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The Analysis of Stochastic Volatility in the Presence of Daily Realised Measures*

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Abstract

We develop a systematic framework for the joint modelling of returns and multiple daily realised measures. We assume a linear state space representation for the log realised measures, which are noisy and biased estimates of the log integrated variance, at least due to Jensen's inequality. We incorporate filtering methods for the estimation of the latent log volatility process. The endogeneity between daily returns and realised measures leads us to develop a consistent two-step estimation method for all parameters in our specification. This method is computationally straightforward even when the stochastic volatility model contains non-Gaussian return innovations and leverage effects. The empirical results reveal that measurement errors become significantly smaller after filtering and that the forecasts from our model outperforms those from a set of recently developed alternatives.

KEYWORDS: Kalman filter, leverage, realised volatility, simulated maximum likelihood.

JEL CLASSIFICATION: C22; C58.

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1 Introduction

In this paper we develop a systematic framework for linking a general class of discrete time stochastic volatility (SV) models to realised measures of volatility such as the two time scales estimator of Zhang, Mykland, and Aït-Sahalia (2005), the realised kernel of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) and the pre-averaging based realised variance estimator of Jacod, Li, Mykland, Podolskij, and Vetter (2009). Our analysis considers a fully specified time series model for both the returns and the realised measures. We model the daily asset return series via a SV specification in which the latent daily log volatility process has a linear dynamic representation. The SV model class accommodates a range of dynamic processes for volatility, leverage effects and non-Gaussian return innovations; see, for example, Ghysels, Harvey, and Renault (1996) and Shephard (2005) for an overview of SV models and their applications. Realised measures are high-frequency based estimators of the integrated variance or the quadratic variation of an asset price over a certain period, so that we also specify an observation equation stating that the log realised measures are noisy and possibly biased estimators of the unobserved daily log variance of the asset. Our assumptions imply a linear state space model for the log realised measures, which we analyse by Kalman filter and smoother (KFS) methods.

We refer to this extension of the SV framework by an explicit measurement equation the *realised stochastic volatility* (RSV) model. Takahashi, Omori, and Watanabe (2009) and Dobrev and Szerszen (2010) propose related approaches within this setting and adopt a Bayesian inference methodology. The realised SV model extends and complements previous methods in several directions. First, it establishes the estimation of all parameters that characterise the conditional distribution of returns in the presence of realised measures. Second, the return data allows the estimation of the bias in the realised measures. Bias is an inevitable problem in this context at least due to Jensen's inequality (via the log transformation) and overnight returns. Third, existing applications of discrete time SV models, such as options pricing, can immediately rely on this framework. Fourth, it improves volatility estimation and forecasting via time series filtering. Finally, it does not require the selection of a single realised measure, but rather it can incorporate as many measures as considered relevant. At the same time, it is a framework for assessing the estimation improvements introduced by different realised measures.

This paper presents two main contributions. First, we propose a simple and consistent estimation method for the realised SV model. The joint likelihood function for the returns and the realised measures consists of two parts: the likelihood of the linear model for the vector of realised measures and the expectation of the product of return densities conditional

on the realised measures. In contrast to previous studies that have proposed the estimation of joint models of returns and realised measures, including Takahashi, Omori, and Watanabe (2009) and Dobrev and Szerszen (2010), we argue that the estimation of such a model must recognise the endogeneity between these two random variables (conditional on the unobserved daily volatility). This problem is due to discretisation effects and jumps in the estimation of the integrated variance or quadratic variation of asset prices from high-frequency data; see Peters and de Vilder (2006), Andersen, Bollerslev, and Dobrev (2007b), Andersen, Bollerslev, Frederiksen, and Nielsen (2010) and Fleming and Paye (2011).

The endogeneity problem implies that the analysis of the realised SV model based on the joint likelihood function is generally infeasible: the joint distribution of returns and the realised measures is only available for specialised cases. Our proposed estimation approach consists of two-steps that mirror the joint estimation method. In the first step, we estimate the parameters of the volatility process using the likelihood function from the Kalman filter only. In the second step we estimate the remaining parameters, including leverage effects, by evaluating integrals based on the deletion smoother of de Jong (1989). This smoothing method provides the distribution of the unobserved log volatility at a certain time period, conditional on the sample of all realised measures except for the one in that day. Because of the deletion sampling scheme, the integrals in the second step do not require the knowledge of the joint distribution of the returns and realised measures. The necessary computations are straightforward even when the model specification includes non-Gaussian return innovations and a copula function for modelling leverage effects. Simulation results suggest that we can expect this method to be almost as efficient as full joint estimation.

Second, we perform a detailed empirical study of the realised SV model using data for nine Dow Jones index stocks in the period between 2001 and 2010. We find that superpositions of three autoregressive processes are able to accurately describe the dynamics of the unobserved log volatility series for these nine stocks. The three processes have clear empirical interpretations as persistent, transitory and noisy volatility components, with leverage effects significantly impacting both the long run and short run dynamics of the series. A large kurtosis is present in the conditional distribution of close-to-close returns even after controlling for stochastic volatility. We therefore reject the hypothesis of Gaussian return innovations for most of the series.

We find that measurement errors account for between 24% and 53% of the variance of daily innovations in the log realised kernel and pre-averaging based realised variance series. Filtering methods prove to be an useful complement to the realised estimates of volatility, leading to 30-45% variance reductions in the estimation of the log volatility signal. Variance improvements are even more pronounced for a simpler subsampled realised variance estimate,

highlighting the robustness of the filtering approach. Our methods also indicate that the realised kernel and pre-averaging based measures significantly overestimate the open-to-close variance of the stocks, confirming that bias correction is an important feature of the realised SV model. Our bias estimates are comparable to the findings in Hansen, Huang, and Shek (2011b).

In an out of sample predictive analysis, we find that the realised SV model outperforms a set of recent models in forecasting the one-day and one-month ahead daily volatility of the nine stocks. The predictive gains are stronger for the one-month horizon. As a consequence of the efficiency the Kalman filter in estimating the persistent log-volatility series, the forecasting gains from incorporating more efficient realised measures into the realised SV model are modest at the daily frequency and disappear as we increase the predictive horizon. The small forecasting benefit we obtain by using the more robust realised measures is consistent with the theoretical analysis of Andersen, Bollerslev, and Meddahi (2011a).

Our methods and findings relate to other contributions. Barndorff-Nielsen and Shephard (2002) have originally studied the use of realised volatility in estimating stochastic volatility models. Examples of joint models of returns and realised volatility outside the SV methodology are the HEAVY model of Shephard and Sheppard (2010) and the Realised GARCH model of Hansen, Huang, and Shek (2011b). Andersen, Bollerslev, and Meddahi (2011a), Ghysels and Sinko (2011) and Asai, McAleer, and Medeiros (2012) consider the impact of measurement noise in forecasting realised volatility. Bollerslev, Kretschmer, Pigorsch, and Tauchen (2009) have proposed a joint model for realised volatility, returns and jumps that does not include a measurement equation. Andersen, Bollerslev, Diebold, and Labys (2003), Andersen, Bollerslev, and Diebold (2007a), Corsi (2009) and Hillebrand and Medeiros (2010), among others, suggest other reduced form approaches for modelling and forecasting realised volatility.

We organise the paper as follows. Section 2 presents the realised stochastic volatility model in detail, discusses its properties, and motivates our empirical specification. Section 3 discusses estimation. Section 4 presents our empirical results.

2 Stochastic Volatility and Realised Measures

2.1 A general discrete time stochastic volatility model

Let $p(t)$ be the logarithmic price of an asset at day t and let $y_1, \ldots, y_n$ denote a sequence of daily continuously compounded returns, defined as $y_t = p(t) - p(t-1)$. Our objective is to model the conditional distribution $p(y_{t+1}|\mathcal{F}_t)$, where $\mathcal{F}_t$ is the information set generated by

the data available up to time t. The model specification for the daily asset return is

$$y_t = \mu_t + \sigma_t \varepsilon_t, \qquad \sigma_t^2 = f(\theta_t), \qquad \theta_t = c + \sum_{i=1}^{k} \theta_{i,t} \tag{1}$$

for $t = 1, \ldots, n$, where μ_t is the expected return, σ_t is the latent daily volatility, ε_t is an independent innovation with mean zero and unit variance, and $f(\cdot)$ is a function with strictly positive support (typically the exponential function). We do not explicitly specify the expected return μ_t in this study, so that we set $\mu_t = 0$ in our simulation and empirical studies below.

Our framework allows for a diversity of stationary and non-stationary specifications for the volatility process. We assume that we can express the signal θ_t and its components $\theta_{i,t}$ as functions of the state vector α_t,

$$\theta_t = c + \sum_{i=1}^{k} \theta_{i,t} = c + Z_t \alpha_t, \qquad \theta_{i,t} = Z_{i,t} \alpha_t, \tag{2}$$

for $i = 1, \ldots, k$ and $t = 1, \ldots, n$, where c is a constant, α_t is an $m \times 1$ state vector, $Z_{i,t}$ is a $1 \times m_i$ fixed vector, $Z_t = (Z_{1,t} \ldots, Z_{k,t})$ is a $1 \times m$ fixed vector, with $m = \sum_{i=1}^{k} m_i$. The state vector α_t is a stochastically time-varying vector which we model as

$$\alpha_{t+1} = T \alpha_t + R \eta_t, \qquad \eta_t \sim \mathrm{N}(0, Q), \tag{3}$$

where η_t represents a normally distributed and serially uncorrelated $r \times 1$ disturbance vector, T is a $m \times m$ transition matrix, R is a $m \times r$ disturbance selection matrix and Q is a $r \times r$ covariance matrix. The specification of these matrices determine the dynamic properties of the state and signal vectors. The state disturbance η_t and the return innovation ε_t may be dependent. We model the initial state vector as $\alpha_1 \sim \mathrm{N}(a_1, P_1)$, where the unconditional properties of the state vector α_t imply the mean vector a_1 and variance matrix P_1. This general framework accommodates combinations of autoregressive moving average, random walk, time-varying regression, and other dynamic components. Harvey (1989) and Durbin and Koopman (2001) provide more details on state space representations and unobserved components time series models.

Tauchen and Pitts (1983), Taylor (1986), Melino and Turnbull (1990) and Harvey, Ruiz, and Shephard (1994) are classical references for stochastic volatility models in the financial econometrics literature. Shephard (1996) and Ghysels, Harvey, and Renault (1996) provide complete reviews on SV models. The collection of papers in Shephard (2005) contains addi-

tional references.

2.2 Stochastic volatility with leverage

The basic Gaussian stochastic volatility model with leverage is

$$\begin{aligned} y_t &= \exp(\theta_t / 2)\, \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0,1), \qquad \theta_t = c + \alpha_t, \\ \alpha_{t+1} &= \phi\, \alpha_t + \eta_t, \qquad \eta_t \sim \mathrm{N}(0, \sigma_\eta^2), \quad \rho(\varepsilon_t, \eta_t) \neq 0, \end{aligned} \tag{4}$$

for $t = 1, \ldots, n$, with stationary condition $|\phi| < 1$ and where $\rho(\varepsilon_t, \eta_t)$ denotes the correlation between the disturbances ε_t and η_t. This specification is a special case of the general state space representation (1), (2) and (3) with $\mu_t = 0$, $f(\theta_t) = \exp(\theta_t)$ $Z_t = 1$, $T = \phi$, $R = 1$ and $Q = \sigma_\eta^2$.

We refer to the negative dependence between returns and volatility as the leverage effect. Bollerslev, Litvinova, and Tauchen (2006) and Asai, McAleer, and Medeiros (2009) present recent evidence on this empirical regularity. In the stochastic volatility model (4), the return innovation at the current time period has an impact on the volatility in the following period. The correlation coefficient $\rho(\varepsilon_t, \eta_t)$ captures the dependence in this Gaussian setting. Alternatively, we can also specify the dependence between returns and volatility via a copula function. Copula functions allow for nonlinear and asymmetric dependence relations and do not rely on the normality of ε_t. We present an application in Section 4.5, where we analyse a SV model with student-t errors and leverage. Joe (1997), among others, provides a comprehensive discussion of copulas.

In our framework we do not consider a possible additional dependence between η_t and ε_{t+1}. In contrast to the predictive formulation in (4), this specification implies that the negative relation between returns and volatility can also be contemporaneous. Although this correlation may be present in some empirical settings due to leverage effects at higher frequencies, Yu (2005) argues that the specification of a negative dependence between η_t and ε_{t+1} brings important theoretical drawbacks to the SV model. For example, expected returns can be highly negative in this setting even when $\mu_t = 0$. We can instead directly account for the properties of the returns due to contemporaneous dependence, such as negative conditional skewness, by changing the distributional assumption for ε_t.

2.3 Stochastic volatility with long range dependence

Ding, Granger, and Engle (1993), Andersen, Bollerslev, Diebold, and Labys (2003) and Lima and Crato (1994), among many others, have documented that long range dependence is a common characteristic of volatility processes in financial markets. The slow decay in the

autocorrelation functions for absolute and squared daily returns and for the realised variance of stocks and exchange rates provide strong evidence of this property for these assets.

Superpositions of independent ARMA processes are a convenient way to account for long range dependence in the present modelling framework. An example of a SV model with this property is

$$\begin{aligned} y_t &= \exp(\theta_t / 2)\, \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0,1), \quad \theta_t = c + \textstyle\sum_{i=1}^{k} \alpha_{i,t}, \\ \alpha_{t+1} &= T\, \alpha_t + \eta_t, \quad \eta_t \sim \mathrm{N}(0, Q), \quad \rho_i = \rho(\varepsilon_t, \eta_{i,t}) \neq 0, \end{aligned} \tag{5}$$

with $k \times 1$ state vector α_t and the $k \times k$ system matrices T and Q given by

$$T = \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \phi_k \end{bmatrix}, \qquad Q = \begin{bmatrix} \sigma^2_{1,\eta} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma^2_{k,\eta} \end{bmatrix},$$

where $\alpha_{i,t}$ is the ith element of α_t, $\eta_{i,t}$ is the ith element of η_t and ρ_i is the correlation coefficient between ε_t and $\eta_{i,t}$, for $i = 1, \ldots, k$ and $t = 1, \ldots, n$. The model specification (5) is a special case of our general model given by the equations (1), (2) and (3) with $\mu_t = 0$, $f(\theta) = \exp(\theta)$, $\theta_{i,t} = \alpha_{i,t}$, $R = I$ and $k = m = r$.

Shephard (1996), Barndorff-Nielsen and Shephard (2002), Liesenfeld and Richard (2003), and other studies estimate multiple component stochastic volatility models. Engle and Lee (1999) propose a generalised autoregressive conditional heteroscedasticity (GARCH) model with short and long run components for volatility. Barndorff-Nielsen (2001) formally studies the application of superpositions in modelling long range dependence. Long memory stochastic volatility models based on fractionally integrated processes are alternative approaches; see for example Breidt, Crato, and de Lima (1998), Harvey (1998) and Mesters, Koopman, and Ooms (2011).

2.4 Realised Stochastic Volatility

The analysis of the stochastic volatility model relies on an information set $\mathcal{F}_n$ consisting of a sequence of daily returns $y_1, \ldots, y_n$. Our objective in this paper is to study the case in which we extend the information set by a sequence of realised measures $RM_1, \ldots, RM_n$, where RM_t is a vector of p noisy nonparametric volatility estimates for day $t = 1, \ldots, n$.

We obtain the realised stochastic volatility (RSV) model by adding measurement equations for the realised measures to the model we have specified in equations (1), (2) and (3),

that is

$$\begin{aligned} f^{-1}(RM_{j,t}) &= \gamma_j + \lambda_j \theta_t + \kappa_{j,t}, \qquad j = 1, \ldots, p, \\ \kappa_t &= (\kappa_{1,t}, \ldots, \kappa_{p,t})' \sim \mathrm{N}(0, \Sigma_\kappa) \end{aligned} \tag{6}$$

where $RM_{j,t}$ is the j^{th} realised measure in RM_t and where we treat constant γ_j and coefficient λ_j as unknown parameters. Since we allow $\gamma_j \neq 0$ and $\lambda_j \neq 1$, we implicitly take the realised measures as possibly biased estimates of daily volatility. The measurement disturbances $\kappa_{j,t}$, $j = 1, \ldots, p$, have variance $\sigma^2_{j,\kappa}$, are correlated with one another and are independent of the state disturbance vector η_t. Due to the construction of the information set $\mathcal{F}_t$, we cannot assume that κ_t is independent from the return innovation ε_t in (1); see Section 2.6.

The measurement equations in (6), together with the specifications (2) for the signal θ_t and (3) for the state vector α_t, lead to a linear Gaussian state space model for the realised measures. We can therefore rely on Kalman filter and smoother (KFS) methods for its analysis; see, for example, the treatment in Durbin and Koopman (2001). The analysis includes the estimation of the unknown coefficients by the method of maximum likelihood, signal extraction of θ_t and volatility forecasting. When considering the measurement equations in (6), the signal extraction of θ_t by filtering methods necessarily leads to improved estimates of volatility. On the other hand, this setup does not enable us to identify all parameters of the realised SV model (including the coefficients γ_j and λ_j), which is why the return equation in (1) remains relevant in our framework.

The asymptotic properties of the realised measures justify their treatment in (6). We therefore regard (6) as an approximation and rely on the optimal mean square error properties of the Kalman filter to provide a robust framework in case of misspecification of the measurement equation, in particular with respect the assumption of a multivariate normal disturbance vector κ_t. Barndorff-Nielsen and Shephard (2002) have argued that the central limit theorem approximation for the log of the standard realised variance estimator has a good finite sample performance in practical settings, making the log transformation a natural choice for the function $f(\cdot)$ in the measurement equations. Gonçalves and Meddahi (2010) have shown on the basis of a Monte Carlo simulation study that specific Box-Cox transformations improve the accuracy of asymptotic approximations for realised estimators. Hence our general choice $f(\cdot)$ for the transformation of $RM_{j,t}$ in the measurement equations of (6). We note that any transformation of RM_t is necessarily a biased estimate of the implied signal because of Jensen's inequality. The coefficient γ_j in equation (6) captures this effect along with the bias caused by other sources.

We assume for simplicity that the covariance matrix of the measurement disturbances Σ_κ

is constant. We estimate the unique elements of Σ_κ together with the other parameters in the RSV model. In a related study, Dobrev and Szerszen (2010) specify the measurement disturbance variance $\sigma^2_{\mathrm{J},\kappa}$ as a function of the estimate of the asymptotic variance of the realised measure $RM_{j,t}$. We do not follow their approach for two reasons. First, the variance of the realised measures depends on quantities we cannot accurately estimate, such as the integrated quarticity; see, for example, Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) and Andersen, Dobrev, and Schaumburg (2011b). The resulting instability will lead to poor filtering in the signal extraction procedures. Second, the variance estimates are clearly endogenous in relation to the realised measures themselves. Accounting for this endogeneity would complicate the analysis further; see also § 2.6.

Equations (1), (2), (3) and (6) therefore give the complete formulation of the realised stochastic volatility model. The model density

$$p(y, RM; \psi), \qquad y = (y_1, \ldots, y_n)', \qquad RM = (RM_1', \ldots, RM_n')', \tag{7}$$

refers to the model equations for a given parameter vector ψ. We partition the parameter vector into three sub-vectors: ψ_{sv} includes the parameters in (1), ψ_{ssf} includes the parameters in (2) and (3), and ψ_{rm} includes the parameters in (6). In our current model formulation we have $\psi = (\psi_{sv}', \psi_{ssf}', \psi_{rm}')'$ where

$$\psi_{sv} = \{\psi_\varepsilon, \rho_1, \ldots, \rho_k\}, \qquad \psi_{ssf} = \{c, Z, T, R, Q\}, \qquad \psi_{rm} = \{\gamma_1, \ldots, \gamma_p, \lambda_1, \ldots, \lambda_p, \Sigma_\kappa\},$$

where $\rho_i = \rho(\varepsilon_t, \eta_{i,t})$, for $i = 1, \ldots, k$, with ψ_ε referring to a vector consisting of the parameters for the density function $p(\varepsilon_t)$ which do not determine the mean and variance of ε_t, which are zero and one respectively.

2.5 Conditional return distribution

The current setting provides an useful characterisation of the full conditional density $p(y_{t+1}|\mathcal{F}_t)$, where $\mathcal{F}_t$ represents the natural filtration $RM_1, \ldots, RM_t$. We define $\sigma_t = \exp(\theta_t/2)$. By considering the model (2), (3) and (6), we can numerically evaluate the mean $\widehat{\theta}_{t+1} = \mathbb{E}(\theta_{t+1}|\mathcal{F}_t; \psi)$ and variance $V_{t+1} = \mathrm{Var}(\theta_{t+1}|\mathcal{F}_t; \psi)$ of the Gaussian density $p(\theta_{t+1}|\mathcal{F}_t; \psi)$ by applying the Kalman filter to $\mathcal{F}_t$. The conditional variance of the returns $\mathrm{Var}(y_{t+1}|\mathcal{F}_t; \psi)$ is equivalent to the conditional expectation of σ_t^2,

$$\mathrm{Var}(y_{t+1}|\mathcal{F}_t; \psi) = \mathbb{E}([\sigma_{t+1}\varepsilon_{t+1}]^2|\mathcal{F}_t; \psi) = \mathbb{E}(\sigma_{t+1}^2|\mathcal{F}_t; \psi) = \exp\{\widehat{\theta}_{t+1} + (1/2)V_{t+1}\}, \tag{8}$$

for $t = 1, \ldots, n$. By defining $S(\cdot)$ and $K(\cdot)$ as the skewness and kurtosis of the density $p(\cdot)$, respectively, we can express the higher conditional moments by

$$S(y_{t+1}|\mathcal{F}_t;\psi) = \frac{\mathbb{E}([\sigma_{t+1}\varepsilon_{t+1}]^3|\mathcal{F}_t;\psi)}{\mathrm{Var}(y_{t+1}|\mathcal{F}_t;\psi)^{3/2}} = \exp\{(3/8)V_{t+1}\} \cdot S(\varepsilon_{t+1}), \tag{9}$$

and

$$K(y_{t+1}|\mathcal{F}_t;\psi) = \frac{\mathbb{E}([\sigma_{t+1}\varepsilon_{t+1}]^4|\mathcal{F}_t;\psi)}{\mathrm{Var}(y_{t+1}|\mathcal{F}_t;\psi)^2} = \exp\left(V_{t+1}\right) \cdot K(\varepsilon_{t+1}), \tag{10}$$

for $t = 1, \ldots, n$.

We refer to V_{t+1} as the volatility risk, which is a function of both the variance of the log-volatility innovations and the variance of past measurement errors. Skewness (9) and kurtosis (10) are functions of volatility risk and the properties of the return innovations. The introduction of the realised measures in the stochastic volatility model has therefore two consequences. First, it directly reduces volatility risk V_{t+1} via improved measurement. Second, it leads to improved estimation of the parameters that determine skewness and kurtosis in the model; see also the discussions in Allen, McAleer, and Scharth (2009) and Dobrev and Szerszen (2010).

2.6 The endogeneity between daily returns and realised measures

The daily return innovations ε_t and the measurement disturbances $\kappa_{j,t}$, for $j = 1, \ldots, p$, are dependent in our realised SV model. To introduce this endogeneity issue, we briefly discuss a continuous-time formulation of the model.

Suppose that the logarithmic price of the asset at day t follows the continuous-time diffusion

$$dp(t+\tau) = \mu(t+\tau) + \sigma(t+\tau)dW(t+\tau), \qquad 0 \leq \tau \leq 1, \qquad t = 1, 2, \ldots, n, \tag{11}$$

where $p(t+\tau)$ is the logarithmic price at time $t+\tau$, $\mu(t+\tau)$ is the drift component, $\sigma(t+\tau)$ is the spot volatility, and $dW(t+\tau)$ is standard Brownian motion. Barndorff-Nielsen and Shephard (2002) and Andersen, Bollerslev, Diebold, and Labys (2003), among others, have shown that for any specification of the dynamics of spot volatility, it holds that

$$y_t|\sigma_t^2 \sim N\left(\int_0^1 \mu(t-1+\tau)d\tau, \sigma_t^2\right), \tag{12}$$

where

$$\sigma_t^2 = \int_0^1 \sigma^2(t-1+\tau)d\tau. \tag{13}$$

The term $\int_0^1 \sigma^2(t-1+\tau)d\tau$ is known as the integrated variance. Barndorff-Nielsen and Shephard (2002) have shown that a standard mean reverting specification for the spot volatility leads to an autoregressive moving average process with Gaussian innovations for σ_t^2. This argument provides an example of a discrete-time model specification for y_t based on equations (1), (2) and (3).

Realised measures are nonparametric estimates of the integrated variance of assets based on asset prices sampled at high-frequency time intervals. Since any given realised measure $RM_{j,t}$ and the daily return y_t are functions of intra-day returns, it follows that these two quantities are dependent conditional on σ_t^2. We can relate this endogeneity issue to the analyses of Peters and de Vilder (2006), Andersen, Bollerslev, and Dobrev (2007b), Andersen, Bollerslev, Frederiksen, and Nielsen (2010) and Fleming and Paye (2011), who study the theoretical and empirical properties of returns standardised by realised measures. These studies have shown that returns scaled in this manner are typically *thin* tailed; see also our empirical results in Section 4.5. Discretisation effects in the estimation of integrated variance and jumps explain this phenomenon: a relatively large return in the numerator implies a large squared return in the realised measure in the denominator.

Using similar arguments to the ones we have used to derive (10), if ε_t and $\kappa_{j,t}$ are independent and ε_t is Gaussian, then returns standardised by volatility estimates based on any set of noisy measures are always leptokurtic. This contradiction shows the relevance of the endogeneity issue in our framework. If we assume that the two innovations are orthogonal, then the analysis of the realised SV model will misleadingly lead to the implication that return innovations are thin tailed and measurement errors are negligible, when the opposite could be true. We discuss the consequences of endogeneity for parameter estimation in Section 3.

2.7 Overnight returns

Realised measures typically estimate the open-to-close variance of asset returns, while we are more generally interested in the volatility of whole day returns. The volatility of stock prices outside trading hours is substantial: in our empirical study below, we estimate that overnight returns account for between 20% and 30% of the total daily variance of stock returns. Hansen and Lunde (2005) discuss general estimates of the type

$$RM_t = \delta_1 \cdot RM_t^{oc} + \delta_2 \cdot \left(y_t^{co}\right)^2, \tag{14}$$

where RM_t^{oc} is the realised measure for the open-to-close period in day t, y_t^{co} is the overnight return at the opening of the market in day t, and δ_1 and δ_2 are predetermined values. We may choose these parameters according to some mean-square error criterion.

Throughout the paper we implicitly assume that there is a daily volatility factor which we can extract from the realised measures only. We therefore let $\delta_1 = 1$ and $\delta_2 = 0$ above and use the bias term γ_j in the realised SV model to estimate the whole day variance of the stocks. This approach has the advantage of preserving the asymptotic approximation argument that justifies (6). Shephard and Sheppard (2010), Hansen, Huang, and Shek (2011b) and Dobrev and Szerszen (2010) follow similar strategies. Alternatively, we can extend the model by separate return equations for the open-to-close and overnight periods.

3 Maximum likelihood estimation

3.1 The likelihood function based on full information

The likelihood for the realised stochastic volatility model (1), (2), (3) and (6) is

$$\begin{aligned} L(y, RM; \psi) &= \int p(\alpha, y, RM; \psi) \mathrm{d}\alpha \\ &= \int p(y|RM, \alpha; \psi) p(RM|\alpha; \psi_{rm}) p(\alpha; \psi_{ssf}) \mathrm{d}\alpha \qquad (15) \\ &= \int \prod_{t=1}^{n} p(y_t|RM_t, \alpha_t, \alpha_{t+1}; \psi_{sv})\, p(RM_t|\alpha_t; \psi)\, p(\alpha_t|\alpha_{t-1}; \psi_{ssf}) \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n, \end{aligned}$$

where we have defined y, RM and $\psi = (\psi'_{sv}, \psi'_{ssf}, \psi'_{rm})'$ in Section 2.4; furthermore, let $\alpha = (\alpha'_1, \dots, \alpha'_n)'$. We simplify the conditional return density to

$$p(y_t|RM_t, \alpha_t, \alpha_{t+1}; \psi_{sv}) = p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv}),$$

where θ_t reflects the dependence on the signal and η_t on the leverage effect. We also have that $p(RM_t|\alpha_t; \psi_{rm}) = p(RM_t|\theta_t; \psi_{rm})$. Hence it follows that

$$L(y, RM; \psi) = \int \prod_{t=1}^{n} p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv})\, p(RM_t|\theta_t; \psi_{rm})\, p(\theta_t|\alpha_{t-1}; \psi_{ssf}) \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n. \quad (16)$$

Endogeneity implies that

$$p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv}) \neq p(y_t|\theta_t, \eta_t; \psi_{sv}),$$

where the density $p(y_t|\theta_t, \eta_t; \psi_{sv})$ refers to the model equation (1). Peters and de Vilder (2006) derives the distribution of $y_t/\sqrt{RM_t}$ for the special case in which RM_t is the realised variance estimator and the underlying diffusion is homogeneous. In all other settings,

$p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv})$ is currently not available.

We therefore propose an estimation approach that does not require the knowledge of $p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv})$. To motivate our new method, we first consider the hypothetical case in which we know $p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv})$ and hence estimation based on the complete likelihood function is feasible. The following discussion may also be useful for the estimation of the volatility of overnight returns, which are not endogenous to the realised measures.

Let $p(RM; \psi)$ be the likelihood of the linear state space model for the realised measure. By multiplying the likelihood function (16) with the term $p(RM; \psi) \cdot p(RM; \psi)^{-1}$, we have

$$L(y, RM; \psi) = p(RM; \psi) \int \prod_{t=1}^{n} p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv}) p(\alpha|RM; \psi) \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n, \qquad (17)$$

since $p(RM|\alpha; \psi)\, p(\alpha; \psi_{ssf}) \,/\, p(RM; \psi) = p(\alpha|RM; \psi)$. The expression in (17) has a straightforward interpretation. The density $p(RM; \psi)$ refers to the likelihood function of the linear Gaussian state space model (6), (2) and (3). We can therefore carry out the evaluation of $p(RM; \psi)$ by the Kalman filter. The integral part in (17) is effectively the expectation of the product of densities $\prod_{t=1}^{n} p(y_t|RM_t, \theta_t, \eta_t; \psi_{sv})$ with respect to density $p(\alpha|RM; \psi)$. We can estimate the integral via a routine application of Monte Carlo integration in which we sample S trajectories $\alpha^{(s)} = (\alpha_1^{(s)\prime}, \dots, \alpha_n^{(s)\prime})'$ from $p(\alpha|RM; \psi)$ and compute the likelihood as

$$\widehat{L}(y, RM; \psi) = p(RM; \psi) \cdot \frac{1}{S} \sum_{s=1}^{S} \prod_{t=1}^{n} p(y_t|RM_t, \theta_t^{(s)}, \eta_t^{(s)}; \psi), \qquad (18)$$

where $\theta_t^{(s)} = c + Z\alpha_t^{(s)}$ and $\eta_t^{(s)} = (R'R)^{-1}(\alpha_{t+1}^{(s)} - T\alpha_t^{(s)})$; see the relations in (2) and (3). The simulation smoothing methods of de Jong and Shephard (1995) and Durbin and Koopman (2002) can carry out the simulation of $\alpha^{(s)}$ from the smoothed density $p(\alpha|RM; \psi)$, for $s = 1, \dots, S$.

Despite the efficiency of the realised measures, the direct implementation of Monte Carlo integration in (18) may require a large number of draws S to ensure a reliable and efficient estimate of (17). To improve computational and numerical efficiency, we can alternatively consider the method of importance sampling; see Durbin and Koopman (2001, Part II) for an exposition of the importance sampling method for this class of models. Koopman, Lucas, and Scharth (2011) propose a method for the construction of efficient importance samplers based on an approximating linear Gaussian state space model. These samplers are designed to minimise the Monte Carlo variance of the resulting likelihood estimates. The computation of the likelihood estimate is similar to (18), with $\alpha^{(s)}$ then becoming a draw from the efficient importance sampler.

3.2 A likelihood function based on selected information

We can alternatively rewrite (17) using a partial prediction error decomposition with respect to y. We obtain

$$\begin{aligned} L(y, RM; \psi) &= p(RM; \psi)p(y|RM; \psi) \\ &= p(RM; \psi)p(y_1|RM; \psi)p(y_2|RM, y_1; \psi) \cdots p(y_n|RM, y_1, \ldots y_{n-1}; \psi). \end{aligned}$$

Due to the fact that the high-frequency information set for calculating the realised measures at day t includes y_t, returns introduce new information about the signal in our framework only via leverage effects and bias correction. The primary role of returns in the estimation of the realised SV model is accordingly the identification of a selection of parameters, rather than the estimation of the signal. We therefore propose the likelihood approximation

$$\begin{aligned} L(y, RM; \psi) \approx\ & p(RM; \psi)p(y_1|RM; \psi)p(y_2|RM; \psi) \cdots p(y_n|RM; \psi) = \\ & p(RM; \psi) \prod_{t=1}^{n} p(y_t|RM; \psi), \end{aligned} \tag{19}$$

where

$$p(y_t|RM; \psi) = \int p(y_t|RM_t, \theta_t, \eta_t; \psi)p(\theta_t, \eta_t|RM; \psi)\mathrm{d}(\theta_t, \eta_t). \tag{20}$$

Due to endogeneity, RM_t remains a conditioning variable.

If an expression for $p(y_t|RM_t, \theta_t, \eta_t; \psi)$ was available, the evaluation of the integral in (20) by low dimensional numerical or Monte Carlo integration would be straightforward. The Kalman filter and smoother provides the mean and variance of $p(\theta_t, \eta_t|RM; \psi)$; see Appendix A for the details. The resulting estimate of ψ is still consistent and the loss of efficiency is small as we only discard redundant information.

3.3 Two-step estimation for model without leverage

Our two-step method for the estimation of ψ builds on the results presented above. We start with the simpler case in which there is no leverage in the model. We adopt the same arguments as for the approximate likelihood function in the previous section. We decompose the likelihood as $p(y, RM; \psi) = p(RM; \psi)p(y|RM; \psi)$. Since in our framework $p(y|RM; \psi)$ adds little information about the volatility signal and $p(RM; \psi)$ does not contain any information about the statistical properties of the return innovations, we treat the two components of the likelihood separately.

We first estimate the linear Gaussian state space model (6), (2) and (3) by maximising

the likelihood function $p(RM_t;\psi)$ with respect to ψ_{ssf} and ψ_{rm}. We evaluate this likelihood by the Kalman filter. For identification, the optimisation is subject to two scaling coefficients in ψ_{rm} which we refer to as ψ_{bias}. We take a particular value ψ^*_{bias} and denote the resulting estimates as $\widehat{\psi}^*_{ssf}$ and $\widehat{\psi}^*_{rm}$. In contrast to joint estimation, this first part of the method is robust to the misspecification of the return equation (1). In the second step, we estimate the remaining part of the parameter vector by maximising

$$p^*(y|RM;\psi) = \prod_{t=1}^{n} p(y_t|RM^{-t};\psi), \tag{21}$$

where RM^{-t} is the deletion set

$$\{RM_1,\ldots,RM_{t-1},RM_{t+1},\ldots,RM_n\}.$$

and

$$p(y|RM^{-t};\psi) = \int p(y_t|\theta_t;\psi_{sv})p(\theta_t|RM^{-t};\psi_{bias},\widehat{\psi}^*_{ssf},\widehat{\psi}^*_{rm})\mathrm{d}\theta_t. \tag{22}$$

The optimisation is with respect to ψ_{sv} and ψ_{bias} only.

The likelihood function (21) is a direct counterpart to the product of conditional return densities in (19). We remove the endogeneity problem by recognising that y_t does not share information with the other realised measures in the sample. In doing so, we still incorporate as much information as possible about θ_t in the estimation. Since the log-volatility process $\theta_1,\ldots,\theta_n$ is highly persistent in empirical settings, we can regard RM_j, with j close to t as being informative about θ_t. We compute the mean and variance of the Gaussian density $p(\theta_t|RM^{-t};\psi_{ssf},\psi_{rm})$ by the deletion smoothing algorithm of de Jong (1989) applied to the model (6), (2) and (3). We give the details in Appendix A. The evaluation of (22) by numerical or Monte Carlo integration is straightforward for any density $p(y_t|\theta_t;\psi_{sv})$. We have found that Gaussian quadratures are the most accurate and computationally efficient methods for this purpose.

The computational simplicity of the two-step estimation method contrasts with the simulation intensive algorithms typically required for the estimation of stochastic volatility models when only return data is available. Examples of such methods are the Markov Chain Monte Carlo approaches of Kim, Shephard, and Chib (1998) and Chib, Nardari, and Shephard (2002) and the importance sampling approaches of Sandmann and Koopman (1998) and Liesenfeld and Richard (2003). In the current framework, parameter estimation for stochastic volatility models with leverage and non-Gaussian daily returns also becomes straightforward.

3.4 Two-step estimation for model with leverage

The first estimation step of the last section is not affected when we let the disturbances ε_t in (1) and η_t in (3) depend on each other. In the second step, the counterpart of (22) we maximise is given by

$$p(y|RM^{-t};\psi) = \prod_{t=1}^{n} \int p(y_t|\theta_t,\eta_t;\psi_{sv},\widehat{\psi}^*_{ssf}) p(\theta_t,\eta_t|RM^{-t};\psi_{bias},\widehat{\psi}^*_{ssf},\widehat{\psi}^*_{rm}) \mathrm{d}(\theta_t,\eta_t). \quad (23)$$

Appendix A provides the details on how we obtain the mean, variance and covariances of the conditional Gaussian density $p(\theta_t,\eta_t|RM^{-t};\psi_{ssf},\psi_{rm})$. The necessary modification is straightforward and the additional computational cost is small. The integral in (23) is multidimensional. We have used quasi-Monte Carlo integration using Halton sequences for its estimation; see, for example, Train (2003) for further details on this method.

The evaluation of $p(y_t|\theta_t,\eta_t;\psi_{sv},\widehat{\psi}^*_{ssf})$ in (23) follows standard results. For example, if we assume that ε_t and η_t are both Gaussian in the SV model (4), the multivariate normal lemma applies and we have

$$p(y_t|\theta_t,\eta_t;\psi_{sv},\widehat{\psi}^*_{ssf}) = \mathrm{N}(m,V), \;\; m = \exp(\theta_t\,/\,2)\frac{\rho(\varepsilon_t,\eta_t)}{\widehat{\sigma}_\eta}\eta_t, \;\; V = [1-\rho(\varepsilon_t,\eta_t)]^2 \exp\theta_t,$$

where $\widehat{\sigma}^2_\eta$ is the estimate of σ^2_η from the first step. When we specify the dependence as a copula function, we adopt the following corollary of Sklar's theorem

$$p(y_t|\theta_t,\eta_t;\psi_{sv},\widehat{\psi}^*_{ssf}) = p(y_t|\theta_t;\psi_{sv}) \cdot C\left[F(y_t|\theta_t;\psi_{sv})\,,\, G(\eta_t;\widehat{\psi}^*_{ssf});\psi_{sv}\right], \quad (24)$$

where $C(\cdot,\cdot)$ is a probability density function for the copula that describes the dependence between ε_t and η_t, $F(\cdot)$ is the cumulative distribution function of the daily returns y_t conditional on θ_t and $G(\cdot)$ is the normal cumulative distribution function of η_t. We can adopt many results on copula functions in this framework; see, for example, the discussions in Joe (1997).

3.5 Simulation study

In order to investigate the performance of the two-step method based on the deletion smoothing scheme, we design a simulation study in which the endogeneity issue does not arise. We simulate 250 series of returns and log volatility measurements using the model (1), (6), (2) and (3), drawing the disturbance series κ_t in (6) independently from ε_t in (1). We generate simulations for two different models for which the transformation function in (1) and (6) is

$f(\cdot) = \exp(\cdot)$. We consider

- Model 1 : the Gaussian stochastic volatility model with leverage of equation (4); we fix the parameters at $c = 0.4$, $\phi = 0.98$, $\sigma_\eta^2 = 0.05$ and $\rho(\varepsilon_t, \eta_t) = -0.5$;
- Model 2 : a stochastic volatility model with a standardised Student's t density with ν degrees of freedom for ε_t; we fix the parameters at $\nu = 10$, $c = 0.4$, $\phi = 0.98$, $\sigma_\eta^2 = 0.05$. This specification does not have leverage, so that $\rho(\varepsilon_t, \eta_t) = 0$.

Equation (6) with $p = 1$ gives the model for the log variance measurement in the two specifications. The bias is $\gamma = 0.1$ and the observation variance is $\sigma_\kappa^2 = 0.05$. We fix the coefficient λ at one and do not treat it as a parameter.

We estimate the parameters as if they are unknown for each of the 250 time series for y_t and RM_t with $t = 1, \ldots, n$ and $n = 2,500$. We repeat the estimation for the full likelihood, approximate likelihood and two-step methods we have discussed above. In case of the two-step method, we consider three different sets of realised measures for the conditioning of the density $p(\theta_t, \eta_t | \chi; \widehat{\psi}_{ssf}, \psi_{rv})$ in the second step : $\chi = \{RM_1, \ldots, RM_{t-1}\}$, $\chi = RM^{-t}$ and $\chi = RM$. The first conditioning set leads to a second step that only requires the Kalman filter. We use this case as a benchmark. The deletion and full sets of RM allows us to determine the efficiency loss when we drop RM_t from the conditioning set. We emphasise that in these three cases the estimate of ψ_{ssf} from the first step remains the same by construction. We compute the integral in (23) by quasi-Monte Carlo integration using Halton sequences with $S = 100$ samples.

Tables 1 and 2 present our simulation results. We report the mean and the standard deviation of the series of 250 parameter estimates for each estimation method. The findings support our discussions above. All methods lead to similar means and standard deviations for the parameters in ψ_{ssf}, confirming that the return information has minimal impact on the estimation of ψ_{ssf}. The five methods also perform equally well in estimating the bias coefficient γ. The only differences arise in the estimation of the leverage effect $\rho(\varepsilon_t, \eta_t)$ in Model 1 and of the degrees of freedom ν in Model 2. Whereas the two joint estimation methods and the efficient two-step method lead similar standard deviations for these two parameters, a small loss of efficiency appears for the deletion method. We conclude that the deletion smoothing approach provides an effective estimation method for the realised SV model when the endogeneity problem is present.

4 Empirical Analysis

4.1 Data and measurement

Our data set consists of NYSE TAQ open-to-close transaction prices for nine Dow Jones index stocks in the period between January 1993 and December 2010. We list the stocks in Table 3 along with their ticker indicators. We remove potential sources of errors from the data set by following the guidelines in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009). We have taken the daily return series from the CRSP database. The quality of the data has markedly improved over the years for purposes of measuring volatility; see the discussion in Hansen and Lunde (2006). To ensure that our results reflect recent and more relevant patterns, we concentrate exclusively on the post-decimalisation years (2001-2010) in the estimation and filtering analysis of Section 4.3. We do however use the earlier period to estimate the model in the rolling window exercise of Section 4.4.

We compute the following realised measures in transaction time: the realised kernel (RK) of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), the pre-averaging based realised variance (PRV) of Jacod, Li, Mykland, Podolskij, and Vetter (2009), the subsampled realised variance (SRV) of Zhang, Mykland, and Aït-Sahalia (2005) and the subsampled median-based realised variance (MedRV) of Andersen, Dobrev, and Schaumburg (2009). Our calculations for the realised kernel and the pre-averaging based realised variance follow the suggested implementations in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009) and Jacod, Li, Mykland, Podolskij, and Vetter (2009) respectively. We base the subsampled RV estimator on subgrids containing every m^{th} transaction, where we select m so that the grid points are 15 minutes apart on average. The median-RV estimator follows a similar scheme with average intervals of 2.5 minutes. We remove stale prices from the sample for computing the median-RV measure.

The realised kernel and pre-averaging measures are among the most efficient estimates currently available. Consistent with the theoretical argument in Christensen, Kinnebrock, and Podolskij (2010a), we have found the two measures to be nearly perfectly correlated in their standard implementations. Since the computational cost of simultaneously adopting the two estimates is small, we do so for completeness. We also include the subsampled realised variance in our analysis because the impact of microstructure noise is small at the low frequency we have used to calculate the measure. In comparison with discretisation errors, microstructure noise contamination is possibly of greater concern for time series analysis due to the nonstationary behaviour of microstructure effects; see the discussions, for example, in Hansen and Lunde (2006). Finally, we consider the median-RV measure to investigate whether jump-robust estimates improve the empirical performance of the realised SV model.

This estimate is a special case of the quantile-based realised variance of Christensen, Oomen, and Podolskij (2010b).

4.2 Model specification

In the empirical study we mostly focus on daily close-to-close returns, using a selection of results for open-to-close returns in complementary analyses. We model the returns by the stochastic volatility specification (5). We therefore let the log-volatility signal be a sum of autoregressive processes. On the basis of the Bayesian information criterion, we have found conclusive support for the inclusion of $k = 3$ autoregressive processes of order one for all the stocks in this study. We initially assume that the daily returns innovations are normally distributed; we investigate the validity of this assumption in Section 4.5. We allow for leverage effects by having non-zero values for $\rho_i = \rho(\varepsilon_t, \eta_{i,t})$ for $i = 1, 2, 3$. We have found no evidence of a leverage effect for the third autoregressive process, so that we fix $\rho_3 = 0$ throughout the study. We consider two vectors of realised measures in Section 4.3: the 3×1 vector $RM_t = (\mathrm{RK}_t\,,\, \mathrm{PRV}_t\,,\, \mathrm{SRV}_t)'$ and $RM_t = \mathrm{MedRV}_t$. We model the realised measures by equation (6) with the restriction that $\lambda_j = 1$. We obtain all estimation results by two-step method of Sections 3.3 and 3.4.

4.3 Estimation and filtering

Tables 4 and 5 report the parameter estimates for the stochastic volatility model (5). The results are similar across the nine stocks and the two realised vectors. We find that the first autoregressive components are near unit root processes with relatively small estimated disturbance variances $\sigma^2_{\eta,1}$. The second autoregressive components are persistent processes with estimated autoregressive coefficients between 0.91 and 0.95. The third autoregressive components are invariably noisy with autoregressive coefficients estimated as low as 0.15 and reaching a maximum of 0.47. Despite their statistical significance, the third volatility states are in practice hard to distinguish from the measurement disturbances due to their low persistence. Figure 1 shows the three estimated volatility components for IBM. With only a few exceptions, leverage effects significantly impact both the long and short run volatility components. The estimated long run effects contrast with previous studies which have found leverage effects for transitory components only; see, for example, Engle and Lee (1999). We attribute our finding to the inclusion of realised measures in the analysis. The long run effect is typically difficult to identify given the small estimated values for the state variances $\sigma^2_{\eta,1}$.

We next consider the added value of including equation (6) for the estimation of the daily volatility signal θ_t in (5). We apply the Kalman filter and smoother to the individual

elements of $RM_t = (\text{RK}_t\ \text{PRV}_t\ \text{SRV}_t)'$ and to the three measures simultaneously using the parameters of Table 4. Table 6 presents estimates of the measurement variances $\sigma^2_{j,\kappa}$, the correlations between the measurement disturbances, the signal-to-noise ratios (defined as the variance of the innovation in the signal θ_t divided by the variance of κ_t) and the variances of the volatility signal θ_t, conditional on $RM_1, \ldots, RM_s$ with $s = t$ for filtering, $s = n$ for smoothing and $s = t-1$ for one-step ahead forecasting. We report the last three variances as the steady-state values computed by the Kalman filter and smoother for $t = 1, \ldots, n$. Table 7 repeats the exercise for the median-RV measure. Due to presence of the noisy volatility state, the measurement variances and correlations for the Exxon and Procter & Gamble stocks were not identified by the estimation. Our discussion therefore focus on the remaining series.

The empirical evidence indicates that the level of noise in the realised measures is relatively high, even for the most efficient estimates. The signal-to-noise ratios for the realised kernel and pre-averaging measures range from 0.9 for Coca-Cola to 3.2 for the JP Morgan stock. The Kalman filter substantially improves the estimation of the unobserved volatility signal in this setting. The filtered variances for the realised kernel and pre-averaging series are between 29% and 46% lower than the corresponding estimated measurement variances σ^2_κ. Similar findings hold for the Median-RV measure. Figure 2 displays the log realised kernel measure and the smoothed estimate of the volatility component θ_t for Coca-Cola. For the subsampled RV measure, our results show that the signal-to-noise ratios are nearly half of those for the other two RV estimates in most cases. However, this less efficient measure appears to benefit even more from filtering. Estimated variance reductions for this measure range between 36% and 53%. We conclude that time series filtering is an useful complement to realised measures for estimating volatility.

As a consequence of the efficiency of the Kalman filter in reducing the noise in the time series of the log realised measures, we find that the predictive variances are similar across all the estimates. Despite the differences in the signal-to-noise ratios, the forecasting variances for the realised kernel and the pre-averaging measures are only between 5% and 9% lower than the variance implied by the subsampled RV estimator. The presence of the third volatility component $\alpha_{3,t}$ in our model (5) illustrates this result. This state is important for in-sample fit but plays a minor role in forecasting, given its low persistence. The additional improvement of treating the three realised measures simultaneously seems negligible.

Table 8 reports the estimates of the bias parameter γ_j in (6) for close-to-close and open-to-close returns. The results suggest that the realised kernel and pre-averaging measures are significantly upwards biased estimates of the open-to-close variance of all stocks except JP Morgan. The bias is around 6% in the case of IBM, but reaches as high as 23% for Procter & Gamble. Our high estimates for some of the stocks are consistent with some of the results in

Hansen, Huang, and Shek (2011b), even though the authors do not emphasise the bias in their discussions. The subsampled RV measure substantially reduces or eliminates this problem. The coefficients for this realised measure are not significant for the majority of stocks, even though relatively large positive biases still appear for Coca-Cola, Procter & Gamble and Wal-Mart. Comparing these results with the coefficients we obtain for close-to-close returns, we find that the period outside the trading hours accounts for between 20% and 30% of the total daily variance of the stocks; the results in Hansen, Huang, and Shek (2011b) lead to similar conclusions.

Table 9 presents a selection of diagnostic statistics based on the one-step ahead prediction residual $v_t = RK_t - \mathrm{E}(RK_t|\mathcal{F}_{t-1})$. The diagnostic statistics for skewness, kurtosis and serial correlation (for both v_t and v_t^2) indicate possibly useful extensions of our current modelling framework. The positive skewness and excess kurtosis in the residuals imply that we strongly reject the Gaussian assumption for the linear model (6). In our framework, we can consider Box-Cox transformations as in Gonçalves and Meddahi (2010) to alleviate the skewness problem. Alternatively, we can also consider a non-Gaussian density for $\kappa_{j,t}$ in (6). We also report the Box-Ljung serial correlation test statistics for v_t and v_t^2, for different lag lengths; they are satisfactory.

4.4 Forecasting

We next analyse out-of-sample forecasts of daily volatility from the realised SV model, comparing them to predictions from three other recently developed models. For all models, including the realised SV model, we base the analysis on the realised kernel measure RK_t (or $RK_t^* = \log RK_t$) and the close-to-close daily returns y_t. In the model specifications below, let u_t be an independently and identically distributed error term and denote the model parameters by $a, b, c, \ldots$, possibly with subscripts i for $i = 1, 2, 3, \ldots$. We consider the following alternatives:

- The heterogeneous autoregressive (HAR) model of Corsi (2009),

$$RK_t^* = a + b_1 RK_{t-1}^* + b_2 RK_{5,t-1}^* + b_3 RK_{22,t-1}^* + c(y_{t-1}/\sqrt{RK_{t-1}}) + u_t,$$

  where $RK_{j,t-1}^* = \sum_{i=t-j}^{t-1} RK_{t-i}^*$ for $j = 1, 2, 3, \ldots$, and with $u_t \sim \mathrm{N}(0, g)$ for $t = 1, \ldots, n$.

- The high-frequency-based volatility (HEAVY) model of Shephard and Sheppard (2010),

$$\begin{aligned} y_t &= h_t^{1/2} u_t, \\ h_t &= \mathrm{Var}(y_t | RK_1, \ldots, RK_{t-1}) = a_1 + b_1 h_{t-1} + c_1 RK_{t-1} + d_1 (y_{t-1}/h_{t-1}), \\ \omega_t &= \mathrm{E}(RK_t | RK_1, \ldots, RK_{t-1}) = a_2 + b_2 \omega_{t-1} + c_2 RK_{t-1} + d_2 (y_{t-1}/h_{t-1}), \end{aligned}$$

  for $t = 1, \ldots, n$.

- The realised GARCH model of Hansen, Huang, and Shek (2011b),

$$\begin{aligned} y_t &= \exp(h_t/2) u_{1,t} \\ h_t &= a_1 + b_1 h_{t-1} + c_1 RK^*_{t,t-1} + d_1 (y_{t-1}/h_{t-1}), \\ RK^*_t &= a_2 + b_2 h_t + u_{2,t}, \qquad u_{2,t} \sim \mathrm{N}(0, g), \end{aligned}$$

  with variance g, for $t = 1, \ldots, n$.

We compute one-day and one-month ahead forecasts of the log realised kernel in the period between January 2001 and December 2010. We update the parameter estimates monthly and calculate the forecasts using a rolling window of the most recent 2,000 observations. In the case of the realised SV model, we first compute the Kalman filter prediction RK^*_{t+1} and subsequently approximate the leverage effect by calculating $\widehat{\varepsilon}_t = y_t \times \mathrm{E}(\exp(-\theta_t/2) | RK_1, \ldots, RK_t)$ and substituting this estimate in the expression for $\mathrm{E}(\eta_t | \varepsilon_t)$. We estimate coefficients in the HAR model by OLS and those in the HEAVY and realised GARCH models by quasi-maximum likelihood, as described in the original papers we reference above.

We validate the forecasts on the basis of the corresponding mean squared errors (MSE) and the model confidence set (MCS) methodology of Hansen, Lunde, and Nason (2011a). The design of the MCS is such that it contains the best model in terms of MSE with a certain level of confidence. We report the MCS p-value, which we denote by p_m^{mcs}, to indicate that model m is in a $(1-\alpha)\%$ confidence set for $\alpha \leq p_m^{\mathrm{mcs}}$. The model with the most accurate forecasts in the results has a p-value of one by construction. We base the test on $10,000$ bootstrap resamples. Since we base the evaluation on RK^*_t itself, the reported MSEs do not take into account the possible bias in the forecast of the log variance. Differences in MSE can be measured relatively to the predictive variances as reported in Table 6.

We report the forecasting results in Tables 10 and 11. We find that the realised stochastic volatility model generates good forecasts, obtaining the lowest MSEs for the two horizons considered across all nine stocks. The realised SV model is the only specification in the one-day ahead 90% MCS for three of the stocks, while the 95% MCS contains all models

for the remaining series. The one-step forecast precision differences are therefore relatively small. When we take the HAR specification as a benchmark against the three remaining models, the superior predictive ability (SPA) test of Hansen (2005) can shed further light on the statistical significance of the results. The SPA test rejects the HAR model at the 5% level for all stocks. The forecasting results are consistent with the findings of Table 6: even large improvements in the realised measures have a modest impact on one-step ahead forecasting.

The relative MSE of the realised SV model improves with the forecasting horizon, validating the specification of multiple autoregressive states in our empirical model. For the one-month ahead forecasts, the RSV is the single model in the 90% confidence set for five of the stocks, sharing the MCS for the other four stocks with the HAR model. The short memory dynamics of the standard HEAVY and realised GARCH specifications do not appear to be well suited for multi-step predictions.

The realised SV model can include multiple realised measures whose choice may have an impact on forecasting. For example, microstructure noise may distort predictive accuracy; see, for example, Andersen, Bollerslev, and Meddahi (2011a), Ghysels and Sinko (2011) and Asai, McAleer, and Medeiros (2012). Additionally, jump robust estimates such as the median-RV measure are less noisy and may produce better predictions compared to quadratic variation measures such as the RK, PRV and SRV estimates; see, for example, Andersen, Bollerslev, and Diebold (2007a). To investigate these issues, we compute out of sample forecasts for the realised SV model based on the realised kernel, subsampled RV and median-RV measures individually. We focus on the variance of forecasts only. We evaluate the rolling window forecasts from these different models using the realised kernel and subsampled RV measures. Here we consider the period between 2006 and 2010 since we can only reliably estimate the median-RV measure from 1998 onwards.

We conclude from Table 12 that the choice between the realised kernel and median-RV measures does not matter for both one-day and one-month ahead predictions. The forecasting variances evaluated using the realised kernel or the subsampled RV are nearly the same: the the model confidence set includes both variants of the realised SV model in all cases. With respect to the subsampled RV measure, the results depend on the horizon and the measure we use to evaluate the forecasts. For the one-day ahead predictions evaluated against the realised kernel, the variance reductions generated by the realised kernel over the subsampled RV measure are small; we report a similar finding in Table 6. However, the MCS excludes the subsampled RV based forecasts in this setting. If we use the subsampled RV measure for evaluation the results are mixed.

For one-month ahead predictions, the subsampled RV based forecasts are similar to those we obtain using the other two measures. We find that the subsampled RV based model

has the lowest variance when forecasting the realised kernel for two of the stocks, but the differences are small and not significant. The three measures are in the one-month MCS for all stocks except Pfizer. We therefore have two conclusions. First, microstructure noise distortions are weak given the small loss in the relative performance of the realised kernel based model when we base the forecasting evaluation on the subsampled RV measure. Second, better volatility measurement appears to have a small impact and affect only the short-term forecasting precision. We should base the selection of realised measures in the RSV model on robustness rather than efficiency.

4.5 Testing the Gaussian SV model

We now investigate whether the Gaussian assumption for (5) holds empirically. A related question is whether we can attribute the excess kurtosis in equation (10) for the conditional distribution of stock returns to volatility risk only. Andersen, Bollerslev, Diebold, and Ebens (2001) argue that the returns standardised by realised volatility measures are approximately normally distributed. However, this type of standardisation is subject to the endogeneity issue we have discussed in Section 2.6. Table 13 presents the sample kurtosis of open-to-close returns standardised by the four realised measures between 2001 and 2010, showing their tendency towards thin tails.

An appropriate way of scaling the returns for purposes of testing the Gaussian assumption is by adopting the conditional variance (8). Equations (9) and (10) allow for the construction of test statistics for testing departures from the normality assumption using the higher moments of standardised returns. We implement a parametric bootstrap procedure. Our test consists of calculating the sample skewness and kurtosis from the standardised returns and comparing them with the corresponding finite sample distributions under the null hypothesis of a Gaussian SV model with the parameter reported in Table 4. We use the conditional volatility for the standardisation rather than the volatility estimates from the deletion smoothing method so that we do not have to control for leverage effects. This choice also facilitates the interpretation of the results.

Table 14 presents the results. The conditional kurtosis implied by the estimated Gaussian SV models range from 3.33 for Coca-Cola to 3.61 for Procter & Gamble. We find that the sample kurtosis estimates for the open-to-close returns mostly agree with the values predicted by the Gaussian model. Even though the sample kurtosis estimates are always higher than the model implied moments, the differences are small and not significant for six of the stocks. We also do not reject the skewness hypothesis for six of the stocks. Figure 3 shows the empirical distribution of the standardised open-to-close returns. We can generally attribute

rejections to a few outliers.

For close-to-close returns, we find evidence of excess kurtosis for seven of the stocks and non-zero skewness for four of the stocks. Figure 4 further these findings. We interpret the higher kurtosis in the close-to-close returns as an indication of the presence of a specific overnight volatility factor that is not well captured by the realised SV model for the open-to-close realised measures. For completeness, we also consider a realised SV specification based on the Student's t distribution for the suitable stocks, where we specify the leverage effects via a Gaussian copula function. We display the resulting parameter estimates in Table 15. We do not reject the kurtosis hypothesis implied by the Student's t model for seven of the stocks. Outliers lead to strong rejections of the Student's t model for the GE and Coca-Cola stocks. We may therefore more appropriately tackle the non-Gaussian features of the data by accounting for jumps in returns or volatility.

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A Deletion Smoothing

Our estimation method of Section 3.3 relies on the deletion smoothing estimate

$$\grave{\alpha}_t = \mathrm{E}(\alpha_t | RM^{-t}; \psi) \tag{25}$$

and the associated variance

$$\grave{V}_t = \mathrm{Var}(\alpha_t | RM^{-t}; \psi), \tag{26}$$

where RM^{-t} is the interpolation set $\{RM_1, \ldots, RM_{t-1}, RM_{t+1}, \ldots, RM_n\}$.

We compute these quantities using the results in de Jong (1989). For the linear state space model

$$\begin{array}{lll} \log RM_t = c + Z\alpha_t + \kappa_t, & \kappa_t \sim \mathrm{N}(0, \Sigma_\kappa), & t = 1, \ldots, n, \\ \alpha_{t+1} = T\alpha_t + R\eta_t, & \alpha_1 \sim \mathrm{N}(a_1, P_1), & \eta_t \sim \mathrm{N}(0, Q_t), \end{array} \tag{27}$$

we first obtain $a_{t+1} = \mathrm{E}(\alpha_{t+1} | RM_1, \ldots, RM_t; \psi)$ and $P_{t+1} = \mathrm{Var}(\alpha_{t+1} | RM_1, \ldots, RM_t; \psi)$ via the Kalman filter recursion

$$\begin{array}{ll} v_t = \log RM_t - c - Za_t, & F_t = Z_t P_t Z' + \Sigma_\kappa, \\ K_t = TP_t Z' F_t^{-1}, & L_t = T - K_t Z, \\ a_{t+1} = Ta_t + K_t v_t, & P_t = TP_t L_t' + RQR. \end{array} \tag{28}$$

Next, we compute $\widehat{\alpha}_t = \mathrm{E}(\alpha_t | RM_1, \ldots, RM_n; \psi)$ and $V_t = \mathrm{Var}(\alpha_t | RM_1, \ldots, RM_n; \psi)$ by the backward state smoothing equations

$$\begin{array}{ll} r_{t-1} = Z' F_t^{-1} v_t + L_t' r_t, & N_{t-1} = Z' F_t^{-1} Z + L_t' N_t L_t, \\ \widehat{\alpha}_t = a_t + P_t r_{t-1}, & V_t = P_t - P_t N_{t-1} P_t, \end{array} \tag{29}$$

initialised with $r_n = 0$ and $N_n = 0$.

We obtain the deletion smoothing mean $\grave{\alpha}_t$ and variance $\grave{V}_t$ as straightforward adjustments to $\widehat{\alpha}_t$ and V_t. Define

$$\begin{array}{l} w_t = F_t^{-1} v_t - K_t' r_t, \\ W_t = F_t^{-1} + K_t' N_t K_t, \\ M_t = L_t N_t K_t - Z' F_t^{-1}. \end{array} \tag{30}$$

Theorem 5 of de Jong (1989) shows that

$$\begin{aligned}\grave{\alpha}_t &= \widehat{\alpha}_t + P_t M_t W_t^{-1} w_t, \\ \grave{V}_t &= \widehat{V}_t + P_t M_t W_t^{-1} M_t' P_t.\end{aligned} \tag{31}$$

In the case in which η_t and ε_t are dependent, we compute the Gaussian density $p(\theta_t, \eta_t | RM^{-t})$ by applying the above result for a redefined state. The state space model becomes

$$\begin{aligned}\log RM_t &= c + Z^* \alpha_t^* + \kappa_t \\ \alpha_{t+1}^* &= T^* \alpha_t^* + R^* \eta_t^*\end{aligned}$$

with

$$\begin{aligned}\alpha_t^* &= (\alpha_t' \ \eta_t')', \\ Z^* &= [Z \ 0^{(p \times r)}], \\ T^* &= \begin{bmatrix} T & R \\ 0^{(r \times m)} & 0^{(r \times r)} \end{bmatrix}, \\ R^* &= \begin{bmatrix} 0^{(m \times r)} \\ I^{(r \times r)} \end{bmatrix}, \\ \eta_t^* &\sim \mathrm{N}(0, Q_t),\end{aligned}$$

where $0^{(.\times.)}$ and $I^{(.\times.)}$ are zero and identity matrices with the indicated dimensions, respectively. We then apply the deletion smoothing algorithm to this model.

B Tables and Figures

Table 1: Simulation results: Gaussian SV model with leverage

We simulate 200 trajectories of the realised SV model $y_t = \exp(\theta_t/2)\varepsilon_t$, $\varepsilon_t \sim \mathrm{N}(0,1)$, $\log RM_t = \gamma + \theta_t + \kappa_t$, $\kappa_t \sim \mathrm{N}(0, \sigma_\kappa^2)$, $\theta_t = c + \alpha_t$, $\alpha_t = \phi\alpha_t + \eta_t$, $\eta_t \sim \mathrm{N}(0, \sigma_\eta^2)$, $t = 1, \ldots, 2500$. This disturbances ε_t and η_t have correlation ρ and are independent from κ_t. The table shows the average estimated parameters, with standard errors in parentheses, for the estimation methods discussed in Sections 3.1 and 3.3.

		Joint		Two-step		
	True	Full	Simple	Smoother	Deletion	Prediction
γ	0.1	0.098	0.098	0.098	0.098	0.098
		(0.0288)	(0.0289)	(0.0289)	(0.0291)	(0.0294)
ρ	-0.3	−0.302	−0.302	−0.301	−0.302	–
		(0.0294)	(0.0300)	(0.0299)	(0.0349)	–
		Full	Simple	Two-step		
σ_κ^2	0.05	0.050	0.050	0.050		
		(0.0028)	(0.0028)	(0.0027)		
c	0.4	0.403	0.401	0.401		
		(0.2376)	(0.2379)	(0.2500)		
ϕ	0.98	0.978	0.978	0.978		
		(0.0047)	(0.0047)	(0.0049)		
σ_η^2	0.05	0.050	0.050	0.050		
		(0.0031)	(0.0031)	(0.0031)		

Table 2: Simulation results: SV-t model

We simulate 200 trajectories of the realised SV model $y_t = \exp(\theta_t/2)\sqrt{\frac{\nu-2}{\nu}}\varepsilon_t$, $\varepsilon_t \sim t(\nu)$, $\log RM_t = \gamma + \theta_t + \kappa_t$, $\kappa_t \sim \mathrm{N}(0, \sigma_\kappa^2)$, $\theta_t = c + \alpha_t$, $\alpha_t = \phi\alpha_t + \eta_t$, $\eta_t \sim \mathrm{N}(0, \sigma_\eta^2)$, $t = 1, \ldots, 2500$. This disturbances ε_t and η_t are independent. The table shows the average estimated parameters, with standard errors in parentheses, for the estimation methods discussed in Sections 3.1 and 3.3.

		Joint		Two-step		
	True	Full	Simple	Smoother	Deletion	Prediction
γ	0.1	0.098	0.098	0.098	0.098	0.099
		(0.0355)	(0.0355)	(0.0356)	(0.0360)	(0.0366)
ν	10	10.547	10.545	10.542	10.614	10.717
		(2.2234)	(2.2233)	(2.2219)	(2.4486)	(2.7363)
		Full	Simple	Two-step		
σ_κ^2	0.05	0.050	0.050	0.050		
		(0.0027)	(0.0027)	(0.0028)		
c	0.4	0.407	0.407	0.407		
		(0.2077)	(0.2077)	(0.2077)		
ϕ	0.98	0.978	0.978	0.978		
		(0.0042)	(0.0042)	(0.0042)		
σ_η^2	0.05	0.050	0.050	0.050		
		(0.0031)	(0.0031)	(0.0031)		

Table 3: Stocks

The table lists the stocks in our empirical analysis and provides their abbreviations.

Ticker	Stock	Sector
GE	General Electric	Conglomerate
IBM	IBM	Computers and technology
JPM	JP Morgan	Banking
KO	Coca-Cola	Beverages
PFE	Pfizer	Pharmaceuticals
PG	Procter & Gamble	Consumer goods
T	AT&T	Telecommunication
WMT	Wal-Mart	Retail
XOM	Exxon	Oil and gas

Table 4: Stochastic Volatility Parameter Estimates I.

The reported estimation results are for the realised stochastic volatility model (5), for close-to-close returns of nine DJIA stocks using data in the years of the post-decimalisation period, 2001-2010. The vector of realised measures include the realised kernel, the pre-averaging RV and the subsampled SV estimators. The model is specified as $y_t = \exp(\theta_t/2)\varepsilon_t$, $\varepsilon_t \sim \mathrm{N}(0,1)$, $\log RM_{j,t} = \gamma_j + \theta_t + \kappa_{j,t}$, $\kappa_t \sim \mathrm{N}(0, \Sigma_\kappa)$, where Σ_κ is a full covariance matrix with diagonal elements $\sigma^2_{j,\kappa}$, $\theta_t = c + \sum_{i=1}^{3} \alpha_{i,t}$, $\alpha_{i,t+1} = \phi_i \alpha_{i,t} + \eta_{i,t}$, $\eta_{ji,t} \sim \mathrm{N}(0, \sigma^2_{i,\eta})$, for $i = 1, \ldots, 3$. The return and state disturbances ε_t and $\eta_{i,t}$ have correlation $\rho_i = \rho(\varepsilon_t, \eta_{i,t})$ for $i = 1, 2$. Parameter estimation is carried out by the two-step method and is based on the deletion smoothing scheme as proposed in Sections 3.3 and 3.4. The standard errors of the parameter estimates are in parentheses.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
c	0.844	0.570	1.215	0.058	0.665	0.015	0.696	0.420	0.502
	(0.480)	(0.370)	(0.549)	(0.362)	(0.241)	(0.299)	(0.434)	(0.369)	(0.222)
ϕ_1	0.997	0.996	0.998	0.997	0.995	0.996	0.998	0.997	0.994
	(0.002)	(0.003)	(0.001)	(0.002)	(0.003)	(0.003)	(0.002)	(0.002)	(0.004)
$\sigma^2_{1,\eta}$	0.005	0.005	0.005	0.004	0.004	0.003	0.004	0.004	0.005
	(0.002)	(0.003)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.004)
ϕ_2	0.929	0.944	0.916	0.925	0.931	0.927	0.942	0.927	0.947
	(0.027)	(0.024)	(0.024)	(0.036)	(0.035)	(0.023)	(0.027)	(0.037)	(0.021)
$\sigma^2_{2,\eta}$	0.023	0.019	0.034	0.017	0.015	0.022	0.019	0.014	0.023
	(0.007)	(0.005)	(0.008)	(0.007)	(0.006)	(0.005)	(0.007)	(0.005)	(0.005)
ϕ_3	0.456	0.366	0.324	0.462	0.420	0.194	0.473	0.401	0.153
	(0.080)	(0.068)	(0.067)	(0.102)	(0.060)	(0.038)	(0.054)	(0.080)	(0.036)
$\sigma^2_{3,\eta}$	0.068	0.066	0.091	0.049	0.087	0.130	0.103	0.059	0.117
	(0.009)	(0.007)	(0.010)	(0.009)	(0.009)	(0.007)	(0.009)	(0.007)	(0.006)
ρ_1	−0.279	−0.340	−0.616	−0.380	−0.371	−0.249	−0.677	−0.237	0.006
	(0.105)	(0.146)	(0.187)	(0.126)	(0.139)	(0.195)	(0.239)	(0.170)	(0.238)
ρ_2	−0.339	−0.472	−0.264	−0.260	−0.107	−0.292	−0.103	−0.279	−0.599
	(0.070)	(0.092)	(0.086)	(0.081)	(0.094)	(0.092)	(0.119)	(0.112)	(0.115)

Table 5: Stochastic Volatility Parameter Estimates II.

Estimation results for the same setting as that of Table 4, but with the vector of realised measures replaced by the subsampled median-based realised variance measure.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
c	0.849	0.567	1.217	0.066	0.698	0.037	0.737	0.426	0.507
	(0.477)	(0.355)	(0.546)	(0.327)	(0.244)	(0.287)	(0.394)	(0.351)	(0.221)
ϕ_1	0.997	0.996	0.998	0.996	0.995	0.996	0.997	0.997	0.994
	(0.002)	(0.003)	(0.001)	(0.002)	(0.003)	(0.003)	(0.002)	(0.002)	(0.004)
$\sigma^2_{1,\eta}$	0.005	0.005	0.005	0.004	0.004	0.003	0.004	0.004	0.005
	(0.002)	(0.003)	(0.002)	(0.002)	(0.002)	(0.002)	(0.003)	(0.002)	(0.004)
ϕ_2	0.932	0.945	0.918	0.924	0.937	0.934	0.950	0.928	0.951
	(0.027)	(0.025)	(0.025)	(0.037)	(0.031)	(0.022)	(0.026)	(0.036)	(0.021)
$\sigma^2_{2,\eta}$	0.019	0.017	0.030	0.016	0.012	0.019	0.016	0.013	0.020
	(0.006)	(0.005)	(0.008)	(0.007)	(0.005)	(0.005)	(0.006)	(0.005)	(0.005)
ϕ_3	0.454	0.352	0.312	0.474	0.446	0.223	0.491	0.342	0.189
	(0.127)	(0.128)	(0.167)	(0.192)	(0.106)	(0.161)	(0.103)	(0.173)	(0.096)
$\sigma^2_{3,\eta}$	0.066	0.072	0.088	0.041	0.064	0.094	0.075	0.065	0.106
	(0.018)	(0.023)	(0.035)	(0.015)	(0.015)	(0.058)	(0.016)	(0.028)	(0.049)
ρ_1	−0.284	−0.337	−0.617	−0.381	−0.357	−0.252	−0.710	−0.248	0.010
	(0.101)	(0.142)	(0.184)	(0.120)	(0.139)	(0.200)	(0.179)	(0.168)	(0.238)
ρ_2	−0.356	−0.461	−0.281	−0.241	−0.132	−0.284	−0.060	−0.254	−0.619
	(0.071)	(0.092)	(0.087)	(0.079)	(0.098)	(0.097)	(0.102)	(0.113)	(0.123)

Table 6: Measuring, Filtering and Forecasting with Different Measures.

We apply the Kalman filter and smoother individually to the realised kernel, pre-averaging RV, subsampled SV as well as to the three measures in combination. Let j index the realised measures. The calculations are based on the parameters of Tables 4 and the measurement variances $\text{Var}(\kappa_{j,t})$ reported below. The signal-to-noise rows indicate the ratio between the variance of the innovations in the latent log-variance process $\sum_{i=1}^{3}\sigma_{i\eta}^2$ and the measurement variances $\text{Var}(\kappa_{j,t})$. Let $\mathcal{F}_{j,t}$ denote the natural filtration $RM_{j,1},...,RM_{j,t}$. $\text{Var}(\theta_t|\mathcal{F}_t)$ is the filtered variance of the log-volatility signal, $\text{Var}(\theta_t|\mathcal{F}_n)$ is the smoothed variance and $\text{Var}(\theta_{t+1}|\mathcal{F}_{j,t})$ is the predictive variance.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
					Realised kernel				
Signal-to-noise	1.116	1.525	3.181	0.856	1.564	-	1.801	1.015	
$\text{Var}(\kappa_{1,t})$	0.086	0.059	0.041	0.082	0.068	0.005	0.070	0.076	0.001
$\text{Var}(\theta_t\|\mathcal{F}_{1,t})$	0.053	0.040	0.033	0.046	0.046	0.005	0.049	0.045	0.001
$\text{Var}(\theta_t\|\mathcal{F}_{1,n})$	0.045	0.035	0.031	0.038	0.041	0.005	0.043	0.038	0.001
$\text{Var}(\theta_{t+1}\|\mathcal{F}_{1,t})$	0.137	0.123	0.166	0.104	0.139	0.186	0.162	0.109	0.177
					Pre-averaging RV				
Signal-to-noise	1.160	1.563	3.244	0.893	1.718 -		1.811	1.034	-
$\text{Var}(\kappa_{2,t})$	0.083	0.058	0.040	0.078	0.062	0.003	0.069	0.074	0.000
$\text{Var}(\theta_t\|\mathcal{F}_{2,t})$	0.051	0.039	0.032	0.044	0.043	0.003	0.048	0.044	0.000
$\text{Var}(\theta_t\|\mathcal{F}_{2,n})$	0.044	0.035	0.030	0.037	0.038	0.003	0.043	0.038	0.000
$\text{Var}(\theta_{t+1}\|\mathcal{F}_{2,t})$	0.136	0.123	0.166	0.103	0.138	0.185	0.162	0.109	0.177
					Subsampled RV				
Signal-to-noise	0.641	0.770	1.245	0.532	0.795	2.465	1.046	0.541	2.930
$\text{Var}(\kappa_{3,t})$	0.150	0.117	0.105	0.131	0.134	0.063	0.120	0.142	0.049
$\text{Var}(\theta_t\|\mathcal{F}_{3,t})$	0.075	0.063	0.067	0.061	0.071	0.048	0.071	0.065	0.039
$\text{Var}(\theta_t\|\mathcal{F}_{3,n})$	0.062	0.053	0.058	0.049	0.061	0.045	0.061	0.054	0.037
$\text{Var}(\theta_{t+1}\|\mathcal{F}_{3,t})$	0.150	0.136	0.182	0.113	0.151	0.197	0.173	0.120	0.187
					Combined				
$\rho(\kappa_{1,t},\kappa_{2,t})$	0.985	0.975	0.958	0.973	0.966	0.394	0.953	0.974	-
$\rho(\kappa_{1,t},\kappa_{3,t})$	0.877	0.806	0.818	0.832	0.845	0.636	0.804	0.847	-
$\rho(\kappa_{2,t},\kappa_{3,t})$	0.856	0.767	0.778	0.788	0.797	0.370	0.756	0.815	-
$\text{Var}(\theta_t\|\mathcal{F}_t)$	0.050	0.039	0.029	0.044	0.041	0.002	0.048	0.043	0.000
$\text{Var}(\theta_t\|\mathcal{F}_n)$	0.043	0.034	0.028	0.037	0.037	0.002	0.043	0.037	0.000
$\text{Var}(\theta_{t+1}\|\mathcal{F}_t)$	0.135	0.123	0.165	0.103	0.137	0.185	0.162	0.108	0.176

Table 7: Measuring, Filtering and Forecasting II

We reproduce the analysis of Table 6 for the median-RV measure. The results reflect the estimated parameters of Table 5.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
Signal-to-noise	1.414	2.406	4.548	0.956	1.582	2.994	1.345	1.685	-
$\text{Var}(\kappa_t)$	0.060	0.038	0.025	0.063	0.047	0.033	0.063	0.047	0.000
$\text{Var}(\theta_t\|\mathcal{F}_t)$	0.040	0.029	0.021	0.037	0.032	0.026	0.041	0.032	0.000
$\text{Var}(\theta_t\|\mathcal{F}_n)$	0.034	0.026	0.020	0.031	0.028	0.025	0.035	0.029	0.000
$\text{Var}(\theta_{t+1}\|\mathcal{F}_t)$	0.117	0.118	0.143	0.088	0.099	0.126	0.116	0.105	0.153

Table 8: Bias Estimates.

The table displays the estimates for the bias term γ_j of the realised stochastic volatility model, where j indexes the realised measures. The standard errors are in parentheses.

	Close-to-close returns				Open-to-close returns			
	RK	PRV	SRV	MedRV	RK	PRV	SRV	MedRV
GE	−0.198	−0.203	−0.269	−0.228	0.071	0.066	0.000	0.040
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
IBM	−0.220	−0.227	−0.301	−0.251	0.066	0.058	−0.016	0.035
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
JPM	−0.213	−0.219	−0.270	−0.247	0.047	0.041	−0.010	0.016
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
KO	−0.017	−0.026	−0.099	−0.060	0.167	0.158	0.085	0.127
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
PFE	−0.132	−0.134	−0.227	−0.144	0.084	0.082	−0.012	0.081
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
PG	0.033	0.023	−0.073	−0.033	0.237	0.228	0.131	0.176
	(0.031)	(0.031)	(0.031)	(0.031)	(0.031)	(0.031)	(0.031)	(0.030)
T	−0.009	−0.029	−0.057	−0.064	0.177	0.157	0.129	0.128
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
WMT	−0.074	−0.081	−0.156	−0.141	0.144	0.137	0.062	0.077
	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)	(0.030)
XOM	−0.133	−0.134	−0.225	−0.178	0.093	0.091	0.000	0.085
	(0.031)	(0.031)	(0.031)	(0.030)	(0.029)	(0.029)	(0.029)	(0.030)

Table 9: Diagnostics for the realised kernel residuals.

The table displays diagnostic statistics for the prediction errors (v_t) from the estimation in Table 4. The prediction errors are outputs of the Kalman filter calculated as $v_t = \log RK_t - \mathrm{E}(\log RK_t | RM_1, ..., RM_{t-1}; \psi)$.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
Skewness	0.43	0.36	0.38	0.46	0.65	0.47	0.46	0.29	0.44
Kurtosis	4.69	5.68	4.49	4.98	5.75	5.72	4.57	5.00	5.34
LB(1) (v_t)	0.71	0.80	0.37	0.99	0.37	0.77	0.92	0.60	0.82
LB(5) (v_t)	0.96	0.12	0.68	0.39	0.22	0.27	0.83	0.14	0.23
LB(22) (v_t)	0.38	0.04	0.38	0.86	0.46	0.90	0.98	0.14	0.00
LB(1) (v_t^2)	0.00	0.26	0.00	0.00	0.01	0.07	0.01	0.10	0.05
LB(5) (v_t^2)	0.00	0.26	0.00	0.00	0.01	0.00	0.01	0.02	0.01
LB(22) (v_t^2)	0.00	0.03	0.00	0.00	0.13	0.00	0.00	0.08	0.00

Figure 1: Estimated log variance signal (top) and individual components for IBM

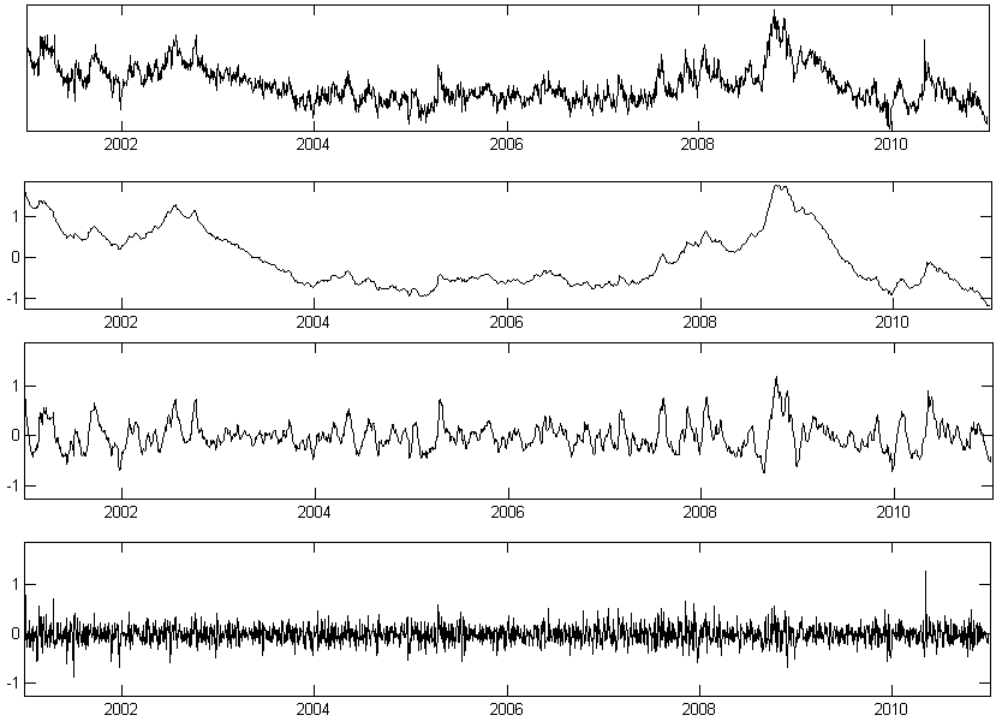

Table 10: Forecasting results: realised kernel, one day ahead.

We compare out of sample predictions for the daily log realised kernel of nine DJIA stocks in the period between January 2002 and December 2010. Parameter estimates are updated monthly in a rolling window of 2000 observations. We use information from the realised kernel and close-to-close return series in the estimations. The RSV model is based on the superposition specification (5). The specifications for the HARX, HEAVY and Realised GARCH models are provided in Section 4.4. MSE is the mean-square error. The p^{mcs} column indicates the p-value of the Hansen, Lunde, and Nason (2011a) model confidence set. A $(1-\alpha)$ MCS is constructed so that it will contain the best model in MSE at a $(1-\alpha)\times 100\%$ confidence level. The model is included in the $(1-\alpha)$ model confidence set for $\alpha \leq \mathrm{p}^{mcs}$. We base the test on $10,000$ bootstrap resamples. R^2 denotes the coefficient of determination of the Mincer-Zarnowitz regression.

	GE			IBM			JPM		
	MSE	p^{mcs}	R^2	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2
HARX	0.220	0.392	0.829	0.177	0.270	0.772	0.206	0.463	0.865
HEAVY	0.218	0.392	0.831	0.181	0.069	0.768	0.205	0.463	0.866
Realised GARCH	0.219	0.392	0.830	0.177	0.270	0.773	0.206	0.463	0.866
Realised SV	0.214	1.000	0.833	0.173	1.000	0.777	0.200	1.000	0.869
	KO			PFE			PG		
	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2
HARX	0.184	0.101	0.734	0.210	0.009	0.676	0.191	0.020	0.685
HEAVY	0.187	0.101	0.731	0.224	0.000	0.655	0.200	0.020	0.673
Realised GARCH	0.183	0.145	0.737	0.217	0.000	0.668	0.194	0.020	0.682
Realised SV	0.180	1.000	0.739	0.207	1.000	0.680	0.187	1.000	0.691
	T			WMT			XOM		
	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2
HARX	0.232	0.077	0.739	0.187	0.184	0.717	0.176	0.081	0.747
HEAVY	0.240	0.006	0.731	0.188	0.184	0.716	0.172	0.886	0.752
Realised GARCH	0.239	0.005	0.732	0.186	0.184	0.719	0.172	0.706	0.752
Realised SV	0.229	1.000	0.742	0.181	1.000	0.726	0.171	1.000	0.753

Table 11: Forecasting results: realised kernel, one month ahead.

We compare out of sample twenty two day ahead predictions for the daily log realised kernel of nine DJIA stocks in the period between January 2002 and December 2010. Parameter estimates are updated monthly in a rolling window of 2000 observations. We use information from the realised kernel and close-to-close return series in the estimations. The RSV model is based on the superposition specification (5). The specifications for the HARX, HEAVY and Realised GARCH models are provided in Section 4.4. MSE is the mean-square error. The p_{mcs} column indicates the p-value of the Hansen, Lunde, and Nason (2011a) model confidence set. A $(1-\alpha)$ MCS is constructed so that it will contain the best model in MSE at a $(1-\alpha)\times 100\%$ confidence level. The model is included in the $(1-\alpha)$ model confidence set for $\alpha \leq \mathrm{p}_{mcs}$. We base the test on $10,000$ bootstrap resamples. R^2 denotes the coefficient of determination of the Mincer-Zarnowitz regression.

	GE			IBM			JPM		
	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2
HARX	0.551	0.044	0.573	0.479	0.032	0.412	0.591	0.049	0.615
HEAVY	0.752	0.000	0.453	0.629	0.000	0.289	0.721	0.000	0.562
Realised GARCH	0.707	0.000	0.482	0.611	0.000	0.311	0.678	0.000	0.583
Realised SV	0.518	1.000	0.598	0.450	1.000	0.437	0.550	1.000	0.641
	KO			PFE			PG		
	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2
HARX	0.423	0.143	0.423	0.443	0.215	0.336	0.444	0.033	0.303
HEAVY	0.502	0.000	0.366	0.615	0.000	0.136	0.561	0.000	0.185
Realised GARCH	0.507	0.000	0.360	0.592	0.000	0.164	0.530	0.000	0.234
Realised SV	0.406	1.000	0.437	0.430	1.000	0.347	0.416	1.000	0.332
	T			WMT			XOM		
	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2	MSE	p_{mcs}	R^2
HARX	0.529	0.242	0.426	0.405	0.070	0.422	0.501	0.193	0.300
HEAVY	0.573	0.012	0.416	0.547	0.000	0.273	0.545	0.004	0.257
Realised GARCH	0.618	0.000	0.370	0.495	0.000	0.351	0.548	0.004	0.262
Realised SV	0.506	1.000	0.447	0.389	1.000	0.436	0.490	1.000	0.317

Table 12: Forecasting with different realised measures (2006-2010).
We compare the out of sample forecasting variances of the realised stochastic volatility model estimated using the realised kernel, subsampled SV and Median-RV measures individually. The forecasts use information from the corresponding realised measure only. The same predictions are separately compared as forecasts of the realised kernel and the subsampled RV measures. Parameter estimates are updated monthly in a rolling window of 2000 observations. In parentheses are the p-values of the Hansen, Lunde, and Nason (2011a) model confidence set. The $(1-\alpha)$ MCS is constructed so that it will contain the best model in MSE at a $(1-\alpha)\times 100\%$ confidence level. The model is included in the $(1-\alpha)$ model confidence set for $\alpha \leq \mathrm{p}_{mcs}$. The test is based on $10,000$ bootstrap resamples.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
	1 day ahead predictions evaluated via RK								
Realised kernel	0.235 (0.774)	0.188 (0.424)	0.202 (1.000)	0.202 (0.502)	0.181 (1.000)	0.197 (1.000)	0.212 (0.822)	0.194 (0.709)	0.180 (1.000)
Subsampled RV	0.251 (0.000)	0.198 (0.004)	0.211 (0.012)	0.215 (0.000)	0.191 (0.001)	0.213 (0.000)	0.225 (0.000)	0.205 (0.002)	0.185 (0.145)
Median-RV	0.235 (1.000)	0.187 (1.000)	0.203 (0.601)	0.201 (1.000)	0.182 (0.597)	0.199 (0.314)	0.211 (1.000)	0.193 (1.000)	0.180 (0.504)
	1 day ahead predictions evaluated via SRV								
Realised kernel	0.304 (0.500)	0.252 (0.853)	0.273 (1.000)	0.252 (0.781)	0.237 (1.000)	0.258 (1.000)	0.257 (0.507)	0.266 (1.000)	0.227 (0.598)
Subsampled RV	0.311 (0.017)	0.250 (1.000)	0.277 (0.431)	0.258 (0.050)	0.243 (0.061)	0.265 (0.203)	0.267 (0.005)	0.267 (0.975)	0.226 (1.000)
Median-RV	0.303 (1.000)	0.252 (0.853)	0.275 (0.466)	0.251 (1.000)	0.238 (0.915)	0.262 (0.203)	0.256 (1.000)	0.267 (0.975)	0.229 (0.395)
	22 days ahead predictions evaluated via RK								
Realised kernel	0.637 (0.826)	0.533 (0.661)	0.649 (1.000)	0.475 (1.000)	0.443 (0.026)	0.467 (1.000)	0.560 (0.550)	0.444 (0.202)	0.568 (1.000)
Subsampled RV	0.641 (0.649)	0.530 (1.000)	0.652 (0.631)	0.481 (0.550)	0.459 (0.005)	0.473 (0.522)	0.556 (1.000)	0.451 (0.202)	0.570 (0.941)
Median-RV	0.636 (1.000)	0.538 (0.366)	0.699 (0.025)	0.483 (0.550)	0.434 (1.000)	0.470 (0.522)	0.566 (0.550)	0.438 (1.000)	0.568 (0.941)
	22 days ahead predictions evaluated via SRV								
Realised kernel	0.671 (0.712)	0.554 (0.346)	0.687 (1.000)	0.509 (1.000)	0.482 (0.018)	0.504 (1.000)	0.580 (0.562)	0.505 (0.287)	0.589 (1.000)
Subsampled RV	0.674 (0.712)	0.547 (1.000)	0.689 (0.596)	0.515 (0.622)	0.495 (0.012)	0.509 (0.651)	0.576 (1.000)	0.510 (0.287)	0.592 (0.963)
Median-RV	0.669 (1.000)	0.558 (0.194)	0.735 (0.024)	0.516 (0.622)	0.472 (1.000)	0.506 (0.651)	0.585 (0.562)	0.499 (1.000)	0.591 (0.963)

Table 13: Sample kurtosis of open-to-close returns standardised by realised measures

	RK	PRV	SRV	MedRV
GE	2.69	2.73	2.52	2.83
IBM	2.79	2.80	2.55	2.91
JPM	2.83	2.84	2.55	2.99
KO	3.12	3.16	2.76	3.33
PFE	2.75	2.82	2.54	3.01
PG	3.08	3.10	2.79	3.20
T	2.83	2.86	2.58	3.10
WMT	2.84	2.86	2.58	3.04
XOM	2.83	2.87	2.56	2.93

Table 14: Testing the Gaussian SV model (2001-2010).

We analyse the series of open-to-close and close-to-close returns standardised by their conditional volatilities estimated from the realised SV model with $\mathrm{RM}_t = (\mathrm{RK}_t\ \mathrm{PRV}_t\ \mathrm{SRV}_t)'$. We calculate the standardised returns as $\widehat{\varepsilon}_t = y_t \times \mathrm{E}(\exp(\theta_t)|\mathrm{RM}_1, \ldots, \mathrm{RM}_{t-1})^{-1/2}$. We report the sample skewness and kurtosis of the standardised returns and compare it to the values implied by the estimated Gaussian SV model of Table 4 and the SV-t model with additional parameters reported in Table 15. The model implied kurtosis is calculated via equation (10). The table shows the p-values of parametric bootstrap tests of the model implied skewness and kurtosis against the data.

	GE	IBM	JPM	KO	PFE	PG	T	WMT	XOM
	Open-to-close returns								
Sample skewness	0.20	-0.10	-0.08	0.03	0.30	-0.08	-0.05	0.12	-0.19
p-value (Gaussian)	0.00	0.12	0.26	0.59	0.00	0.27	0.46	0.06	0.01
Sample Kurtosis	4.25	3.60	3.85	4.07	5.54	3.85	3.77	3.58	3.58
Gaussian SV kurtosis	3.43	3.39	3.54	3.33	3.44	3.61	3.53	3.34	3.58
p-value (Gaussian)	0.00	0.18	0.10	0.00	0.00	0.25	0.18	0.11	1.00
	Close-to-close returns								
Sample skewness	-0.23	-0.25	-0.02	-0.15	-0.48	-0.21	0.03	-0.02	-0.24
p-value (Gaussian)	0.00	0.00	0.77	0.01	0.00	0.01	0.64	0.70	0.00
p-value (SV-t)	0.02	0.05	0.78	0.24	0.00	0.11	0.72	0.82	0.00
Sample Kurtosis	6.44	5.42	3.90	5.45	9.12	5.19	4.38	4.43	3.63
Gaussian SV kurtosis	3.43	3.39	3.54	3.33	3.44	3.61	3.53	3.34	3.58
p-value (Gaussian)	0.00	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.80
SV-t kurtosis	4.11	4.66	3.61	4.90	4.54	4.74	3.98	3.94	3.58
p-value (SV-t)	0.00	0.13	0.14	0.41	0.00	0.38	0.17	0.08	0.80

Table 15: SV-t parameters.

We extend the results in Tables 4 and 5 to allow for the return innovations to follow the (standardised) t-distribution with ν degrees of freedom. The leverage effects are modelled via a Gaussian copula. Because of the two-step estimation method, the parameters of the linear model for the log realised measures remain the same. We report the new parameters. The standard errors are in parentheses.

	GE	IBM	JPM	KO	PFE	PG	T	WMT
	Realised kernel, pre-averaging RV and subsampled RV estimation							
γ_{rk}	−0.213	−0.249	−0.214	−0.047	−0.157	−0.003	−0.025	−0.089
	(0.034)	(0.037)	(0.031)	(0.038)	(0.036)	(0.038)	(0.033)	(0.033)
γ_{prv}	−0.218	−0.257	−0.221	−0.056	−0.159	−0.012	−0.045	−0.095
	(0.034)	(0.037)	(0.031)	(0.038)	(0.036)	(0.038)	(0.033)	(0.033)
γ_{srv}	−0.284	−0.331	−0.272	−0.129	−0.253	−0.109	−0.073	−0.170
	(0.034)	(0.037)	(0.031)	(0.038)	(0.036)	(0.038)	(0.033)	(0.033)
ν	14.19	9.35	106.20	8.21	10.28	10.36	19.65	15.16
	(3.45)	(1.56)	(243.75)	(1.25)	(1.79)	(2.27)	(7.87)	(4.38)
ρ_1	−0.497	−0.503	−0.610	−0.457	−0.512	−0.252	−0.441	−0.245
	(0.141)	(0.193)	(0.180)	(0.171)	(0.155)	(0.205)	(0.202)	(0.173)
ρ_2	−0.265	−0.431	−0.266	−0.232	−0.078	−0.294	−0.212	−0.276
	(0.081)	(0.113)	(0.083)	(0.099)	(0.102)	(0.095)	(0.107)	(0.114)
	Median-RV estimation							
γ_{med}	−0.243	−0.282	−0.249	−0.089	−0.166	−0.070	−0.042	−0.156
	(0.034)	(0.037)	(0.031)	(0.038)	(0.036)	(0.038)	(0.035)	(0.034)
ν	13.51	9.06	92.25	7.76	9.21	8.82	12.16	14.29
	(3.10)	(1.47)	(189.74)	(1.10)	(1.41)	(1.59)	(2.88)	(3.91)
ρ_1	−0.500	−0.514	−0.612	−0.490	−0.527	−0.254	−0.500	−0.255
	(0.139)	(0.190)	(0.176)	(0.167)	(0.162)	(0.205)	(0.219)	(0.173)
ρ_2	−0.277	−0.412	−0.283	−0.202	−0.094	−0.290	−0.167	−0.253
	(0.087)	(0.116)	(0.084)	(0.100)	(0.110)	(0.099)	(0.125)	(0.115)

Figure 2: Log realised kernel (top) and estimated log variance signal for Coca-Cola

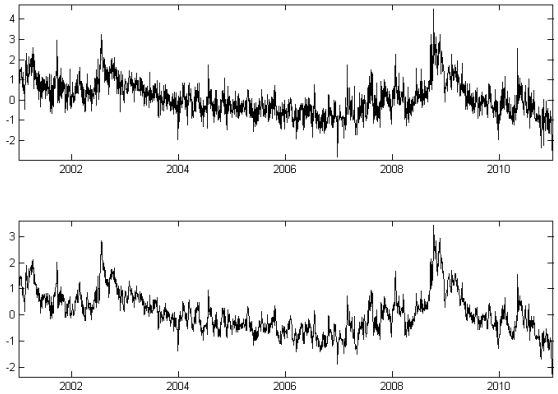

Figure 3: Open-to-close returns standardised by the conditional volatilities

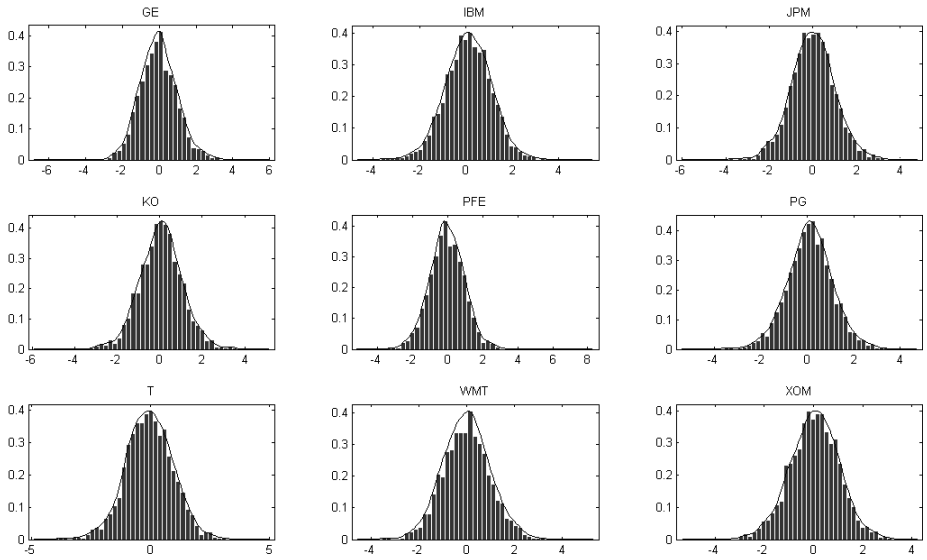

Figure 4: Close-to-close returns standardised by the conditional volatilities