OPTIMAL ORDER QUANTITIES WITH VOLUME DISCOUNTS BEFORE AND AFTER PRICE INCREASE

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Abstract

An inventory problem in which annual demand is normally distributed with known means and standard deviations is considered. A purchase price increase is imminent before the next order is placed. Volume discounts are also given in accordance to the size of the order. A model to compute an optimal order quantity and an optimal delivery point is presented. This model can also account for any price change that may occur from time to time.

1 Introduction

Models involving price increase have been discussed (Tersine (1976), Naddor(1966), Huang and Kulkarni (2003)). In these models constant annual demand rate is assumed.

Tersine (1976) assumes no stock when the order needs to be made and the special order coincides with the end of a cycle. Naddor (1966) on the other hand assumes that the special order is a multiple of the economic order quantity (EOQ) after the price increase. Huang and Kulkarni (2003) discussed an infinite horizon model in which the special order size is not necessarily a multiple of the new EOQ. In all these models, the approach is to determine the special order size by minimising some cost difference function of when the special order is made and when it is not. Naddor

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(1966) and Huang and Kulkarni (2003) achieve this by using a so called Cesaro limit and conclude by finding a (s, S) type strategy.

The problem of volume discounts has also been discussed separately in the literature (Taha (1992), Naddor (1966) and Winston (1997)). These models are based on an assumption of no price increases and hence no consideration of special order quantities. Algorithms for minimising total annual costs have been proposed for use in such a case to determine an optimal order quantity that takes advantage of the volume discounts.

Here we propose a model that can incorporate both volume discounts and price changes. Annual demand is assumed to be normally distributed with known mean and standard deviation. The order size and cycle length between delivery points are also assumed to be variable. Central to this model is that the demand rate in one cycle is not equal to that of the others. This allows for such practical cases as seasonal demand, about which the retailers would have an idea of how the stock can deplete in a particular cycle. Shortages are allowed in any cycle but no backlogging is assumed. The optimal order quantity is dependent on the price.

2 The model

2.1 Model assumptions

We assume a finite planning period of one year, during which a predetermined number of orders are to be made.Annual demand is stochastic with a known mean annual demand (D). The order quantity, q_i, and the length, T_i, of each cycle are variable. Order quantity q_1 is placed at $t = 0$. For the i-th cycle the retailer has a good knowledge of how the stock is depleted and can assign a mean demand rate, which may vary from cycle to cycle. The inventory level is stochastic with constant variance over a time cycle. Backlogging is not allowed. Hence all shortages account for lost sales. On time delivery of an order is assumed. Ordering cost, O_i, has two components: the fixed ordering cost and variable ordering costs. The variable cost depends on the quantity ordered, so that $O_i = F + cq_i$. This is in addition to the purchase cost, $P_i = p_i q_i$, associated with the unit price of the stock. The times when an order is delivered are allowed to vary within certain ranges; i.e. $t_i \in [l_i, u_i], i \geq 1$. The sum of the periodic order quantities equals annual mean demand; i.e. $\sum q_i = D$.

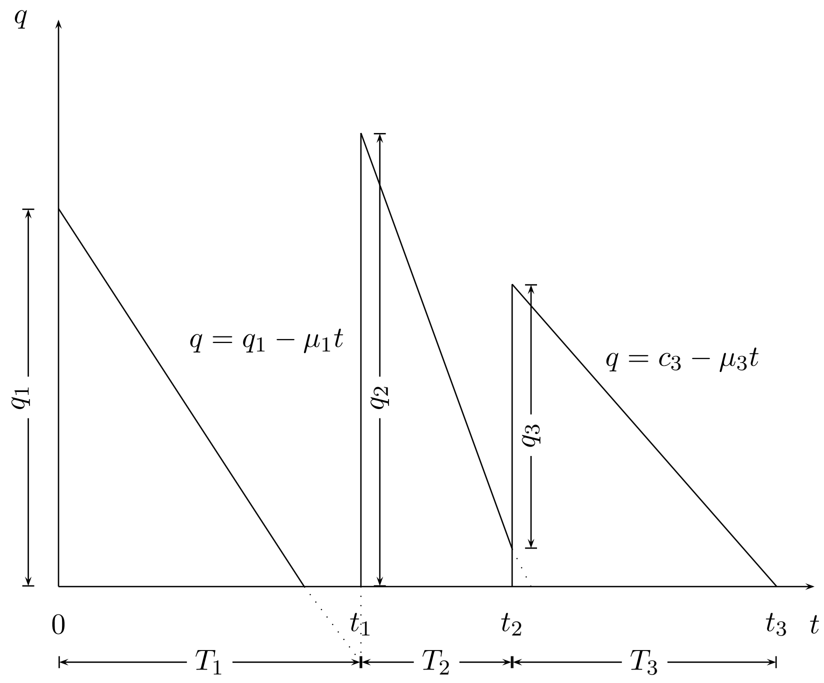

Figure 1: Operational structure ($n = 3$).

2.2 Description

Let the mean demand rate μ_i for each cycle be known. Then the inventory level in each cycle will be determined by

$$q = c_i - \mu_i t + \nu_i \ , \tag{2.1}$$

where $\nu_i \sim N(0, \sigma_i^2)$ and $c_i = c_i(q_i, \nu_{i-1})$. Therefore the inventory $q \sim N(c_i - \mu_i t, \sigma_i^2)$.

We normalise the demand rate so that if $\mu_i = 0$, then $\nu_i = Z_i \sigma_i$. We consider the case where σ_i is proportional to the order quantity, i.e. $\sigma_i = \epsilon q_i$, ϵ small. Thus $\nu_i = \nu_i(q_i)$ and in every reference thereto such dependence shall be understood.

Figure 1 shows some of the features of the operation for a planning horizon involving 3 orders/cycles. The initial order quantity, q_1, is delivered at $t_0 = 0$. At any stage, the next delivery point occurs at $t_i = t_{i-1} + \dfrac{q_i}{\mu_i}$, $i = 1, \ldots, n-1$, with $t_n = 1$ marking the end of the planning horizon and hence no delivery at that point.

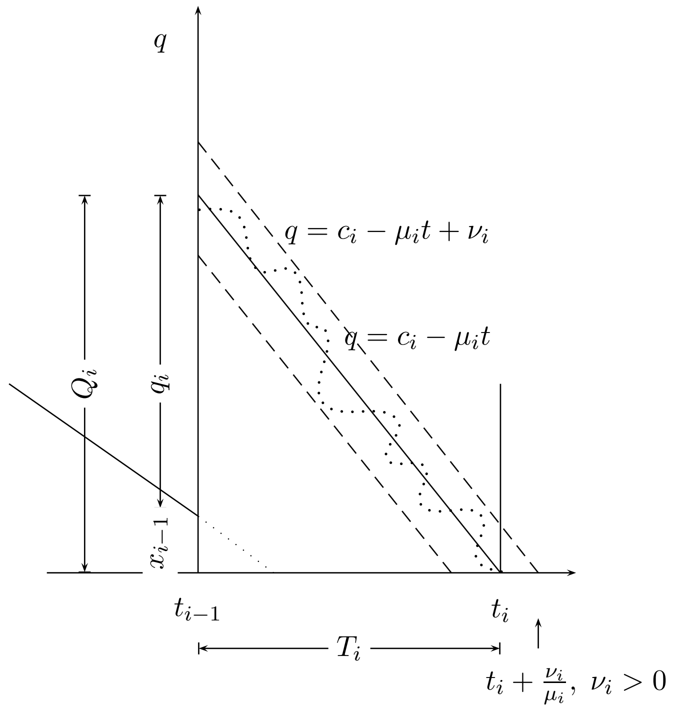

Figure 2: Interval T_i

A more detailed representation of a typical cycle i is shown in Figure 2. Each cycle begins with a delivery of an order quantity $q_i,\ i = 1, 2, \ldots, n,$ which is depleted at a mean demand rate $\mu_i,\ i = 1, 2, \ldots, n,$ over a period $T_i = \frac{q_i}{\mu_i}$. The stock level during the i-th cycle is given by the stochastic function $q = q(t) = c_i - \mu_i t + \nu_i$.

Also, the admissible range for the delivery point is $l_i \leq t_i \leq u_i$, where l_i and u_i are respectively the lower and upper bounds for the variation.

The costs likely to be incurred in cycle T_i are:

Holding cost:

If $\nu_{i-1} > 0$, then there will be stock, x_{i-1}, on hand when the next delivery q_i occurs. The holding cost is taken up to t_{i-1} and the rest is incorporated into the next cycle. If we let $Q_i = x_{i-1} + q_i$ be the total stock at the beginning of the i-th cycle, then the average inventory during T_i is

$$I_i = \frac{\mu_i}{q_i + \nu_i} \int_{t_{i-1}}^{t_i + \frac{\nu_i}{\mu_i}} (Q_i - \mu_i t + \nu_i)\, dt\ . \tag{2.2}$$

The holding costs incurred between t_{i-1} and $t_i = t_{i-1} + \dfrac{q_i}{\mu_i}$ are therefore given by

$$H_i = \left(\frac{q_i + \nu_i}{\mu_i}\right) hI_i \ , \tag{2.3}$$

where h is the cost of holding a unit in stock for a year. The last inventory holding cost incurred if $\nu_n > 0$ is

$$H_n' = h \int_{t_n}^{t_n + \frac{\nu_n}{\mu_n}} \left[Q_n - \mu_n t + \nu_n\right] dt \ . \tag{2.4}$$

Shortage cost:

If $\nu_i < 0$, then a shortage cost is incurred between $t_i + \dfrac{\nu_i}{\mu_i}$ and t_i and is given by

$$S_i = s \int_{\frac{q_i + \nu_i}{\mu_i}}^{\frac{q_i}{\mu_i}} (Q_i - \mu_i t + \nu_i) \ dt \ , \tag{2.5}$$

where s is the shortage cost per unit per year.

Ordering cost:

This is given by

$$O_i = F + cq_i \ . \tag{2.6}$$

Purchasing cost:

This is the cost of the stock given by

$$P_i = p_i q_i \ . \tag{2.7}$$

2.2.1 Total inventory cost:

The total cost in interval T_i is therefore

$$C_i = \begin{cases} P_i + O_i + H_i & \text{if} \quad \nu_i > 0 \ , \\ P_i + O_i + H_i + |S_i| & \text{if} \quad \nu_i < 0 \ . \end{cases} \tag{2.8}$$

2.3 Total annual cost

The main objective is to minimise the total annual cost,

$$C = \sum_{i=1}^{n} C_i = f(q_1, q_2, \ldots, q_n) \ .$$

The optimisation problem can be stated as

$$\text{Minimise} \quad C = f(q_1, q_2, \ldots, q_n) \tag{2.9}$$
$$\text{subject to} \quad l_1 \le t_i \le u_i, \qquad i = 1, 2, \ldots, n \tag{2.10}$$
$$q_1 + q_2 + \ldots + q_n = D \tag{2.11}$$

where

$$t_1 = \frac{q_1}{\mu_1},\ t_i = t_{i-1} + \frac{q_i}{\mu_i},\ i = 2, \ldots, n \ .$$

The total annual costs depends on a number of cases. Let the total cost in cycle T_i be

$$C_i = P_i + O_i + H_i, \quad \text{if} \quad S_i = 0 \ , \tag{2.12}$$

and

$$C_i^s = P_i + I_i + H_i + |S_i|, \quad \text{if} \quad S_i \neq 0 \ . \tag{2.13}$$

For example the following total annual costs for $n = 3$ arise (see Ali and Masinga [2004] for details):

Case 1: $\nu_1, \nu_2, \nu_3 < 0$

$$C = C_1^s + C_2^s + C_3^s \ , \tag{2.14}$$

Case 2: $\nu_1, \nu_2 < 0, \nu_3 > 0$

$$C = C_1^s + C_2^s + C_3 + H_3' \ , \tag{2.15}$$

Case 3: $\nu_1, \nu_3 > 0, \nu_2 < 0$

$$C = C_1 + C_2^s + C_3 + H_3' \ , \tag{2.16}$$

Case 4: $\nu_1, \nu_3 < 0, \nu_2 > 0$

$$C = C_1^s + C_2 + C_3^s \ , \tag{2.17}$$

Case 5: $\nu_1, \nu_2 > 0, \nu_3 < 0$

$$C = C_1 + C_2 + C_3^s \ , \tag{2.18}$$

Inputs	Run 1	Run 2	Run 3
p_1	50	50	90
p_2	50	60	80
p_3	50	70	70
Output			
q_1	269	979	267
q_2	99	415	99
q_3	1132	105	1133
$\sum q_i$	1500	1499	1499
Total costs	8850	9186	118835
t_1	0.1571	0.5498	0.1506
t_2	0.2013	0.7612	0.2007
t_3	0.7403	0.8113	0.7406

Table 1: Results for three runs for a set of data.

Case 6: $\nu_1 < 0, \nu_2, \nu_3 > 0$

$$C = C_1^s + C_2 + C_3 + H_3' \, , \tag{2.19}$$

Case 7: $\nu_1 > 0, \nu_2, \nu_3 < 0$

$$C = C_1 + C_2^s + C_3^s \, , \tag{2.20}$$

Case 8: $\nu_1, \nu_2, \nu_3 > 0$

$$C = C_1 + C_2 + C_3 + H_3' \, . \tag{2.21}$$

The model is elaborated further in Ali and Masinga (2004), where a detailed breakdown of the computation is given.

3 Conclusions and recommendations

The model was tried for the case $n = 3$, using some data previously used for other models for different values of the unit price in different intervals. Some data taken from Yong-Wu (2004), with $F = 500$, $h = 10$, $s = 15$, $c = 4$ and $\mu_1 = 1780$, $\mu_2 = 1970$, $\mu_3 = 2100$, $D = 1500$ were used. The results obtained for different runs are shown in Table 1.

The results show that the model adapts well to variations in the mean periodic demand μ_i and the respective prices p_i. For example, we can deduce from Run 1, where the price is fixed, that the optimal strategy would be to leave the highest order quantity to the last, which is in line with the fact that the mean demand is highest in cycle three. In Run 2 mean demand values are the same, and the price increases from cycle to cycle. The optimal order quantities are $q_1 = 979$, $q_2 = 415$, $q_3 = 105$. Thus when the price is highest, the order quantity is lowest.

With such results, it seems possible that the model can be modified to accommodate the case of quantity discounts. This and other such considerations are presented in Ali and Masinga (2004).

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References

Ali, M.M. and Masinga, L.C. (2004). A nonlinear optimisation model with stochastic demand for optimal order quantities with price change. *In preparation*

Huang, W. and Kulkarni, V.G. (2003). Optimal EOQ for announced price increases in infinite horizon. *Operations Research* **51**, 336-339.

Naddor, E. (1966). *Inventory Systems,* John Wiley, New York.

Taha,H.A. (1992). *Operations research: An Introduction,* MacMillan Publishing Co., New York.

Taylor, S.G. and Bradley, C.E. (1985). Optimal ordering strategies for announced price increases. *Operations Research,* **33**, 312-325.

Tersine, R.J. (1982). *Principles of Inventory and Materials Management,* 2nd ed., North Holland, New York.

Winston, W.L. (1997). *Operations Research: Applications and Algorithms.* 3rd ed., Pacific Grove, Duxbury.

Yong-Wu, Z., Hon-Shiang, L. and Shan-Lin, Y. (2004). A finite horizon lot-sizing problem with time-varying deterministic demand and waiting-time-dependent partial backlogging. *International Journal of Production Economics,* **91**, 109-119.