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Temporal aggregation in first order cointegrated vector autoregressive

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Temporal aggregation in first order cointegrated vector autoregressive models

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Abstract
We study aggregation - or sample frequencies - of time series, e.g. aggregation from weekly to monthly or quarterly time series. Aggregation usually gives shorter time series but spurious phenomena, in e.g. daily observations, can on the other hand be avoided. An important issue is the effect of aggregation on the adjustment coefficient in cointegrated systems. We study only first order vector autoregressive processes for n dimensional time series X_t, and we illustrate the theory by a two dimensional and a four dimensional model for prices of various grades of gasoline.

1 Introduction

Assume that a multivariate process X_t for integer t is observed only every k'th point in time; that is consider the time series X_{ks}, for integer s. Intuitively it is clear, that the dependence in the series is weakened by aggregation, as e.g. autocorrelations over lag k are usually smaller than first order autocorrelations. Moreover the aggregated series are usually shorter as they contain a smaller number of observations than the basic high frequency series. A smaller number of observations and a less distinct autocorrelation structure imply less significantly estimated parameters and weaker tests. On the other hand spurious phenomena in high frequency series, which are of no practical interest, are avoided by the aggregation.

The problems arising when multivariate time series are aggregated are studied by many authors, see e.g. Lütkepohl(1987) and Marcellino(1999). Assuming a known model for the time series observed at the highest frequency makes it possible to derive an explicit description of the model for the aggregated series. The most general result is given by Marcellino(1999), covering situations with much more general models than those considered in the present paper. His expression of the model for the aggregated process is, however, not very useful in practical situations as it is rather involved. In section 2 we present an explicit formula in the simple situation of vector autoregressive models of order one and present a simple example. This model is extended in section 3 to the similar cointegration models for VAR(1) processes. Even though this model is very simple it is often economically relevant as quite a few prices and price ratios will follow it. In section 4 the two dimensional case is considered in details and an example is presented. In section 5 this example is extended to a four dimensional model. The practical examples consider data for price series of various grades of gasoline, for which thousands of daily observations are aggregated to weekly, monthly and quarterly series.

2 Aggregation in a vector autoregressive model of order one

In this section we shortly consider the theory for aggregation in vector autoregressive models of order one. Following the notation of Marcellino(1999) consider a vector autoregressive model of order one:

$\mathbf{G}(L)X_t = (1 - \mathbf{G}_1 L)X_t = E_t$,

where E_t denotes a white noise stochastic error term and $\mathbf{G}(L)$ is the autoregressive polynomial.

Pre multiplication by the lag polynomial

$\mathbf{B}(L) = (\mathbf{I} + \mathbf{G}_1 L + \mathbf{G}_1^2 L^2 + .. + \mathbf{G}_1^{k-1} L^{k-1})$

gives

$\mathbf{C}(L) = \mathbf{B}(L)\mathbf{G}(L) = (\mathbf{I} - \mathbf{G}_1^k L^k)$

and hence we have the expression

$(1 - \mathbf{G}_1^k L^k)X_t = \mathbf{B}(L)E_t$.

If we consider the series X^*_t which is point-in-time aggregated from the series X_t, that is the series X_t sampled as every k'th observation, we obtain the expression:

$(1 - \mathbf{G}_1^k L)X^*_t = E^*_t$.

Here E^*_t is a white noise series as it equals $B(L)E_t$, which is a moving average of order k -1 sampled every k'th time unit. It is possible to extend the first order autoregressive model by moving average terms, but for a more precise discussion of the moving average terms see Marcellino(1999).

First we assume that the singular value decomposition of $\mathbf{G}_1$ exist, that is

$\mathbf{G}_1 = \mathbf{A}\mathbf{\Lambda}\mathbf{A}^{-1}$ or $\mathbf{\Lambda} = \mathbf{A}^{-1}\mathbf{G}_1\mathbf{A}$.

Here the matrix $\mathbf{\Lambda}$ is diagonal having the eigenvalues of $\mathbf{G}_1$ in the diagonal and the matrix $\mathbf{A}$ consists of the eigenvectors of $\mathbf{G}_1$ that is the roots of the determinant equation taken in the same order as the eigenvalues appearing in the diagonal of the matrix $\mathbf{\Lambda}$. This decomposition could involve complex numbers and it does not necessarily exist as seen in section 3.2.

Notice that the characteristic equation for $\mathbf{G}_1$ is related to the determinant equation for the lambda polynomial $\mathbf{G}(\lambda)$ as

$\det(\mathbf{G}_1 - \lambda\mathbf{I}) = \det(\mathbf{G}(\lambda))\,(-\lambda)^{-n}$.

The matrix $\mathbf{G}_1^k$ is for any power k then decomposed as $\mathbf{G}_1^k = \mathbf{A}\mathbf{\Lambda}^k\mathbf{A}^{-1}$. This means that the eigenvalues of $\mathbf{G}_1^k$ simply equal the eigenvalues of $\mathbf{G}_1$ raised to the power k.

Of course this decomposition is not generally valid for roots of multiplicity greater than one, but it follows from a mathematical result, e.g. Dhrymes (1984) page 47, that every square matrix is very close to matrix with distinct characteristic roots and hence very close to a matrix for which it is valid. If the matrix $\mathbf{G}_1$ is estimated as a stationary autoregressive model, this decomposition will be valid.

3 Cointegration in a first order VAR model

In a cointegration model we consider the situation where the matrix $\mathbf{G}_1$ has one or more unit roots. Following the well known theory of cointegration, see e.g. Johansen(1995), for an n dimensional series $\mathbf{X}_t$ systems we write

$$\Delta\mathbf{X}_t = (\mathbf{G}_1 - \mathbf{I})\mathbf{X}_{t-1} + \mathbf{E}_t = \mathbf{\Pi}\mathbf{X}_{t-1} + \mathbf{E}_t .$$

When the rank of the n × n matrix $\mathbf{\Pi} = -\mathbf{G}(1)$ is denoted r, the matrix $\mathbf{\Pi}$ admits the rank factorization

$$\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}^T ,$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are n × r matrices of full rank r.

As the autoregressive polynomial for the aggregated process is

$$\mathbf{C}(L) = (\mathbf{I} - \mathbf{G}_1^k L) = \mathbf{B}(L)\mathbf{G}(L)$$

the similar rank factorization of the aggregated series is

$$\mathbf{\Pi}^* = \boldsymbol{\alpha}^*\boldsymbol{\beta}^{*T} = -\mathbf{C}(1) = -\mathbf{B}(1)\mathbf{G}(1) = \mathbf{B}(1)\mathbf{\Pi} = \mathbf{B}(1)\boldsymbol{\alpha}\boldsymbol{\beta}^T .$$

Hence the beta part of the factorization is independent of the aggregation, The alpha part - the adjustment factors - of the model for the aggregated series simply equals $\mathbf{B}(1)$ times the alpha part of the original model.

$$\boldsymbol{\alpha}^* = \mathbf{B}(1)\boldsymbol{\alpha}$$

for $\mathbf{B}(1)$ given as above (Marcellino(1999)).

3.1 The structure of B(1)

Assume first that a singular value decomposition of $\mathbf{G}_1$ exists

$$\mathbf{G}_1 = \mathbf{A}\mathbf{\Lambda}\mathbf{A}^{-1} \text{ or } \mathbf{\Lambda} = \mathbf{A}^{-1}\mathbf{G}_1\mathbf{A}$$

In which case the matrix $\mathbf{\Lambda}$ is diagonal having the eigenvalues of $\mathbf{G}_1$ in the diagonal and the matrix $\mathbf{A}$ is composed by the eigenvectors of $\mathbf{G}_1$ in the right order. As

$$\mathbf{\Pi} = -\mathbf{G}(1) = \mathbf{G}_1 - \mathbf{I}$$

the n × n matrices $\mathbf{\Pi}$, $\mathbf{G}(1)$ and $\mathbf{G}_1$ all have the same eigenvectors and the corresponding eigenvalues for $\mathbf{\Pi}$ and $\mathbf{G}(1)$ are the eigenvalues of $\mathbf{G}_1$ minus one.

Using this, we get

$$\mathbf{B}(1) = \mathbf{I} + \mathbf{G}_1 + \mathbf{G}_1^2 + .. + \mathbf{G}_1^{k-1} = \mathbf{A}\sum_{i=0}^{k-1}\mathbf{\Lambda}^i\mathbf{A}^{-1} = \mathbf{A}\sum_{i=0}^{k-1}\begin{pmatrix}\lambda_1^i & 0 & . & . & 0\\ 0 & . & . & . & .\\ . & . & . & . & .\\ . & . & . & . & 0\\ 0 & . & . & 0 & \lambda_n^i\end{pmatrix}\mathbf{A}^{-1} =$$

$$\mathbf{A}\begin{pmatrix}\frac{1-\lambda_1^k}{1-\lambda_1} & 0 & . & . & 0\\ 0 & . & . & . & .\\ . & . & . & . & .\\ . & . & . & . & 0\\ 0 & . & . & 0 & \frac{1-\lambda_n^k}{1-\lambda_n}\end{pmatrix}\mathbf{A}^{-1}$$

In this relation the λ's denote the eigenvalues of the matrix $\mathbf{G}_1$. If an eigenvalue λ is one, the corresponding diagonal element of course equals k as a sum of k ones. In section 4 we in some detail consider how the eigenvalues and the matrix $\mathbf{A}$ are related to the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

However, as is well known from standard matrix theory, this decomposition is not generally valid. In practical situations the dimension of the time series is usually very low, leaving no "space" for the special possibilities in which the validity is violated, but for completeness these situations are to some extent discussed in the next section.

3.2 Special situations

a)Complex eigenvalues

One situation is the possibility of complex eigenvalues. This situation could only happen for systems of dimension of at least three, as this requires at least one cointegrating relation and a pair of complex conjugate roots. As the complex roots fall in pairs of complex conjugates and hence the roots raised to any power also give pairs of complex conjugates, this possibility does not affect the general formula.

Example

Consider the three dimensional VAR(1) model with

$$\mathbf{G}_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0.5 & 0.5\\ 0 & -0.5 & 0.5\end{bmatrix} \quad \Pi = \begin{bmatrix}0 & 0 & 0\\ 0 & 0.5 & 0.5\\ 0 & -0.5 & 0.5\end{bmatrix}$$

The matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ in the decomposition $\mathbf{G}_1 = \mathbf{A}^{-1}\mathbf{\Lambda}\mathbf{A}$ are

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.5i \\ 0 & 0.5 & 0.5i \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5+0.5i & 0 \\ 0 & 0 & 0.5-0.5i \end{pmatrix}$$

We find

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & 0 \\ -0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For k = 3, that is aggregation of monthly data to quarterly data, we have

$$\mathbf{B}(1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1.5 & 1 \\ 0 & -1 & 1.5 \end{bmatrix}$$

And hence

$$\boldsymbol{\alpha}^*(k=3) = \begin{bmatrix} 0 & 0 \\ -1.25 & 0.25 \\ -0.25 & -1.25 \end{bmatrix}$$

Similarly, for k = 12 we get

$$\mathbf{B}(1) = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 1.015625 & 1.015625 \\ 0 & -1.015625 & 1.015625 \end{bmatrix}$$

And hence

$$\boldsymbol{\alpha}^*(k=12)=\begin{bmatrix} 0 & 0 \\ -1.015625 & 0 \\ 0 & -1.015625 \end{bmatrix}$$

The main point in this example is that the adjustment for the disaggregated series and for the series aggregated to lag k = 3 is done in both series x_2 and x_3, while it for aggregation period k = 12 is done within the series x_2 and x_3 separately. It is seen that in a situation where the complex root raised to some power is real, that is the root is of the form

$$a \pm \frac{2\pi}{k}$$

for integer k, the adjustment matrix $\boldsymbol{\alpha}^*$ could reduce to a matrix containing many zeroes.

b)Jordan
Another possibility arises if some characteristic roots of the matrix $\mathbf{G}_1$ are equal. In this situation factorizations like the ones above are not generally valid.

The situation of characteristic roots of the matrix $\boldsymbol{\Pi}$ different from zero having multiplicity greater than one is hardly of practical relevance, as every matrix is approximated arbitrarily close to a matrix with different characteristic roots, see Dhrymes(1984). The case of the characteristic root one for $\mathbf{G}_1$ (zero for $\boldsymbol{\Pi}$) having larger root multiplicity than the cointegration rank could however happen. For this reason the following remarks are necessary.

In this case the matrix $\mathbf{G}_1$ admits decomposition into a Jordan form

$$\mathbf{G}_1 = \mathbf{AJA}^{-1}$$

in which case the Jordan matrix has the form of a block diagonal matrix with so called Jordan blocks along the main diagonal. A Jordan block is a b × b quadratic matrix of the form

$$\mathbf{J}=\begin{pmatrix} \lambda & 1 & . & . & 0 \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & \lambda & 1 \\ 0 & . & . & 0 & \lambda \end{pmatrix}$$

in which the number λ in the diagonal is an eigenvalue of the matrix $\mathbf{G}_1$ and the superdiagonal consists of ones, all other entries being zero. The eigenvalues could form pairs of complex conjugates and the matrix **A** could similarly have complex entries.

Following the notation in section 2 the matrix $\mathbf{B}(1)$ is given by

$$\mathbf{B}(1) = \mathbf{I} + \mathbf{G}_1 + \mathbf{G}_1^2 + .. + \mathbf{G}_1^{k-1} = \mathbf{A}\sum_{i=0}^{k-1}\mathbf{J}^i\,\mathbf{A}^{-1}$$

The i'th power of a b × b Jordan block is

$$\mathbf{J}^i = \begin{pmatrix} \lambda^i & i\lambda^{i-1} & . & \binom{i}{a}\lambda^{i-a} & . & \binom{i}{b}\lambda^{i-b} \\ 0 & . & . & . & . & . \\ . & . & . & . & . & \binom{i}{a}\lambda^{i-a} \\ 0 & . & . & . & . & . \\ . & . & . & . & . & i\lambda^{i-1} \\ 0 & . & 0 & . & 0 & \lambda^i \end{pmatrix}$$

That is an upper triangular matrix having bands of the form $\binom{i}{a}\lambda^{i-a}$ in the a'th superdiagonal. Here the binomial is to be interpreted as zero for a > i.

For a Jordan block we see that the sum

$$\sum_{i=0}^{k-1}\mathbf{J}^i = \sum_{i=0}^{k-1}\begin{pmatrix} \lambda & 1 & . & . & 0 \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & \lambda & 1 \\ 0 & . & . & 0 & \lambda \end{pmatrix}^i$$

is a band matrix of terms of the form

$$\sum_{i=a}^{k-1}\binom{i}{a}\lambda^{i-a} = \sum_{i=0}^{k-a-1}\binom{i+a}{a}\lambda^i$$

in the a'th superdiagonal, which gives a polynomial in λ of order k – a and a polynomial in the aggregation period k of order a.

For a unit root this term equals

$$\sum_{i=a}^{k-1}\binom{i}{a} = \sum_{i=0}^{k-a-1}\binom{i+a}{a}$$

This is a polynomial in k of order a+1.

Example 1

Consider the two dimensional VAR(1) model

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

This is a stationary AR(1) model

$y_t = \lambda y_{t-1} + \varepsilon_{2t}$

for the series y_t when $|\lambda| < 1$ and an AR(1) with $y_{t-1,}$ the lagged value of the first AR(1) series, added of the form

$x_t = y_{t-1} + \lambda x_{t-1} + \varepsilon_{1t}$

for the series x_t.

For this process the aggregated model is

$$\begin{pmatrix} x^*_{t\,t} \\ y^*_t \end{pmatrix} = \begin{pmatrix} \lambda^k & (k-1)\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix} \begin{pmatrix} x^*_{t-1} \\ y^*_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon^*_{1t} \\ \varepsilon^*_{2t} \end{pmatrix} + \text{MA-terms}$$

and nothing particular happens.

Example 2

As a more interesting example we consider the situation $\lambda = 1$. In this case the model becomes

$x_t = x_{t-1} + y_{t-1} + \varepsilon_{1t}$ or $\Delta x_t = y_{t-1} + \varepsilon_{1t}$

$y_t = y_{t-1} + \varepsilon_{2t}$ or $\Delta y_t = \varepsilon_{2t}$

The series of first differences of the series x_t then equals a random walk y_t and we then see that the series x_t is in fact I(2). A situation with such different order of differencing is hardly to appear in practical modelling. This model has no parameters to be estimated.

For this model

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\mathbf{B}(1) = \sum_{i=0}^{k-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i = \begin{pmatrix} k & \frac{k(k+1)}{2} \\ 0 & k \end{pmatrix}$$

and hence the adjustment matrix is a polynomial of degree two in k.

$$\boldsymbol{\alpha}^* = \begin{pmatrix} k & \frac{k(k+1)}{2} \\ 0 & k \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{k(k+1)}{2} \\ k \end{pmatrix}$$

3.3 Eigenvalues of Π for r = 1

As the (n × r) matrix $\boldsymbol{\beta}$ is of rank r the n - r independent solutions to the equation $\boldsymbol{\beta}^T\mathbf{z} = 0$ forms n - r eigenvectors of $\boldsymbol{\Pi}$ corresponding to the eigenvalue zero. If all other roots of the characteristic polynomial of $\boldsymbol{\Pi}$ are different, i.e. have root multiplicity one, the matrix $\boldsymbol{\Pi}$ admits a diagonalization.

If r = 1 the only solution to the eigenvalue problem except for the unit roots is the eigenvector $\boldsymbol{\alpha}$ of dimension (n × 1) and the corresponding eigenvalue of the matrix $\boldsymbol{\Pi}$ is the number $\lambda = \boldsymbol{\beta}^T\boldsymbol{\alpha}$ (of dimension (1 × 1)) as we trivially see

$\boldsymbol{\Pi\alpha} = \boldsymbol{\alpha\beta}^T\boldsymbol{\alpha}$.

The corresponding eigenvalue of $\mathbf{G}_1$ is $\lambda = 1 + \boldsymbol{\beta}^T\boldsymbol{\alpha}$. If r = 1 the (n × n) matrix $\mathbf{A}$ has the form

$\mathbf{A}$ = (The n -1 eigenvectors for 1 | $\boldsymbol{\alpha}$)

in which the bar | means concatenation.

We then have

$$\boldsymbol{\alpha}^* = \mathbf{A} \begin{pmatrix} k & 0 & . & . & 0 \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & k & 0 \\ 0 & . & . & 0 & \frac{1-\lambda^k}{1-\lambda} \end{pmatrix} \mathbf{A}^{-1}\boldsymbol{\alpha}$$

As $\boldsymbol{\alpha}$ is the last column of the matrix $\mathbf{A}$ we have

$$\mathbf{A}^{-1}\boldsymbol{\alpha} = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

Hence

$$\boldsymbol{\alpha}^{*} = \mathbf{A}\begin{pmatrix} k & 0 & . & . & 0 \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & k & 0 \\ 0 & . & . & 0 & \frac{1-\lambda^{k}}{1-\lambda} \end{pmatrix}\begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix} = \frac{1-\lambda^{k}}{1-\lambda}\boldsymbol{\alpha}$$

as $\boldsymbol{\alpha}$ is the last column of the matrix **A.**

Note that the ratio $\frac{1-\lambda^{k}}{1-\lambda}$ is an exponentially increasing function of the aggregation period k as the eigenvalue $\lambda = 1 + \boldsymbol{\beta}^{T}\boldsymbol{\alpha}$ of $\mathbf{G}_1$ is numerically greater than one.

3.4 Eigenvalues of Π for r > 1

By diagonalisation of the (r × r) matrix $\boldsymbol{\beta}^{T}\boldsymbol{\alpha}$ there exists a matrix **F** (perhaps complex) and a diagonal matrix **M** such that

$$\mathbf{F}^{-1}\boldsymbol{\beta}^{T}\boldsymbol{\alpha}\mathbf{F} = \mathbf{M} = \begin{pmatrix} \mu_1 & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \mu_r \end{pmatrix}$$

(perhaps of Jordan form in the special situations of root multiplicity larger than the eigenspace multiplicity)

The diagonal elements of **M** are nonzero as the matrix $\boldsymbol{\beta}^{T}\boldsymbol{\alpha}$ is of full rank.

We then see that letting

$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}\mathbf{F}$ and $\boldsymbol{\beta}_1 = \boldsymbol{\beta}\,\mathbf{F}^{-1T}$

defines another parameterization of

$\boldsymbol{\Pi} = \boldsymbol{\alpha}_1\boldsymbol{\beta}_1^{T} = \boldsymbol{\alpha}\mathbf{F}(\boldsymbol{\beta}\,\mathbf{F}^{-1T})^{T}$.

As $\boldsymbol{\alpha}_1\boldsymbol{\beta}_1{}^{\mathrm{T}}\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_1\mathbf{M}$, we see that the non-zero eigenvalues of the matrix $\boldsymbol{\alpha}_1\boldsymbol{\beta}_1{}^{\mathrm{T}}$ are the diagonal elements in $\mathbf{M} = \boldsymbol{\beta}_1{}^{\mathrm{T}}\boldsymbol{\alpha}_1$ with the columns of $\boldsymbol{\alpha}_1$ as the corresponding eigenvectors.

We further see that the diagonalisation of $\mathbf{G}_1 = \boldsymbol{\Pi} + \mathbf{I}$ is

$\mathbf{G}_1 = \mathbf{A}(\mathbf{I} + \mathbf{D})\mathbf{A}^{-1}$

where the matrix $\mathbf{D}$ is of the form

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & 0 & . & . & . \\ . & . & . & \mu_1 & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & 0 & \mu_r \end{pmatrix}$$

and

$\mathbf{A}$ = (n - r eigenvectors of $\boldsymbol{\Pi}$ for zero | $\boldsymbol{\alpha}_1$)

such that

$$\mathbf{A}^{-1}\,\boldsymbol{\alpha}_1 = \begin{pmatrix} \mathbf{0}_{n-r} \\ \mathbf{I}_r \end{pmatrix}.$$

$$\mathbf{B}(1) = \mathbf{A}\sum_{i=0}^{k-1}\left(\mathbf{I} + \mathbf{D}\right)^i\mathbf{A}^{-1} = \mathbf{A}\begin{pmatrix} k & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & k & 0 & . & 0 \\ 0 & . & 0 & \dfrac{(1+\mu_1)^k - 1}{\mu_1} & 0 & 0 \\ . & . & . & 0 & . & 0 \\ 0 & . & 0 & 0 & 0 & \dfrac{(1+\mu_r)^k - 1}{\mu_r} \end{pmatrix}\mathbf{A}^{-1}$$

This means that the α-matrix for the aggregated series corresponding to $\boldsymbol{\beta}_1$ is

$$\boldsymbol{\alpha}_1^* = \mathbf{B}(1)\,\boldsymbol{\alpha}_1 = \mathbf{A}\sum_{i=0}^{k-1}\left(\mathbf{I}+\mathbf{D}\right)^i \mathbf{A}^{-1}\boldsymbol{\alpha}_1 = \mathbf{A}\begin{pmatrix} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \\ \dfrac{(1+\mu_1)^k-1}{\mu_1} & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & \dfrac{(1+\mu_r)^k-1}{\mu_r} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{(1+\mu_1)^k-1}{\mu_1} & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & \dfrac{(1+\mu_r)^k-1}{\mu_r} \end{pmatrix}\boldsymbol{\alpha}_1$$

We have

$$\boldsymbol{\Pi}^* = \boldsymbol{\alpha}_1^*\,\boldsymbol{\beta}_1^{\mathrm{T}} = \boldsymbol{\alpha}_1^*\,(\boldsymbol{\beta}\,\mathbf{F}^{-1\mathrm{T}})^{\mathrm{T}} = \boldsymbol{\alpha}_1^*\,\mathbf{F}^{-1}\,\boldsymbol{\beta}^{\mathrm{T}}$$

and then as $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}\mathbf{F}$ we arrive at the main result for the α-matrix of the aggregated series corresponding to the original $\boldsymbol{\beta}$

$$\boldsymbol{\alpha}^* = \begin{pmatrix} \dfrac{(1+\mu_1)^k-1}{\mu_1} & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & \dfrac{(1+\mu_r)^k-1}{\mu_r} \end{pmatrix}\boldsymbol{\alpha}$$

This has the form of $\boldsymbol{\alpha}$ multiplied by a diagonal matrix, which for one thing implies that eventual zeroes in the matrix $\boldsymbol{\alpha}$ are also present in the matrix $\boldsymbol{\alpha}$ for the aggregated series.

4 Cointegration in the two dimensional VAR model of order one

In this section we present in detail the parameter values for a two dimensional time series.

In case n = 2 and with one cointegration vector present, all possible parameterizations are of the form

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{pmatrix} 1 \\ -\beta \end{pmatrix}.$$

The matrix $\boldsymbol{\Pi} = -\mathbf{G}(1)$ has the form

$$\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}^T = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \begin{pmatrix} 1 & -\beta \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha\beta \\ \gamma & -\gamma\beta \end{pmatrix}$$

and

$$\mathbf{G}(\mathrm{L}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1+\alpha & -\alpha\beta \\ \gamma & 1-\gamma\beta \end{pmatrix} L \ .$$

The Error Correction Model form for this model is

$$\begin{pmatrix} \Delta x_{1t} \\ \Delta x_{2t} \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \begin{pmatrix} 1 & -\beta \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha\beta \\ \gamma & -\gamma\beta \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

In this situation the hypothesis $H_0 : \beta = 1$ means that the series of differences $x_{1t} - x_{2t}$ is stationary. The hypothesis $H_0 : \gamma = 0$ means that the series x_{2t} is weakly exogenous. If $\alpha < 0$ then if x_{1t-1} is larger than equilibrium then we expect $\Delta x_{1t} < 0$ such that we observe a movement back toward the long run relation. Usually we normalise the β-vector with respect to the variable that we believe do the adjustment if we are out of equilibrium (here x_{1t}).

The diagonalisation of $\boldsymbol{\Pi}$ is given by

$$\boldsymbol{\Pi} = \begin{pmatrix} \beta & \alpha \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \alpha - \beta\gamma \end{pmatrix} \begin{pmatrix} \beta & \alpha \\ 1 & \gamma \end{pmatrix}^{-1}$$

as the non zero eigenvalue of the matrix $\boldsymbol{\Pi}$ is 0 and

$$\boldsymbol{\beta}^T\boldsymbol{\alpha} = \begin{pmatrix} 1 & -\beta \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \alpha - \beta\gamma.$$

and we assume $\alpha - \beta\gamma \neq 0$. The corresponding eigenvectors are

$$\begin{pmatrix} \beta \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} .$$

As this is point in time aggregated for aggregation period k, we obtain the autoregressive polynomial

$$\mathbf{C}(\mathrm{L}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1+\alpha & -\alpha\beta \\ \gamma & 1-\gamma\beta \end{pmatrix}^k L \ .$$

Using the diagonalisation of $\mathbf{G}_1$ we see that the coefficient matrix in $\mathbf{C}(L)$ equals

$$\begin{pmatrix}\beta & \alpha\\ 1 & \gamma\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & (1+\alpha-\beta\gamma)^k\end{pmatrix}\begin{pmatrix}\beta & \alpha\\ 1 & \gamma\end{pmatrix}^{-1}$$

$$= \frac{1}{\beta\gamma-\alpha}\begin{pmatrix}\beta\gamma-\alpha(1+\alpha-\beta\gamma)^k & -\alpha\beta+\alpha\beta(1+\alpha-\beta\gamma)^k\\ \gamma-\gamma(1+\alpha-\beta\gamma)^k & -\alpha+\beta\gamma(1+\alpha-\beta\gamma)^k\end{pmatrix}$$

The polynomial $\mathbf{C}(L)$ admits the factorisation

$\mathbf{C}(L) = \mathbf{B}(L)\mathbf{G}(L)$,

where

$$\mathbf{B}(L) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1+\alpha & -\alpha\beta\\ \gamma & 1-\gamma\beta\end{pmatrix}L + \begin{pmatrix}1+\alpha & -\alpha\beta\\ \gamma & 1-\gamma\beta\end{pmatrix}^2 L^2 + \ldots + \begin{pmatrix}1+\alpha & -\alpha\beta\\ \gamma & 1-\gamma\beta\end{pmatrix}^{k-1} L^{k-1}$$

For the polynomial $\mathbf{B}(L)$ we then obtain

$$\mathbf{B}(1) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1+\alpha & -\alpha\beta\\ \gamma & 1-\gamma\beta\end{pmatrix} + \begin{pmatrix}1+\alpha & -\alpha\beta\\ \gamma & 1-\gamma\beta\end{pmatrix}^2 + \ldots + \begin{pmatrix}1+\alpha & -\alpha\beta\\ \gamma & 1-\gamma\beta\end{pmatrix}^{k-1}.$$

As the two eigenvalues of $\mathbf{G}_1$ are $\lambda_1 = 1$ and $\lambda_2 = 1 + \alpha - \beta\gamma$ we by application of the general formula arrive at

$$\mathbf{B}(1) = \begin{pmatrix}\beta & \alpha\\ 1 & \gamma\end{pmatrix}\begin{pmatrix}k & 0\\ 0 & \dfrac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma}\end{pmatrix}\begin{pmatrix}\beta & \alpha\\ 1 & \gamma\end{pmatrix}^{-1}$$

$$= \frac{1}{\beta\gamma-\alpha}\begin{pmatrix}k\beta\gamma-\alpha\dfrac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma} & -k\alpha\beta+\alpha\beta\dfrac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma}\\ k\gamma-\gamma(1+\alpha-\beta\gamma)^k & -k\alpha+\beta\gamma\dfrac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma}\end{pmatrix}$$

The matrix (vector) $\boldsymbol{\alpha}^*$ for the aggregated process is

$$\boldsymbol{\alpha}^* = \mathbf{B}(1)\boldsymbol{\alpha} = \begin{pmatrix}\beta & \alpha\\ 1 & \gamma\end{pmatrix}\begin{pmatrix}k & 0\\ 0 & \dfrac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma}\end{pmatrix}\begin{pmatrix}\beta & \alpha\\ 1 & \gamma\end{pmatrix}^{-1}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$$

$$= \begin{pmatrix} \beta & \alpha \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & \dfrac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1-(1+\alpha-\beta\gamma)^k}{-\alpha+\beta\gamma} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$$

For the coefficient of adjustment it is seen that aggregation over k observations multiplies the α-matrix by the factor

$$\frac{1-(1+\alpha-\beta\gamma)^k}{\alpha-\beta\gamma} .$$

If $\gamma = 0$, the α-matrix simplifies to

$$\begin{pmatrix} (1+\alpha)^k - 1 \\ 0 \end{pmatrix} .$$

4.1 An empirical example of cointegration in a two dimensional VAR model of order one
This example considers the log transformed price series of two grades of gasoline. Both of these are New York wholesale gasoline prices as quoted by Dow Jones for the period 2. January 1995 to 30. December 2005. Gas1 is (GSUNLRG) regular non oxygenated gasoline while GAS2 (GSOXPRE) is premium oxygenated gasoline - oxygenated meaning that ethanol has been added. Both are quoted in US cents per gallon. Later the data set is extended by GAS3 (GASSUPRE) which is unleaded premium, non-oxygenated gasoline and GAS4 (GASUREG) which is unleaded regular oxygenated gasoline. The source of the data is Data Stream.

The daily and the monthly series are shown in figures A.1 and A.2 in the appendix. Both series are obviously integrated of order one and therefore it is relevant to study the cointegration behaviour of the system.. Figure A.3 and A.4 presents the log transformed relative price difference between the two price series. These graphs could be seen as an indication that the relative price is in fact stationary. In the following we follow this idea.

From the leaflet "A Primer on Gasoline Prices" by the US Energy Information Administration it is stated that "Price levels vary by grade, but the price differential between grades is generally constant" and this provides us with the hypothesis that the difference of the log prices is stationary. This is the hypothesis of a cointegration rank of one with coefficients ± 1 in the **β**-matrix.

The hypothesis of a cointegration rank of one is accepted at the 5% level for the daily data even though the series is very long as seen in Table 1. The trace test has the value 1.77 for cointegration rank r = 1which is below the 5% critical value of 3.84.

Table 1 Trace test for the daily series

P-r	r	λ	Trace	Frac.	P-val
2	0	0.004	117.9	15.41	0.00
1	1	0.001	1.8	3.84	0.18

Normalising using the notation applied in the section on cointegration in a two dimensional system

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{pmatrix} 1 \\ -\beta \end{pmatrix}$$

gives the estimates α =-0.04909, β = 0.98607 and γ = 0.03027 for the daily series.

Table 2 presents the estimated values of β, α and γ for daily, weekly, monthly and quarterly data. The estimation is performed by Proc Varmax in SAS using maximum likelihood estimation. In accordance with the theory the value of the parameter β is almost independent of the aggregation lag. The hypothesis H_0: $\beta = 1$ is accepted for all levels of aggregation. The numerical value of α is steadily increasing keeping the same negative sign.

Table 2 Parameter estimates and p-values for various levels of aggregation.

	Number of observations	β	p-value H_0: $\beta = 1$	α	γ	p-value H_0: $\gamma = 0$	p-value H_0: $\gamma = 0$ and $\beta = 1$
Daily	2870	0.98607	0.2602	-0.04909	0.03027	0.0043	0.006
Weekly	574	0.98765	0.4874	-0.16654	0.01515	0.7719	0.719
Monthly	132	0.98044	0.3403	-0.66036	-0.12453	0.5926	0.601
Quaterly	44	0.97104	0.1412	-0.67205	0.29614	0.6019	0.272

The parameter γ is also numerically increasing, but with changing sign and in fact the hypothesis of $\gamma = 0$ is only rejected for the daily series. The hypothesis $\gamma = 0$ is the hypothesis of exogeneity of the second series - the regular grade.

When the series is aggregated using each aggregation level k = 1, .., 100, the parameters estimated under the restrictions $\beta = 1$ and $\gamma = 0$ are presented graphically in Figures 1 and 2. Under these restrictions the parameter α is a function

$$\alpha^* = (1+\alpha)^k - 1$$

of the aggregation period k. Estimating α using a linear regression of $\log(\alpha^* + 1)$ as endogenous and k as exogenous, as shown in Figure 2 weighted according to the actual length T/k of the aggregated

series as above, gives α = -0.03120. This value of α is closer to zero, than the actual estimated value -0.04909 for the daily data. The corresponding curve is also presented in Figure 2 giving an acceptable fit to the actual estimated α*. The theory is seen to apply as the linear fit in Figure 2 is met and the fit of the curve in Figure 1 is within the 95% confidence limits.

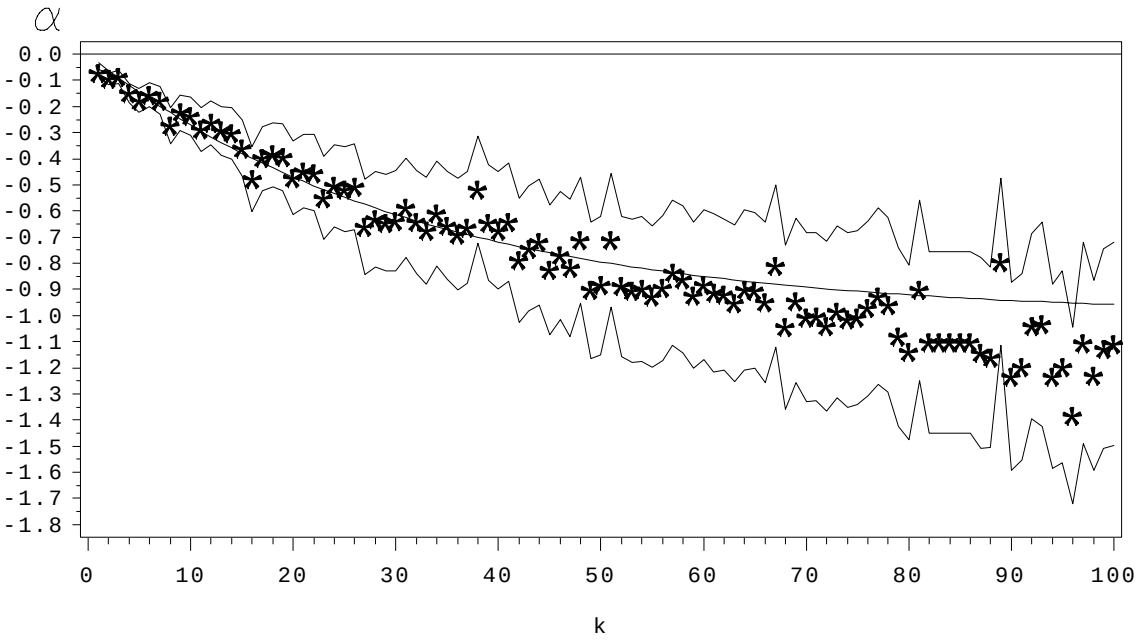

Figure 1 Estimate for α* for aggregation for k = 1, ..100.

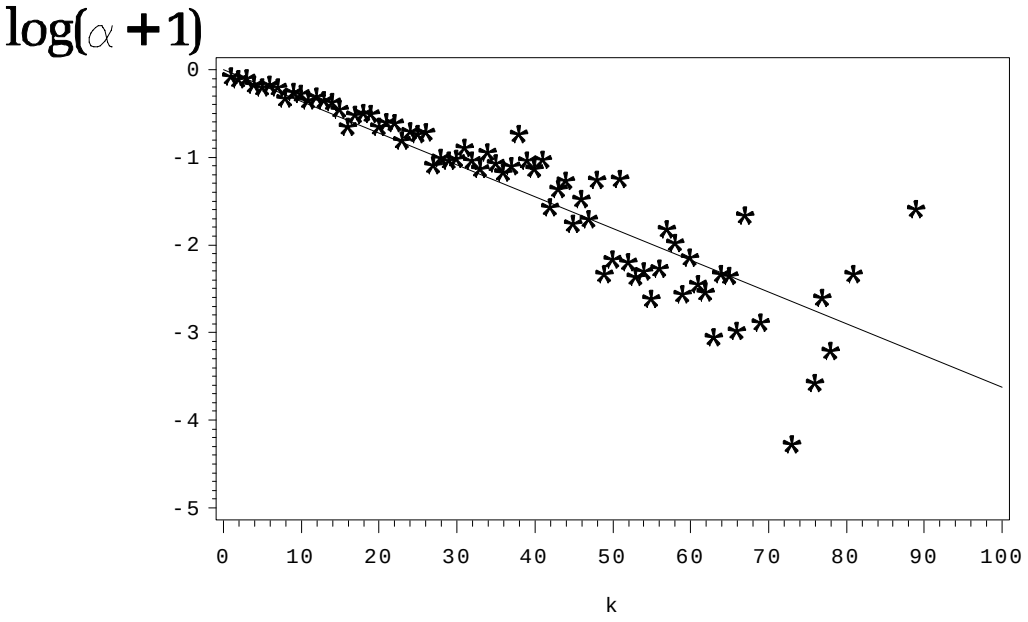

Figure 2 Plot of log(α* + 1) to k for k = 1

5 An empirical example of cointegration in a four dimensional VAR model of order one

For the four dimensional vector of prices we calculate the roots of a first order fitted autoregressive model for aggregation periods k = 1, .., 25. The figure shows clearly that one root should be considered as one, while the other roots are stationary. Also the trace tests support our conclusion of three cointegration vectors as seen from Table 3.

Figure 3 The modulus of roots of the VAR(1) polynomial for level of aggregation k = 1, .., 25.

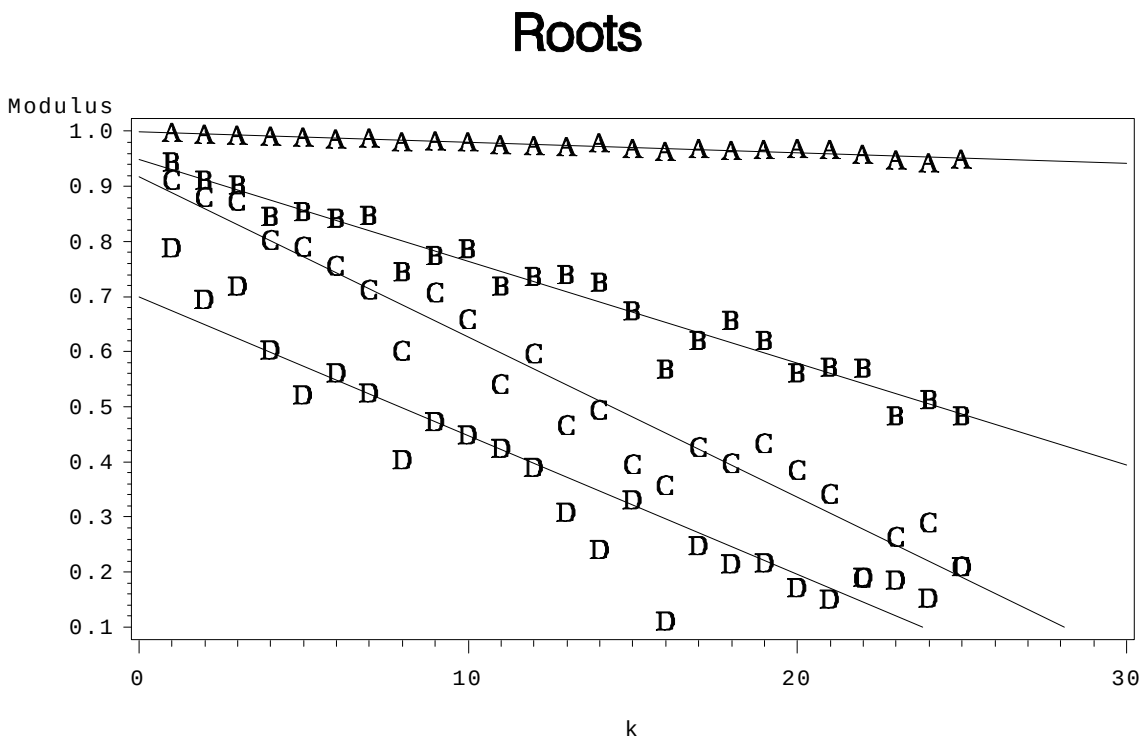

Table 3 Trace test for the daily series

P-r	r	λ	Trace	Frac.	P-val
4	0	0.111	551.8	47.71	0.00
3	1	0.044	215.2	29.80	0.00
2	2	0.029	85.1	15.41	0.00
1	3	0.001	1.8	3.84	0.184

Hence we study a model with cointegration rank three, meaning that the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are of dimension (4 ×3).

Again our hypothesis is that the series GAS1 is the driving series leaving the other series as followers. This means that we test that the log relative prices of GAS2, GAS3 and GAS4 to GAS1 are stationary. Also our hypothesis implies that GAS1 is weakly exogenous.

This leaves us with the following restriction on the **β** matrix when the series are taken in the order GAS4, GAS3, GAS2, and GAS1

$$x = \begin{pmatrix} \text{GAS2} \\ \text{GAS3} \\ \text{GAS4} \\ \text{GAS1} \end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

The tests for this hypothesis for various levels of aggregations are given in Table 4.

Table 4 Tests for the hypothesis for the β-matrix for various levels of aggregations.

Frequency	Number of observations.	H_0: relative prices stationary (the β-matrix)	H_0: weak exogeneity (the α-matrix)
Daily	2870	0.137*	0.164*
Weekly	574	0.327*	0.654*
Monthly	132	0.749*	0.969*
Quarterly	44	0.852*	0.965*

* P-values for Bartlett corrected test statistics, see Johansen(2000).

For the **α** matrix we first consider the most general form – first without imposing the exogeneity restriction.

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \end{bmatrix}$$

The **α** matrix is estimated using aggregation levels k=1, 2, .., 25 as presented in Figures B.1 – B.12. Note that the entries in the **α**-matrix could be estimated using ordinary least squares because of super consistency of the β-estimates. With the present restrictions on the **β**-matrix we do not even need this (as all the β's are constants) and we have a system with the same set of explanatory variables which allows the use of OLS equation wise. The figures also present 95% confidence limits.

The hypothesis of the price of regular non-oxygenated gasoline being the driving force is the hypothesis that the last row in the α-matrix consists of zeroes.

These graphs indicate that the entries α_{41}, α_{42} and α_{43} in the **α**-matrix, B.10 – B.12, are clearly zero indicating that the fourth series GSUNLRG is the driving force. This conclusion is also supported by the tests of the hypothesis of weak exogeneity presented in the last column of Table 4. From the graphs we also conclude that the entries α_{12} α_{13} and α_{32} could be restricted to zero.

This leaves us with the α-matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & 0 & \alpha_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrices

$$\mathbf{\Pi} = \mathbf{\alpha\beta}^{\mathrm{T}} = \begin{pmatrix} \alpha_{11} & 0 & 0 & -\alpha_{11} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & -\alpha_{21} - \alpha_{22} - \alpha_{23} \\ \alpha_{31} & 0 & \alpha_{33} & -\alpha_{31} - \alpha_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{G}_1 = \mathbf{I} + \mathbf{\Pi} = \begin{bmatrix} 1+\alpha_{11} & 0 & 0 & -\alpha_{11} \\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} & -\alpha_{21} - \alpha_{22} - \alpha_{23} \\ \alpha_{31} & 0 & 1+\alpha_{33} & -\alpha_{31} - \alpha_{33} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has eigenvalues 0, α_{11}, α_{22} and α_{33} (resp. 1, $1+\alpha_{11}$, $1+\alpha_{22}$ and $1+\alpha_{33}$) and the eigenvectors written as columns form the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \dfrac{\alpha_{11}-\alpha_{33}}{\alpha_{31}} & 0 & 0 \\ 1 & \dfrac{\alpha_{11}\alpha_{21}-\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23}}{\alpha_{31}(\alpha_{11}-\alpha_{22})} & 1 & 1 \\ 1 & 1 & 0 & \dfrac{\alpha_{33}-\alpha_{22}}{\alpha_{23}} \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Using the general formulas we now see

$$\mathbf{B}(1) = \mathbf{A}\sum_{i=0}^{k-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (1+\alpha_{11})^i & 0 & 0 \\ 0 & 0 & (1+\alpha_{22})^i & 0 \\ 0 & 0 & 0 & (1+\alpha_{33})^i \end{bmatrix}\mathbf{A}^{-1} =$$

$$\mathbf{A}\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & \dfrac{(1+\alpha_{11})^k-1}{\alpha_{11}} & 0 & 0 \\ 0 & 0 & \dfrac{(1+\alpha_{22})^k-1}{\alpha_{22}} & 0 \\ 0 & 0 & 0 & \dfrac{(1+\alpha_{33})^k-1}{\alpha_{33}} \end{bmatrix}\mathbf{A}^{-1}$$

and

$$\mathbf{B(1)}\ \boldsymbol{\alpha} = \begin{pmatrix} \dfrac{(1+\alpha_{11})^k-1}{\alpha_{11}} & 0 & 0 \\ 0 & \dfrac{(1+\alpha_{22})^k-1}{\alpha_{22}} & 0 \\ 0 & 0 & \dfrac{(1+\alpha_{33})^k-1}{\alpha_{33}} \end{pmatrix}\boldsymbol{\alpha}$$

$$= \begin{bmatrix} (1+\alpha_{11})^k - 1 & 0 & 0 \\ \alpha_{21}\dfrac{(1+\alpha_{11})^k - 1}{\alpha_{11}} & (1+\alpha_{22})^k - 1 & \alpha_{23}\dfrac{(1+\alpha_{33})^k - 1}{\alpha_{33}} \\ \alpha_{31}\dfrac{(1+\alpha_{11})^k - 1}{\alpha_{11}} & 0 & (1+\alpha_{33})^k - 1 \\ 0 & 0 & 0 \end{bmatrix}$$

For k = 1 we obtain the original matrix $\boldsymbol{\alpha}$.

These expressions could be used as a basis for estimating some of the α's. For instance the value for α^*_{22} the value of α_{22} is found by a regression of $\log(1 + \alpha^*_{22})$ on k as indicated in the plot where the regression coefficient is $\log(1 + \alpha_{22})$. These plots, see Figures B.13 – B.15 in Appendix B, for the entries (1,1), (2,2) and (3,3), give a reasonable fit as an indication that the entries (1,1), (2,2) and (3,3) are clearly nonzero. The estimated values obtained by the linear regressions are

$\alpha_{11} \approx -0.02941$ $\alpha_{22} \approx -0.05840$ and $\alpha_{33} \approx -0.05542$

which are actually closer to zero than the estimated values for the (disaggregated) series of daily prices.

The other entries (2,1), (1,3) and (3,1) could at least for a moderate order of aggregation be accepted as being zero. One argument in favour of this point of view is that the numerical value of e.g.

$$\alpha^*_{21} = \alpha_{21}\frac{(1+\alpha_{11})^k - 1}{\alpha_{11}}$$

is not, as shown on Figure B.17, increasing towards an upper limit of the form $-\alpha_{21}/\alpha^*_{11}$ according to the factor

$$\frac{(1+\alpha_{11})^k - 1}{\alpha_{11}}$$

in the formula. This could be seen as an indication that α_{21} could be restricted to zero. The same comments apply for α_{23} and α_{31} as well, see Figures B.19 and B.20.

In a model with these further restrictions added we have

$$\alpha = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

and hence

$$\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}^{\mathrm{T}} = \begin{bmatrix} \alpha_{11} & 0 & 0 & -\alpha_{11} \\ 0 & \alpha_{22} & 0 & -\alpha_{22} \\ 0 & 0 & \alpha_{33} & -\alpha_{33} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

and

$$\mathbf{G}_1 = \begin{bmatrix} 1+\alpha_{11} & 0 & 0 & -\alpha_{11} \\ 0 & 1+\alpha_{22} & 0 & -\alpha_{22} \\ 0 & 0 & 1+\alpha_{33} & -\alpha_{33} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here the eigenvalues of are obviously the diagonal elements. From the general expression we then see

$$\boldsymbol{\alpha}^* = \mathbf{B}(1)\boldsymbol{\alpha} = \begin{pmatrix} (1+\alpha_{11})^k - 1 & 0 & 0 \\ 0 & (1+\alpha_{22})^k - 1 & 0 \\ 0 & 0 & (1+\alpha_{33})^k - 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Again the plots of $\log(1 + \alpha^*_{ii})$ against k with fitted straight lines are shown in Appendix B, Figures B.22 and B.24. From these lines the values of the parameters α_{ii} are found and the expression using these values are drawn on the plots of the estimated α^*_{ii}. The fit is seen to be acceptable.

6 Conclusions

In the paper cointegrating properties in VAR(1) models are studied for various levels of aggregations. A simple formula for the speed of adjustment matrix is presented as a function of the level of aggregation. The speed of adjustment of course increases for level of aggregation. By applying the formula the researcher could compare the actual estimation and test results for the data series with the theoretical features for aggregation of the cointegrated VAR(1) model. This gives an idea of the fit of the model and could give the result that the model structure is in fact consistent with the actual level of aggregation.

The two empirical examples of price series demonstrate that the theory works in practice. As they are of a fairly general nature this is hopefully confirmed by many other examples in the future.

7 References

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Johansen, S. (2000). A Bartlertt correction factor for tests on the cointegrating relations, *Econometric Theory*, 16, pp 740 – 778.

Lütkepohl, H. (1987). Forecasting Aggregated Vector ARMA Processes. *Springer-Verlag.* Berlin Heidelberg New York.

Marcellino, M. (1999). Some Consequences of Temporal Aggregation in Empirical Analysis. *Journal of Business and Economic Statistics,* 17, pp. 129-136.

Appendix A. The data.

Figure A1: Daily data.

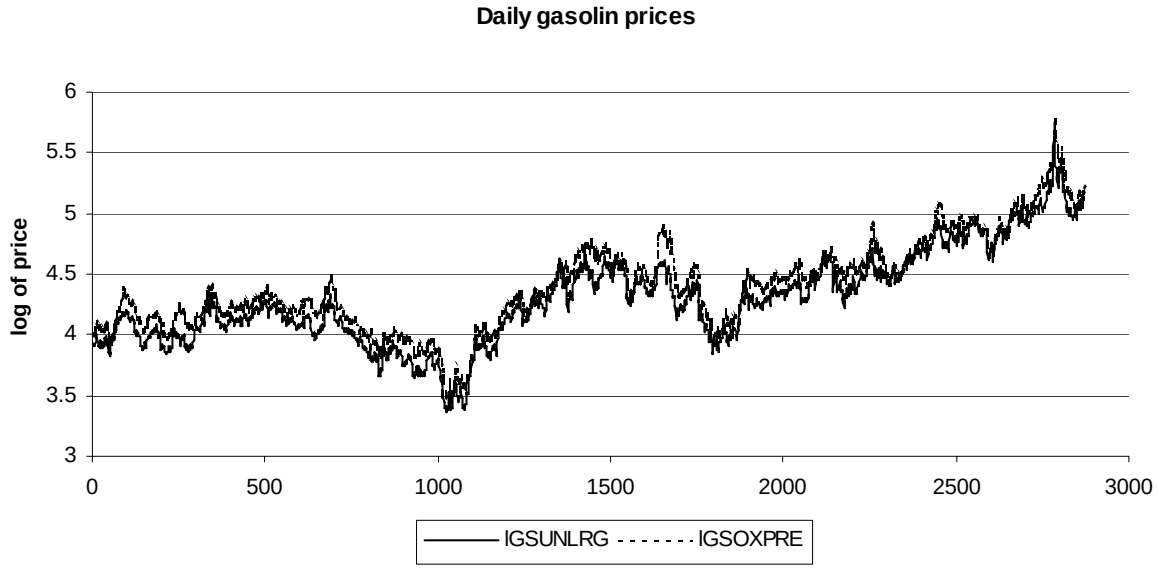


Figure A2: Monthly data.

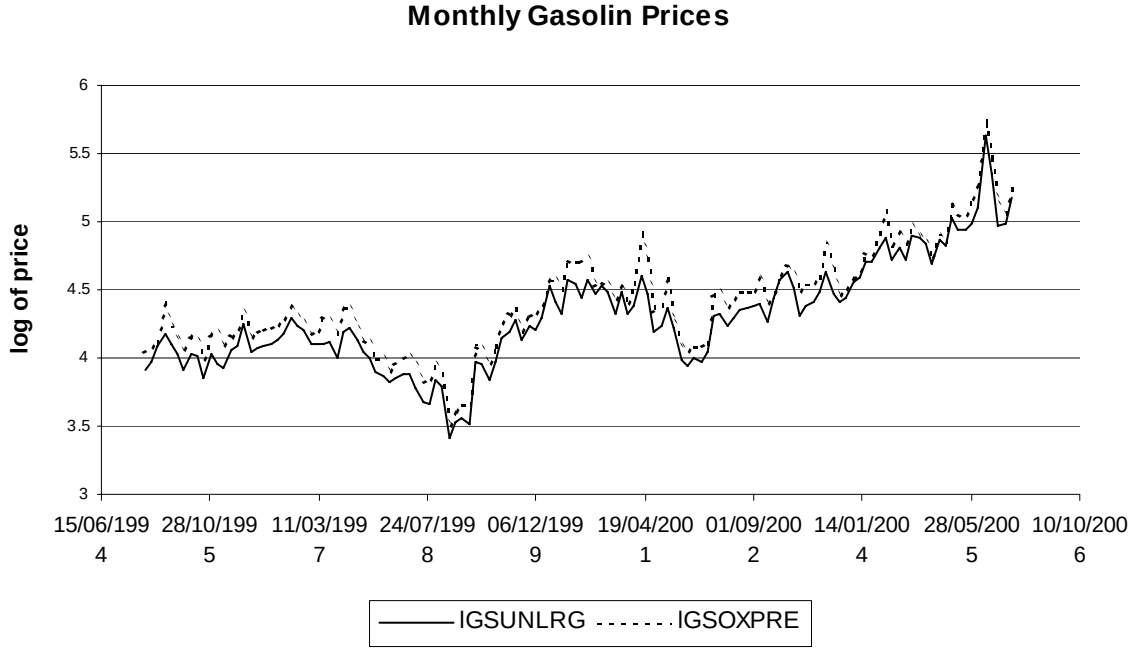


Figure A3: Daily data.

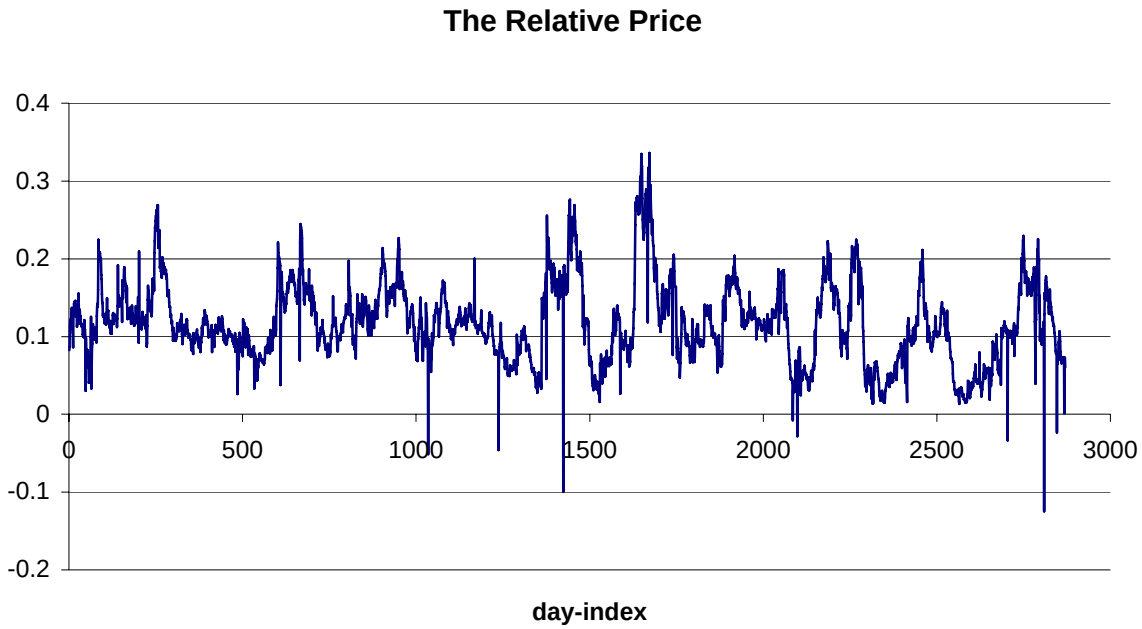
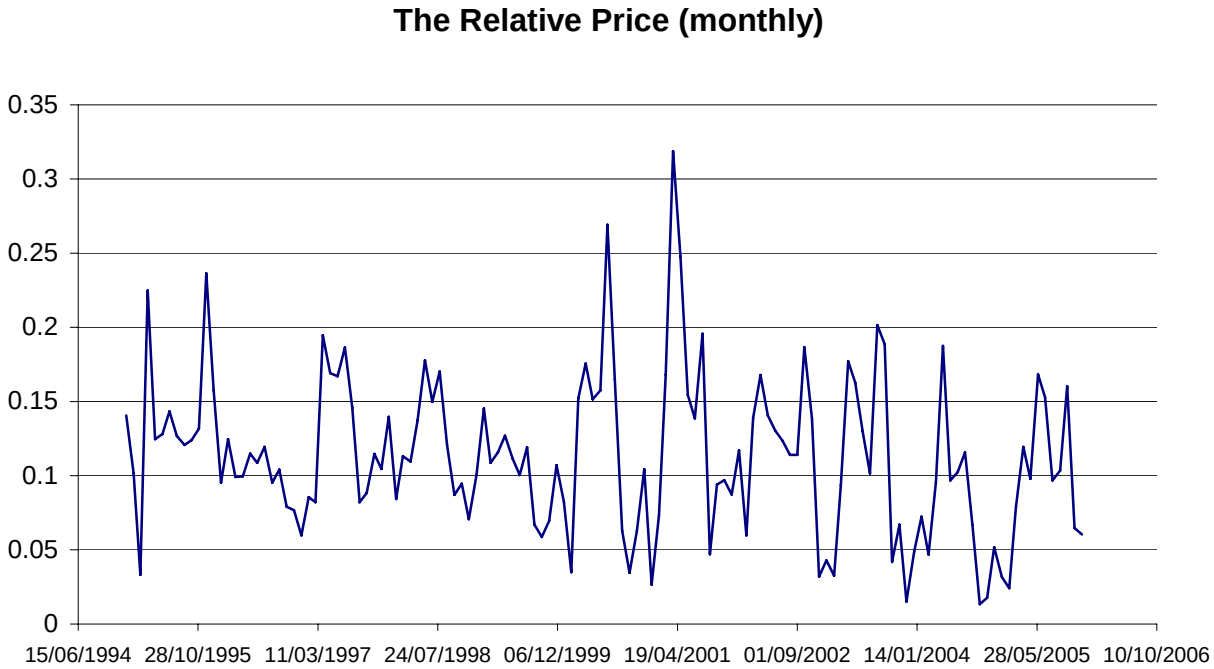


Figure A4: Monthly data.



Appendix B: Graphs of estimated α's from the 4-dimensional example.

Figure B1: No restrictions on α.

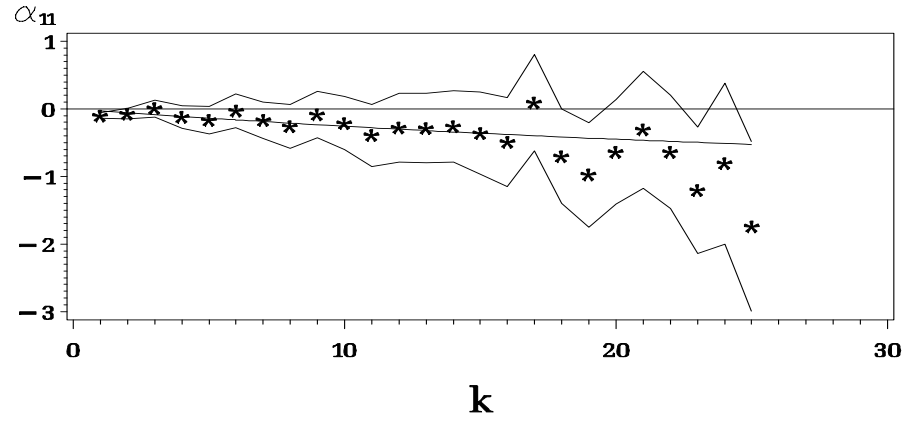


Figure B2: No restrictions on α.

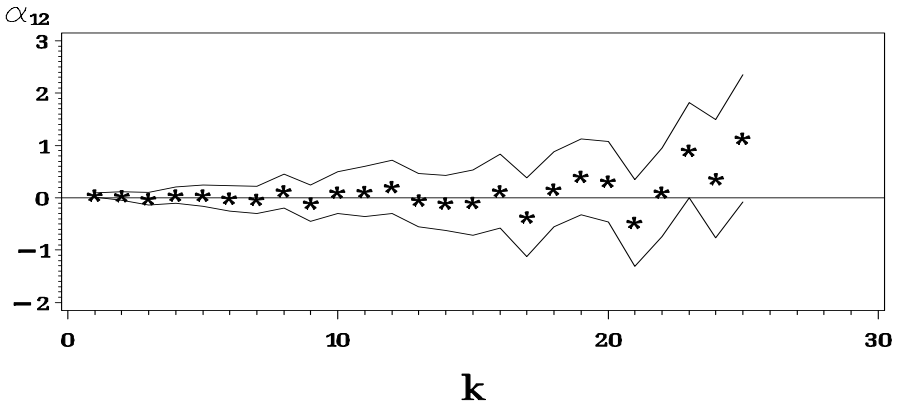


Figure B3: No restrictions on α.

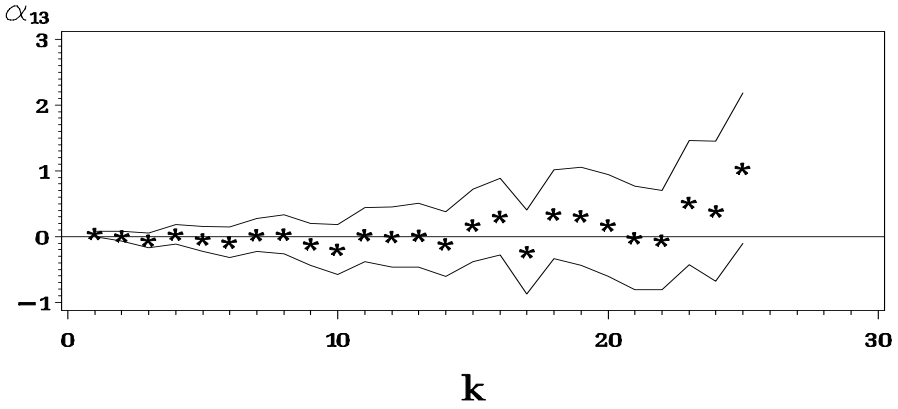


Figure B4: No restrictions on α.

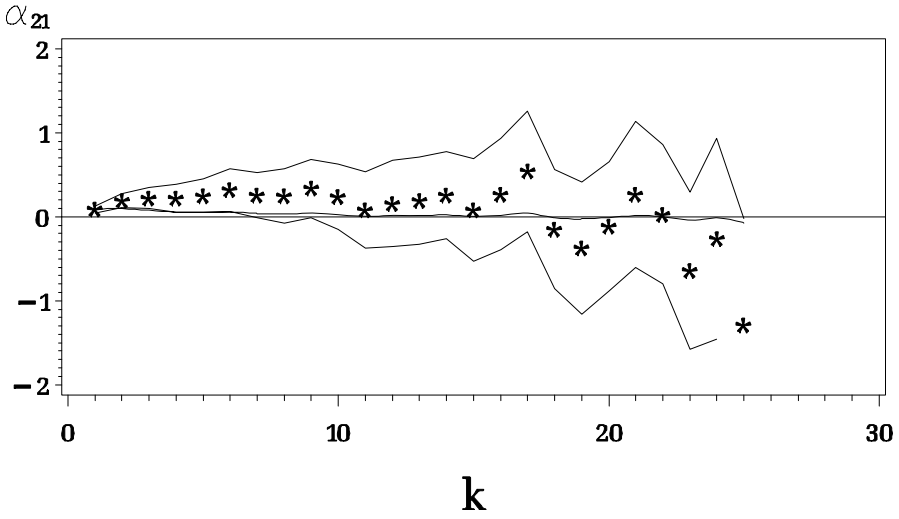


Figure B5: No restrictions on α.

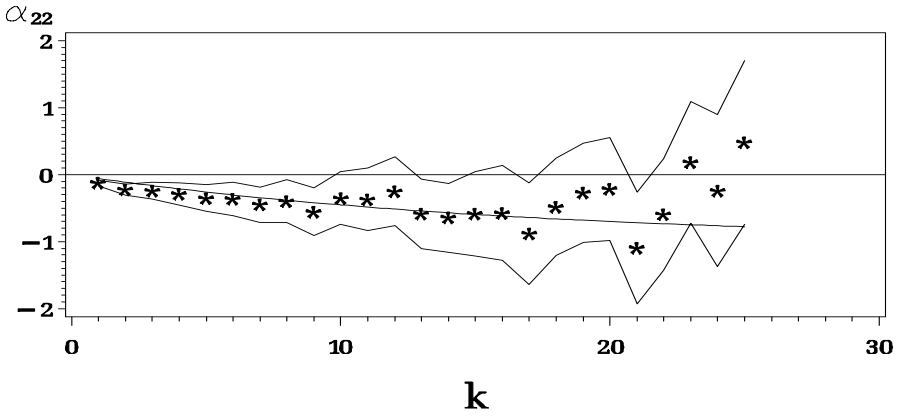


Figure B6: No restrictions on α.

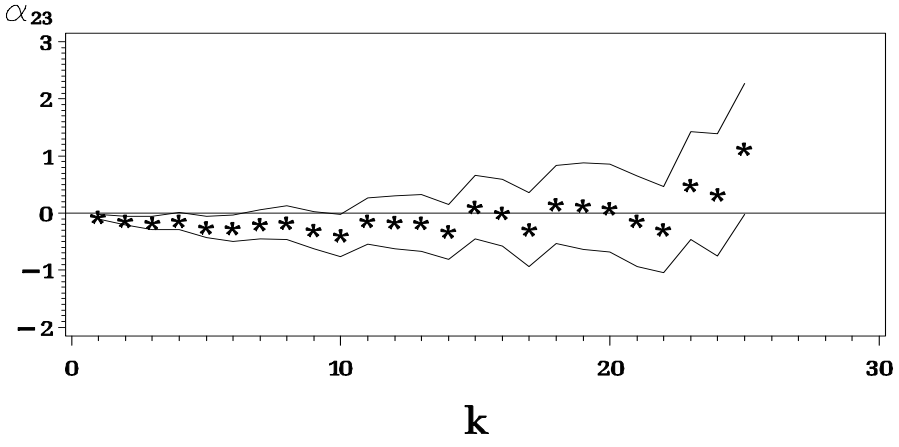


Figure B7: No restrictions on α.

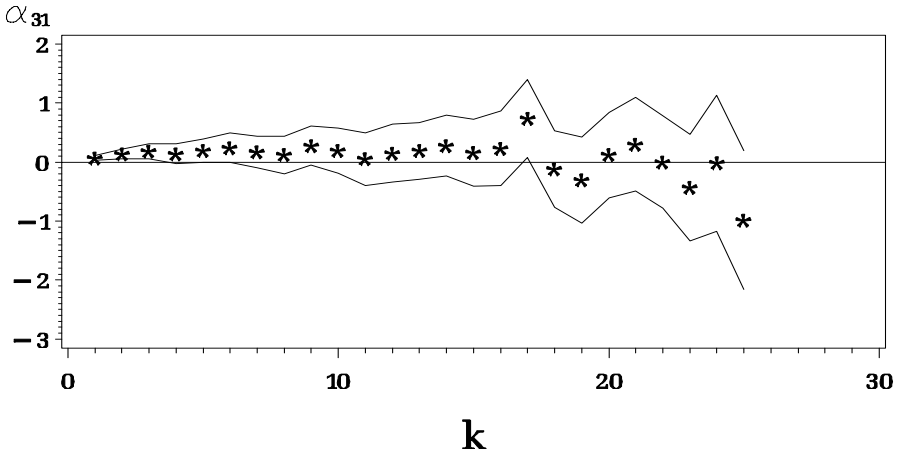


Figure B8: No restrictions on α.

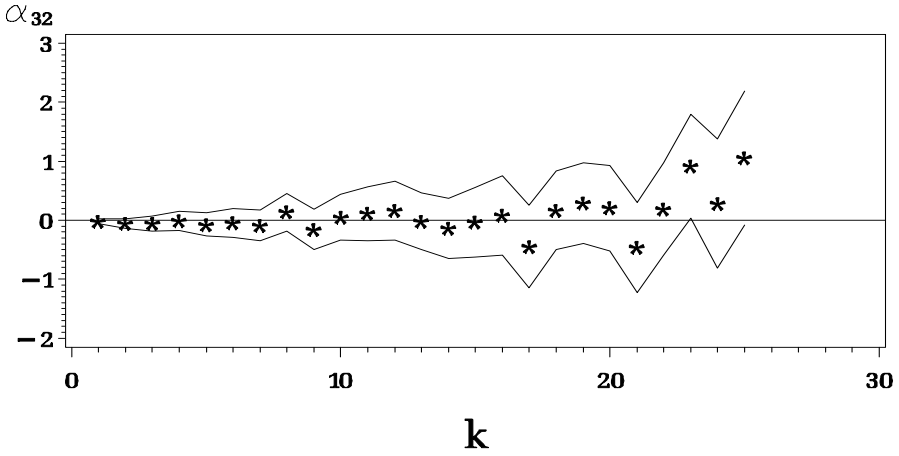


Figure B9: No restrictions on α.

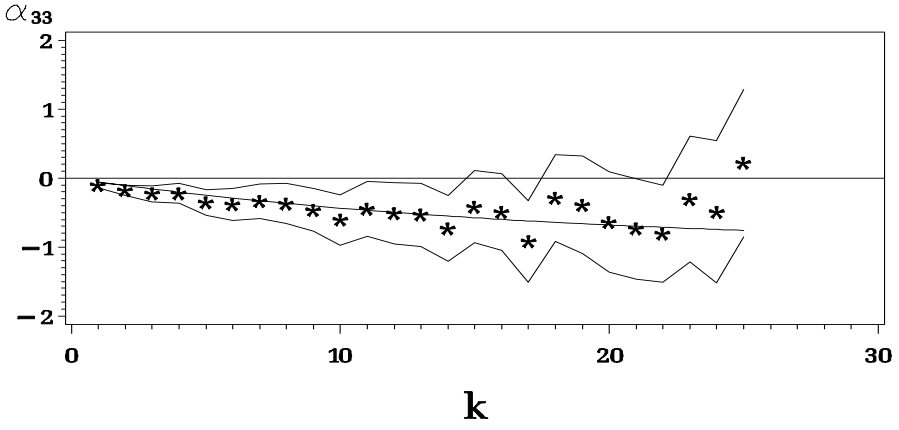


Figure B10: No restrictions on α.

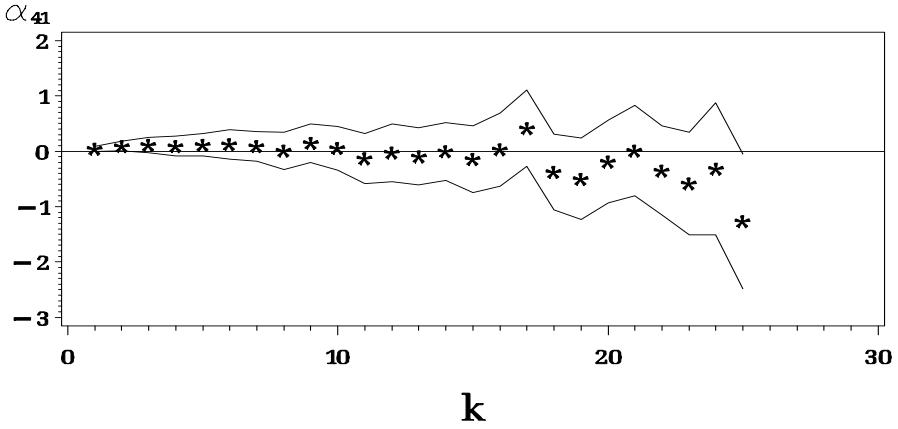


Figure B11: No restrictions on α.

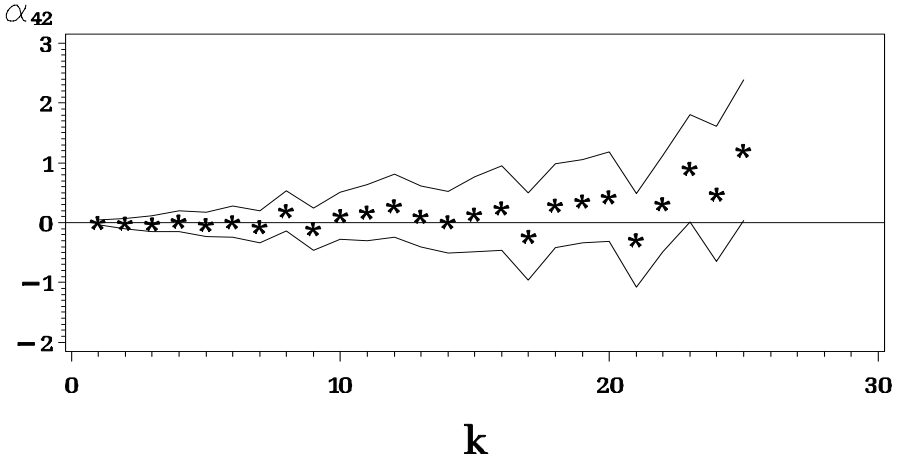


Figure B12: No restrictions on α.

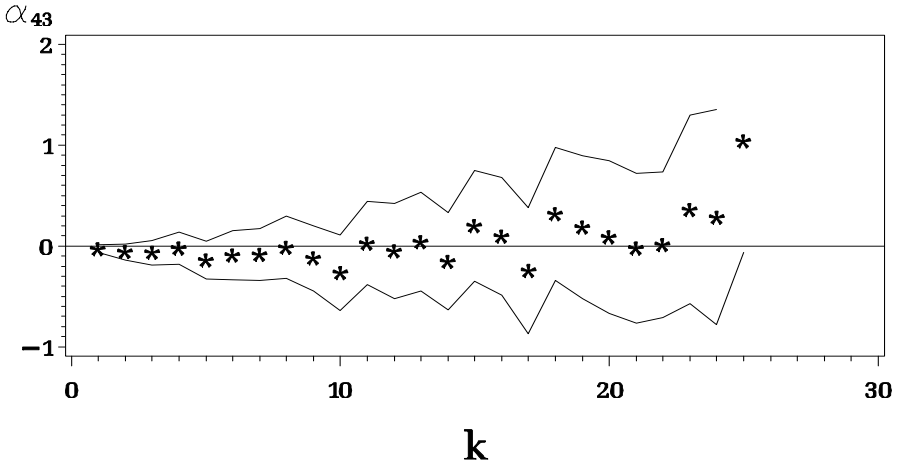


Figure B13: Log transform of diagonal α's of the model with 6 zero restrictions.

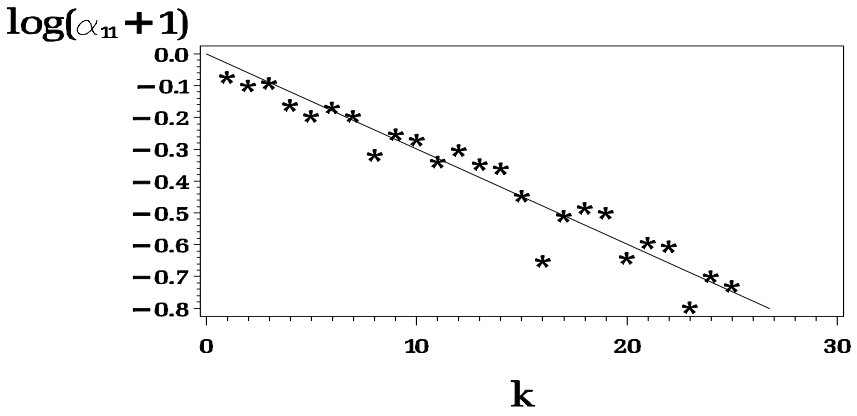


Figure B14: Log transform of diagonal α's of the model with 6 zero restrictions.

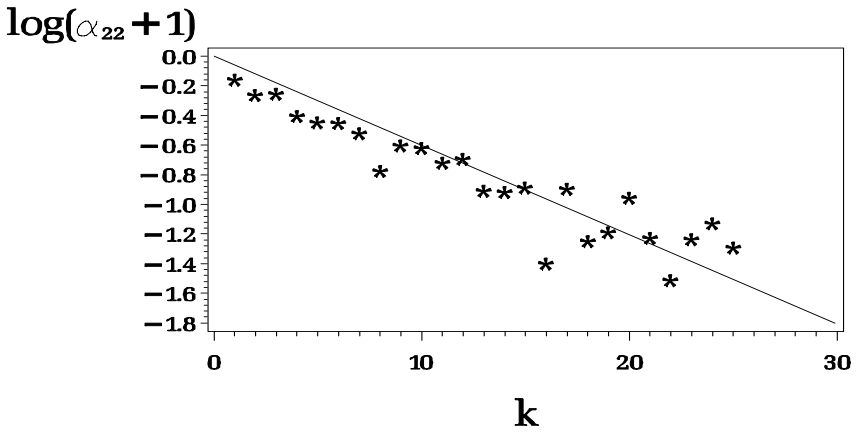


Figure B15: Log transform of diagonal α's of the model with 6 zero restrictions.

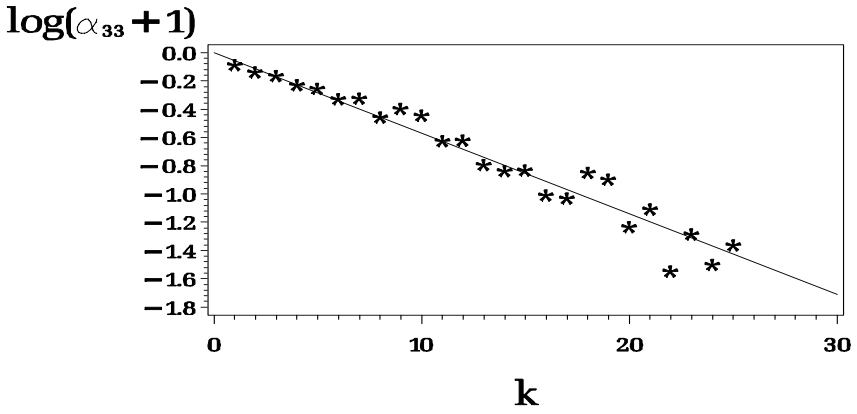


Figure B16: α's of the model with 6 zero restrictions.

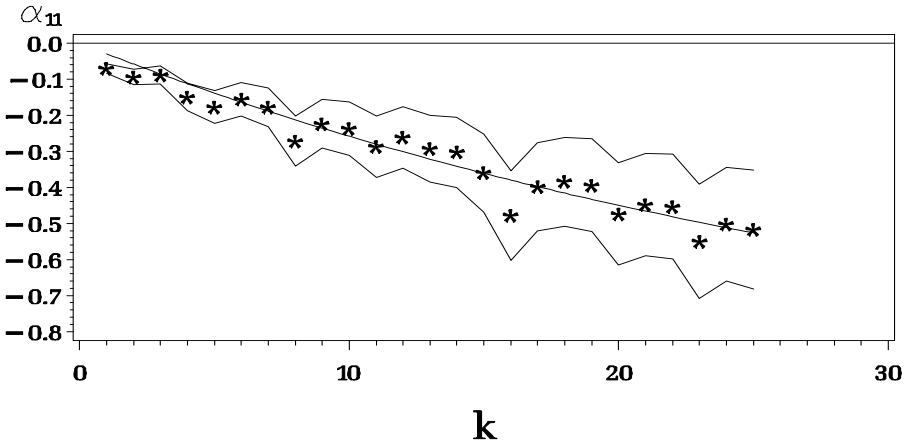


Figure B17: α's of the model with 6 zero restrictions.

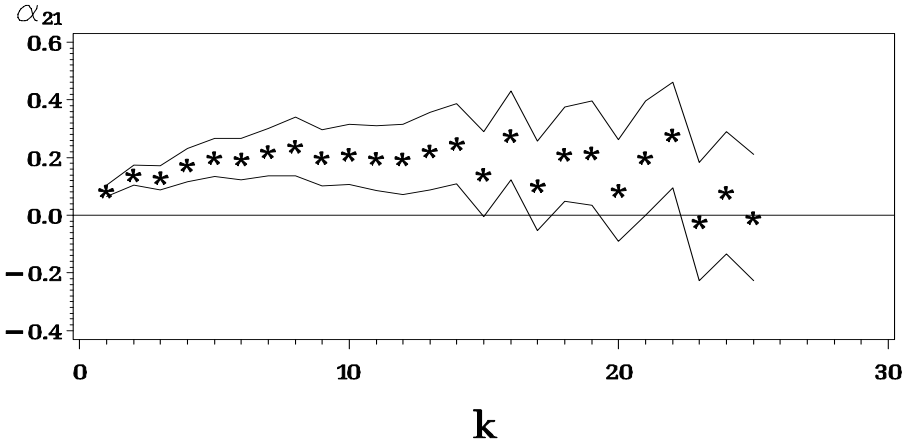


Figure B18: α's of the model with 6 zero restrictions.

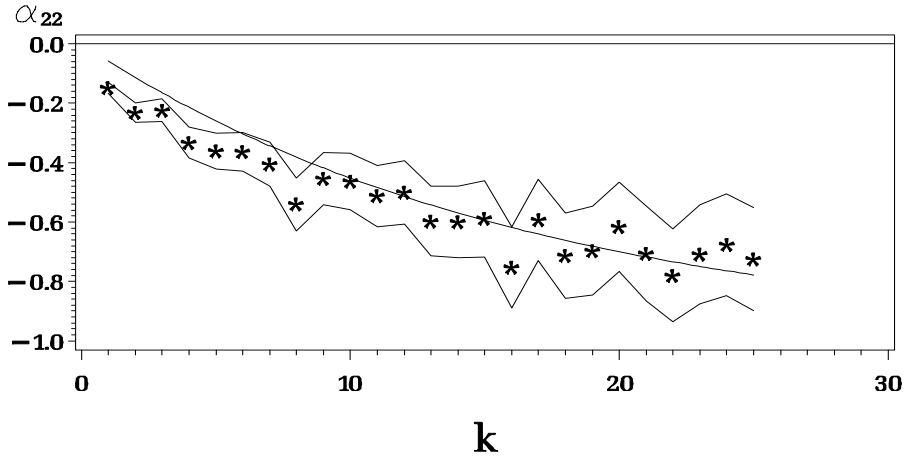


Figure B19: α's of the model with 6 zero restrictions.

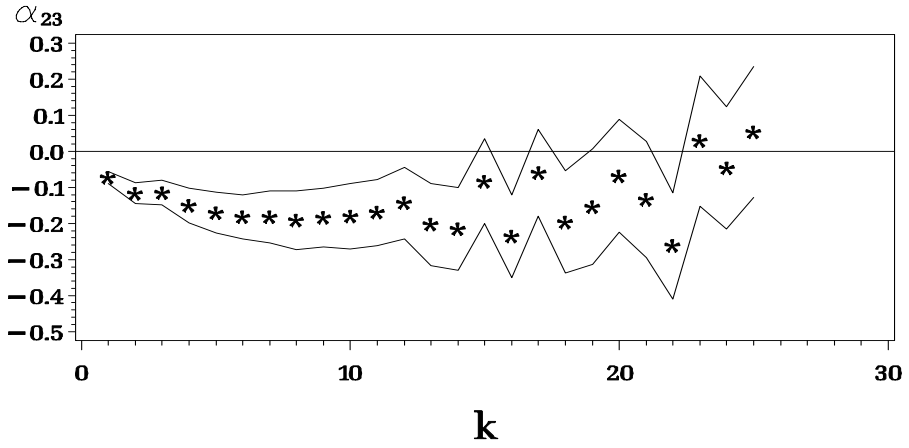


Figure B20: α's of the model with 6 zero restrictions.

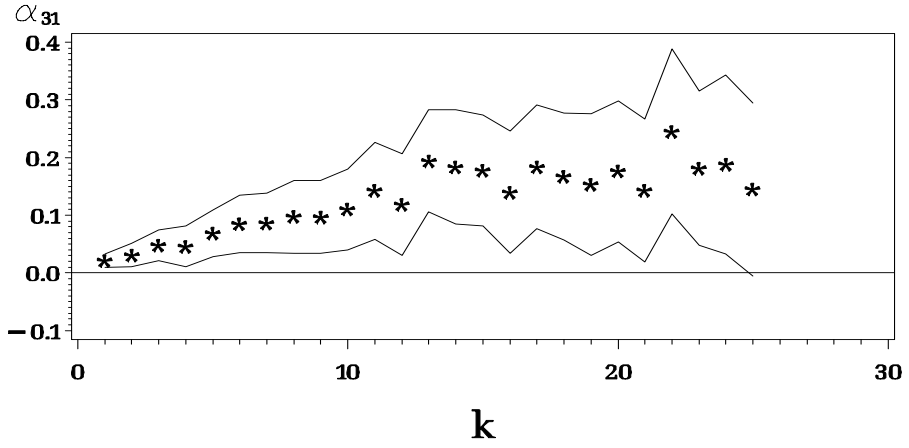


Figure B21: α's of the model with 6 zero restrictions.

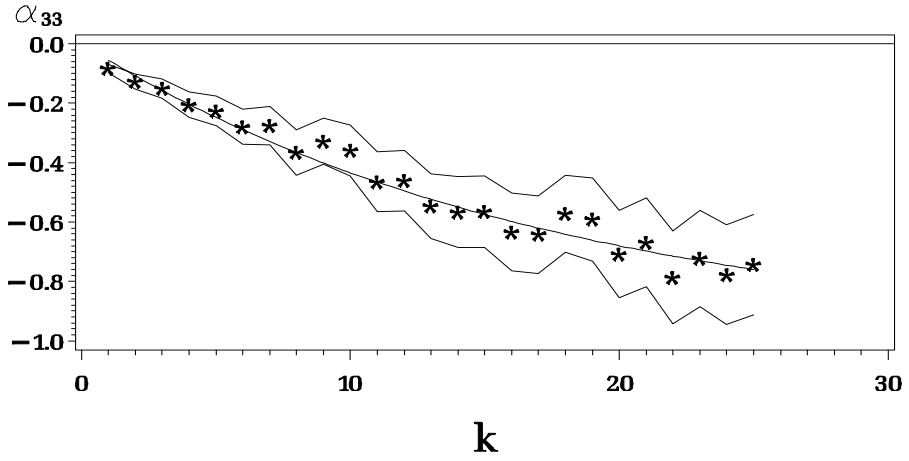


Figure B22: Log transform of α's of the model with 9 zero restrictions.

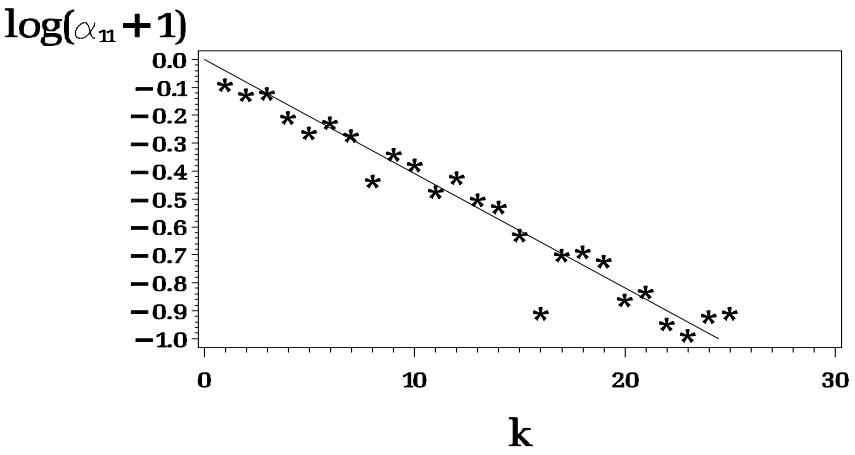


Figure B23: Log transform of α's of the model with 9 zero restrictions.

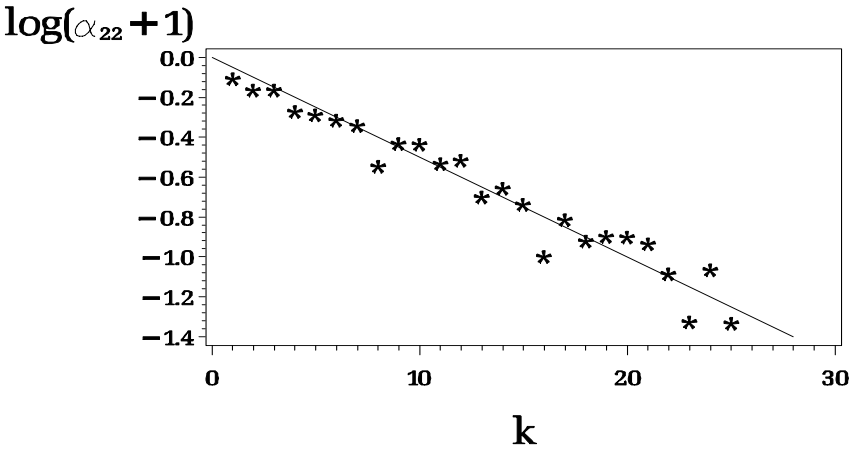


Figure B24: Log transform of α's of the model with 9 zero restrictions.

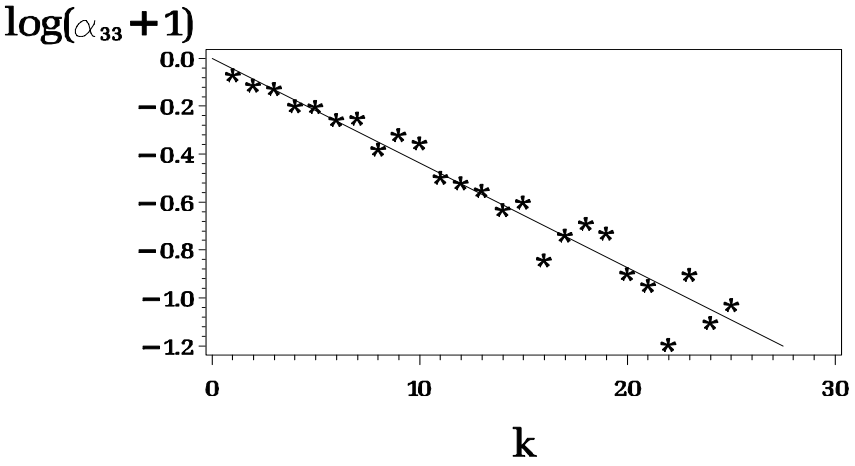


Figure B25: α's of the model with 9 zero restrictions.

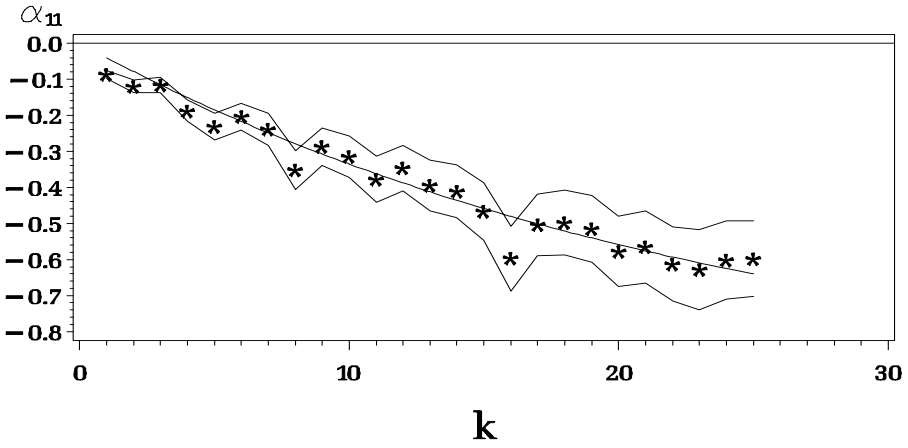


Figure B26: α's of the model with 9 zero restrictions.

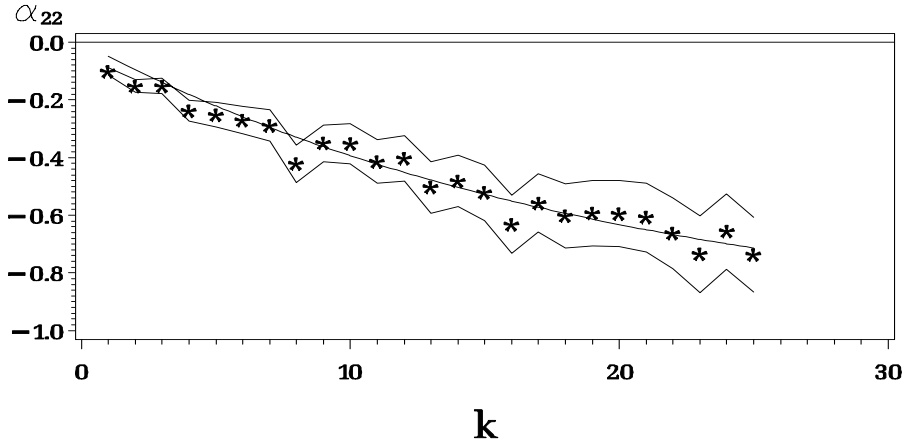


Figure B27: α's of the model with 9 zero restrictions.