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On the Phase Dependence in Time-Varying Correlations Between Time-Series

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Abstract

This paper proposes the use of a double correlation coefficient as a nonparametric measure of phase-dependence in time-varying correlations. An asymptotically Gaussian test statistic for the null hypothesis of no phase-dependence is derived from the proposed measure. Finite-sample distributions, power and size are analyzed in a Monte-Carlo exercise. An application of this test provides evidence that correlation strength between major macroeconomic aggregates is both time-varying and phase dependent in the business cycle.

Keywords: nonparametric; phase-dependence; time-varying correlation.

JEL Codes: C01, C14, C32.

1 Introduction

Many economic time-series exhibit correlation strengths that change over time. Parametric state-space models like the dynamic conditional correlation (DCC) model of Engle (2000) attempt to capture these time-variations in correlation strength between financial time-series. Structural parametric models with nonlinearities and time-varying parameters in macroeconomics and microeconomics are also capable of

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generating time-varying correlations. Nonparametric methods are however desirable in cases where the researcher does not wish to impose a potentially restrictive parametric model structure on the data. Evidence of model misspecification in certain applications is thus a possible reason to adopt, or at least complement, econometric analysis with nonparametric methods. Nonparametric methods are also appealing when searching for 'descriptive statistics' that reflect simple and agnostic 'stylized facts'. For example, in empirical macroeconomics, considerable effort has been invested since Kydland and Prescott (1982) in the design and estimation of structural macroeconomic models capable of describing the correlation structure of the data. Even estimated macroeconomic structural models are usually evaluated by their ability to accurately describe a matrix of correlations between contemporaneous macroeconomic variables and their lags; see e.g. Dave and Dejong (2007).

This paper proposes the use of a double correlation coefficient as a nonparametric measure of phase-dependence in the correlation between two time-series. Under appropriate regularity conditions the proposed statistic is easily shown to be consistent a asymptotically normal. The asymptotic distribution of a derived test statistic is also obtained. The small sample distributions, power and sizes of the phase-dependence measure and associated test statistic are obtained by means of a Monte Carlo exercise. In an application, the proposed test is used to provide evidence that the autocorrelation structure of macroeconomic aggregates for the US economy evolves over time, and most importantly, that these time-varying correlations are systematically linked to the phase of the business cycle, i.e. that they are phase-dependent. In particular, the data reveals that correlations between aggregate output and aggregate investment in the US are significantly stronger during recessions and weaker during expansions.

Section 2 motivates the problem by analyzing some stylized evidence of time-varying correlation and phase-dependence in US macroeconomic aggregates. Section 3 proposes the nonparametric measure of phase-dependence and related test of no phase-dependence and obtains their asymptotic properties. Section 4 provides a Monte Carlo description of their small sample behavior. Finally, Section 5 assesses the existence of phase-dependence in correlations between macroeconomic aggregates.

2 Time-Varying and Phase-Dependent Correlations in Macroeonomic Aggregates

Since Kydland and Prescott (1982) the characterization of the business cycle in macroeconomics has been typically restricted to the second-order unconditional moments of filtered data. The following table contains a standard characterization of the business cycle for US quaterly data (1947 $Q1$ - 2012 $Q1$) based on the variance and cross-autocorrelation between HP-filtered logarithms of three macroeconomic aggregates: real private investment (i_t), rel private consumption (c_t) and real gross domestic product (y_t).[1]

	σ/σ_y	r_y	r_c	r_i	$r_{y_{-1}}$	$r_{c_{-1}}$	$r_{i_{-1}}$
y_t	1.00	-	0.69	0.78	0.82	0.62	0.66
c_t	1.18	0.69	-	0.55	0.69	0.91	0.54
i_t	3.20	0.78	0.55	-	0.62	0.52	0.81

Table 1: Log US HP-filtered business cycle characterization of auto-correlation structure. Column σ/σ_y shows ratio of estimated variance with output's estimated variance, r_x shows estimated correlation with variable x, and $r_{x_{-1}}$ shows estimated correlation with the lag of x.

Table 1 provides however a rather incomplete picture of the correlation between these aggregates. In particular, it ignores the rich time-variation that occurs in the correlation structure, and most importantly, the fact that this time-variation depends

[1]Data obtained from the Federal Reserve Bank of Saint Louis in billions of chained 2005 dollars.

itself on the business cycle; i.e. that correlations are phase-dependent w.r.t. the business cycle. For example, Figure 1 reveals the striking fact that the correlation strength between output and investment is phase-dependent in NBER cycles. In particular, NBER recession periods are marked by very high correlations and large drops in correlation strength seem to occur only during NBER expansions.

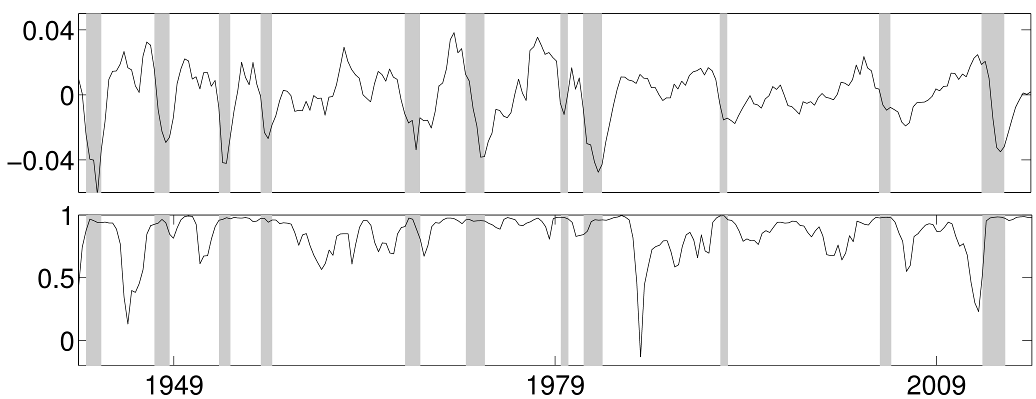

Figure 1: Time series of HP-filtered log US output $\{y_t\}$ (above) and an estimated s-shifted w-window correlation series $\{\hat{\rho}_t^{w,s}\}$ between aggregate output $\{y_t\}$ and aggregate investment $\{i_t\}$ (below) with NBER recession indicator in shaded areas.[2]

A careful inspection of the data shows that correlations between output and investment never dropped below 0.5 in recession periods. Even correlations below 0.8 are rare during recessions. In contrast, in expansion periods, drops of correlation below 0.5 do occur, and correlations below 0.8 are quite frequent. This apparent relation can however be spurious and requires statistical verification. It is thus important to assess the statistical significance of such findings. In particular, it is crucial to understand the properties of the estimated correlation series and to find a statistically sound method of evaluating the presence of phase-dependence. We now address this issue.

[2]The series plotted in Figure 1 make use of the s-shifted w-window uniformly weighted rolling window defined in (1) with a small positive shift to guarantee centering $s = 2$. The span of the kernel $w = 4$ was selected to be no larger than the average length of the NBER output cycle.

3 A Measure of Phase-Dependence

The estimated time-varying correlation series $\{\hat{\rho}_t^{w,s}\}$ in Figure 1 are Pearson correlations calculated over a rolling window. In its simplest form, given a bivariate time-series $\{(y_t, x_t)\}$, the sequence $\{\hat{\rho}_t^{w,s}\}$ of s-shifted w-window correlations between y and x is defined, for every t, as the sample Pearson correlation $\mathbb{C}\mathrm{orr}\left(\mathbf{y}_t^{w,s}, \mathbf{x}_t^{w,s}\right)$ between $\mathbf{y}_t^{w,s} = (y_{t-w+s}, ..., y_{t+s})$ and $\mathbf{x}_t^{w,s} = (x_{t-w+s}, ..., x_{t+s})$ where s is a shift index that defines the centering of the window over the time period of interest,

$$\hat{\rho}_t^{w,s} := \frac{\sum_{j=t-w+s}^{j=t+s}(x_j - \bar{x}_j^{w,s})(y_j - \bar{y}_j^{w,s})}{\sqrt{\sum_{j=t-w+s}^{j=t+s}(x_j - \bar{x}_j^{w,s})^2}\sqrt{\sum_{j=t-w+s}^{j=t+s}(y_t - \bar{y}_j^{w,s})^2}} \tag{1}$$

where $\bar{x}_j^{w,s} = \sum_{j=t-w+s}^{j=t+s} x_j/w$ and $\bar{y}_j^{w,s} = \sum_{j=t-w+s}^{j=t+s} y_j/w$.

Figure 2 (right) illustrates various window sizes with shift parameter $s = 0$ (dash-dotted line), $s > 0$ (dashed line), and $s = w/2$ (solid line) centering the window at t and including $\pm n$ observations around (x_t, y_t) so that $w = 2n$. In a more general form, Figure 2 (left) shows how different data points can be weighted by using an appropriate kernel and shift parameter.

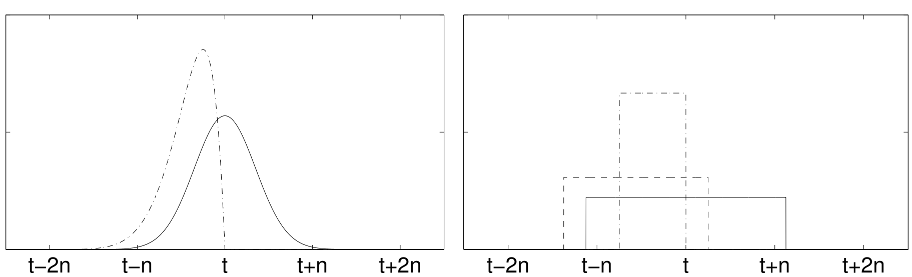

Figure 2: Alternative weighing Kernels for estimation of local window correlations.

As a measure of phase-dependence for time-varying correlations, we propose the calculation of a double correlation coefficient: the Pearson correlation coefficient between the sequence of estimated local s-shifted w-window correlations $\{\hat{\rho}_t^{w,s}\}$ and an appropriate indicator of the business cycle $\{z_t\}$. This correlation coefficient summarizes the strength of agreement and linear co-movement between the business cycle indicator z and the correlation strength between the macroeconomic aggregates x and y. This allows us to conduct statistical inference on the relation between $\hat{\rho}_t^{w,s}$ and z_t. Proposition 1 below reveals the statistical soundness of such a measure. In particular, it shows that the consistency and asymptotic normality of the estimator $\hat{\rho}_T^{w,s,z}$ of the double correlation coefficient $\rho^z_{w,s} = \mathbb{C}\mathrm{orr}(\hat{\rho}_t^{w,s}, z_t)$ follows easily under standard regularity conditions. Proposition 1 proposes also a test statistic with known asymptotic distribution under the null hypothesis of no phase-dependence in $\{\hat{\rho}_t^{w,s}\}$.[3]

Proposition 1. *Let $\{x_t\}_{t\in\mathbb{Z}}$, $\{y_t\}_{t\in\mathbb{Z}}$ and $\{z_t\}_{t\in\mathbb{Z}}$ be strictly stationary and ergodic (SE) stochastic sequences. Then $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ with elements defined in (1) is a subset of an SE random sequence satisfying $\mathbb{E}|\hat{\rho}_t^{w,s}|^k < \infty \ \forall \ (k, w, s) \in \mathbb{N} \times \{1, ..., T\} \times \{1, ..., w-1\}$. Suppose furthermore that $\mathbb{E}|z_t|^2 < \infty$ and define*

$$\hat{\rho}_T^{w,s,z} := \frac{\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s} - \bar{\hat{\rho}}^{w,s})(z_t - \bar{z})}{\sqrt{\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s} - \bar{\hat{\rho}}_t^{w,s})^2}\sqrt{\sum_{t=w-s}^{t=T-s}(z_t - \bar{z})^2}} \qquad \text{and} \qquad \rho^z_{w,s} := \mathbb{C}\mathrm{orr}(\hat{\rho}_T^{w,s,z}, z_t)$$

where $\bar{\hat{\rho}}^{w,s} = \sum_{t=w-s}^{t=T-s} \hat{\rho}_t^{w,s}/(T-w)$ and $\bar{z}_t^{w,s} = \sum_{t=w-s}^{t=T-s} z_t/(T-w)$. Then $\exists\ \sigma^2_\rho > 0$ such that $\hat{\rho}_T^{w,s,z} \xrightarrow{a.s.} \rho^z_{w,s}$ and $\sqrt{T}(\hat{\rho}_T^{w,s,z} - \rho^z_{w,s}) \xrightarrow{d} N(0, \sigma^2_\rho) \ \forall \ (w, s) \in \mathbb{N} \times \{1, ..., w-1\}$ as $T \to \infty$. Finally, define the test statistic,[4]

$$\tilde{\rho}_T^{w,s,z} := \left(\sum_{t=w-s}^{t=T-s}(z_t - \bar{z})^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s} - \bar{\hat{\rho}}_t^{w,s})^2\right)^{\frac{1}{2}} \times \sqrt{T-w}\ \hat{\rho}_T^{w,s,z}$$

[3]Note that while for Gaussian data the finite sample distribution of the Pearson correlation coefficient is known since Fisher (1925), this well known result will not extend to the double correlation considered here as the Gaussianity assumption cannot hold for the sequence $\{\hat{\rho}_t^{w,s}\}$ with elements in $[0, 1]$. The same is true for the approximate variance stabilizing Fisher's z transformation $F(\hat{\rho}_T^{w,s,z}) = \tanh^{-1}(\hat{\rho}_T^{w,s,z})$ whose finite sample distribution is known to be approximately $N(\tanh^{-1}(\rho^z_{w,s}, 1/(T-3))$ but only under a Gaussianity assumption; see Pearson (1931), Rider (1932), Kowalski (1972) and Duncan and Layard (1973).

[4]Normalization by $T-w$ instead of T is asymptotically equivalent but preferable on finite samples.

Then $\tilde{\rho}_T^{w,s,z} \xrightarrow{d} N(0,1)\ \forall\ (w,s) \in \mathbb{N}\times\{1,...,w-1\}$ *as* $T \to \infty$ *under the null hypothesis of no phase-dependence* $\mathrm{H}_0:\ \rho^z_{w,s} = 0$ *and* $\tilde{\rho}_T^{w,s,z} \to \infty\ \forall\ (w,s) \in \mathbb{N} \times \{1,...,w-1\}$ *as* $T \to \infty$ *under the alternative* $\mathrm{H}_1:\ \rho^z_{w,s} \neq 0$.

Proposition 1 defines implicitly a 'two-sided' size α test that rejects the null hypothesis $\mathrm{H}_0 : \rho^z_{w,s} = 0$ against the alternative $\mathrm{H}_1 : \rho^z_{w,s} \neq 0$ if $|\tilde{\rho}_T^{w,s,z}| > z^*_{\alpha/2}$ where $z^*_{\alpha/2}$ is taken from the standard normal table. The 'one-sided' size α tests for $\mathrm{H}_0 : \rho^z_{w,s} > 0$ or $\mathrm{H}_0 : \rho^z_{w,s} < 0$ against $\mathrm{H}_1 : \rho^z_{w,s} \leq 0$ or $\mathrm{H}_1 : \rho^z_{w,s} \geq 0$ are naturally rejected if $\tilde{\rho}_T^{w,s,z} > z^*_\alpha$ or $\tilde{\rho}_T^{w,s,z} < -z^*_\alpha$ respectively.

Proposition 1 above is limited in that (i) it does not provide guidance on the selection of window-size w; (ii) it is silent about the small sample behavior of $\hat{\rho}_T^{w,s,z}$ and $\tilde{\rho}_T^{w,s,z}$, and (iii) it does not convey a structural interpretation to the correlation $\mathbb{C}\mathrm{orr}(\hat{\rho}_t^{w,s}, z_t) = \rho^z_{w,s}$ between $\{z_t\}$ and the constructed rolling-window correlation series. The following Monte Carlo exercise reveals that (i) the optimal window size may depend on the average cycle length, (ii) the asymptotic results derived above constitute a reasonable approximation to the finite sample distribution of $\hat{\rho}_T^{w,s,z}$ and $\tilde{\rho}_T^{w,s,z}$; and (iii) the proposed test statistic can be used as a test of no phase-dependence in the unobserved sequence $\{\rho_t\}$ of 'true correlations' between $\{y_t\}$ and $\{x_t\}$.

4 Monte Carlo Evidence of Small Sample Behavior

Let $\{y_t\}$ and $\{x_t\}$ be sequences with time-varying correlation $\big\{\rho_t = \mathbb{C}\mathrm{orr}(y_t, x_t)\big\}$ between them that is phase dependent (i.e. correlated with some sequence $\{z_t\}$). The following Monte Carlo shows that estimates of $\rho^z_{w,s} = \mathbb{C}\mathrm{orr}(\hat{\rho}_t^{w,s}, z_t)$ approximate $\rho^z = \mathbb{C}\mathrm{orr}(\rho_t, z_t)$. In other words, despite $\rho^z_{w,s}$ being a measure of phase-dependence in the approximate constructed sequence $\{\hat{\rho}_t^{w,s}\}$, estimates of phase-dependence in $\rho^z_{w,s}$ provide information about phase-dependence in the true underlying and unobserved correlation sequence $\{\rho_t\}$. Accordingly, the Monte Carlo also reveals that the statistic

$\tilde{\rho}_T^{w,s,z}$ introduced in Proposition 1 can be used as a nonparametric test statistic for the hypothesis of no phase-dependence in the true unobserved sequence $\{\rho_t\}$. The optimal choice of w is shown to be related with the autocorrelation in the business cycle indicator z_t (and hence with the average cycle length). Finally, the Monte Carlo reveals that the asymptotic distribution of $\tilde{\rho}_T^{w,s,z}$ derived above constitutes a reasonable approximation to the finite sample distribution in moderate sample sizes.

For the Monte Carlo study we consider stochastic sequences $\{y_t\}$ and $\{x_t\}$ generated according to a state space model with a single time-varying parameter $\{\phi_t\}$

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \phi_y \\ \phi_x \end{bmatrix} + \begin{bmatrix} \phi_{yx} & \phi_t \\ \phi_{xy} & \phi_{xx} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ w_t \end{bmatrix} \tag{2}$$

where $\{\epsilon_t\}$ and $\{w_t\}$ are iid Gaussian sequences with $\epsilon_t \sim N(0, \sigma_\epsilon)$ and $w_t \sim N(0, \sigma_w)$, the scalar parameters ϕ_y, ϕ_x, ϕ_{yx} and ϕ_{xx} are time invariant and $\{\phi_t\}$ is a time-varying two-regime threshold parameter generated according to,

$$\phi_t = \begin{cases} \alpha + v_t & \text{if } \ z_t < \bar{z} \\ \alpha + \beta + v_t & \text{if } \ z_t \geq \bar{z} \end{cases} \qquad \text{and} \qquad z_t = \theta_0 + \theta z_{t-1} + u_t \ , \tag{3}$$

with $\{v_t\}$ and $\{u_t\}$ also iid Gaussian $v_t \sim N(0, \sigma_v)$ and $u_t \sim N(0, \sigma_u)$ and α, β, θ_0 and θ fixed scalar parameters. For $\beta > 0$ the time-varying parameter $\{\phi_t\}$ has a 'low regime' (when $z_t < \bar{z}$) where it fluctuates around a mean α, and and 'high regime' (when $z_t \geq \bar{z}$) where it fluctuates around $\alpha + \beta$. If $\beta = 0$ then there is no phase-dependence in the 'observed data' $\{(y_t, x_t)\}$ simulated using (1) and (2).

It is trivial to show that, under certain parameter restrictions, the simulated data satisfies the conditions required by Proposition 1 for the consistency and asymptotic normality results. The following lemma is thus given without proof. For simplicity, we set $\phi_{yx} = \phi_{xy} = 0$ so that $\{x_t\}$ is an exogenous process whose influence on $\{y_t\}$

depends on the business cycle indicator z_t through the time-varying parameter $\{\phi_t\}$.

LEMMA 1. *Let* $0 < \sigma_\epsilon < \infty$, $0 < \sigma_w < \infty$, $0 \leq \sigma_v < \infty$, $0 < \sigma_u < \infty$, $|\phi_y| < \infty$, $|\phi_x| < \infty$, $|\theta_0| < \infty$, $|\alpha| < \infty$, $|\beta| < \infty$, $\phi_{yx} = \phi_{xy} = 0$, $|\phi_{xx}| < 1$ *and* $|\theta| < 1$. *Then* $\{(y_t, x_t, z_t)\}$ *is SE and satisfies* $\mathbb{E}|x_t|^2 < \infty$, $\mathbb{E}|y_t|^2 < \infty$ *and* $\mathbb{E}|z_t|^2 < \infty$.

Data simulated using (1) and (2) is now used to assess the small sample distribution of $\hat{\rho}_T^{w,s,z}$ and $\tilde{\rho}_T^{w,s,z}$ in Proposition 1 for alternative parameter values various choices of sample size T and window-size w. Unless stated otherwise, the following parameter values have been selected $\sigma_\epsilon = 0.1$, $\sigma_w = 0.1$, $\sigma_v = 0.02$, $\sigma_u = 0.1$, $\phi_y = \phi_x = \theta_0 = 0$, $\alpha = 0.5$, $\theta = 0.8$ and $s = w/2$. Figure 3 shows the small sample distribution of the double correlation coefficient under no phase-dependence ($\beta = 0$) in the true unobserved correlation process $\{\rho_t\}$ and reveals how the measure $\hat{\rho}_T^{w,s,z}$ based on the approximate sequence $\{\hat{\rho}_t^{w,s}\}$ is correctly centered at zero.

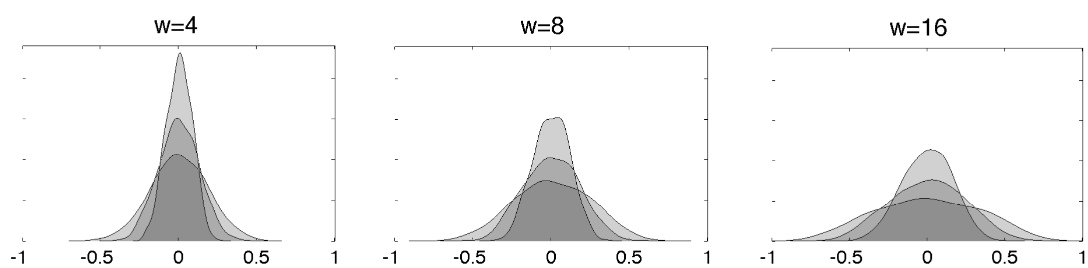

Figure 3: Density plots of $\hat{\rho}_T^{w,s,z}$ obtained from $S = 1000$ Monte Carlo simulated paths of $\{(y_t, x_t, z_t)\}_{t=1}^T$ under no-phase-dependence ($\beta = 0$), for window size $w = 4$ (left), $w = 8$ (center) and $w = 16$ (right), and sample size $T = 100$, 250 and 500. The larger T densities are identifiable by the higher concentration of mass around the origin.

On the contrary, Figure 4 plots the small sample distribution of $\hat{\rho}_T^{w,s,z}$ when there is phase-dependence ($\beta = -0.3$). Figure 4 reveals once again that the measure $\hat{\rho}_T^{w,s,z}$ based on the approximate sequence $\{\hat{\rho}_t^{w,s}\}$ reflects the phase-dependence in the true underlying and unobserved correlation sequence $\{\rho_t\}$.

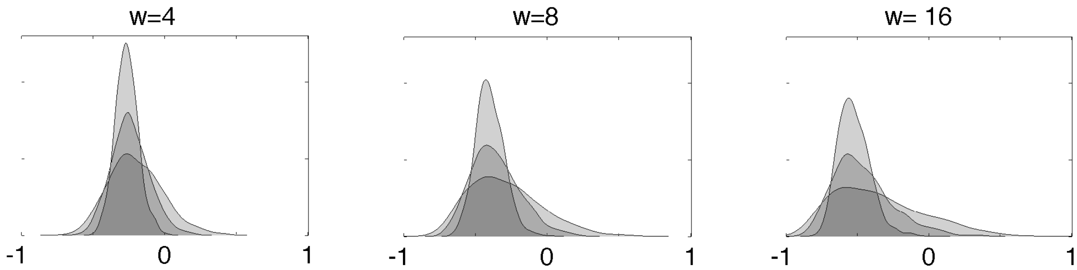

Figure 4: Density plots of $\hat{\rho}_T^{w,s,z}$ obtained from $S = 1000$ Monte Carlo simulated paths of $\{(y_t, x_t, z_t)\}_{t=1}^T$ under phase-dependence ($\beta = -0.3$), for window size $w = 4$ (left), $w = 8$ (center) and $w = 16$ (right), and sample size $T = 100$, 250 and 500. The larger T densities are identifiable by the smaller variance.

Table 2 provides information about the finite sample power of the no phase-dependence test statistic $\tilde{\rho}_T^{w,s,z}$ by stating rejection frequencies for the null hypothesis of no phase dependence when $\beta = 1$ and $\beta = 2$. Naturally, the power of the test increases with sample size, and rejection frequencies are better at $\beta = 2$ than $\beta = 1$.

		$\beta = 1$				$\beta = 2$			
	T	$w = 2$	$w = 4$	$w = 8$	$w = 16$	$w = 2$	$w = 4$	$w = 8$	$w = 16$
$\theta = .4$	100	0.23	0.30	0.24	0.19	0.38	0.46	0.35	0.23
	200	0.42	0.52	0.45	0.30	0.71	0.77	0.64	0.42
	500	0.68	0.79	0.69	0.47	0.94	0.96	0.89	0.65
	1000	0.93	0.98	0.93	0.72	1.00	1.00	1.00	0.92
$\theta = .8$	100	0.40	0.56	0.63	0.52	0.65	0.78	0.80	0.65
	200	0.71	0.86	0.91	0.83	0.95	0.98	0.98	0.93
	500	0.95	0.99	1.00	0.97	1.00	1.00	1.00	1.00
	1000	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\theta = .97$	100	0.39	0.55	0.66	0.70	0.59	0.73	0.77	0.78
	200	0.72	0.85	0.93	0.94	0.91	0.95	0.96	0.97
	500	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1000	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 2: Power of $\tilde{\rho}_T^{w,s,z}$ as measured by rejection frequencies from $S = 1000$ Monte Carlo draws at 95% confidence level for the null hypothesis of no phase-dependence $\mathrm{H}_0 : \rho_{w,s}^z = 0$ and a two-sided alternative $\mathrm{H}_1 : \rho_{w,s}^z \neq 0$ using the asymptotic $N(0,1)$ critical values derived in Proposition 1.

The optimal choice of w depends on the amount of temporal dependence in $\{z_t\}$ as measured by θ. In particular, smaller window sizes perform better under low dependence (e.g. $w = 4$ under $\theta = 0.4$), while larger window sizes perform better under high dependence (e.g. $w = 16$ at $\theta = 0.97$).

The relation between the optimal choice of window-size w and the temporal dependence in $\{z_t\}$ is made clear in Figure 5. As expected, large window sizes perform badly in low dependence setting where the average cycle length in $\{z_t\}$ is quite short and the large window choice ends up averaging over various periods of high and low correlation. For low dependence, small windows have higher power. Strong dependence in $\{z_t\}$ is favorable to large window sizes that calculate correlations with a larger number of observations and hence a smaller degree of uncertainty. When cycles in z_t are large, the advantage of large window correlation sequences in better filtering the signal from the noise is reflected in their higher power.

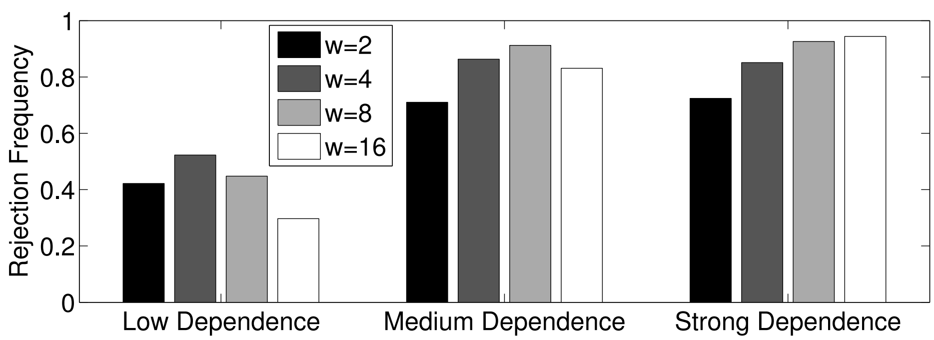

Figure 5: Null hypothesis rejection frequencies from $S = 1000$ Monte Carlo draws at 95% confidence level for the null hypothesis of no phase-dependence $\mathrm{H}_0 : \ \rho^z_{w,s} = 0$ and a two-sided alternative $\mathrm{H}_1 : \ \rho^z_{w,s} \neq 0$ using asymptotic $N(0,1)$ critical values derived in Proposition 1 for $\beta = 1$ and $T = 200$ under low ($\theta = 0.4$), medium ($\theta = 0.8$) and strong ($\theta = 0.97$) dependence.

Table 3 now shows the finite sample size of the test $\tilde{\rho}^{w,s,z}_T$. The Monte Carlo reveals that size is always better with small window size w and naturally improving with sample size T. Furthermore, it is clear that the finite sample size is reasonably close to nominal size for small θ and small w, yet considerably distorted under a local-to-unit root strong dependence ($\theta = 0.97$) and large w.

	$\beta = 0$				
	T	$w = 2$	$w = 4$	$w = 8$	$w = 16$
$\theta = .4$	100	0.084	0.093	0.109	0.125
	200	0.062	0.082	0.084	0.099
	500	0.057	0.074	0.079	0.091
	1000	0.057	0.068	0.076	0.088
$\theta = .8$	100	0.093	0.131	0.205	0.264
	200	0.082	0.113	0.167	0.232
	500	0.073	0.106	0.159	0.214
	1000	0.071	0.104	0.143	0.201
$\theta = .97$	100	0.104	0.148	0.246	0.381
	200	0.084	0.124	0.221	0.350
	500	0.078	0.117	0.206	0.332
	1000	0.076	0.110	0.191	0.312

Table 3: Size of $\tilde{\rho}_T^{w,s,z}$ as measured by rejection frequencies (under $\beta = 0$) from $S = 4000$ Monte Carlo draws at 95% confidence level for the null hypothesis of no phase-dependence $\mathrm{H}_0 : \ \rho^z_{w,s} = 0$ and a two-sided alternative $\mathrm{H}_1 : \ \rho^z_{w,s} \neq 0$ using asymptotic results in Proposition 1.

A conservative test will thus favor a small window size w despite the cost that this might bring in terms of power evidenced by Table 2 and Figure 5. The trade-off between power and size for $w = 2$ and $w = 4$ might however be worth consideration depending on the application, since under strong dependence ($\theta = 0.97$) the gains in power can be larger than 15% (see e.g. Table 2 under $\beta = 1$).

Alternative parameter choices in terms of $0 < \sigma_\epsilon < \infty$, $0 < \sigma_w < \infty$, $0 \leq \sigma_v < \infty$, $0 < \sigma_u < \infty$, $|\phi_y| < \infty$, $|\phi_x| < \infty$, $|\theta_0| < \infty$, $|\alpha| < \infty$ seem to have only marginal effects on the results presented in Tables 2 and 3.

Finally, Figure 6 shows a finite sample power function that summarizes the test behavior under local-to-null-hypothesis parameter values and the effects of temporal dependence in $\{z_t\}$ for various sample sizes T. In accordance to the previous results, power increases as β diverges from 0, as T increases, and size distortion is worse under strong dependence.

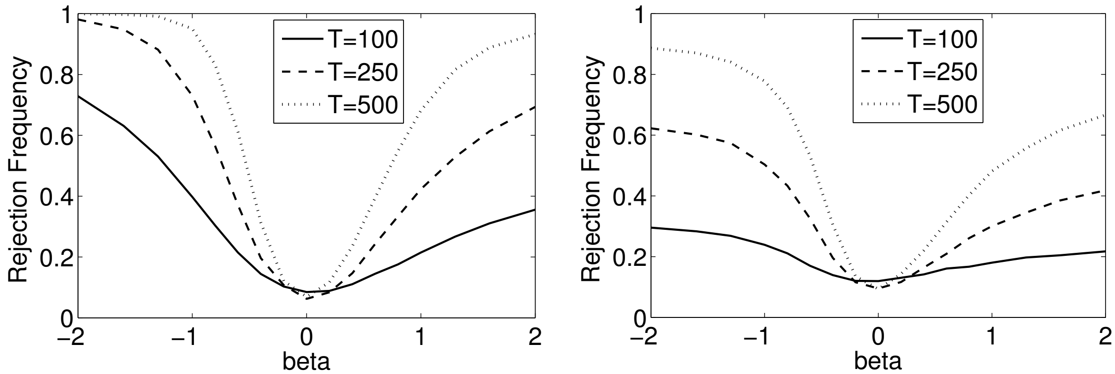

Figure 6: Finite sample power as function of β obtained from $S = 2000$ Monte Carlo draws for $(w, \theta) = (2, 0.4)$ (left) and $(w, \theta) = (16, 0.97)$ (right).

5 Phase Dependence in the US Business Cycle

Making use of the theory established in Section 3, we now turn our attention back to the data presented in Section 2 and assess wether the presence of phase-dependence in the correlation between US macroeconomic aggregates is a statistically significant.

The phase-dependence measures and test statistics are obtained using a window size of $w = 4$ and a shift of $s = 2$ for centering. According to the small sample Monte Carlo evidence collected in Section 4, this will give us conservative testing procedures with small sample size that very close to nominal at the cost of loosing some power.

Table 4 shows that the presence of phase-dependence in the correlation between US macroeconomic aggregates is statistically significant at standard confidence levels. Correlation strengths between macroeconomic aggregates are generally negatively related with the NBER business cycle. In particular, correlations are statistically stronger during recessions and weaker during expansions. Indeed, except for the correlation between consumption and investment which shows no sign of phase dependence with the business cycle, all other correlations (both contemporaneous and lagged) appear significantly different from zero. Table 4 thus documents a rich time-varying correlation structure of macroeconomic data that is usually ignored.

	y	i	c	y_{-1}	i_{-1}	c_{-1}
y	-	-0.23^{**}	-0.18^{*}	-0.29^{**}	-0.26^{**}	-0.30^{**}
	-	(-5.44)	(-2.45)	(-6.28)	(-5.41)	(-6.87)
i	-0.23^{**}	-	-0.08	-0.24^{**}	-0.25^{**}	-0.25^{**}
	(-5.44)	-	(0.20)	(-4.21)	(-4.47)	(-5.19)
c	-0.18^{*}	-0.08	-	-0.18^{**}	-0.19^{**}	-0.21^{**}
	(-2.45)	(0.20)	-	(-2.85)	(-2.88)	(-2.91)

Table 4: NBER business cycle phase-dependence characterization of log US HP-filtered auto-covariance structure. Table shows values of nonparametric phase-dependence measure $\hat{\rho}_T^{w,s,z}$ with associated test statistic $\tilde{\rho}_T^{w,s,z}$ in brackets. $*$ and $**$ indicate rejection at 5% and 1% significance levels.

A Proof of Proposition 1

This proof follows from standard well known results. Continuity of the sample s-shifted w-window correlation $\hat{\rho}_t^{w,s}$ on $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$ $\forall$ (t,w,s) : $t-w+s \geq 1 \wedge t+s \leq T$ ensures measurability when relevant sets are equipped with a Borel sigma-algebra. As a result, $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ is SE by Proposition 4.3 of Krengel (1985, p.26), the SE nature of $\{x_t\}_{t=1}^T$ and $\{y_t\}_{t=1}^T$ and $\hat{\rho}_t^{w,s}$ being a continuous function of a finite subset of $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$ $\forall$ (t,w,s). Naturally, $\mathbb{E}|\hat{\rho}_t^{w,s}|^k < \infty$ $\forall$ $(k,w,s) \in \mathbb{N} \times \{1,...,T\} \times \{1,...,w-1\}$ holds for one t and hence all t by the SE nature of $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ and the fact that its elements take values in $[0,1]$. Since $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ and $\{z_t\}_{t=1}^{t=T}$ are both subsets of SE sequences with $\mathbb{E}|\hat{\rho}_t^{w,s}|^2 < \infty$ and $\mathbb{E}|z_t|^2 < \infty$, it follows that $\{\hat{\rho}_t^{w,s} - \mu_{\hat{\rho},w,s}\}_{t=w-s}^{t=T-s}$ and $\{z_t - \mu_z\}_{t=1}^{t=T}$ are SE with $\mathbb{E}|\hat{\rho}_t^{w,s} - \mu_{\hat{\rho},w,s}|^2 < \infty$ and $\mathbb{E}|z_t - \mu_z|^2 < \infty$ (by Cauchy-Schwartz inequality) where $\mu_{\hat{\rho},w,s} := \mathbb{E}\hat{\rho}_t^{w,s}$ and $\mu_z := \mathbb{E}z_t$, and furthermore, by Minkowsky's inequality

$$\mathbb{E}\Big|(\hat{\rho}_t^{w,s} - \mu_{\hat{\rho},w,s})(z_t - \mu_z)\Big| \leq \Big(\mathbb{E}|\hat{\rho}_t^{w,s} - \mu_{\hat{\rho},w,s}|^2\Big)^{\frac{1}{2}}\Big(\mathbb{E}|z_t - \mu_z|\Big)^{\frac{1}{2}} < \infty.$$

Application of the ergodic theorem (see e.g. Davidson (1994, Theorem 13.12)) yields

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}\hat{\rho}_t^{w,s} \overset{a.s.}{\rightarrow} \mathbb{E}\hat{\rho}_t^{w,s} \quad , \quad \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} z_t \overset{a.s.}{\rightarrow} \mathbb{E}z_t \ ,$$

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2 \overset{a.s.}{\rightarrow} \mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2 = \mathbb{V}\text{ar}(\hat{\rho}_t^{w,s}) \ ,$$

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z)^2 \overset{a.s.}{\rightarrow} \mathbb{E}(z_t-\mu_z)^2 = \mathbb{V}\text{ar}(z_t) \quad \text{and}$$

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z) \overset{a.s.}{\rightarrow} \mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z) = \mathbb{C}\text{ov}(\hat{\rho}_t^{w,s}, z_t) \ ,$$

as $T \to \infty \ \forall \ (w,s) \in \mathbb{N} \times \{1,...,w-1\}$. Hence,

$$\begin{aligned}
&\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})(z_t-\bar{z}) \\
&= \frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}+\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})(z_t-\mu_z+\mu_z-\bar{z}) \\
&= \frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z) + (\mu_z-\bar{z})\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}) \\
&\quad + (\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z) + (\mu_z-\bar{z})\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s}) \\
&\overset{a.s.}{\rightarrow} \mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z)+0+0+0 = \mathbb{C}\text{ov}(\hat{\rho}_t^{w,s}, z_t) \quad \text{as} \quad T \to \infty \quad \text{and}
\end{aligned}$$

$$
\begin{aligned}
&\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})^2\Big)\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\bar{z})^2\Big)\\
&=\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}+\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})^2\Big)\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z+\mu_z-\bar{z})^2\Big)\\
&=\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2+\frac{T-w}{T-w}(\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})^2\\
&\quad+(\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})\frac{2}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})\Big)\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z)^2\\
&\quad+\frac{T-w}{T-w}(\mu_z-\bar{z})^2+(\mu_z-\bar{z})\frac{2}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z)\Big)\\
&\quad\overset{a.s.}{\to}\ \Big(\mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2+0^2+0\Big)\Big(\mathbb{E}(z_t-\mu_z)^2+0^2+0\Big)=\mathbb{V}\mathrm{ar}(\hat{\rho}_t^{w,s})\mathbb{V}\mathrm{ar}(z_t)
\end{aligned}
$$

as $T\to\infty$. Application of a continuous mapping theorem yields the desired consistency result as $T\to\infty$ and for every $(w,s)\in\mathbb{N}\times\{1,...,w-1\}$

$$
\hat{\rho}_T^{w,s,z}:=\frac{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})(z_t-\bar{z})}{\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}_t^{w,s})^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\bar{z})^2}}\ \overset{a.s.}{\to}\ \rho_{w,s}^z:=\mathbb{C}\mathrm{orr}(\hat{\rho}_T^{w,s,z},z_t).
$$

Asymptotic normality of $\sqrt{T-w}(\hat{\rho}_T^{w,s,z}-\rho_{w,s}^z)\ \forall\ (w,s)\in\mathbb{N}\times\{1,...,w-1\}$ as $T\to\infty$ follows from

$$
\begin{aligned}
\sqrt{T-w}(\hat{\rho}_T^{w,s,z}-\rho_{w,s}^z)&=\frac{\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s}r_ts_t}{\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}r_t^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}s_t^2}}-\frac{rs}{\sqrt{r^2}\sqrt{s^2}}\\
&=\frac{\sqrt{r^2}\sqrt{s^2}\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s}r_ts_t-rs\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}r_t^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}s_t^2}}{\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}r_t^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}s_t^2}\sqrt{r^2}\sqrt{s^2}}
\end{aligned}
$$

where $r_t:=(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})$, $s_t=(z_t-\bar{z})$, $rs:=\mathbb{C}\mathrm{ov}(\hat{\rho}_T^{w,s,z},z_t)$, $r^2:=\mathbb{V}\mathrm{ar}(\hat{\rho}_T^{w,s,z})$ and

$s^2 := \mathbb{V}\mathrm{ar}(s_t^2)$. Now the numerator satisfies,

$$
\begin{aligned}
&\sqrt{r^2}\sqrt{s^2}\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2} \\
&= \sqrt{r^2}\sqrt{s^2}\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs\sqrt{r^2}\sqrt{s^2} \\
&\quad + rs\sqrt{r^2}\sqrt{s^2} - rs\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2} \\
&= \sqrt{r^2}\sqrt{s^2}\Big(\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs\Big) \\
&+ rs\left(\sqrt{r^2}\sqrt{s^2} - \sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2}\right).
\end{aligned}
$$

Asymptotic normality of the numerator is thus obtained by application of the central limit theorem Billingsley (1961) to the SE martingale difference sequence $\{r_t s_t - rs\}$ in to obtain, for some $0 < \sigma_{rs} := \mathbb{E}(r_t s_t)^2 < \infty$,

$$
\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs \overset{d}{\to} N(0,\sigma_{rs}^2) \quad \text{as} \quad T \to \infty
$$

and an ergodic theorem in Davidson (1994, Theorem 13.12) to obtain a denominator,

$$
\sqrt{r^2}\sqrt{s^2}\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2} \overset{a.s.}{\to} r^2 s^2 \quad \text{as} \quad T \to \infty
$$

and hence, application of a continuous mapping theorem and Slutsky's theorem yields,

$$
\sqrt{T-w}(\hat{\rho}_T^{w,s,z} - \rho_{w,s}^z) \overset{d}{\to} N(0,\sigma_\rho^2) \quad \text{as} \quad T \to \infty \quad \text{where} \quad \sigma_\rho^2 := \sigma_{rs}^2/(r^2 s^2)^2.
$$

The claim that $\tilde{\rho}_T^{w,s,z} \overset{d}{\to} N(0,1)\ \forall\ (w,s) \in \mathbb{N} \times \{1,...,w-1\}$ as $T \to \infty$ under the null hypothesis of no phase-dependence $\mathrm{H}_0 :\ \rho_{w,s}^z = 0$ now follows immediately since under H_0 we have $rs := \mathbb{C}\mathrm{ov}(\hat{\rho}_T^{w,s,z}, z_t) = 0$ and hence, $\sqrt{T-w}\ \hat{\rho}_T^{w,s,z} \overset{d}{\to} N(0,\sigma_\rho^2)$

as $T \to \infty$, and furthermore and $\sigma_{rs}^2 = \mathbb{E}(r_t s_t)^2 = \mathbb{E}r_t^2 \mathbb{E}s_t^2 = r^2 s^2$, and hence by Slutsky's theorem,

$$\tilde{\rho}_T^{w,s,z} := \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{\frac{1}{2}} \times \sqrt{T-w}\ \hat{\rho}_T^{w,s,z} \xrightarrow{d} N(0,1) \quad \text{as} \quad T \to \infty.$$

Finally, the claim that $\tilde{\rho}_T^{w,s,z} \to \infty$ as $T \to \infty$ under the alternative $\mathrm{H}_1 :\ \rho_{w,s}^z \neq 0$ is obtained since under H_1 we have $rs := \mathbb{C}\mathrm{ov}(\hat{\rho}_T^{w,s,z}, z_t) \neq 0$ and hence $\sqrt{T-w} rs \to \infty$ and $\sqrt{T-w}\rho_{w,s}^z \to \infty$ as $T \to \infty$ and hence,

$$\begin{aligned}
\tilde{\rho}_T^{w,s,z} :&= \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{-\frac{1}{2}} \times \sqrt{T-w}\ \hat{\rho}_T^{w,s,z} \\
&= \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{-\frac{1}{2}} \times \sqrt{T-w}\ (\hat{\rho}_T^{w,s,z} - \rho_{w,s}^z) \\
&\quad + \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{-\frac{1}{2}} \times \sqrt{T-w}\rho_{w,s}^z \to \infty \quad \text{as} \quad T \to \infty. \quad \square
\end{aligned}$$

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On the Phase Dependence in Time-Varying Correlations Between Time-Series*

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Abstract

This paper proposes the use of a double correlation coefficient as a nonparametric measure of phase-dependence in time-varying correlations. An asymptotically Gaussian test statistic for the null hypothesis of no phase-dependence is derived from the proposed measure. Finite-sample distributions, power and size are analyzed in a Monte-Carlo exercise. An application of this test provides evidence that correlation strength between major macroeconomic aggregates is both time-varying and phase dependent in the business cycle.

Keywords: nonparametric; phase-dependence; time-varying correlation.

JEL Codes: C01, C14, C32.

1 Introduction

Many economic time-series exhibit correlation strengths that change over time. Parametric state-space models like the dynamic conditional correlation (DCC) model of Engle (2000) attempt to capture these time-variations in correlation strength between financial time-series. Structural parametric models with nonlinearities and time-varying parameters in macroeconomics and microeconomics are also capable of

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generating time-varying correlations. Nonparametric methods are however desirable in cases where the researcher does not wish to impose a potentially restrictive parametric model structure on the data. Evidence of model misspecification in certain applications is thus a possible reason to adopt, or at least complement, econometric analysis with nonparametric methods. Nonparametric methods are also appealing when searching for 'descriptive statistics' that reflect simple and agnostic 'stylized facts'. For example, in empirical macroeconomics, considerable effort has been invested since Kydland and Prescott (1982) in the design and estimation of structural macroeconomic models capable of describing the correlation structure of the data. Even estimated macroeconomic structural models are usually evaluated by their ability to accurately describe a matrix of correlations between contemporaneous macroeconomic variables and their lags; see e.g. Dave and Dejong (2007).

This paper proposes the use of a double correlation coefficient as a nonparametric measure of phase-dependence in the correlation between two time-series. Under appropriate regularity conditions the proposed statistic is easily shown to be consistent a asymptotically normal. The asymptotic distribution of a derived test statistic is also obtained. The small sample distributions, power and sizes of the phase-dependence measure and associated test statistic are obtained by means of a Monte Carlo exercise. In an application, the proposed test is used to provide evidence that the autocorrelation structure of macroeconomic aggregates for the US economy evolves over time, and most importantly, that these time-varying correlations are systematically linked to the phase of the business cycle, i.e. that they are phase-dependent. In particular, the data reveals that correlations between aggregate output and aggregate investment in the US are significantly stronger during recessions and weaker during expansions.

Section 2 motivates the problem by analyzing some stylized evidence of time-varying correlation and phase-dependence in US macroeconomic aggregates. Section 3 proposes the nonparametric measure of phase-dependence and related test of no phase-dependence and obtains their asymptotic properties. Section 4 provides a Monte Carlo description of their small sample behavior. Finally, Section 5 assesses the existence of phase-dependence in correlations between macroeconomic aggregates.

2 Time-Varying and Phase-Dependent Correlations in Macroeonomic Aggregates

Since Kydland and Prescott (1982) the characterization of the business cycle in macroeconomics has been typically restricted to the second-order unconditional moments of filtered data. The following table contains a standard characterization of the business cycle for US quaterly data (1947 $Q1$ - 2012 $Q1$) based on the variance and cross-autocorrelation between HP-filtered logarithms of three macroeconomic aggregates: real private investment (i_t), rel private consumption (c_t) and real gross domestic product (y_t).[1]

	σ/σ_y	r_y	r_c	r_i	$r_{y_{-1}}$	$r_{c_{-1}}$	$r_{i_{-1}}$
y_t	1.00	-	0.69	0.78	0.82	0.62	0.66
c_t	1.18	0.69	-	0.55	0.69	0.91	0.54
i_t	3.20	0.78	0.55	-	0.62	0.52	0.81

Table 1: Log US HP-filtered business cycle characterization of auto-correlation structure. Column σ/σ_y shows ratio of estimated variance with output's estimated variance, r_x shows estimated correlation with variable x, and $r_{x_{-1}}$ shows estimated correlation with the lag of x.

Table 1 provides however a rather incomplete picture of the correlation between these aggregates. In particular, it ignores the rich time-variation that occurs in the correlation structure, and most importantly, the fact that this time-variation depends

[1]Data obtained from the Federal Reserve Bank of Saint Louis in billions of chained 2005 dollars.

itself on the business cycle; i.e. that correlations are phase-dependent w.r.t. the business cycle. For example, Figure 1 reveals the striking fact that the correlation strength between output and investment is phase-dependent in NBER cycles. In particular, NBER recession periods are marked by very high correlations and large drops in correlation strength seem to occur only during NBER expansions.

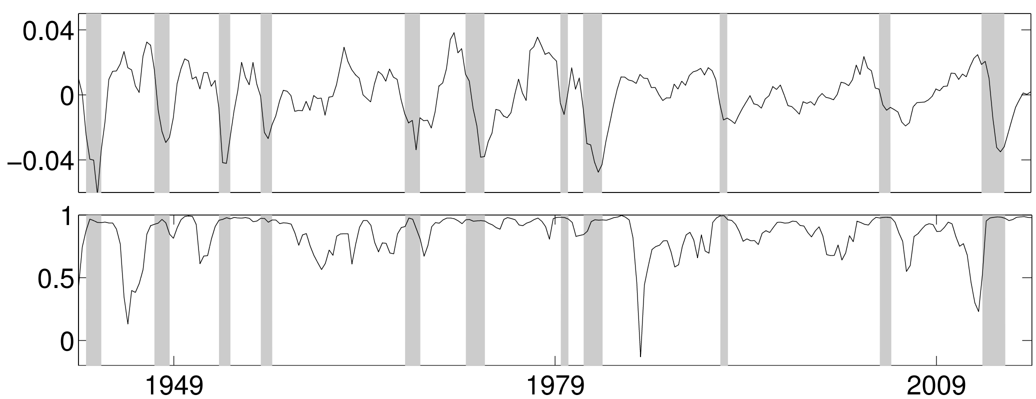

Figure 1: Time series of HP-filtered log US output $\{y_t\}$ (above) and an estimated s-shifted w-window correlation series $\{\hat{\rho}_t^{w,s}\}$ between aggregate output $\{y_t\}$ and aggregate investment $\{i_t\}$ (below) with NBER recession indicator in shaded areas.[2]

A careful inspection of the data shows that correlations between output and investment never dropped below 0.5 in recession periods. Even correlations below 0.8 are rare during recessions. In contrast, in expansion periods, drops of correlation below 0.5 do occur, and correlations below 0.8 are quite frequent. This apparent relation can however be spurious and requires statistical verification. It is thus important to assess the statistical significance of such findings. In particular, it is crucial to understand the properties of the estimated correlation series and to find a statistically sound method of evaluating the presence of phase-dependence. We now address this issue.

[2]The series plotted in Figure 1 make use of the s-shifted w-window uniformly weighted rolling window defined in (1) with a small positive shift to guarantee centering $s = 2$. The span of the kernel $w = 4$ was selected to be no larger than the average length of the NBER output cycle.

3 A Measure of Phase-Dependence

The estimated time-varying correlation series $\{\hat{\rho}_t^{w,s}\}$ in Figure 1 are Pearson correlations calculated over a rolling window. In its simplest form, given a bivariate time-series $\{(y_t, x_t)\}$, the sequence $\{\hat{\rho}_t^{w,s}\}$ of s-shifted w-window correlations between y and x is defined, for every t, as the sample Pearson correlation $\mathbb{C}\mathrm{orr}\Big(\mathbf{y}_t^{w,s}, \mathbf{x}_t^{w,s}\Big)$ between $\mathbf{y}_t^{w,s} = (y_{t-w+s}, ..., y_{t+s})$ and $\mathbf{x}_t^{w,s} = (x_{t-w+s}, ..., x_{t+s})$ where s is a shift index that defines the centering of the window over the time period of interest,

$$\hat{\rho}_t^{w,s} := \frac{\sum_{j=t-w+s}^{j=t+s}(x_j - \bar{x}_j^{w,s})(y_j - \bar{y}_j^{w,s})}{\sqrt{\sum_{j=t-w+s}^{j=t+s}(x_j - \bar{x}_j^{w,s})^2}\sqrt{\sum_{j=t-w+s}^{j=t+s}(y_t - \bar{y}_j^{w,s})^2}} \tag{1}$$

where $\bar{x}_j^{w,s} = \sum_{j=t-w+s}^{j=t+s} x_j/w$ and $\bar{y}_j^{w,s} = \sum_{j=t-w+s}^{j=t+s} y_j/w$.

Figure 2 (right) illustrates various window sizes with shift parameter $s = 0$ (dash-dotted line), $s > 0$ (dashed line), and $s = w/2$ (solid line) centering the window at t and including $\pm n$ observations around (x_t, y_t) so that $w = 2n$. In a more general form, Figure 2 (left) shows how different data points can be weighted by using an appropriate kernel and shift parameter.

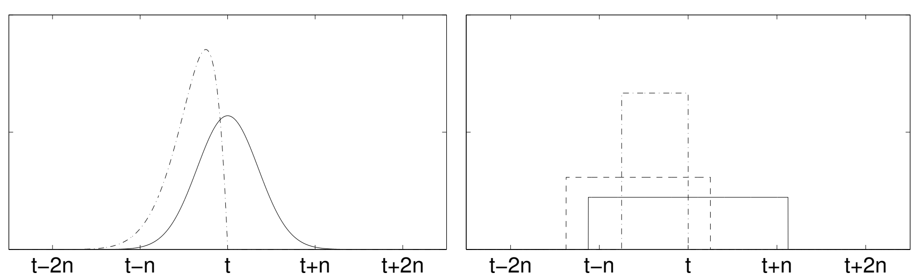

Figure 2: Alternative weighing Kernels for estimation of local window correlations.

As a measure of phase-dependence for time-varying correlations, we propose the calculation of a double correlation coefficient: the Pearson correlation coefficient between the sequence of estimated local s-shifted w-window correlations $\{\hat{\rho}_t^{w,s}\}$ and an appropriate indicator of the business cycle $\{z_t\}$. This correlation coefficient summarizes the strength of agreement and linear co-movement between the business cycle indicator z and the correlation strength between the macroeconomic aggregates x and y. This allows us to conduct statistical inference on the relation between $\hat{\rho}_t^{w,s}$ and z_t. Proposition 1 below reveals the statistical soundness of such a measure. In particular, it shows that the consistency and asymptotic normality of the estimator $\hat{\rho}_T^{w,s,z}$ of the double correlation coefficient $\rho_{w,s}^z = \mathbb{C}\mathrm{orr}(\hat{\rho}_t^{w,s}, z_t)$ follows easily under standard regularity conditions. Proposition 1 proposes also a test statistic with known asymptotic distribution under the null hypothesis of no phase-dependence in $\{\hat{\rho}_t^{w,s}\}$.[3]

Proposition 1. *Let $\{x_t\}_{t\in\mathbb{Z}}$, $\{y_t\}_{t\in\mathbb{Z}}$ and $\{z_t\}_{t\in\mathbb{Z}}$ be strictly stationary and ergodic (SE) stochastic sequences. Then $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ with elements defined in (1) is a subset of an SE random sequence satisfying $\mathbb{E}|\hat{\rho}_t^{w,s}|^k < \infty \ \forall \ (k,w,s) \in \mathbb{N} \times \{1,...,T\} \times \{1,...,w-1\}$. Suppose furthermore that $\mathbb{E}|z_t|^2 < \infty$ and define*

$$\hat{\rho}_T^{w,s,z} := \frac{\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s} - \bar{\hat{\rho}}^{w,s})(z_t - \bar{z})}{\sqrt{\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s} - \bar{\hat{\rho}}_t^{w,s})^2}\sqrt{\sum_{t=w-s}^{t=T-s}(z_t - \bar{z})^2}} \quad \text{and} \quad \rho_{w,s}^z := \mathbb{C}\mathrm{orr}(\hat{\rho}_T^{w,s,z}, z_t)$$

where $\bar{\hat{\rho}}^{w,s} = \sum_{t=w-s}^{t=T-s} \hat{\rho}_t^{w,s}/(T-w)$ and $\bar{z}_t^{w,s} = \sum_{t=w-s}^{t=T-s} z_t/(T-w)$. Then $\exists \ \sigma_\rho^2 > 0$ such that $\hat{\rho}_T^{w,s,z} \xrightarrow{a.s.} \rho_{w,s}^z$ and $\sqrt{T}(\hat{\rho}_T^{w,s,z} - \rho_{w,s}^z) \xrightarrow{d} N(0,\sigma_\rho^2) \ \forall \ (w,s) \in \mathbb{N} \times \{1,...,w-1\}$ as $T \to \infty$. Finally, define the test statistic,[4]

$$\tilde{\rho}_T^{w,s,z} := \left(\sum_{t=w-s}^{t=T-s}(z_t - \bar{z})^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s} - \bar{\hat{\rho}}_t^{w,s})^2\right)^{\frac{1}{2}} \times \sqrt{T-w}\ \hat{\rho}_T^{w,s,z}$$

[3] Note that while for Gaussian data the finite sample distribution of the Pearson correlation coefficient is known since Fisher (1925), this well known result will not extend to the double correlation considered here as the Gaussianity assumption cannot hold for the sequence $\{\hat{\rho}_t^{w,s}\}$ with elements in $[0,1]$. The same is true for the approximate variance stabilizing Fisher's z transformation $F(\hat{\rho}_T^{w,s,z}) = \tanh^{-1}(\hat{\rho}_T^{w,s,z})$ whose finite sample distribution is known to be approximately $N(\tanh^{-1}(\rho_{w,s}^z, 1/(T-3))$ but only under a Gaussianity assumption; see Pearson (1931), Rider (1932), Kowalski (1972) and Duncan and Layard (1973).

[4] Normalization by $T-w$ instead of T is asymptotically equivalent but preferable on finite samples.

Then $\tilde{\rho}_T^{w,s,z} \xrightarrow{d} N(0,1)\ \forall\ (w,s) \in \mathbb{N}\times\{1,...,w-1\}$ *as* $T \to \infty$ *under the null hypothesis of no phase-dependence* $\mathrm{H}_0:\ \rho^z_{w,s} = 0$ *and* $\tilde{\rho}_T^{w,s,z} \to \infty\ \forall\ (w,s) \in \mathbb{N}\times\{1,...,w-1\}$ *as* $T \to \infty$ *under the alternative* $\mathrm{H}_1:\ \rho^z_{w,s} \neq 0$.

Proposition 1 defines implicitly a 'two-sided' size α test that rejects the null hypothesis $\mathrm{H}_0 : \rho^z_{w,s} = 0$ against the alternative $\mathrm{H}_1 : \rho^z_{w,s} \neq 0$ if $|\tilde{\rho}_T^{w,s,z}| > z^*_{\alpha/2}$ where $z^*_{\alpha/2}$ is taken from the standard normal table. The 'one-sided' size α tests for $\mathrm{H}_0 : \rho^z_{w,s} > 0$ or $\mathrm{H}_0 : \rho^z_{w,s} < 0$ against $\mathrm{H}_1 : \rho^z_{w,s} \leq 0$ or $\mathrm{H}_1 : \rho^z_{w,s} \geq 0$ are naturally rejected if $\tilde{\rho}_T^{w,s,z} > z^*_\alpha$ or $\tilde{\rho}_T^{w,s,z} < -z^*_\alpha$ respectively.

Proposition 1 above is limited in that (i) it does not provide guidance on the selection of window-size w; (ii) it is silent about the small sample behavior of $\hat{\rho}_T^{w,s,z}$ and $\tilde{\rho}_T^{w,s,z}$, and (iii) it does not convey a structural interpretation to the correlation $\mathbb{C}\mathrm{orr}(\hat{\rho}_t^{w,s}, z_t) = \rho^z_{w,s}$ between $\{z_t\}$ and the constructed rolling-window correlation series. The following Monte Carlo exercise reveals that (i) the optimal window size may depend on the average cycle length, (ii) the asymptotic results derived above constitute a reasonable approximation to the finite sample distribution of $\hat{\rho}_T^{w,s,z}$ and $\tilde{\rho}_T^{w,s,z}$; and (iii) the proposed test statistic can be used as a test of no phase-dependence in the unobserved sequence $\{\rho_t\}$ of 'true correlations' between $\{y_t\}$ and $\{x_t\}$.

4 Monte Carlo Evidence of Small Sample Behavior

Let $\{y_t\}$ and $\{x_t\}$ be sequences with time-varying correlation $\big\{\rho_t = \mathbb{C}\mathrm{orr}(y_t, x_t)\big\}$ between them that is phase dependent (i.e. correlated with some sequence $\{z_t\}$). The following Monte Carlo shows that estimates of $\rho^z_{w,s} = \mathbb{C}\mathrm{orr}(\hat{\rho}_t^{w,s}, z_t)$ approximate $\rho^z = \mathbb{C}\mathrm{orr}(\rho_t, z_t)$. In other words, despite $\rho^z_{w,s}$ being a measure of phase-dependence in the approximate constructed sequence $\{\hat{\rho}_t^{w,s}\}$, estimates of phase-dependence in $\rho^z_{w,s}$ provide information about phase-dependence in the true underlying and unobserved correlation sequence $\{\rho_t\}$. Accordingly, the Monte Carlo also reveals that the statistic

$\tilde{\rho}_T^{w,s,z}$ introduced in Proposition 1 can be used as a nonparametric test statistic for the hypothesis of no phase-dependence in the true unobserved sequence $\{\rho_t\}$. The optimal choice of w is shown to be related with the autocorrelation in the business cycle indicator z_t (and hence with the average cycle length). Finally, the Monte Carlo reveals that the asymptotic distribution of $\tilde{\rho}_T^{w,s,z}$ derived above constitutes a reasonable approximation to the finite sample distribution in moderate sample sizes.

For the Monte Carlo study we consider stochastic sequences $\{y_t\}$ and $\{x_t\}$ generated according to a state space model with a single time-varying parameter $\{\phi_t\}$

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \phi_y \\ \phi_x \end{bmatrix} + \begin{bmatrix} \phi_{yx} & \phi_t \\ \phi_{xy} & \phi_{xx} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ w_t \end{bmatrix} \tag{2}$$

where $\{\epsilon_t\}$ and $\{w_t\}$ are iid Gaussian sequences with $\epsilon_t \sim N(0, \sigma_\epsilon)$ and $w_t \sim N(0, \sigma_w)$, the scalar parameters ϕ_y, ϕ_x, ϕ_{yx} and ϕ_{xx} are time invariant and $\{\phi_t\}$ is a time-varying two-regime threshold parameter generated according to,

$$\phi_t = \begin{cases} \alpha + v_t & \text{if } \ z_t < \bar{z} \\ \alpha + \beta + v_t & \text{if } \ z_t \geq \bar{z} \end{cases} \qquad \text{and} \qquad z_t = \theta_0 + \theta z_{t-1} + u_t \ , \tag{3}$$

with $\{v_t\}$ and $\{u_t\}$ also iid Gaussian $v_t \sim N(0, \sigma_v)$ and $u_t \sim N(0, \sigma_u)$ and α, β, θ_0 and θ fixed scalar parameters. For $\beta > 0$ the time-varying parameter $\{\phi_t\}$ has a 'low regime' (when $z_t < \bar{z}$) where it fluctuates around a mean α, and and 'high regime' (when $z_t \geq \bar{z}$) where it fluctuates around $\alpha + \beta$. If $\beta = 0$ then there is no phase-dependence in the 'observed data' $\{(y_t, x_t)\}$ simulated using (1) and (2).

It is trivial to show that, under certain parameter restrictions, the simulated data satisfies the conditions required by Proposition 1 for the consistency and asymptotic normality results. The following lemma is thus given without proof. For simplicity, we set $\phi_{yx} = \phi_{xy} = 0$ so that $\{x_t\}$ is an exogenous process whose influence on $\{y_t\}$

depends on the business cycle indicator z_t through the time-varying parameter $\{\phi_t\}$.

LEMMA 1. *Let* $0 < \sigma_\epsilon < \infty$, $0 < \sigma_w < \infty$, $0 \leq \sigma_v < \infty$, $0 < \sigma_u < \infty$, $|\phi_y| < \infty$, $|\phi_x| < \infty$, $|\theta_0| < \infty$, $|\alpha| < \infty$, $|\beta| < \infty$, $\phi_{yx} = \phi_{xy} = 0$, $|\phi_{xx}| < 1$ *and* $|\theta| < 1$. *Then* $\{(y_t, x_t, z_t)\}$ *is SE and satisfies* $\mathbb{E}|x_t|^2 < \infty$, $\mathbb{E}|y_t|^2 < \infty$ *and* $\mathbb{E}|z_t|^2 < \infty$.

Data simulated using (1) and (2) is now used to assess the small sample distribution of $\hat{\rho}_T^{w,s,z}$ and $\tilde{\rho}_T^{w,s,z}$ in Proposition 1 for alternative parameter values various choices of sample size T and window-size w. Unless stated otherwise, the following parameter values have been selected $\sigma_\epsilon = 0.1$, $\sigma_w = 0.1$, $\sigma_v = 0.02$, $\sigma_u = 0.1$, $\phi_y = \phi_x = \theta_0 = 0$, $\alpha = 0.5$, $\theta = 0.8$ and $s = w/2$. Figure 3 shows the small sample distribution of the double correlation coefficient under no phase-dependence ($\beta = 0$) in the true unobserved correlation process $\{\rho_t\}$ and reveals how the measure $\hat{\rho}_T^{w,s,z}$ based on the approximate sequence $\{\hat{\rho}_t^{w,s}\}$ is correctly centered at zero.

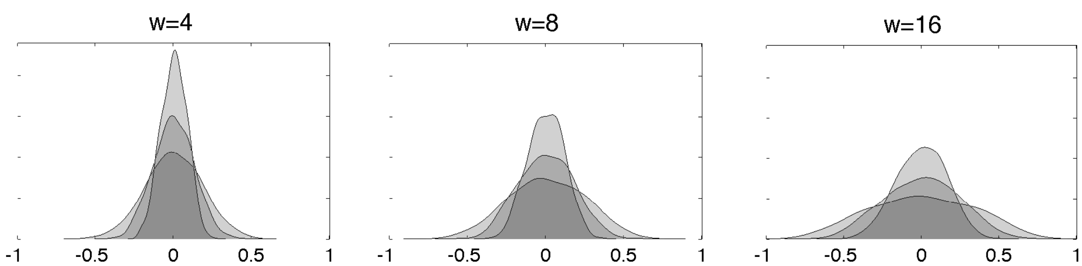

Figure 3: Density plots of $\hat{\rho}_T^{w,s,z}$ obtained from $S = 1000$ Monte Carlo simulated paths of $\{(y_t, x_t, z_t)\}_{t=1}^T$ under no-phase-dependence ($\beta = 0$), for window size $w = 4$ (left), $w = 8$ (center) and $w = 16$ (right), and sample size $T = 100$, 250 and 500. The larger T densities are identifiable by the higher concentration of mass around the origin.

On the contrary, Figure 4 plots the small sample distribution of $\hat{\rho}_T^{w,s,z}$ when there is phase-dependence ($\beta = -0.3$). Figure 4 reveals once again that the measure $\hat{\rho}_T^{w,s,z}$ based on the approximate sequence $\{\hat{\rho}_t^{w,s}\}$ reflects the phase-dependence in the true underlying and unobserved correlation sequence $\{\rho_t\}$.

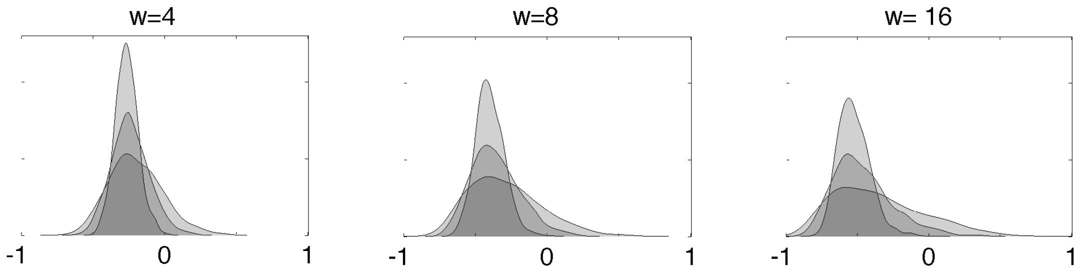

Figure 4: Density plots of $\hat{\rho}_T^{w,s,z}$ obtained from $S = 1000$ Monte Carlo simulated paths of $\{(y_t, x_t, z_t)\}_{t=1}^T$ under phase-dependence ($\beta = -0.3$), for window size $w = 4$ (left), $w = 8$ (center) and $w = 16$ (right), and sample size $T = 100$, 250 and 500. The larger T densities are identifiable by the smaller variance.

Table 2 provides information about the finite sample power of the no phase-dependence test statistic $\tilde{\rho}_T^{w,s,z}$ by stating rejection frequencies for the null hypothesis of no phase dependence when $\beta = 1$ and $\beta = 2$. Naturally, the power of the test increases with sample size, and rejection frequencies are better at $\beta = 2$ than $\beta = 1$.

		$\beta = 1$				$\beta = 2$			
	T	$w = 2$	$w = 4$	$w = 8$	$w = 16$	$w = 2$	$w = 4$	$w = 8$	$w = 16$
$\theta = .4$	100	0.23	0.30	0.24	0.19	0.38	0.46	0.35	0.23
	200	0.42	0.52	0.45	0.30	0.71	0.77	0.64	0.42
	500	0.68	0.79	0.69	0.47	0.94	0.96	0.89	0.65
	1000	0.93	0.98	0.93	0.72	1.00	1.00	1.00	0.92
$\theta = .8$	100	0.40	0.56	0.63	0.52	0.65	0.78	0.80	0.65
	200	0.71	0.86	0.91	0.83	0.95	0.98	0.98	0.93
	500	0.95	0.99	1.00	0.97	1.00	1.00	1.00	1.00
	1000	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\theta = .97$	100	0.39	0.55	0.66	0.70	0.59	0.73	0.77	0.78
	200	0.72	0.85	0.93	0.94	0.91	0.95	0.96	0.97
	500	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1000	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 2: Power of $\tilde{\rho}_T^{w,s,z}$ as measured by rejection frequencies from $S = 1000$ Monte Carlo draws at 95% confidence level for the null hypothesis of no phase-dependence $\mathrm{H}_0 :\ \rho_{w,s}^z = 0$ and a two-sided alternative $\mathrm{H}_1 :\ \rho_{w,s}^z \neq 0$ using the asymptotic $N(0,1)$ critical values derived in Proposition 1.

The optimal choice of w depends on the amount of temporal dependence in $\{z_t\}$ as measured by θ. In particular, smaller window sizes perform better under low dependence (e.g. $w = 4$ under $\theta = 0.4$), while larger window sizes perform better under high dependence (e.g. $w = 16$ at $\theta = 0.97$).

The relation between the optimal choice of window-size w and the temporal dependence in $\{z_t\}$ is made clear in Figure 5. As expected, large window sizes perform badly in low dependence setting where the average cycle length in $\{z_t\}$ is quite short and the large window choice ends up averaging over various periods of high and low correlation. For low dependence, small windows have higher power. Strong dependence in $\{z_t\}$ is favorable to large window sizes that calculate correlations with a larger number of observations and hence a smaller degree of uncertainty. When cycles in z_t are large, the advantage of large window correlation sequences in better filtering the signal from the noise is reflected in their higher power.

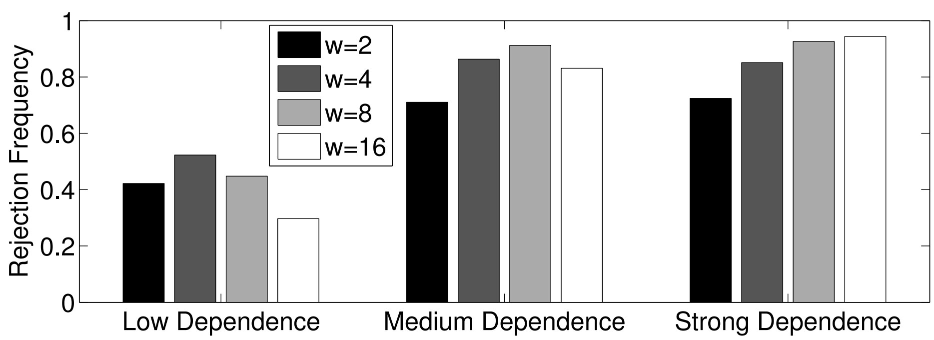

Figure 5: Null hypothesis rejection frequencies from $S = 1000$ Monte Carlo draws at 95% confidence level for the null hypothesis of no phase-dependence $\mathrm{H}_0 : \rho^z_{w,s} = 0$ and a two-sided alternative $\mathrm{H}_1 : \rho^z_{w,s} \neq 0$ using asymptotic $N(0, 1)$ critical values derived in Proposition 1 for $\beta = 1$ and $T = 200$ under low ($\theta = 0.4$), medium ($\theta = 0.8$) and strong ($\theta = 0.97$) dependence.

Table 3 now shows the finite sample size of the test $\tilde{\rho}_T^{w,s,z}$. The Monte Carlo reveals that size is always better with small window size w and naturally improving with sample size T. Furthermore, it is clear that the finite sample size is reasonably close to nominal size for small θ and small w, yet considerably distorted under a local-to-unit root strong dependence ($\theta = 0.97$) and large w.

		$\beta = 0$			
	T	$w = 2$	$w = 4$	$w = 8$	$w = 16$
$\theta = .4$	100	0.084	0.093	0.109	0.125
	200	0.062	0.082	0.084	0.099
	500	0.057	0.074	0.079	0.091
	1000	0.057	0.068	0.076	0.088
$\theta = .8$	100	0.093	0.131	0.205	0.264
	200	0.082	0.113	0.167	0.232
	500	0.073	0.106	0.159	0.214
	1000	0.071	0.104	0.143	0.201
$\theta = .97$	100	0.104	0.148	0.246	0.381
	200	0.084	0.124	0.221	0.350
	500	0.078	0.117	0.206	0.332
	1000	0.076	0.110	0.191	0.312

Table 3: Size of $\tilde{\rho}_T^{w,s,z}$ as measured by rejection frequencies (under $\beta = 0$) from $S = 4000$ Monte Carlo draws at 95% confidence level for the null hypothesis of no phase-dependence $\mathrm{H}_0: \ \rho^z_{w,s} = 0$ and a two-sided alternative $\mathrm{H}_1: \ \rho^z_{w,s} \neq 0$ using asymptotic results in Proposition 1.

A conservative test will thus favor a small window size w despite the cost that this might bring in terms of power evidenced by Table 2 and Figure 5. The trade-off between power and size for $w = 2$ and $w = 4$ might however be worth consideration depending on the application, since under strong dependence ($\theta = 0.97$) the gains in power can be larger than 15% (see e.g. Table 2 under $\beta = 1$).

Alternative parameter choices in terms of $0 < \sigma_\epsilon < \infty$, $0 < \sigma_w < \infty$, $0 \leq \sigma_v < \infty$, $0 < \sigma_u < \infty$, $|\phi_y| < \infty$, $|\phi_x| < \infty$, $|\theta_0| < \infty$, $|\alpha| < \infty$ seem to have only marginal effects on the results presented in Tables 2 and 3.

Finally, Figure 6 shows a finite sample power function that summarizes the test behavior under local-to-null-hypothesis parameter values and the effects of temporal dependence in $\{z_t\}$ for various sample sizes T. In accordance to the previous results, power increases as β diverges from 0, as T increases, and size distortion is worse under strong dependence.

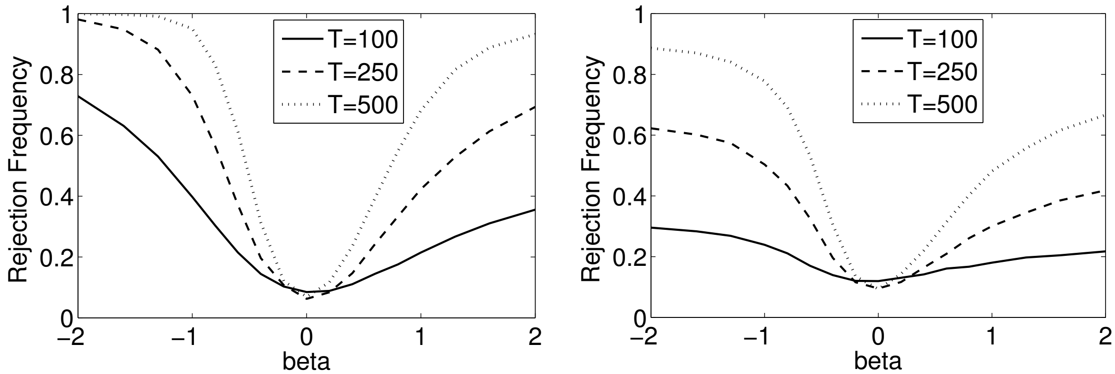

Figure 6: Finite sample power as function of β obtained from $S = 2000$ Monte Carlo draws for $(w, \theta) = (2, 0.4)$ (left) and $(w, \theta) = (16, 0.97)$ (right).

5 Phase Dependence in the US Business Cycle

Making use of the theory established in Section 3, we now turn our attention back to the data presented in Section 2 and assess wether the presence of phase-dependence in the correlation between US macroeconomic aggregates is a statistically significant.

The phase-dependence measures and test statistics are obtained using a window size of $w = 4$ and a shift of $s = 2$ for centering. According to the small sample Monte Carlo evidence collected in Section 4, this will give us conservative testing procedures with small sample size that very close to nominal at the cost of loosing some power.

Table 4 shows that the presence of phase-dependence in the correlation between US macroeconomic aggregates is statistically significant at standard confidence levels. Correlation strengths between macroeconomic aggregates are generally negatively related with the NBER business cycle. In particular, correlations are statistically stronger during recessions and weaker during expansions. Indeed, except for the correlation between consumption and investment which shows no sign of phase dependence with the business cycle, all other correlations (both contemporaneous and lagged) appear significantly different from zero. Table 4 thus documents a rich time-varying correlation structure of macroeconomic data that is usually ignored.

	y	i	c	y_{-1}	i_{-1}	c_{-1}
y	-	-0.23^{**}	-0.18^{*}	-0.29^{**}	-0.26^{**}	-0.30^{**}
	-	(-5.44)	(-2.45)	(-6.28)	(-5.41)	(-6.87)
i	-0.23^{**}	-	-0.08	-0.24^{**}	-0.25^{**}	-0.25^{**}
	(-5.44)	-	(0.20)	(-4.21)	(-4.47)	(-5.19)
c	-0.18^{*}	-0.08	-	-0.18^{**}	-0.19^{**}	-0.21^{**}
	(-2.45)	(0.20)	-	(-2.85)	(-2.88)	(-2.91)

Table 4: NBER business cycle phase-dependence characterization of log US HP-filtered auto-covariance structure. Table shows values of nonparametric phase-dependence measure $\hat{\rho}_T^{w,s,z}$ with associated test statistic $\tilde{\rho}_T^{w,s,z}$ in brackets. $*$ and $**$ indicate rejection at 5% and 1% significance levels.

A Proof of Proposition 1

This proof follows from standard well known results. Continuity of the sample s-shifted w-window correlation $\hat{\rho}_t^{w,s}$ on $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$ $\forall$ (t,w,s) : $t-w+s\geq 1 \wedge t+s\leq T$ ensures measurability when relevant sets are equipped with a Borel sigma-algebra. As a result, $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ is SE by Proposition 4.3 of Krengel (1985, p.26), the SE nature of $\{x_t\}_{t=1}^T$ and $\{y_t\}_{t=1}^T$ and $\hat{\rho}_t^{w,s}$ being a continuous function of a finite subset of $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$ $\forall$ (t,w,s). Naturally, $\mathbb{E}|\hat{\rho}_t^{w,s}|^k<\infty$ $\forall$ $(k,w,s)\in\mathbb{N}\times\{1,...,T\}\times\{1,...,w-1\}$ holds for one t and hence all t by the SE nature of $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ and the fact that its elements take values in $[0,1]$. Since $\{\hat{\rho}_t^{w,s}\}_{t=w-s}^{t=T-s}$ and $\{z_t\}_{t=1}^{t=T}$ are both subsets of SE sequences with $\mathbb{E}|\hat{\rho}_t^{w,s}|^2<\infty$ and $\mathbb{E}|z_t|^2<\infty$, it follows that $\{\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}\}_{t=w-s}^{t=T-s}$ and $\{z_t-\mu_z\}_{t=1}^{t=T}$ are SE with $\mathbb{E}|\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}|^2<\infty$ and $\mathbb{E}|z_t-\mu_z|^2<\infty$ (by Cauchy-Schwartz inequality) where $\mu_{\hat{\rho},w,s}:=\mathbb{E}\hat{\rho}_t^{w,s}$ and $\mu_z:=\mathbb{E}z_t$, and furthermore, by Minkowsky's inequality

$$\mathbb{E}\Big|(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z)\Big|\leq\Big(\mathbb{E}|\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}|^2\Big)^{\frac{1}{2}}\Big(\mathbb{E}|z_t-\mu_z|\Big)^{\frac{1}{2}}<\infty.$$

Application of the ergodic theorem (see e.g. Davidson (1994, Theorem 13.12)) yields

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}\hat{\rho}_t^{w,s} \overset{a.s.}{\to} \mathbb{E}\hat{\rho}_t^{w,s} \quad , \quad \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} z_t \overset{a.s.}{\to} \mathbb{E}z_t \ ,$$

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2 \overset{a.s.}{\to} \mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2 = \mathbb{V}\text{ar}(\hat{\rho}_t^{w,s}) \ ,$$

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z)^2 \overset{a.s.}{\to} \mathbb{E}(z_t-\mu_z)^2 = \mathbb{V}\text{ar}(z_t) \quad \text{and}$$

$$\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z) \overset{a.s.}{\to} \mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z) = \mathbb{C}\text{ov}(\hat{\rho}_t^{w,s}, z_t) \ ,$$

as $T \to \infty \ \forall \ (w,s) \in \mathbb{N} \times \{1,...,w-1\}$. Hence,

$$\begin{aligned}
&\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})(z_t-\bar{z}) \\
&= \frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}+\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})(z_t-\mu_z+\mu_z-\bar{z}) \\
&= \frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z) + (\mu_z-\bar{z})\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}) \\
&\quad + (\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z) + (\mu_z-\bar{z})\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s}) \\
&\overset{a.s.}{\to} \mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})(z_t-\mu_z)+0+0+0 = \mathbb{C}\text{ov}(\hat{\rho}_t^{w,s}, z_t) \quad \text{as} \quad T\to\infty \quad \text{and}
\end{aligned}$$

$$
\begin{aligned}
&\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})^2\Big)\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\bar{z})^2\Big)\\
&=\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s}+\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})^2\Big)\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z+\mu_z-\bar{z})^2\Big)\\
&=\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2+\frac{T-w}{T-w}(\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})^2\\
&\quad+(\mu_{\hat{\rho},w,s}-\bar{\hat{\rho}}^{w,s})\frac{2}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})\Big)\Big(\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z)^2\\
&\quad+\frac{T-w}{T-w}(\mu_z-\bar{z})^2+(\mu_z-\bar{z})\frac{2}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\mu_z)\Big)\\
&\quad\overset{a.s.}{\to}\Big(\mathbb{E}(\hat{\rho}_t^{w,s}-\mu_{\hat{\rho},w,s})^2+0^2+0\Big)\Big(\mathbb{E}(z_t-\mu_z)^2+0^2+0\Big)=\mathbb{V}\mathrm{ar}(\hat{\rho}_t^{w,s})\mathbb{V}\mathrm{ar}(z_t)
\end{aligned}
$$

as $T\to\infty$. Application of a continuous mapping theorem yields the desired consistency result as $T\to\infty$ and for every $(w,s)\in\mathbb{N}\times\{1,...,w-1\}$

$$
\hat{\rho}_T^{w,s,z}:=\frac{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})(z_t-\bar{z})}{\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}_t^{w,s})^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}(z_t-\bar{z})^2}}\overset{a.s.}{\to}\rho_{w,s}^z:=\mathbb{C}\mathrm{orr}(\hat{\rho}_T^{w,s,z},z_t).
$$

Asymptotic normality of $\sqrt{T-w}(\hat{\rho}_T^{w,s,z}-\rho_{w,s}^z)\ \forall\ (w,s)\in\mathbb{N}\times\{1,...,w-1\}$ as $T\to\infty$ follows from

$$
\begin{aligned}
\sqrt{T-w}(\hat{\rho}_T^{w,s,z}-\rho_{w,s}^z)&=\frac{\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s}r_ts_t}{\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}r_t^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}s_t^2}}-\frac{rs}{\sqrt{r^2}\sqrt{s^2}}\\
&=\frac{\sqrt{r^2}\sqrt{s^2}\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s}r_ts_t-rs\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}r_t^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}s_t^2}}{\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}r_t^2\frac{1}{T-w}\sum_{t=w-s}^{t=T-s}s_t^2}\sqrt{r^2}\sqrt{s^2}}
\end{aligned}
$$

where $r_t:=(\hat{\rho}_t^{w,s}-\bar{\hat{\rho}}^{w,s})$, $s_t=(z_t-\bar{z})$, $rs:=\mathbb{C}\mathrm{ov}(\hat{\rho}_T^{w,s,z},z_t)$, $r^2:=\mathbb{V}\mathrm{ar}(\hat{\rho}_T^{w,s,z})$ and

$s^2 := \mathbb{V}\mathrm{ar}(s_t^2)$. Now the numerator satisfies,

$$\begin{aligned}
&\sqrt{r^2}\sqrt{s^2}\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2} \\
&= \sqrt{r^2}\sqrt{s^2}\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs\sqrt{r^2}\sqrt{s^2} \\
&\quad + rs\sqrt{r^2}\sqrt{s^2} - rs\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2} \\
&= \sqrt{r^2}\sqrt{s^2}\Big(\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs\Big) \\
&+ rs\left(\sqrt{r^2}\sqrt{s^2} - \sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2}\right).
\end{aligned}$$

Asymptotic normality of the numerator is thus obtained by application of the central limit theorem Billingsley (1961) to the SE martingale difference sequence $\{r_t s_t - rs\}$ in to obtain, for some $0 < \sigma_{rs} := \mathbb{E}(r_t s_t)^2 < \infty$,

$$\frac{1}{\sqrt{T-w}}\sum_{t=w-s}^{t=T-s} r_t s_t - rs \xrightarrow{d} N(0, \sigma_{rs}^2) \quad \text{as} \quad T \to \infty$$

and an ergodic theorem in Davidson (1994, Theorem 13.12) to obtain a denominator,

$$\sqrt{r^2}\sqrt{s^2}\sqrt{\frac{1}{T-w}\sum_{t=w-s}^{t=T-s} r_t^2 \frac{1}{T-w}\sum_{t=w-s}^{t=T-s} s_t^2} \xrightarrow{a.s.} r^2 s^2 \quad \text{as} \quad T \to \infty$$

and hence, application of a continuous mapping theorem and Slutsky's theorem yields,

$$\sqrt{T-w}(\hat{\rho}_T^{w,s,z} - \rho_{w,s}^z) \xrightarrow{d} N(0, \sigma_\rho^2) \quad \text{as} \quad T \to \infty \quad \text{where} \quad \sigma_\rho^2 := \sigma_{rs}^2/(r^2 s^2)^2.$$

The claim that $\tilde{\rho}_T^{w,s,z} \xrightarrow{d} N(0,1)\ \forall\ (w,s) \in \mathbb{N} \times \{1, ..., w-1\}$ as $T \to \infty$ under the null hypothesis of no phase-dependence $\mathrm{H}_0 :\ \rho_{w,s}^z = 0$ now follows immediately since under H_0 we have $rs := \mathbb{C}\mathrm{ov}(\hat{\rho}_T^{w,s,z}, z_t) = 0$ and hence, $\sqrt{T-w}\ \hat{\rho}_T^{w,s,z} \xrightarrow{d} N(0, \sigma_\rho^2)$

as $T \to \infty$, and furthermore and $\sigma_{rs}^2 = \mathbb{E}(r_t s_t)^2 = \mathbb{E}r_t^2 \mathbb{E}s_t^2 = r^2 s^2$, and hence by Slutsky's theorem,

$$\tilde{\rho}_T^{w,s,z} := \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{\frac{1}{2}} \times \sqrt{T-w}\ \hat{\rho}_T^{w,s,z} \xrightarrow{d} N(0,1) \quad \text{as} \quad T \to \infty.$$

Finally, the claim that $\tilde{\rho}_T^{w,s,z} \to \infty$ as $T \to \infty$ under the alternative $\mathrm{H}_1 :\ \rho_{w,s}^z \neq 0$ is obtained since under H_1 we have $rs := \mathbb{C}\mathrm{ov}(\hat{\rho}_T^{w,s,z}, z_t) \neq 0$ and hence $\sqrt{T-w} rs \to \infty$ and $\sqrt{T-w}\rho_{w,s}^z \to \infty$ as $T \to \infty$ and hence,

$$\begin{aligned}
\tilde{\rho}_T^{w,s,z} :&= \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{-\frac{1}{2}} \times \sqrt{T-w}\ \hat{\rho}_T^{w,s,z} \\
&= \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{-\frac{1}{2}} \times \sqrt{T-w}\ (\hat{\rho}_T^{w,s,z} - \rho_{w,s}^z) \\
&\quad + \left(\sum_{t=w-s}^{t=T-s} s_t^2\right)^{\frac{1}{2}} \left(\sum_{t=w-s}^{t=T-s} r_t^2\right)^{-\frac{1}{2}} \times \sqrt{T-w}\rho_{w,s}^z \to \infty \quad \text{as} \quad T \to \infty. \quad \square
\end{aligned}$$

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