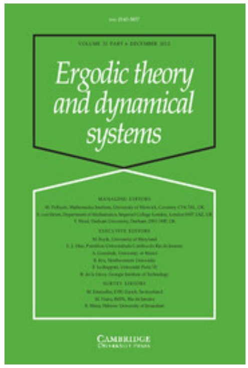

Circle diffeomorphisms forced by expanding circle maps

ALE JAN HOMBURG

Ergodic Theory and Dynamical Systems / Volume 32 / Issue 06 / December 2012, pp 2011 - 2024
DOI: 10.1017/S014338571100068X, Published online: 01 November 2011

Ergod. Th. & Dynam. Sys. (2012), **32**, 2011–2024
doi:10.1017/S014338571100068X

Circle diffeomorphisms forced by expanding circle maps

ALE JAN HOMBURG

KdV Institute for Mathematics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands
and
Department of Mathematics, VU University Amsterdam, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

(*Received* 28 *June* 2010 *and accepted in revised form* 15 *June* 2011)

Abstract. We discuss the dynamics of skew product maps defined by circle diffeomorphisms forced by expanding circle maps. We construct an open class of such systems that are robustly topologically mixing and for which almost all points in the same fiber converge under iteration. This property follows from the construction of an invariant attracting graph in the natural extension, a skew product of circle diffeomorphisms forced by a solenoid homeomorphism.

1. *Introduction*

We investigate the dynamics of a class of circle diffeomorphisms that are forced by expanding circle maps. We start with a numerical experiment on the skew product map

$$(y, x) \mapsto (3y, x + \tfrac{1}{8}\sin(2\pi x) + y) \quad \bmod 1 \tag{1}$$

defined on the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, the results of which are presented in Figure 1. Note that this map is given by a circle diffeomorphism $x \mapsto x + \frac{1}{8}\sin(2\pi x) + y \bmod 1$ in the fiber forced by an expanding circle map $y \mapsto 3y \bmod 1$ in the base. The left panel of Figure 1 shows ten-thousand points of an orbit, which seem to be dense in the torus. The right panel shows time series of the second coordinate of twenty different orbits, for equidistantly distributed initial points in the same fiber (i.e. with identical first coordinate). There appears to be a fast contraction inside the fiber.

Numerical experiments as described above may be explained by the following result. Endow the space of smooth skew product systems $(y, x) \mapsto F(y, x) = (g(y), f_y(x))$ on $\mathbb{T}^2$, considered as a subset of the smooth endomorphisms, with the C^k topology for $k \geq 2$. A smooth endomorphism g on the circle is said to be expanding if $|g'| > 1$. Recall that F is topologically mixing if for any non-empty open $U, V \subset \mathbb{T}^2$, $F^n(U)$ intersects V for all large enough positive integers n; this implies the existence of dense positive orbits.

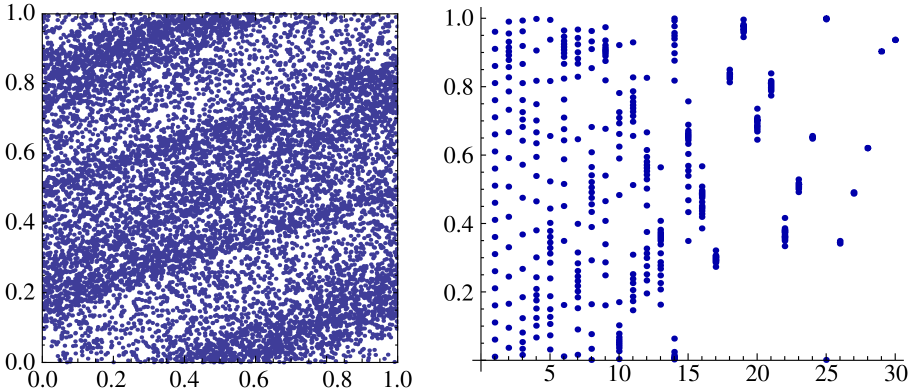

FIGURE 1. Numerical experiments on $(y, x) \mapsto (3y, x + \frac{1}{8}\sin(2\pi x) + y) \bmod 1$. The left panel shows ten-thousand points of an orbit. The right panel shows time series for the x-coordinate, starting from twenty different initial conditions with identical y-coordinates.

THEOREM 1.1. *There is an open class of forced circle diffeomorphisms $(y, x) \mapsto F(y, x) = (g(y), f_y(x))$, forced by expanding circle maps $y \mapsto g(y)$, with the following properties:*

(i) *each map F is topologically mixing;*

(ii) *there is a subset $\Lambda \subset \mathbb{T}^2$ of full Lebesgue measure such that for any $(y, x_1), (y, x_2) \in \Lambda$ we have*

$$\lim_{n\to\infty} |F^n(y, x_1) - F^n(y, x_2)| = 0. \tag{2}$$

To prove the convergence property in Theorem 1.1, we apply the natural extension of the endomorphism F to a homeomorphism on the product of a solenoid and a circle (this construction is described in §2). This homeomorphism is also a skew product map, formed by a circle diffeomorphism forced by a solenoid map. It is shown to admit an attracting invariant graph (see Theorem 5.1 below), from which the result follows.

In the physics literature, a convergence phenomenon as in (2) falls under the study of synchronization; see [**4**] for a review. Forced circle maps appear in various contexts which feature some sort of convergence of orbits. Here we give some pointers to the literature for these different contexts and discuss how they relate to our result.

(i) Quasiperiodically forced circle diffeomorphisms, where the circle diffeomorphisms $f_y(x)$ are forced by $g(y) = y + \alpha \bmod 1$ with α irrational: a large body of work exists in this area of research, related to the existence of strange non-chaotic attractors; see [**18**] and references therein. A notable difference from the context studied in this paper is that the forcing consists of ergodic, but not mixing, dynamics.

(ii) Randomly perturbed circle diffeomorphisms, including iterated function systems [**1, 11, 20**] and circle diffeomorphisms with absolutely continuous noise [**21, 35**]: such systems allow formulation as skew product systems. The aforementioned references give precise classifications of the dynamics in the fibers for both iterated function systems and circle diffeomorphisms with absolutely continuous i.i.d. noise. For an iterated function system consisting of m circle diffeomorphisms $f_1, \ldots, f_m$, this yields circle diffeomorphisms forced by a shift on m symbols: consider $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ endowed with the product topology and, for $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots) \in \Sigma$, the left shift $\sigma\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$. The skew product system F acting on $\Sigma \times \mathbb{T}$ is then given by

$$F(\boldsymbol{\omega}, x) = (\sigma\boldsymbol{\omega}, f_{\omega_0}(x)).$$

The left shift is a topologically mixing map. The dependence of the circle diffeomorphisms on $\boldsymbol{\omega}$ is of a restricted form: they depend only on ω_0 and not on ω_i for $i > 0$ (in [**14**] the term 'step skew product' is used). Our result can be seen as an extension where this restriction is removed, and also as an extension to more general topologically mixing base dynamics.

(iii) Volume-preserving skew products over hyperbolic torus automorphisms [**28, 29, 31**]: one may think of small perturbations of $F : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2 \times \mathbb{T}$,

$$F(y, x) = (Ay, x),$$

where A is a hyperbolic torus automorphism. This research relates to the phenomenon of stable ergodicity. It also relates to work on partially hyperbolic systems with mostly contracting central directions [**7, 26**]. The above references contain results on delta measures in fibers, which go in the direction of the convergence result in Theorem 1.1. Reference [**17**] combines the approaches of [**29, 31**] and this paper, and contains a result akin to Theorem 1.1. As reviewed in §3, one may embed the solenoid from the natural extension of the expanding circle map as a hyperbolic attractor for a smooth diffeomorphism on a manifold, so that the natural extension of the skew product system is partially hyperbolic on $\mathcal{S} \times \mathbb{T}$.

Finally, skew product systems of circle diffeomorphisms over horseshoes and solenoids are also treated in [**14**], with a different emphasis, where the robust occurrence of dense sets of hyperbolic periodic orbits with different indices (attracting or repelling in the fiber) is proved. In [**15**], and continued in [**6, 12**], the existence of ergodic measures with zero Lyapounov exponent is investigated in the related contexts of skew product systems and partially hyperbolic systems.

2. *Natural extensions*

In this section we collect some facts, mostly known, on extensions of skew product torus endomorphisms to skew product homeomorphisms. These facts are used in the arguments of the subsequent sections.

Consider a smooth expanding endomorphism $g : \mathbb{T} \to \mathbb{T}$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We note that g possesses an absolutely continuous invariant measure ν^+, equivalent to Lebesgue measure; see, for instance, [**22**, §III.1]. In fact, g is topologically conjugate to a linear expanding circle map for which Lebesgue measure is invariant [**30**]. The measure ν^+ has density that is bounded and bounded away from zero. We will consider skew product systems of circle diffeomorphisms $x \mapsto f_y(x)$ forced by the expanding circle map $y \mapsto g(y)$. Write

$$F(y, x) = (g(y), f_y(x)) \tag{3}$$

for the skew product map on the torus $\mathbb{T}^2$. For iterates of F, we write

$$F^n(y, x) = (g^n(y), f_{g^{n-1}(y)} \circ \cdots \circ f_y(x)) = (g^n(y), f_y^n(x)).$$

The inverse limit construction [**34**] extends g to a homeomorphism on the solenoid, i.e. the space

$$\mathcal{S} = \{(\ldots, y_{-1}, y_0) \in \mathbb{T}^{-\mathbb{N}} \mid y_{-i} = g(y_{-i-1})\}$$

endowed with the product topology. We will also write g for the extended map, and the context will make clear whether g acts on $\mathbb{T}$ or $\mathcal{S}$. So, for $\mathbf{y} = (\ldots, y_{-1}, y_0)$,

$$g(\mathbf{y}) = (\ldots, y_{-1}, y_0, g(y_0)).$$

The induced skew product map on $\mathcal{S} \times \mathbb{T}$ will also be denoted by F, and we write

$$F(\mathbf{y}, x) = (g(\mathbf{y}), f_{\mathbf{y}}(x)).$$

The inverse map is given by

$$F^{-1}(\mathbf{y}, x) = (\ldots, y_{-2}, y_{-1}, (f_{y_{-1}})^{-1}(x)).$$

On $\mathbb{T}$ and $\mathcal{S}$ we use Borel σ-algebras $\mathcal{F}^+$ and $\mathcal{F}$, respectively. Define the projection $\psi : \mathcal{S} \to \mathbb{T}$, $\psi(\mathbf{y}) = y_0$. Then, with $\mathcal{G} = \psi^{-1}(\mathcal{F}^+)$ we have $g^n \mathcal{G} \uparrow \mathcal{F}$, and $g : \mathcal{S} \to \mathcal{S}$ is a natural extension of $g : \mathbb{T} \to \mathbb{T}$ (see [**2**, Appendix A]). The solenoid $\mathcal{S}$ has an invariant measure ν inherited from the invariant measure ν^+ for g on the circle:

$$\nu(\{y_{-r} \in I_r, \ldots, y_0 \in I_0\}) = \nu^+(g^{-r}(I_0) \cap g^{-r+1}(I_1) \cap \cdots \cap I_r).$$

We will write λ for Lebesgue measure on $\mathbb{T}$. We also write $|I|$ for the length of an interval $I \subset \mathbb{T}$.

Let μ^+ be an invariant measure for $F : \mathbb{T}^2 \to \mathbb{T}^2$ with marginal ν^+; existence is guaranteed by [**10**, Lemma 2.3]. Write μ_y^+ for the disintegrations of μ^+. Occasionally, we shall also write $\mu_{\mathbf{y}}^+$, with the understanding that $\mu_{\mathbf{y}}^+$ depends only on the coordinate y_0 in $\mathbf{y} = (\ldots, y_{-1}, y_0)$. Invariance of μ^+ means that

$$\int_{g^{-1}(A)} f_y \mu_y^+ \, d\nu^+(y) = \int_{g^{-1}(A)} \mu_{g(y)}^+ \, d\nu^+(y) = \int_A \mu_y^+ \, d\nu^+(y)$$

for $A \in \mathcal{F}^+$ (the second equality comes from invariance of ν^+ under g); see [**10**] and [**2**, Theorem 1.4.5].

The following lemma, originating from [**10**], relates invariant measures for the skew product system with one- and two-sided time.

LEMMA 2.1. *Given the invariant measure μ^+ for F acting on $\mathbb{T}^2$, with marginal ν^+ on $\mathbb{T}$, there is an invariant measure μ for F acting on $\mathcal{S} \times \mathbb{T}$, with marginal ν on $\mathcal{S}$. For ν-almost all $\mathbf{y} = (\ldots, y_{-1}, y_0) \in \mathcal{S}$, the limit*

$$\mu_{\mathbf{y}} = \lim_{n \to \infty} f_{y_{-n}}^n \mu_{y_{-n}}^+ \tag{4}$$

gives its disintegrations.

Proof. The lemma is implied by [**2**, Theorem 1.7.2]. We describe briefly the line of reasoning. To avoid confusion, we write $\mathcal{B}$ (rather than $\mathcal{F}^+$ again) for the Borel σ-algebra on the circle of x-coordinates. For fixed $B \in \mathcal{B}$ and for $\mathbf{y} = (\ldots, y_{-n}, \ldots, y_0) \in \mathcal{S}$, define

$$\nu^n_{\mathbf{y}}(B) = f^n_{g^{-n}(\mathbf{y})}\mu^+_{y_{-n}}(B)$$

as the push-forward by $f^n_{g^{-n}(\mathbf{y})}$ of $\mu^+_{y_{-n}}$, evaluated in B. Recall that $\mathcal{G} = \psi^{-1}\mathcal{F}^+$, and with $\mathcal{G}_n = g^n\mathcal{G}$ we have $\mathcal{G}_n \uparrow \mathcal{F}$ as $n \to \infty$. One can compute that $\mathbb{E}(\nu^n_{\mathbf{y}}(B)|\mathcal{G}_m) = \nu^m_{\mathbf{y}}(B)$, i.e. $y \mapsto \nu^n_{\mathbf{y}}(B)$ is a martingale with respect to the filtration $\mathcal{G}_n$. As this holds for all fixed B, $\mu_{\mathbf{y}}(B) = \lim_{n\to\infty} f^n_{y_{-n}}\mu^+_{y_{-n}}(B)$ defines a probability measure for ν-almost all $\mathbf{y}$. □

Conversely, given an invariant measure μ for F on $\mathcal{S} \times \mathbb{T}$,

$$\mu^+_y = \mathbb{E}(\mu|\mathcal{F}^+)_y \tag{5}$$

is an invariant measure for F on $\mathbb{T}^2$ (see [**2**, Theorem 1.7.2]). Moreover, the correspondence maps ergodic measures to ergodic measures in either direction [**10**, §3].

We will also need to study iterates of the inverse map F^{-1} on $\mathcal{S} \times \mathbb{T}$. Noting that this interchanges stable and unstable directions, one obtains a convergence result similar to Lemma 2.1. To state the result, it will be convenient to think of g as acting on [0, 1]; one can identify 0 with 1 to obtain the expanding circle map. The inverse limit construction extends g to a map, also denoted by g, on

$$\mathcal{I} = \{(\ldots, y_{-1}, y_0) \in [0, 1]^{-\mathbb{N}} \mid y_{-i} = g(y_{-i-1})\}.$$

We may think of g as acting on $\Sigma \times [0, 1]$ for a Cantor set $\Sigma = \{0, \ldots, m-1\}^{\mathbb{N}}$. The solenoid $\mathcal{S}$ is then given as a quotient

$$\mathcal{S} = \Sigma \times [0, 1]/\sim, \tag{6}$$

identifying points $(\boldsymbol{\omega}, 0)$ and $(\boldsymbol{\nu}, 1)$ in $\Sigma \times \{0, 1\}$ for which $g^{-1}(\boldsymbol{\omega}, 0) = g^{-1}(\boldsymbol{\nu}, 1)$. We write $\mathbf{y} = (\boldsymbol{\omega}, y_0) \in \Sigma \times \mathbb{T}$.

Consider the projection $\psi : \Sigma \times [0, 1] \to \Sigma$, $\psi(\boldsymbol{\omega}, y_0) = \boldsymbol{\omega}$. The Borel σ-algebra on $\Sigma \times [0, 1]$ is $\mathcal{F} = \mathcal{F}^- \otimes \mathcal{F}^+$. The inverse map g^{-1} on $\Sigma \times [0, 1]$ induces an expanding map on Σ with an invariant measure ν^- (where ν is the invariant measure for g on $\Sigma \times [0, 1]$). Write $\mathcal{G} = \psi^{-1}\mathcal{F}^-$. The measure ν^- is computable from ν^+: for a cylinder $C = C_{\nu_1\ldots\nu_k} = \{\boldsymbol{\omega} \mid \omega_i = \nu_i \text{ for } i = 1, \ldots, k\}$, it satisfies $\nu^-(C) = \nu(F^{-k}(C \times [0, 1])) = \nu^+(J)$ with $F^{-k}(C \times [0, 1]) = \Sigma \times J$. Now $g^{-n}\mathcal{G} \uparrow \mathcal{F}$, and $g^{-1} : \mathcal{I} \to \mathcal{I}$ is the natural extension of $g^{-1} : [0, 1] \to [0, 1]$. By a continuously differentiable coordinate change, the strong unstable lamination $\mathcal{F}^{uu}$ is affine: $\mathcal{F}^{uu} = \{(\boldsymbol{\omega}, y, x) \mid \boldsymbol{\omega}, x \text{ constant}\}$. This makes F^{-1} like F up to interchanging strong stable and strong unstable directions. In the resulting coordinates, write $F^{-1}(\mathbf{y}, x) = (g^{-1}(\mathbf{y}), k^{-1}_{\mathbf{y}}(x))$ (where $k^{-1}_{\mathbf{y}}(x)$ depends only on $\boldsymbol{\omega}$ and x). Suppose that ζ^- is an invariant measure for F^{-1} on $\Sigma \times [0, 1] \times \mathbb{T}$ with σ-algebra $\mathcal{G} \otimes \mathcal{B}$ and with marginal ν^-. We write $\zeta^-_{\boldsymbol{\omega}}$, $\boldsymbol{\omega} \in \Sigma$, or sometimes $\zeta^-_{\mathbf{y}}$, for its disintegrations.

LEMMA 2.2. *Given the invariant measure ζ^- for F^{-1} acting on $\Sigma \times \mathbb{T}$, with marginal ν^- on Σ, there is an invariant measure ζ for F^{-1} acting on $\Sigma \times [0, 1] \times \mathbb{T}$, with marginal ν on $\Sigma \times [0, 1]$. For ν-almost all $\mathbf{y} \in \mathcal{S}$, the limit*

$$\zeta_{\mathbf{y}} = \lim_{n\to\infty} k^{-n}_{g^n(\mathbf{y})}\zeta^-_{g^n(\mathbf{y})} \tag{7}$$

gives its disintegrations.

Proof. As for Lemma 2.1, one can apply [**2**, Theorem 1.7.2] to prove the assertion. □

3. *Partial hyperbolicity*

See, for example, [**19**, §17.1] for the standard construction of the solenoid as an attractor for a diffeomorphism on $(-1, 1)^2 \times \mathbb{T}$. Likewise, the solenoid can appear as an attractor for a diffeomorphism on $(-1, 1)^d \times \mathbb{T}$ with $d \geq 2$. Under the assumption

$$m = \max_{y,x} f_y'(x) < \min_y g'(y) = M, \tag{8}$$

one may embed the solenoid as a hyperbolic attractor, so that the class of skew product systems is partially hyperbolic [**7**] on $\mathcal{S} \times \mathbb{T}$. This results in a partially hyperbolic splitting in one-dimensional strong unstable directions, one-dimensional center directions (the fibers) and the remaining d-dimensional strong stable directions. Write $N = (-1, 1)^d \times \mathbb{T}^2$ for the (open neighborhood in the) manifold that contains $\mathcal{S} \times \mathbb{T}$ as hyperbolic attractor; the map F on $\mathcal{S} \times \mathbb{T}$ is extended to a diffeomorphism F on N.

Write $W^{ss}(\mathbf{y}, x)$ for the strong stable manifold of $(\mathbf{y}, x)$ and $W^{uu}(\mathbf{y}, x)$ for the strong unstable manifold of $(\mathbf{y}, x)$. The strong stable and strong unstable manifolds form laminations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$.

LEMMA 3.1. *Assuming (8), there exists an embedding of $\mathcal{S}$ as a hyperbolic attractor for a smooth diffeomorphism on a manifold, so that the class of skew product systems is partially hyperbolic on $\mathcal{S} \times \mathbb{T}$. For $\max_{y,x}\{f_y'(x), 1/f_y'(x)\}$ sufficiently close to 1 and $M > 2$, such an embedding exists for which $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ are continuously differentiable laminations.*

Proof. In the strong stable directions, taking the dimension d to be sufficiently large (depending on the degree of the expanding circle map g), distances can be assumed to be contracted by a factor close to $1/2$.

Observe that, forced by the form of the map F, the local strong stable manifold $W^{ss}_{\text{loc}}(\mathbf{y}, a_0)$ for any $\mathbf{y} \in \psi^{-1}(y_0)$ equals $\psi^{-1}(y_0) \times \{a_0\}$. The strong stable lamination is therefore continuously differentiable. If $f_y'(x)$ is near 1 for all x and y, then with $M > 2$ (the expanding map g has to be of degree three or higher) spectral gap conditions are satisfied which imply that the strong unstable lamination is continuously differentiable. This is checked by going through the construction of the strong unstable lamination by graph transform techniques [**16**], as we will indicate.

One obtains the strong unstable lamination by integrating the line field formed by the strong unstable directions. Write $TN = N \times E^{uu} \times E^{ss,c}$ so that the strong unstable directions at a point $x \in \mathcal{S} \times \mathbb{T}$ are given as the graph of a linear map in $\mathcal{L}(E^{uu}, E^{ss,c})$. The strong unstable directions are then given by the graph of a section $\mathcal{S} \times \mathbb{T} \mapsto \mathcal{L}(E^{uu}, E^{ss,c})$ that is invariant under the induced diffeomorphism $\hat{F} : \mathcal{S} \times \mathbb{T} \times \mathcal{L}(E^{uu}, E^{ss,c}) \to \mathcal{S} \times \mathbb{T} \times \mathcal{L}(E^{uu}, E^{ss,c})$,

$$\hat{F}(\mathbf{y}, x, \alpha) = (F(\mathbf{y}, x), \beta), \quad \text{graph}\, \beta = DF(\mathbf{y}, x)\, \text{graph}\, \alpha. \tag{9}$$

It is possible to construct strong unstable directions on N that extend those on $\mathcal{S} \times \mathbb{T}$ by choosing a lamination on a fundamental domain in its basin of attraction and iterating under the graph transform [**24**, Appendix 1]. This produces a graph V^{uu} of a section $N \mapsto \mathcal{L}(E^{uu}, E^{ss,c})$ that is invariant under $\hat{F}$.

If λ^{ss} is the strongest rate of contraction, i.e. for some $C > 0$ and $i \in \mathbb{N}$ we have

$$|DF^i(n)v| \geq C(\lambda^{ss})^i |v|$$

for each $n \in N$ and $v \in T_x N$, then such a graph V^{uu} is normally hyperbolic for $m/M < \lambda^{ss}$. Indeed, the contraction of $\hat{F}$ along the fibers $\mathcal{L}(E^{uu}, E^{ss,c})$ is estimated by

$$D\hat{F}^i(n, \alpha)(0, w) \leq C(m/M)^i |w|; \tag{10}$$

cf. [**24**, Appendix 1]. Normal hyperbolicity holds for λ^{ss} near 1/2, m near 1 and $M > 2$. Normal hyperbolicity implies that V^{uu} is continuously differentiable, and this, in turn, implies that the strong unstable lamination is continuously differentiable [**27**]. □

4. *Robust transitivity*

We observe that

$$F_{i,j}(y, x) = (iy, x + jy) \mod 1,$$

with integers j and $i > 1$, is not topologically transitive; it leaves all circles parallel to $jx = (i-1)y$ mod 1 invariant. Note that $F_{i,j}$ induces a homeomorphism on $\mathcal{S} \times \mathbb{T}$, namely $F_{i,j}(\mathbf{y}, x) = (i\mathbf{y}, x + iy_0)$, with inverse $F_{i,j}^{-1}(\mathbf{y}, x) = (\ldots, y_{-2}, y_{-1}, x - jy_{-1})$.

The following result provides a class of robustly topologically mixing skew product maps. We use ad hoc arguments, relying on the skew product structure with topologically mixing base dynamics, to prove it, but the arguments bear a resemblance to the technique of blenders introduced in [**5**].

THEOREM 4.1. *There exist arbitrarily small smooth perturbations F, $F(y, x) = (g(y), f_y(x))$, of $F_{i,0}$, $i > 1$, that are robustly topologically mixing skew product maps (considered on either $\mathbb{T}^2$ or $\mathcal{S} \times \mathbb{T}$). Moreover:*

(i) *there exist $k \in \mathbb{N}$ and $\hat{y} \in \mathbb{T}$ such that $g^k(\hat{y}) = \hat{y}$ and $f_{\hat{y}}^k$ possesses a unique hyperbolic attracting and hyperbolic repelling fixed point;*

(ii) *for any $(\mathbf{y}, x) \in \mathcal{S} \times \mathbb{T}$, the strong stable and strong unstable manifolds $W^{ss}(\mathbf{y}, x)$ and $W^{uu}(\mathbf{y}, x)$ are dense in $\mathcal{S} \times \mathbb{T}$.*

Proof. Consider $\Sigma_n^+ = \{0, \ldots, n\}^{\mathbb{N}}$ endowed with the product topology, and let $\sigma : \Sigma_n^+ \to \Sigma_n^+$ be the left shift. The base map g (or some iterate thereof) admits invariant Cantor sets on which the dynamics is topologically conjugate to $\sigma : \Sigma_n^+ \mapsto \Sigma_n^+$. This observation and the following lemma imply the existence of robustly topologically mixing maps F acting on $\mathbb{T}^2$, as stated in the theorem.

LEMMA 4.1. *There exists a skew product map*

$$H(\boldsymbol{\omega}, x) = (\sigma\boldsymbol{\omega}, h_{\boldsymbol{\omega}}(x))$$

on $\Sigma_n^+ \times \mathbb{T}$, $n \geq 4$, that is robustly topologically mixing under continuous perturbations of $\boldsymbol{\omega} \mapsto h_{\boldsymbol{\omega}}$ in the C^1 topology.

Proof. Following [**13**], take circle diffeomorphisms h_0, h_1 and h_2 such that:

(i) for each $i = 0, 1, 2$, h_i has a unique hyperbolic attracting fixed point p_i and a unique hyperbolic repelling fixed point q_i, and the fixed points are mutually disjoint;

(ii) p_0 and p_1 are close to each other, and h_0 and h_1 are affine on $[p_0, p_1]$;

(iii) $p_2 \in (p_0, p_1)$;

(iv) $1/2 < (h_0)'(p_0), (h_1)'(p_1) < 1$.

The iterated function system generated by h_0, h_1, h_2 and $h_3 = h_2^{-1}$ is robustly minimal under C^1-small perturbations of $h_0, \ldots, h_3$. We present the main steps in the reasoning, referring to [**13**] for details. Consider the iterated function system generated by h_0 and h_1. For a compact subset $S \subset \mathbb{T}$, write $\mathcal{L}(S) = h_0(S) \cup h_1(S)$. Let $E_{\rm in} \subset [p_0, p_1] \subset E_{\rm out}$ be intervals close to $[p_0, p_1]$ on which h_0 and h_1 are affine. Then

$$E_{\rm in} \subset \mathcal{L}(E_{\rm in}) \subset [p_0, p_1] \subset \mathcal{L}(E_{\rm out}) \subset E_{\rm out}, \tag{11}$$

and $\mathcal{L}^i(E_{\rm in})$ and $\mathcal{L}^i(E_{\rm out})$ converge to $[p_0, p_1]$ in the Hausdorff topology as $i \to \infty$. Since h_0 and h_1 are contractions, this shows that the iterated function system generated by h_0 and h_1 is minimal on $[p_0, p_1]$. From the properties of h_2 and h_3, it is easily concluded that the iterated function system generated by h_0, h_1, h_2 and h_3 is minimal on $\mathbb{T}$.

The skew product system $H(\boldsymbol{\omega}, x) = (\sigma\boldsymbol{\omega}, h_{\omega_0}(x))$ is topologically mixing. Indeed, write $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}} \subset \Sigma_n^+$ and take an open set $U \subset \Sigma_2^+ \times [p_0, p_1]$. A high iterate $H^n(U)$ contains a strip $\Sigma_2^+ \times J$ in $\Sigma_2^+ \times [p_0, p_1]$. Now $H^{n+1}(U)$ maps $\Sigma_2^+ \times J$ to two strips with total width greater than $c|J|$, where $c = (h_0)'(p_0) + (h_1)'(p_1) > 1$. Further iterates $H^{n+k}(U)$ contain 2^k strips of increasing total width so that for some $k > 0$, $H^{n+k}(U)$ is dense in $\Sigma_2^+ \times I$ for any $I \subset [p_0, p_1]$. Iterates of $\Sigma_n^+ \times [p_0, p_1]$ under H are dense in $\Sigma_n^+ \times \mathbb{T}$ since the repelling fixed point q_2 of h_2 lies inside $[p_0, p_1]$. This shows that H is topologically mixing.

This reasoning also applies to small perturbations of H, where the fiber maps may also depend on all of $\boldsymbol{\omega}$ rather than just ω_0. We note the following changes in the argument. The inclusions (11) get replaced by

$$\Sigma_2^+ \times E_{\rm in} \subset H(\Sigma_2^+ \times E_{\rm in}), \qquad H(\Sigma_2^+ \times E_{\rm out}) \subset \Sigma_2^+ \times E_{\rm out}.$$

The map H acting on $\Sigma_2^+ \times E_{\rm out}$ acts by contractions in the fibers $\boldsymbol{\omega} \times E_{\rm out}$. A high iterate $H^n(U)$ may not contain a product $\Sigma_2^+ \times J$ but will contain a strip of some width ε lying between the graphs of two maps $\Sigma_2^+ \to \mathbb{T}$. Again, $H^{n+1}(U)$ contains two strips of total width exceeding $c\varepsilon$ for some $c > 1$, and the $H^{n+k}(U)$ contain 2^k strips of increasing total width. We conclude that there is an interval $[\tilde{p}_0, \tilde{p}_1]$ near $[p_0, p_1]$ such that for some $k > 0$, $H^{n+k}(U)$ is dense in $\Sigma_2^+ \times I$ for any $I \subset [\tilde{p}_0, \tilde{p}_1]$.

If $\boldsymbol{\omega}$ starts with a sequence of i symbols 2, then $h_{\sigma^i\boldsymbol{\omega}} \circ \cdots \circ h_{\boldsymbol{\omega}}$ maps an interval $I \subset \mathbb{T}$ that contains q_2 to an interval with length approaching 1 as $i \to \infty$. Also, any point in $\mathbb{T}$ can be mapped into $[\tilde{p}_0, \tilde{p}_1]$ by an iterate that involves $\boldsymbol{\omega}$ with a long sequence of symbols 3.

The lemma follows. □

As a consequence, F acting on $\mathbb{T}^2$ is topologically mixing. Indeed, take an open set U in $\Sigma_n^+ \times \mathbb{T}$. The construction in Lemma 4.1 gives that $\bigcup_{n\in\mathbb{N}} F^n(U)$ is open and dense in $\Sigma_n^+ \times \mathbb{T}$. Now take open sets $U, V \subset \mathbb{T}^2$. As g is expanding, some iterate of U under F intersects $\Sigma_n^+ \times \mathbb{T}$. Again, as g is expanding, a higher iterate will intersect V, establishing the topological mixing of $F : \mathbb{T}^2 \to \mathbb{T}^2$.

It easily follows that F acting on $\mathcal{S} \times \mathbb{T}$ is also topologically mixing. Just note that $g^n(\ldots, y_{-1}, y_0) = (\ldots, g^{n-1}(y_0), g^n(y_0))$, and that an open set in the product topology is of the form $\mathcal{S} \cap (\cdots U_{-2} \times U_{-1} \times U_0)$ with $U_{-i} \subset \mathbb{T}$ open, and is proper only for finitely many values of i.

Property (i) of the theorem follows as part of the above construction. The construction also implies that strong unstable manifolds are dense in $\mathcal{S} \times \mathbb{T}$. To see this, consider the periodic points P and Q for $F : \mathcal{S} \times \mathbb{T} \to \mathcal{S} \times \mathbb{T}$, where P corresponds to p_0 in the proof of the lemma and Q corresponds to q_2. The two-dimensional unstable manifold of Q is dense in $\mathcal{S} \times \mathbb{T}$ since unstable manifolds for g are dense in $\mathcal{S}$. Note that the stable manifold of Q contains points arbitrarily close to P. We claim that $W^u(Q) \subset \overline{W^{uu}(P)}$ (cf. [**5**, Lemma 1.9]): take a point $x \in W^u(Q)$ and a neighborhood V of it; iterate backwards and note that $F^{-m}(V)$ intersects $W^{uu}(P)$. Thus $W^{uu}(P)$, and therefore each strong unstable manifold, must be dense in $\mathcal{S} \times \mathbb{T}$.

Finally, use these arguments for inverse diffeomorphisms, making further small perturbations, to show that there are skew product maps for which the strong stable manifolds are also dense in $\mathcal{S} \times \mathbb{T}$. □

Definition 4.1. The skew product map F is said to be strongly contractive if for all $\varepsilon > 0$ there exist $\hat{y} \in \mathbb{T}$, an interval $V \subset \mathbb{S}^1$ and $n \in \mathbb{N}$ such that $|V| > 1 - \varepsilon$ and $|f^n_{\hat{y}}(I)| < \varepsilon$.

The following lemma, which provides a robust condition for F to be strongly contractive, is immediate.

LEMMA 4.2. *Suppose that there exist $k \in \mathbb{N}$ and $\hat{y} \in \mathbb{T}$ with $g^k(\hat{y}) = \hat{y}$ and $f^k_{\hat{y}}$ possessing a unique hyperbolic attracting and hyperbolic repelling fixed point. Then F is strongly contractive.*

5. *Attracting invariant graphs*

Contraction of positive orbits starting in the same fiber is explained by the following result. Theorem 1.1 follows from it. The arguments that establish random fixed points in iterated function systems—see [**11**, Proposition 5.7] (compare also [**23**, §2.3]) and [**35**]—are based on pushing forward a stationary measure by the circle diffeomorphisms and identifying limit measures. Although there is no stationary measure in our context, our proof of Theorem 5.1 is inspired by this approach. Alternative approaches, which use the theory of non-uniform hyperbolic systems to provide invariant delta measures in forced circle diffeomorphisms, are followed in [**10, 21, 31**]. Such approaches do not determine the number of points in each fiber and would therefore not enable us to explain Theorem 1.1.

THEOREM 5.1. *Let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be robustly topologically mixing as in Theorem 4.1, so that F is also strongly contractive. Then F acting on $\mathcal{S} \times \mathbb{T}^2$ admits an invariant graph $\{(\mathbf{y}, \omega^+(\mathbf{y})) \mid \mathbf{y} \in \mathcal{S}\}$, for a measurable function ω^+, that attracts the positive orbits of $\nu \times \lambda$-almost all initial points.*

Proof. The main steps in the proof are the following. We show that for ν-almost all $\mathbf{y}$, the push-forwards $f^n_{g^{-n}(\mathbf{y})}\lambda$ of Lebesgue measure λ contain delta measures in the fiber over $\mathbf{y}$ as accumulation points in the weak-star topology. Invoking Lemma 2.1, we establish that $f^n_{g^{-n}(\mathbf{y})}\lambda$ in fact converges to a delta measure, thus proving the existence of an invariant graph for F acting on $\mathcal{S} \times \mathbb{T}$. For the attraction properties, we must analogously consider F^{-1} and construct an invariant graph for F^{-1}.

We start with a lemma.

LEMMA 5.1. *Given $\varepsilon > 0$, for ν-almost all $\mathbf{y} \in \mathcal{S}$ there exist an interval $I \subset \mathbb{T}$ with $|I| > 1 - \varepsilon$ and $n \in \mathbb{N}$ such that $|f^n_{g^{-n}(\mathbf{y})}(I)| < \varepsilon$.*

Proof. The lemma will be a consequence of a construction in which we provide $\delta > 0$ and $L \in \mathbb{N}$ so that the following holds. Given an interval $J \subset \mathbb{T}$ and $n \in \mathbb{N}$, we construct for each $y \in J$ an interval $I \subset \mathbb{T}$ with $|I| > 1 - \varepsilon$, an open subset $J' \subset \psi^{-1}(J) \subset \mathcal{S}$ with $\nu(J')/\nu^+(J) = \nu(J')/\nu(\psi^{-1}(J)) > \delta$ and a positive integer $l \leq L$ such that for $\mathbf{y} \in J'$, $|f^{n+l}_{g^{-n-l}(\mathbf{y})}(I)| < \varepsilon$.

Fix $\varepsilon > 0$.

Step 1. There are intervals $K \subset \mathbb{T}$ and $V \subset \mathbb{T}$ with $|V| > 1 - \varepsilon$ and an integer $N \in \mathbb{N}$ such that $|f^N_y(V)| < \varepsilon$ for $y \in K$. We make this more explicit. By Theorem 4.1, $f^k_{\hat{y}}$ has a hyperbolic attracting fixed point p.

By taking suitable smooth coordinates near the forward orbit of $(\hat{y}, p)$, we may assume that the local unstable manifold $W^u_{\text{loc}}(\hat{y}, p)$ of $(\hat{y}, p)$ is contained in $\mathbb{T} \times \{p\}$.

The interval K can be taken to be a fundamental domain (i.e. an interval for which g^k maps one boundary point to the other), so that $g^i(K)$ stays close to the forward orbit of $\hat{y}$ for $0 \leq i \leq N$. We let N be a multiple of k, so that $g^N(K)$ is close to $\hat{y}$. By replacing K with $g^{-ki}(K)$ for some $i > 0$, and replacing N by $N + ki$, we can decrease the size of the image $f^N_y(V)$ while keeping $g^N(K)$ fixed. Write $U_y = f^N_y(V)$ and also $V_z = f^N_y(V)$ with $z = g^N(y)$.

Step 2. Take $q > 1/\varepsilon$. Take $\zeta_0 = y_{-n}$ in J. Write $\hat{\mathbf{y}}$ for the periodic point of $g : \mathcal{S} \to \mathcal{S}$ in $\psi^{-1}(\hat{y})$. Iterates of an interval $N \subset W^{uu}_{\text{loc}}(\hat{\mathbf{y}}, p)$ are dense in $\mathcal{S} \times \mathbb{T}$, by Theorem 4.1 and the expansion properties of g. One can therefore take a point $\boldsymbol{\zeta} = (\ldots, \zeta_0) \in \mathcal{S}$ so that:

(i) there are positive integers $M_1 < \cdots < M_q$ so that $\zeta_{-M_i} \in g^N(K)$ for $1 \leq i \leq q$;

(ii) with a_i and b_i given by $(\zeta_{-M_i}, b_i) \in W^{uu}_{\text{loc}}(\hat{\mathbf{y}}, p)$ and $(\zeta_{-M_1}, a_i) = F^{M_i - M_1}(\zeta_{-M_i}, b_i)$, the points a_i, $1 \leq i \leq q$, are disjoint.

Step 3. Take neighborhoods $L_i \subset g^N(K)$ of ζ_{-M_i} so that $f^{M_i}_z(V_z)$, $z \in L_i$, are disjoint for different i. Consider $\bigcap_{1 \leq i \leq q} g^{M_i}(L_i)$. Since finitely many such intervals (for varying y_0 and J) cover the circle $\mathbb{T}$, the numbers N and M_i are bounded (i.e. depend only on ε and the dynamical system F).

Step 4. Let $L = \bigcup_{1 \leq i \leq q} L_i \subset g^N(K)$. Let $O = \bigcup_{1 \leq i \leq q} g^{M_i}(L_i)$. For $y \in J \cap O$, there is a j, with $1 \leq j \leq q$, such that $f^n_y(V_z)$ with $z = g^{-M_j}(y) \cap L_j$ has length smaller than ε (since there are $q > 1/\varepsilon$ such disjoint intervals). This defines a set of y values for which one of the $f^{N+M_j+n}_y(U_y)$ is small.

For given $\varepsilon > 0$, there is a bound $\delta > 0$ such that $|\bigcup_{1 \leq i \leq q} L_i|/|J| > \delta$. A similar bound holds with ν^+ replacing the length of intervals, since ν^+ has density that is bounded and bounded away from zero. This ends the construction. Now define

$$\Delta_N = \{\mathbf{y} \in \mathcal{S} \mid \text{for each interval } I \text{ with } |I| > 1 - \varepsilon \text{ and each } i \leq N,\ |f^i_{g^{-i}(\mathbf{y})}(I)| > \varepsilon\}.$$

The above construction yields the estimate $\nu(\Delta_{tl}) \leq (1 - \delta)^t$. Thus $\nu(\Delta) = 0$, where

$$\Delta = \{\mathbf{y} \in \mathcal{S} \mid \text{for each interval } I \text{ with } |I| > 1 - \varepsilon \text{ and each } i,\ |f^i_{g^{-i}(\mathbf{y})}(I)| > \varepsilon\}. \qquad \square$$

Lemma 5.1 implies that the push-forwards $f^n_{g^{-n}(\mathbf{y})}\lambda$ contain a delta measure $\delta_{\omega^+(\mathbf{y})}$, concentrated at $\omega^+(\mathbf{y})$, as accumulation point. This yields an invariant graph $\{(\mathbf{y}, \omega^+(\mathbf{y})) \mid \mathbf{y} \in \mathcal{S}\}$ for $F : \mathcal{S} \times \mathbb{T} \to \mathcal{S} \times \mathbb{T}$. Let $\mu^+_{y_0}$ be obtained from $\delta_{\omega^+(\mathbf{y})}$, $\mathbf{y} = (\ldots, y_0)$ as in (5). The following lemma will be applied to yield that $f^n_{g^{-n}(\mathbf{y})}\lambda$ and $f^n_{g^{-n}(\mathbf{y})}\mu^+_{y_{-n}}$ converge to the delta measure $\delta_{\omega^+(\mathbf{y})}$ for ν-almost all $\mathbf{y} \in \mathcal{S}$. We refer to [**32, 33**] for general results on invariant measures for partially hyperbolic endomorphisms. Recall that a measure is diffuse if it has no atoms.

LEMMA 5.2. *For each $y_0 \in \mathbb{T}$,*

$$\operatorname{supp} \mu^+_{y_0} = \mathbb{T}. \tag{12}$$

Moreover, $\mu^+_{y_0}$ is diffuse and depends continuously on y_0 in the weak-star topology.

Proof. We use an estimate, $m = \max_{y,x}\{f'_y(x), 1/f'_y(x)\} < \min_y g'(y) = M$, which is implicit in Theorem 4.1. Consider $\mathbf{z}$ close to $\mathbf{y}$, i.e. z_i close to y_i for all $i \in -\mathbb{N}$. The branch of g defined near y_{-i-1} for which $g(y_{-i-1}) = g_{-i}$ has an inverse; in the following we denote it by g^{-1}, with the understanding that we consider orbits near $\mathbf{y}$. Consider $f^n_{g^{-n}(y)}(x) = f_{g^{-1}(y)} \circ \cdots \circ f_{g^{-n}(y)}(x)$ and compute

$$\frac{\partial}{\partial y} f^n_{g^{-n}(y)}(x) = \sum_{i=1}^{n} \big(f^{i-1}_{g^{-i+1}(\mathbf{y})}\big)'(f^{n-i+1}_{g^{-n}(\mathbf{y})}(x)) \frac{\partial}{\partial y} f_{g^{-i}(\mathbf{y})}(f^{n-i}_{g^{-n}(\mathbf{y})}(x))(g^{-i})'(\mathbf{y}),$$

which is uniformly bounded by (8). Likewise,

$$\frac{\partial}{\partial y} f^{n-l}_{g^{-n}(y)}(x) M^l = \mathcal{O}(1).$$

Now, for a subsequence $n_i \to \infty$, $f^{n_i}_{y_{-n_i}}\lambda$ converges in the weak-star topology to a delta measure $\delta_{\omega^+(\mathbf{y})}$. Take l so that $g^{-l}(\mathbb{T})$ is an interval of length $\mathcal{O}(\varepsilon^s)$ for a positive s. Recall that $f^n_{g^{-n}(\mathbf{y})}$ maps an interval V of length $1 - \varepsilon$ to an interval I of length ε. Then $f^{-l}_{\mathbf{y}}(I)$ is an interval of length ε^t for some $t > 0$. These estimates imply that for $\mathbf{z}$ near $\mathbf{y}$, $f^{n_i}_{g^{-n_i}(z)}\lambda$ converges to a delta measure $\delta_{\omega^+(\mathbf{z})}$ depending continuously on $\mathbf{z}$. The graph transform construction of the strong unstable lamination shows, in fact, that $(\mathbf{z}, \omega^+(\mathbf{z}))$ is in $W^{uu}(\mathbf{y}, \omega^+(\mathbf{y}))$. See also [**25**] and [**7**, Ch. 11].

Consider the local center stable manifolds $W^{ss,c}(y) = \{((\ldots, y_{-1}, y_0), x) \in \mathcal{S} \times \mathbb{T},\ y_0 = y\}$ in $\mathcal{S} \times \mathbb{T}$. The invariant measure μ has disintegrations μ_y along $W^{ss,c}(y)$, $y \in \mathbb{T}$. If π^{ss} denotes the projection onto the fiber $\mathbb{T}$,

$$\pi^{ss}((\ldots, y_{-1}, y_0), x) = (y_0, x),$$

then

$$\mu^+_{y_0} = \pi^{ss}\mu_{y_0}. \tag{13}$$

We claim that the disintegrations μ_{y_0} are u-invariant, meaning that the disintegrations μ_{y_0} are invariant under the holonomy along strong unstable leaves.

Consider F acting on $\Sigma \times \mathbb{T}^2$ (see §2) and take coordinates in which the strong unstable lamination is affine. Take a product measure $m = \nu_2 \times \nu$. A Césaro accumulation point of push-forwards $F^n m$ is a Gibbs u-measure [**9, 25**], which is unique [**8**]. The Césaro accumulation point is a product measure and hence u-invariant (see also [**3**, Remark 4.1]).

Equation (12) follows from (5), since strong unstable manifolds are dense in $\mathcal{S} \times \mathbb{T}$. If an open set has positive measure, the image under F also has positive measure. Since the measure μ is invariant and the strong unstable lamination is minimal, together with (13) this yields (12). Continuous dependence of $\mu^+_{y_0}$ on y_0 is implied by (13) and the u-invariance of μ_{y_0}. Since $\mu_{\mathbf{y}}$ is ergodic, the measure $\mu^+_{y_0}$ is ergodic. In view of (12), it is therefore diffuse. □

LEMMA 5.3. *For ν-almost all $\mathbf{y} \in \mathcal{S}$,*

$$\lim_{n\to\infty} f^n_{g^{-n}(\mathbf{y})}\mu^+_{g^{-n}(\mathbf{y})} = \lim_{n\to\infty} f^n_{g^{-n}(\mathbf{y})}\lambda = \delta_{\omega^+(\mathbf{y})}$$

for a delta measure $\delta_{\omega^+(\mathbf{y})}$.

Proof. Recall that $f^n_{g^{-n}(\mathbf{y})}\lambda$ has a delta measure $\delta_{\omega^+(\mathbf{y})}$ as accumulation point. Further, $f^n_{g^{-n}(\mathbf{y})}\mu^+_{g^{-n}(\mathbf{y})}$ converges by Lemma 2.1. By Lemma 5.2, $f^n_{g^{-n}(\mathbf{y})}\lambda$ and $f^n_{g^{-n}(\mathbf{y})}\mu^+_{g^{-n}(\mathbf{y})}$ converge to $\delta_{\omega^+(\mathbf{y})}$. □

We have constructed an invariant graph $\{(\mathbf{y}, \omega^+(\mathbf{y})\}$ for $F : \mathcal{S} \times \mathbb{T} \to \mathcal{S} \times \mathbb{T}$. To prove its attraction property, we need to consider iterates from time zero to time $n > 0$. Invertibility of the maps in the fibers implies that if $f^n_{\mathbf{y}}\lambda$ is close to a delta measure, then $f^{-n}_{g^n(\mathbf{y})}\lambda$ is also close to a delta measure. So we can also consider iterates from time $n > 0$ to time 0, for which we take the inverse skew product map F^{-1}. One can largely follow the previous reasoning to construct an invariant graph $\{(\mathbf{y}, \omega^-(\mathbf{y})\}$ for $F^{-1} : \mathcal{S} \times \mathbb{T} \to \mathcal{S} \times \mathbb{T}$.

We give the lemmas that correspond to Lemmas 5.2 and 5.3. Recall the last part of §2 on ergodic properties of F^{-1}.

LEMMA 5.4. *For each $\boldsymbol{\omega} \in \Sigma$,*

$$\operatorname{supp} \zeta^-_{\boldsymbol{\omega}} = \mathbb{T}. \tag{14}$$

Moreover, $\zeta^-_{\boldsymbol{\omega}}$ is diffuse and depends continuously on $\boldsymbol{\omega}$ in the weak-star topology.

LEMMA 5.5. *For ν-almost all $\mathbf{y} \in \mathcal{S}$,*

$$\lim_{n\to\infty} k^{-n}_{g^n(\mathbf{y})}\zeta^-_{g^n(\mathbf{y})} = \lim_{n\to\infty} f^{-n}_{g^n(\mathbf{y})}\lambda = \delta_{\omega^-(\mathbf{y})}$$

for a delta measure $\delta_{\omega^-(\mathbf{y})}$.

Lemma 5.5 implies that the graph of ω^+, whose existence is given by Lemma 5.3, is attracting. It attracts all points lying outside the graph of ω^-. This is true even if $\omega^+ = \omega^-$, but in fact $\omega^+(\mathbf{y}) \neq \omega^-(\mathbf{y})$ for ν-almost all $\mathbf{y}$. This can be seen by writing $\mathcal{S} = \Sigma \times I/\!\sim$ as in (6), so that in $\mathbf{y} = \boldsymbol{\omega} \times y$ the 'past' $\boldsymbol{\omega}$ and the 'future' y are independent. The resulting positions $\omega^+(\mathbf{y})$ and $\omega^-(\mathbf{y})$ depend, respectively, on the past and future only, and vary according to Lemmas 5.3 and 5.5.

This finishes the proof of Theorem 5.1. □

REFERENCES

[1] V. A. Antonov. Modeling of processes of cyclic evolution type: synchronization by a random signal. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **2** (1984), 67–76.

[2] L. Arnold. *Random Dynamical Systems (Springer Monographs in Mathematics)*. Springer, Berlin, 1998.

[3] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.* **181** (2010), 115–174.

[4] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares and C. S. Zhou. The synchronization of chaotic systems. *Phys. Rep.* **366**(1–2) (2002), 1–101.

[5] C. Bonatti and L. J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math.* (2) **143**(2) (1996), 357–396.

[6] C. Bonatti, L. J. Díaz and A. S. Gorodetskiĭ. Non-hyperbolic ergodic measures with large support. *Nonlinearity* **23**(3) (2010), 687–706.

[7] C. Bonatti, L. J. Díaz and M. Viana. *Dynamics Beyond Uniform Hyperbolicity (Encyclopaedia of Mathematical Sciences, 102)*. Springer, Berlin, 2005.

[8] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115** (2000), 157–193.

[9] K. Burns, D. Dolgopyat, Ya. Pesin and M. Pollicott. Stable ergodicity for partially hyperbolic attractors with negative central exponents. *J. Mod. Dyn.* **2**(1) (2008), 63–81.

[10] H. Crauel. Extremal exponents of random dynamical systems do not vanish. *J. Dynam. Differential Equations* **2** (1990), 245–291.

[11] B. Deroin, V. A. Kleptsyn and A. Navas. Sur la dynamique unidimensionnelle en régularité intermédiaire. *Acta Math.* **199**(2) (2007), 199–262.

[12] L. J. Díaz and A. S. Gorodetskiĭ. Non-hyperbolic ergodic measures for non-hyperbolic homoclinic classes. *Ergod. Th. & Dynam. Sys.* **29**(5) (2009), 1479–1513.

[13] G. H. Ghane, A. J. Homburg and A. Sarizadeh. C^1 robustly minimal iterated function systems. *Stoch. Dyn.* **10**(1) (2010), 155–160.

[14] A. S. Gorodetskiĭ and Yu. S. Il'yashenko. Certain properties of skew products over a horseshoe and a solenoid. *Dynamical Systems, Automata, and Infinite Groups (Proceedings of the Steklov Institute of Mathematics, 231)*. Ed. R. I. Grigorchuk. MAIK Nauka/Interperiod. Publ., Moscow, 2000, pp. 90–112.

[15] A. S. Gorodetskiĭ, Yu. S. Il'yashenko, V. A. Kleptsyn and M. B. Nal'skiĭ. Nonremovability of zero Lyapunov exponents. *Funct. Anal. Appl.* **39**(1) (2005), 21–30.

[16] M. W. Hirsch, C. C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, Berlin, 1977.

[17] A. J. Homburg. Atomic disintegrations for partially hyperbolic diffeomorphisms. *Preprint*, 2011, available at http://www.science.uva.nl/∼alejan/publications.html.

[18] T. H. Jäger. Strange non-chaotic attractors in quasiperiodically forced circle maps. *Comm. Math. Phys.* **289**(1) (2009), 253–289.

[19] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems, with a Supplement by Anatole Katok and Leonardo Mendoza (Encyclopedia of Mathematics and Its Applications, 54)*. Cambridge University Press, Cambridge, 1997.

[20] V. A. Kleptsyn and M. B. Nalskii. Contraction of orbits in random dynamical systems on the circle. *Funct. Anal. Appl.* **38**(4) (2004), 267–282.

[21] Y. Le Jan. Equilibre statistique pour les produits de difféomorphismes aléatoires indépendants. *Ann. Inst. Henri. Poincaré Probab. Stat.* **23** (1987), 111–120.

[22] R. Mañé. *Ergodic Theory and Differentiable Dynamics (Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, 8)*. Springer, Berlin, 1987.

[23] A. Navas. *Groups of Circle Diffeomorphisms*. University of Chicago Press, Chicago, IL, 2011.

[24] J. Palis and F. Takens. *Hyperbolicity & Sensitive Chaotic Dynamics at Homoclinic Bifurcations*. Cambridge University Press, Cambridge, 1993.

[25] Ya. B. Pesin and Ya. G. Sinaĭ. Gibbs measures for partially hyperbolic attractors. *Ergod. Th. & Dynam. Sys.* **2**(3–4) (1983), 417–438, 1982.

[26] Ya. B. Pesin. *Lectures on Partial Hyperbolicity and Stable Ergodicity (Zurich Lectures in Advanced Mathematics)*. European Mathematical Society, Zürich, 2004.

[27] C. Pugh, M. Shub and A. Wilkinson. Hölder foliations. *Duke Math. J.* **86**(3) (1997), 517–546, Correction. *Duke Math. J.* **105**(1) (2000) 105–106.

[28] D. Ruelle. Perturbation theory for Lyapunov exponents of a toral map: extension of a result of Shub and Wilkinson. *Israel J. Math.* **134** (2003), 345–361.

[29] D. Ruelle and A. Wilkinson. Absolutely singular dynamical foliations. *Comm. Math. Phys.* **219**(3) (2001), 481–487.

[30] M. Shub. Endomorphisms of compact differentiable manifolds. *Amer. J. Math.* **91**(1) (1969), 175–199.

[31] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.* **139**(3) (2000), 495–508.
[32] M. Tsujii. Fat solenoidal attractors. *Nonlinearity* **14**(5) (2001), 1011–1027.
[33] M. Tsujii. Physical measures for partially hyperbolic surface endomorphisms. *Acta Math.* **194**(1) (2005), 37–132.
[34] R. F. Williams. Expanding attractors. *Publ. Math. Inst. Hautes Études Sci.* **43** (1974), 169–203.
[35] H. Zmarrou and A. J. Homburg. Dynamics and bifurcations of random circle diffeomorphisms. *Discrete Contin. Dyn. Syst. Ser. B* **10**(2–3) (2008), 719–731.