For the stochastic part $\sum_{t=1}^{T} E(\sum_{i=1}^{N} h_{it} I_{it}^{+})$ in the objective function (1′), using Mathematica 6.0 (Wolfram 2007), we can obtain $E(\sum_{i=1}^{N} h_{it} I_{it}^{+}) = \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - r) dF_{i,(1,t)}(r) = \frac{1}{2}((e^{-(I_{i,0}+\sum_{s=1}^{t} d_{is}-tu_i)^2/(2t\sigma_i^2)} - e^{-tu_i^2/(2\sigma_i^2)})\sigma_i\sqrt{\frac{2t}{\pi}} + 2(I_{i,0} + \sum_{s=1}^{t} d_{is} - tu_i)(\Phi(\frac{u_i\sqrt{t}}{\sigma_i}) - \Phi(\frac{(tu_i-(I_{i,0}+\sum_{s=1}^{t} d_{is}))}{\sigma_i\sqrt{t}})))).$

For constraints (3) or (3′), we have $F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is}) - F_{i,(1,t)}(0) \geq \alpha_{it}$, that is,

$$\Phi\left(\frac{((I_{i,0}+\sum_{s=1}^{t} d_{is}) - tu_i)}{\sigma_i\sqrt{t}}\right) - \Phi\left(\frac{-u_i\sqrt{t}}{\sigma_i}\right) \geq \alpha_{it} \quad \text{and}$$

$$\Phi\left(\frac{((I_{i,0}+\sum_{s=1}^{t} d_{is}) - tu_i)}{\sigma_i\sqrt{t}}\right) \geq \Phi\left(\frac{-u_i\sqrt{t}}{\sigma_i}\right) + \alpha_{it}$$

After reformulating, we have $\sum_{s=1}^{t} d_{is} \geq tu_i + \sigma_i\sqrt{t}\Phi^{-1}(\Phi(-u_i\sqrt{t}/\sigma_i) + \alpha_{it}) - I_{i,0}$. In practice, if $\sigma_i \leq 30\%u_i$, $F_{i,(1,t)}(0) = \Phi(-u_i\sqrt{t}/\sigma_i) \leq 4.3 \times 10^{-4}$ in constraints (3). In such a case, we can therefore omit $F_{i,(1,t)}(0)$ (i.e., $\Phi(-u_i\sqrt{t}/\sigma_i)$) and reformulate (3) as $\sum_{s=1}^{t} d_{is} \geq tu_i + \sigma_i\sqrt{t}\Phi^{-1}(\alpha_{it}) - I_{i,0}$. Note that I_{it} is a function of the product deliveries and stochastic demands of customer i in all the previous t periods. The longer t is, the larger standard deviation of $\zeta_{i,(1,t)}$ (i.e., $\sqrt{t}\sigma_i$ from (33)) we have. As a result, the larger $\sum_{s=1}^{t} d_{is}$ is required to satisfy the service level of the depot.

Similarly, for constraints (4) or (4′), we have $F_{i,(1,t-1)}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i) \leq 1 - \beta_{it}$. According to (33), $F_{i,(1,t-1)}(I_{i,0}+\sum_{s=1}^{t} d_{is} - V_i) = \Phi(\frac{(I_{i,0}+\sum_{s=1}^{t} d_{is})-V_i-(t-1)u_i}{\sigma_i\sqrt{t-1}})$. Constraints (4′) then become $(I_{i,0} + \sum_{s=1}^{t} d_{is}) - V_i - (t-1)u_i \leq \sigma_i\sqrt{t-1}\Phi^{-1}(1 - \beta_{it})$. That is $\sum_{s=1}^{t} d_{is} \leq V_i + (t-1)u_i + \sigma_i\sqrt{t-1}\Phi^{-1}(1 - \beta_{it}) - I_{i,0}$.

5.2 Solution evaluation

In Sect. 4, a feasible solution of the SIRPSD is obtained and its corresponding total cost provides an upper bound of the minimum total cost of the studied SIRPSD. The quality of the solution can then be evaluated by the relative gap between the lower and upper bounds, e.g., (the upper bound-the lower bound)/the lower bound $\times 100\%$. The smaller the gap is, the better the solution is. A lower bound of the minimum total cost is already available as the result of Sect. 3.4 for the evaluation in this paper.

6 Numerical examples

This section aims to evaluate the performance of our developed approach. The stochastic demand ζ_{it} is subject to a normal distribution with $\zeta_{it} \sim N(u_i, \sigma_i^2)$. We first consider a base example with the total number of customers and the depot, N_0, being 100. In the example, the length of the time horizon is taken as $T = 5$, which corresponds to five working days every week. Parameters u_{it}, C, f_t, h_{it}, I_{i0}, and V_i are randomly and uniformly generated from the intervals [50, 400], [100, 300], [400, 700], [0.5, 2], [50, 400], and [600, 1000] respectively; For c_{ij}, to ensure that the triangle inequality condition is satisfied, we first generate the coordinates of all customers and the central depot from a 10×10 square, and then calculate c_{ij} as the geometrical distance between customers i and j. $\sigma_{it} = 0.2u_{it}$, $\alpha_{it} = \beta_{it} = 95\%$, and $c_{i0}^b = 10 \times c_{i0}$. The domains of the above parameters are mainly taken from Yu et al. (2008). We generate 10 random examples for the base example with corresponding

Table 1 Notations used in numerical results

UB	Upper bound of the SIRPSD (given by the near optimal solution of the SIRPSD)
LB	Lower bound of the SIRPSD (found by the Lagrangian relaxation approach in Sect. 3)
Gap	Value of $(\text{UB} - \text{LB})/\text{LB} \times 100\%$
CT	Computational time (minutes: seconds)

Table 2 Computational results for the base example

Instance	LB (10^5)	UB (10^5)	Gap (%)	CT
1	10.36	10.69	3.24	12:43
2	8.06	8.51	5.57	11:59
3	9.14	9.94	8.76	10:25
4	7.41	7.90	6.60	11:13
5	12.27	12.36	0.78	13:01
6	11.89	12.03	1.16	12:54
7	10.24	10.61	3.63	11:22
8	7.00	7.59	8.35	11:10
9	8.52	9.05	6.20	11:18
10	10.25	10.51	2.49	11:05
Average	9.51	9.85	4.68	11:47

results presented in Table 2. The notations used for presenting the results are shown in Table 1.

In order to evaluate the robustness of our approach, based on the parameters of the base example, we generate 10 instances for each of other four scenarios: (a) the service levels of customer demands α_{it}, and the customers' warehouses β_{it} are both changed from 95% to 99% (see results in Table 3), (b) σ_{it} is changed from $0.2u_{it}$ to $0.3u_{it}$ (the results shown in Table 4), (c) T is changed from 5 to 10 (the results shown in Table 5), and (d) N_0 is changed from 100 to 200 (the results shown in Table 6).

The approach is coded in C++ using callable library of Lingo 6.0. To obtain a high quality solution of the SIRPSD, we construct a feasible solution of the Model **P′** based on the solution of its relaxed problem in every iteration. The best feasible solution of Model **P′** is repaired to be the final feasible solution of the SIRPSD. The test is conducted on an Intel (R) Core (TM) Due CPU 2.4 GHz notebook PC with 2 G RAM and the termination condition of each instance for the Lagrangian relaxation approach is 150 iterations.

From Tables 2, 3, 4, 5, 6, we obtain the following observations:

(1) Our algorithm can obtain high quality near-optimal solutions to the studied SIRPSD with the average gap between the upper bound and the lower bound of the problem less than 5.3% for all considered scenarios.
(2) For all the scenarios, the largest gap is 9.73% (in Table 3) and the smallest gap is 0.78% (in Table 2). This shows that our approach is robust since the gap for a scenario does not change much with the change of parameter values.
(3) Our approach can solve large instances of the SIRPSD in a reasonable computational time on an ordinary PC, with the average computational time of the instances of the base example being only 11 minutes and 47 seconds. With the increase of the problem size from $N_0 = 100$ to 200, our approach can obtain near optimal solutions within 70

Table 3 Results for the example with service levels (α_{it} and β_{it}) no less than 99%

Instance	LB (10^5)	UB (10^5)	Gap (%)	CT
1	11.80	12.01	1.78	12:40
2	12.78	13.04	2.04	13:11
3	11.03	11.31	2.54	12:37
4	11.73	12.16	3.69	11:04
5	8.49	8.95	5.42	12:39
6	8.42	8.81	4.67	11:27
7	9.12	9.59	5.19	12:13
8	8.70	9.33	7.23	11:52
9	9.66	10.20	5.51	11:17
10	7.59	8.33	9.73	11:10
Average	9.93	10.60	4.78	12:07

Table 4 The results for the example with $\sigma_{it} = 0.3u_{it}$

Instance	LB (10^5)	UB (10^5)	Gap (%)	CT
1	9.90	10.38	4.85	10:33
2	9.29	9.74	4.84	10:04
3	9.48	10.00	5.45	11:31
4	13.89	14.07	1.27	10:46
5	8.80	9.44	7.37	12:17
6	9.42	9.75	3.46	10:15
7	10.14	10.59	4.42	11:01
8	9.59	10.04	4.65	10:33
9	8.79	9.36	6.49	11:09
10	9.09	9.49	4.35	11:23
Average	9.84	10.37	4.71	10:54

minutes on average. Although the number of decision variables increases from about 99 thousands to about 400 thousands and the number of integer variables from about 50 thousands to about 200 thousands in each instance, similar results can be obtained when T is changed from 5 to 10.

(4) With the increase of the problem size, the average gap increases slightly. For instance, with the increase of N_0 from 100 to 200 or the increase of T from 5 to 10, the average gap only increases 0.22% and 0.61% respectively from Tables 5, 6.

(5) If other criterions, like the service level, β_{it} for each customer i's warehouse or α_{it} for each customer i's demand, increase, the average total cost for a corresponding scenario also increases but the average gap between the upper and lower bounds only changes slightly. This can be seen from the comparison between Table 2 and 3.

(6) With the increase of the uncertainty of the customers' demands by changing $\sigma_{it} = 0.2u_{it}$ to $\sigma_{it} = 0.3u_{it}$, we can find that the average gap increases slightly by 0.04%. This may be because meeting the required service levels becomes more difficult with the increase of the demand uncertainty.

Table 5 The results for the example with $T = 10$

Instance	LB (10^5)	UB (10^5)	Gap (%)	CT
1	21.20	21.59	1.84	29:52
2	17.33	18.31	5.65	30:27
3	18.18	18.95	4.23	27:44
4	20.03	20.83	3.99	28:45
5	16.93	18.45	8.93	28:54
6	18.01	18.64	3.50	31:02
7	17.84	19.08	6.94	27:37
8	19.84	20.70	4.34	26:60
9	15.77	17.10	8.43	28:52
10	17.56	18.44	5.01	31:23
Average	18.27	19.29	5.29	28:55

Table 6 The results for the example with $N_0 = 200$

Instance	LB (10^5)	UB (10^5)	Gap (%)	CT
1	19.61	20.23	3.15	75:27
2	14.98	15.93	6.37	67:41
3	18.55	19.85	7.01	63:18
4	17.01	17.49	2.84	73:36
5	15.83	16.86	6.46	65:09
6	21.51	22.11	2.77	77:34
7	15.34	16.19	5.59	65:21
8	16.27	17.34	6.56	68:42
9	22.17	22.65	2.18	67:57
10	17.27	18.32	6.04	68:03
Average	16.23	18.74	4.90	69:25

7 Conclusion

This paper studies a stochastic inventory routing problem with split delivery where the service level to satisfy each customer's demand measured in stockout probability and the service level to each customer's warehouse measured in its overfilling probability are considered. The complexity of the SIRPSD with service levels motivates us to develop a hybrid approach which uses techniques such as the transformation of stochastic components of a model of the SIRPSD into deterministic ones, the use of Lagrangian relaxation to decompose the model into submodels, the partial linearization of the nonlinear objective function of the model, and local search improvement of feasible solutions of the studied SIRPSD to solve it. The numerical examples demonstrate that our proposed approach can obtain high quality solutions in a reasonable computational time on an ordinary personal computer.

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