RICE UNIVERSITY

Locally Mass-Conservative Method With Discontinuous Galerkin In Time For Solving Miscible Displacement Equations Under Low Regularity

by

Jizhou Li

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE
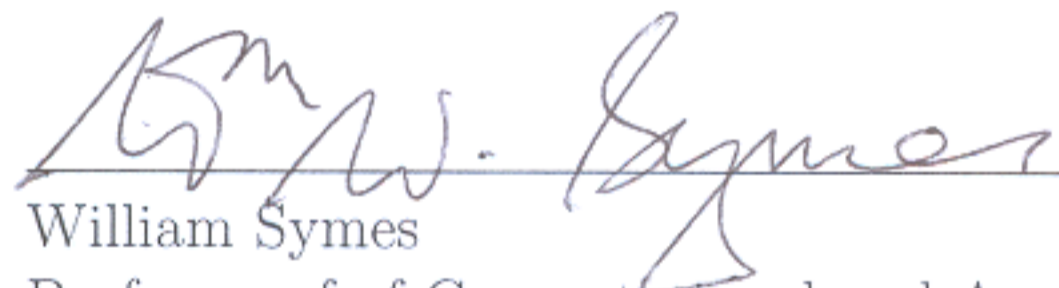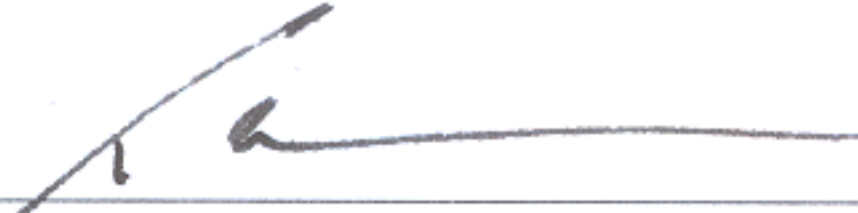
Master of Arts

APPROVED, THESIS COMMITTEE:

Béatrice Rivière, Chair
Associate Professor of Computational and
Applied Mathematics

Matthias Heinkenschloss
Professor of of Computational and Applied
Mathematics

William Symes
Professor of of Computational and Applied
Mathematics

Timothy Warburton
Associate Professor of Computational and
Applied Mathematics

Houston, Texas

March, 2013

Rice University

Locally Mass-Conservative Method With Discontinuous Galerkin In Time For Solving Miscible Displacement Equations Under Low Regularity

by

Jizhou Li

A Thesis Submitted
in Partial Fulfillment of the
Requirements for the Degree

Master of Arts

Approved, Thesis Committee:

Béatrice Rivière, Chair
Associate Professor of Computational and
Applied Mathematics

Matthias Heinkenschloss
Professor of of Computational and Applied
Mathematics

William Symes
Professor of of Computational and Applied
Mathematics

Timothy Warburton
Associate Professor of Computational and
Applied Mathematics

Houston, Texas

March, 2013

ABSTRACT

Locally Mass-Conservative Method With Discontinuous Galerkin In Time For Solving
Miscible Displacement Equations Under Low Regularity

by

Jizhou Li

The miscible displacement equations provide the mathematical model for simulating the displacement of a mixture of oil and miscible fluid in underground reservoirs during the Enhance Oil Recovery(EOR) process. In this thesis, I propose a stable numerical scheme combining a mixed finite element method and space-time discontinuous Galerkin method for solving miscible displacement equations under low regularity assumption. Convergence of the discrete solution is investigated using a compactness theorem for functions that are discontinuous in space and time. Numerical experiments illustrate that the rate of convergence is improved by using a high order time stepping method. For petroleum engineers, it is essential to compute finely detailed fluid profiles in order to design efficient recovery procedure thereby increase production in the EOR process. The method I propose takes advantage of both high order time approximation and discontinuous Galerkin method in space and is capable of providing accurate numerical solutions to assist in increasing the production rate of the miscible displacement oil recovery process.

Acknowledgments

Above all, I give thank to my heavenly Father for His divine countenance and sovereign hand that providentially guide and sustain; to the Son who continually makes intercession on my behalf daily; and to the Spirit who imparts, sanctifies, renews and brings me under a firm religious commitment and conviction.

I dedicate this thesis to my beloved parents who devoted 24 years of their lives into my up-bringing. Their supports, encouragements, prayers and disciplines will forever be inspirational in my life.

It is my greatest privilege to work with under Béatrice Rivière. Her insight on the subject always dazzles me and leads me to the right direction. Her patience and kindness are particularly encouraging when facing difficulty. I also benefit a lot from all the wonderful opportunities she has so graciously provided. For that I am forever grateful.

Furthermore, I would like to thank my master committee: Matthias Heinkenschloss, William Symes and Timothy Warburton for carrying out a laborious tasks of carefully reading through my thesis and their insightful comments and critics. I would also like to thank Jan Hewitt for her helps from the thesis writing class.

Outside Rice community, I want to first thank Noel Walkington from Carnegie Mellon University for his tremendous aid on providing theoretical insight and rigorous feedback for my thesis. Also, this thesis cannot come to pass without the helps from a group of enthusiastic engineers from the other side of Atlantic. I have been given a wonderful opportunity to work at Department of Hydromechanics and Modelling of Hydrosystems in University of Stuttgart, Germany last summer. In particular, I would like to express my gratitude to Rainer Helmig for patiently explaining to me the physics and motivation of the subject. Also, to Bernd Flemisch for pointing me to the right software package for the numerical implementation. And many thanks to Christoph Grüninger and Nicolas Schwenck for their instructions on using DUNE-PDELab. Their helps are truly overwhelming.

Contents

Illustrations

Tables

Chapter 1

Introduction

According to a report [27] in 2007, 58% of the entire original oil reserved in U.S. is trapped in discovered reservoirs, but was unrecoverable by current technology. With this staggering percentage and the increasing demand of energy, engineers have designed Enhanced Oil Recovery (EOR) techniques after the secondary recovery process. EOR seeks to alter the properties of reservoir and the remaining oil including the pressure and fluid displacement. One of the most important EOR techniques is called the miscible displacement technique.

Original Oil in Place (U.S.): 649 Billion Barrels

Discovered Unrecoverable by Current Technology: 337 Billion Barrels
Cumulative Production: 183 Billion Barrels
Proved Reserves: 22 Billion Barrels
Undiscovered Recoverable by Current Technology: 30 Billion Barrels
Undiscovered Unrecoveravle by Current Technology: 37 Billion Barrels
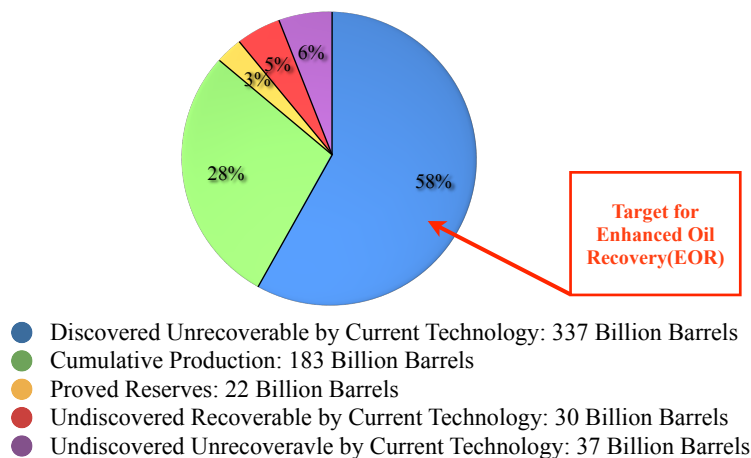
Figure 1.0.1 : original U.S. oil reserves

During the miscible displacement recovery process, instead of using water, a solvent is injected into the reservoir to mix with the remaining oil, and eventually the fluid mixture is forced out of the reservoir. This process is governed by a system of non-linear partial differential equations (PDE) called the miscible displacement equations. Therefore, providing high resolution numerical simulations by solving the PDE is essential for engineers to design

and optimize the recovery strategy. One of the biggest challenges involves incorporating highly sophisticated reservoir formations and fluid properties into the design of the numerical simulation tools.

The contribution of my thesis is a novel numerical method to solve the miscible displacement equations that is locally mass-conservative and high order in space. Also, the numerical method uses discontinuous Galerkin in time which allows arbitrary order of approximation in time.

An outline of the thesis is as follows. First, I give a literature review for the miscible displacement problem and related numerical methods. In chapter 3, I introduce the mathematical model and numerical scheme for solving the problem. Afterwards in chapter 4, I dive into the theoretical analysis of the numerical method and prove stability and convergence of the numerical solutions. In chapter 5, numerical examples both for analytical and physical problems are presented. Finally, I draw conclusion in the last chapter and propose a full discontinuous Galerkin method in time and space for the miscible displacement problem.

Chapter 2

Literature Review

Over the last five decades, the miscible displacement problem has gained enormous attention in the fields of science and engineering, in particular petroleum engineering, environmental science, hydrology, and geophysics. Hundreds of papers have been published on the subject with interests ranging from physical principles, mathematical analysis, experimental results and economics.

The quantitative studies of the miscible displacement process depend on the mathematical model which has been derived in [5, 6, 20, 38, 45]. The derived mathematical model is a system of coupled nonlinear partial differential equations describing the displacement of a fluid mixture in porous media. The coupled system consists of an elliptic equation for the pressure and a convection-dominant convection-diffusion equation for the concentration of the solvent. There are still many open questions concerning the solutions and the well-posedness of the miscible displacement equations. The coupled system of equations is of great theoretical interest itself and extremely challenging to provide analytic solutions.

Numerical simulations, on the other hand, provide insight and offer a systematic way to study the miscible displacement equations. On one hand the miscible displacement processes guarantee virtually complete recovery, but on the other hand the miscible fluids are in general more expensive than oil. Hence, the oil production must exceed far more than the injected miscible fluids to assure profitability. Driven by this economical consideration, one of the main concerns from numerical perspective is to provide accurate numerical approximations to the physical problems with real-world parameters where the analytic solutions generally are unknown. And as a consequence, the miscible displacement equations have been extensively

investigated by numerical analysts over the last fifty years.

As early as 1962, Peaceman and Rachford introduce the mathematical formulation based on the source-sink approach, and also propose a finite difference method for solving the equations [39]. For the detail of this mathematical formulation proposed by Peaceman and Rachford, one can refer to [55]. Soon after the use of the finite difference method for solving the problem, Garder et al introduce the method of the characteristics [29] for the miscible displacement simulations. In 1971, based on the result in [39], Chaudhari propose an improved high-order finite difference method eliminating most of the numerical smearing in previous cases. In the next decade, in the engineering community, the numerical simulations of the miscible displacement are mostly done by using finite difference approach. This approach, however, might not be accurate, especially when working with real-world parameters that are heterogeneous such as permeability and porosity. A significant amount of numerical diffusion is often observed.

Until 1980, Ewing and Wheeler propose a finite element method to handle more complex geometry and to better approximate solutions that lack certain regularity [26]. Following the Ewing and Wheeler's analysis, these authors and Darlow introduce the mixed finite element method for solving the pressure equation [17]. In their analysis, they show that the mixed finite element method is able to produce very accurate Darcy's velocity. In addition, this work shows that by solving the pressure equation in one term reduces the difficulty of differentiation comparing to the traditional finite difference method. The concentration equation, however, is still solved by using the finite element method. But, due to the convection-dominant nature of the problem and that the conforming finite element is not mass-conservative, global oscillations will occur in the numerical solutions if no stabilization technique is applied. A stabling technique for the finite element method is introduced later by Wei [56] to reduce the nonphysical oscillations caused by using the finite element method. In this result the author uses discontinuous Galkerin in time. However, no numerical examples

are presented to illustrate the reduction of the oscillations.

The years between 1980 and 1990, the study of the miscible displacement equations from numerical perspective mainly dwells on the methodologies and the related error estimation. Methods such as finite element method, mixed finite element method, method of characteristics, collocation method, the combinations thereof and their variations, have been introduced in [46, 19, 51, 21, 24, 25, 18, 57].

In the next decade, the efficient implementations of those methods become one of the main concerns. On one hand, using the mixed finite element method for the Darcy's law, one can obtain very accurate approximations for the pressure and velocity of the same order unlike the classical finite element method where the velocity is one order lower. Yet, on the other hand, the linear system becomes indefinite which poses a big challenge to solve for the iterative solver for large-scale simulation. Yang et al [58, 35] propose methods to simplify computation by replacing it with an iterative process. They also show that the number of the iterations is small. For the concentration equation, it is solved by using the method of the characteristics which is not mass-conservative. The parallel implementations are done by Coutinho et al both in shared memory machines [15] and distributed memory machines [36]. Both pressure-velocity and concentration equations are solved by finite element method with post processing procedure to enhance the stability and accuracy.

Around the beginning of this century, discontinuous Galerkin (DG) method have gained a renewed interest for providing numerical solutions for the partial differential equations, largely due to the advancement on high performance computing and the highly-parallelizable nature of the method. In particular for the miscible displacement equations, Rivière and Wheeler [40, 42] conduct numerical experiments and show that DG is well-suited for the problem because of the local mass conservation property of the method, the ability to handle unstructured grids and to capture fluid instability. The quality of the numerical solutions from the results of those two authors suggests DG is a good alternative for space discretization

for the miscible displacement problem comparing with methods such as finite element method and Godunov method. Following the numerical experiments, convergence and stability have been shown by Epshteyn and Rivière for a fully discrete DG scheme they introduce [23]. While developing efficient and accurate solutions to this real-world problem, we also like to maintain a solid theoretical base. Yet, one of the major drawbacks in the analysis of convergence and stability for the numerical methods mentioned so far is the assumption that the diffusion/dispersion tensor is uniformly bounded above in L^∞. However, from a problem formulation point of view, there is no theoretical guarantee for this condition to hold because the fluid velocity might not be bounded and in fact one can construct such a problem [3]. This condition is known as the low regularity condition.

Under low regularity assumption, Sun, Rivière and Wheeler [50] introduce a stable numerical scheme with mixed method and DG in space using a "cut-off" operator. An error bound is derived to show the convergence of the numerical solutions to the strong solutions whose existence is still unknown. The weak solution, on the other hand, is proven to exist by Feng [28] in 2D and extend to 3D by Chen and Ewing [13]. This theoretical result, therefore, gives grounds for us to approximate the weak solutions with methods such as finite element and DG, though the weak solutions might not even be unique under low regularity assumption. The work done by Bartels, Jensen and Müller [3] establishes the convergence and stability of the numerical solutions to the weak solutions with mixed method and DG in space. The Aubin-Lions compactness theorem is used in this case to prove the convergence since under low regularity condition one cannot obtain the error estimators between the exact and numerical solutions. In their analysis, they bypass the difficulty of the unboundness of the diffusion/dispersion tensor by using its L^2 projection in their numerical scheme which enables them show stability and convergence of the solutions. The DG form for the convection term has been modified into a skew symmetric form as opposed to the upwind DG convection operator in order to prove the coercivity of the bilinear form. Again,

their numerical results demonstrate the advantage of using the DG method by showing the robustness of the DG solutions in L-shaped domain with a singularity point. Different from scheme in [50], no "cut-off" operator is required in this case. Nevertheless, their resulting DG discretization only addresses symmetric interior penalty Galerkin method (SIPG).

There are many advantages for combining the mixed finite element method and DG in space. Using the mixed finite element method, one can obtain fluid pressure and flux at the same time and it is more accurate than methods such as finite difference and finite volume; the method is locally mass-conservative; and is capable of handling discontinuous parameter while producing the flux that is continuous between the interface of two neighboring elements [37, 22]. DG is also locally mass-conservative and is known for its flexibility and higher order approximation [32]. Apart from the incompressible miscible displacement equations, the method is also used to solve two-phase incompressible immiscible fluid flow problem [12], convection-diffusion problem [48], incompressible single-phase flow in porous media [10], single-phase flow of compressible and multicomponent fluid in fractured media [31], and two-phase compressible multicomponent fluid flow in porous media [32], compressible miscible displacement equations [16].

While the discretization in space is under on-going analysis, the time stepping method has often been overlooked. Rivière and Walkington [41] propose a scheme with mixed finite method for the pressure and finite element method in space and DG in time for the concentration equation. They prove stability and convergence of the scheme up to arbitrary order of approximation in time. They establish a generalized compactness theorem to show the convergence that allows the approximation using the discontinuous functions both in space and time. For the low order DG in time, stability and convergence is proved by obtaining the exact integral over time. Radau quadrature is used for proving the stability and convergence in time for higher order approximation in time.

Following up the analysis done by Bartels, Jensen and Müller, Jensen and Müller intro-

duced a stable second-order Crank-Nicolson time approximation [33] within the context of the scheme they have derived earlier [3]. Most of their analysis is built upon the existing results in [3]. Their result is the highest order time stepping until now with regard to DG discretization in space. They observe a second-order convergence in time from their numerical experiment.

Motivated by the results from [3, 33, 41], I intend to solve the concentration for transport equation using a space-time discontinuous Galerkin method which not only allows arbitrary order of approximation in space, but also arbitrary order of approximation in time. The space-time discontinuous Galerkin method itself has attracted considerable attention recently due to its inherit nature for handling hp-adaptation. It seeks to localize the problem that results in a local conservative and highly parallelizable method which is a very important numerical method for today's numerical simulation in science and engineering. The method is used by Van der Vegt and Van der Ven [52, 53] for problem concerning the inviscid flow in 2002. Soon, this method started to be implemented and analysed for problems such as compressible Navier-Stokes equations [34], convection-diffusion equation [49], shallow water equation [2]. Yet, problems concerning porous media flow like miscible displacement equations have not been addressed using space-time DG.

This thesis aims at developing and analyzing a numerical method using the mixed finite element method for the pressure equation and space-time discontinuous Galerkin method for the concentration equation. My analysis avoids projecting the diffusion/dispersion tensor onto polynomial space as what has been done in [3, 33]. Apart from SIPG, I also will address NIPG and IIPG discretization in the thesis. For the new higher order discretization in time, Radau quadrature will no longer be required. Stability and convergence of the numerical solutions to the weak solutions will be covered. In the following section, I will start by introducing the weak formulation and the numerical scheme for the problem.

Chapter 3

Mathematical Model

3.1 Introduction

In this chapter, I will first introduce the mathematical model for the miscible displacement.
Then I will move on to the numerical scheme and to show its consistency. Consider the
miscible displacement equation in a porous medium Ω modelling the displacement of oil
in underground reservoirs by mixing fluids with oil over the time interval $[0, T]$. With the
assumption of incompressibility, we need to determine the pressure p, velocity $\mathbf{u}$, and the
concentration c satisfy:

$$\phi \partial_t c - \mathrm{div}(\mathbb{D}(\mathbf{u})\nabla c) + \mathbf{u} \cdot \nabla c + q^I c = \hat{c} q^I, \ \text{in } \Omega \times [0, T] \tag{3.1}$$

$$\mathrm{div}(\mathbf{u}) = q^I - q^P, \ \text{in } \Omega \times [0, T] \tag{3.2}$$

$$\mathbf{u} = -\mathbb{K}(x, c)(\nabla p - \rho(c)\mathbf{g}), \ \text{in } \Omega \times [0, T] \tag{3.3}$$

with the boundary conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0, \ \mathbb{D}(\mathbf{u})\nabla c \cdot \mathbf{n} = 0, \ \text{on } \partial\Omega \times [0, T]$$

and initial condition:

$$c(\cdot, 0) = c_0, \ \text{in } \Omega.$$

The coefficients of the PDEs are: ϕ is the porosity of the porous medium; $\mathbb{K}(x, c) = \dfrac{\mathcal{K}(x)}{\mu(c)}$
where $\mathcal{K}(x)$ is the absolute permeability of the porous media and $\mu(c)$ is the viscosity of the

fluid; ρ is the density of the fluid mixture; the constant vector g describes the gravity; $\mathbb{D}$ is the diffusion dispersion coefficient; c_0 and $\hat{c}$ are initial and injected concentration respectively; And last, $q^I, q^P \geq 0$ are the injection source and production sinks.

We shall have following assumptions on the input data:

- $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$, is a bounded Lipschitz domain.

- $\mathbb{K} : \Omega \times \mathbb{R} \to \mathbb{R}^{d \times d}$ is symmetric, Carathéodory (measurable in first argument and continuous almost everywhere in the second), uniformly bounded and elliptic. And there exist constants $0 < k_0 < k_1$ such that

$$k_0 \left|\xi\right|^2 \leq \xi^T \mathbb{K}(x, c)\xi \leq k_1 \left|\xi\right|^2 , \quad \xi \in \mathbb{R}^d, \ \forall x, c \in \mathbb{R}^d \times \mathbb{R}$$

- $\mathbb{D} : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is symmetric, Lipschitz continuous. There exist constants $0 < d_0 < d_1$ such that

$$d_0(1 + |\mathbf{u}|) \left|\xi\right|^2 \leq \xi^T \mathbb{D}(\mathbf{u})\xi \leq d_1(1 + |\mathbf{u}|) \left|\xi\right|^2 , \quad \mathbf{u}, \xi \in \mathbb{R}^d \tag{3.4}$$

We note that $\mathbb{D}(\mathbf{u})$ is not assumed to be bounded.

- $\phi \in L^\infty(\Omega)$ and $\phi_0 < \phi < \phi_1$ for some positive constants ϕ_0, ϕ_1.

- $q^I, q^P \in L^\infty(0, T, L^2(\Omega))$ with $q^I, q^P \geq 0$ and $\int_\Omega q^I(x, t) = \int_\Omega q^P(x, t)$ for $t \in [0, T]$.

- There exist positive constants ρ_0, ρ_1 such that the function $\rho : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $\rho_0 \leq \rho \leq \rho_1$.

3.2 Discretization in Time and Space

Set

$$H_0(\Omega, \mathrm{div}) = \{\mathbf{v} \in L^2(\Omega)^d : \mathrm{div}(\mathbf{v}) \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ in } H^{-1/2}(\partial\Omega)\}$$

and

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega qdx = 0\}$$

We denote the L^2 inner-product on Ω by $(\cdot, \cdot)$.

The weak formulation of the problem is as follows:

We need to find the triple $(\mathbf{u}, p, c) \in L^\infty[0, T; H_0(\Omega, \text{div})] \times L^\infty[0, T; L_0^2(\Omega)] \times L^2[0, T; H^1(\Omega)]$

such that

$$\int_0^T (\mathbb{K}^{-1}(c)\mathbf{u}, \mathbf{v}) - (p, \text{div}(\mathbf{v})) = \int_0^T (\rho(c)\mathbf{g}, \mathbf{v}) \tag{3.5}$$

$$\int_0^T (q, \text{div}(\mathbf{u})) = \int_0^T (q^I - q^P, q) \tag{3.6}$$

for all $(\mathbf{v}, q) \in L^1[0, T; H(\Omega, \text{div})] \times L^1[0, T; L_0^2(\Omega)]$ and

$$\int_0^T -(\phi c, \partial_t w) + (\mathbb{D}(\mathbf{u})\nabla c, \nabla w) + (\mathbf{u} \cdot \nabla c, w) + (q^I c, w) - (\phi c_0, w(0)) - (\hat{c}q^I, w) = 0 \tag{3.7}$$

for all $w \in \{w \in L^4[0, T; W^{1,4}(\Omega)] \cap H^1[0, T; H^1(\Omega)'] : w(T) = 0\}$

The requirement that $w \in L^4[0, T; W^{1,4}(\Omega)]$ is needed since $\mathbb{D}(\mathbf{u})$ is not known to be bounded which is also the major challenge when solving the equation. We know from [14], [28] the weak solutions $(\mathbf{u}, p, c) \in L^\infty[0, T, H_0(\Omega, \text{div})] \times L^\infty[0, T, L_0^2(\Omega)] \times L^2[0, T, H^1(\Omega)]$ exist with $\mathbb{D}(\mathbf{u})^{1/2}\nabla c \in L^2[0, T; L^2(\Omega)]$.

We use mixed finite element method for the first two equations and discontinuous Galerkin in both time and space for solving the last equation. Thus, we let $\{\mathcal{E}_h\}_{\{h>0\}}$ be a family of regular mesh of Ω and $\{\Gamma_h\}_{\{h>0\}}$ be the corresponding interior edges. Define the Raviart-Thomas space

$$RT_k(\mathcal{E}_h) = \{\mathbf{u} \in H(\Omega; \text{div}) \mid \mathbf{u}|_E \in (\mathcal{P}_k(E))^d + x\mathcal{P}_k(E), E \in \mathcal{E}_h\},$$

where $\mathcal{P}_k(E)$ is the set of all polynomials of degree less or equal to k over the element E. One should notice that the mixed finite element is not restricted to Raviart-Thomas space. Any classical mixed finite element space such as $BDM_k(\mathcal{E}_h)$ and $BDFM_k(\mathcal{E}_h)$ for the spacial discretization will suffice. Then we define the related finite element subspaces:

$$\mathbf{U}_h = RT_k(\mathcal{E}_h)$$

$$P_h = \{q_h \in L^2(\Omega) : q_h|_E \in \mathcal{P}_k(E), E \in \mathcal{E}_h\}$$

$$C_h = \{c_h \in H^1(\mathcal{E}_h) : c_h|_E \in \mathcal{P}_\ell(E), E \in \mathcal{E}_h\}$$

where $H^1(\mathcal{E}_h) = \{c \in L^2(\Omega) : c|_E \in H^1(E), E \in \mathcal{E}_h\}$ is the H^1 broken Sobolev space. Before we introduce the numerical scheme we define our notation: Let e denote the face between two elements. We fix a normal vector $\mathbf{n}_e$, and we let E_1^e and E_2^e denote two neighboring elements sharing the face e. If $\mathbf{n}_e$ is oriented from E_1^e to E_2^e, then

$$v_+^n = \lim_{\epsilon \downarrow 0} v(\cdot, t_n + \epsilon), \;\; v_-^n = \lim_{\epsilon \downarrow 0} v(\cdot, t_n - \epsilon), \;\; [v^n]_t = v_+^n - v_-^n$$

$$\{v\} = \frac{v|_{E_1^e} + v|_{E_2^e}}{2}, \;\; \text{and} \;\; [v] = v|_{E_1^e} - v|_{E_2^e}$$

Vice versa for if the normal vector $\mathbf{n}_e$ is pointing from E_2^e to E_1^e.

We derive the numerical scheme as follows:

$$\int_{t_{n-1}}^{t_n} \left((\mathbb{K}^{-1}(c_h)\mathbf{u}_h, \mathbf{v}_h) - (p_h, \text{div}(\mathbf{v}_h)) \right) = \int_{t_{n-1}}^{t_n} (\rho(c_h)\mathbf{g}, \mathbf{v}_h) \tag{3.8}$$

$$\int_{t_{n-1}}^{t_n} (q_h, \text{div}(\mathbf{u}_h)) = \int_{t_{n-1}}^{t_n} ((q^I - q^P), q_h) \tag{3.9}$$

$$\int_{t_{n-1}}^{t_n} \left((\phi\partial_t c_h, w_h) + B_d(c_h, w_h; \mathbf{u}_h) + B_{cq}(c_h, w_h; \mathbf{u}_h) \right) + \left([c_h^{n-1}]_t, \phi w_{h+}^{n-1} \right) = \int_{t_{n-1}}^{t_n} (\hat{c}q^I, w_h)$$

$$\tag{3.10}$$

for all $\mathbf{v}_h \in \mathcal{P}_\ell[t^{n-1}, t^n; \mathbf{U}_h], q_h \in \mathcal{P}_\ell[t^{n-1}, t^n; P_h], w_h \in P_\ell[t^{n-1}, t^n; C_h].$

The form B_d is the discretization of the operator $-\nabla \cdot (\mathbb{D}(\mathbf{u})\nabla c)$:

$$B_d(c_h, w_h; \mathbf{u}_h) = (\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h)_\mathcal{E} - ([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_{\Gamma_h}$$

$$+\epsilon([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} + (\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h} \quad (3.11)$$

We should recognize that $\{|\mathbf{u}_h|\} = \frac{1}{2}\left(\left|\mathbf{u}_h^+\right| + \left|\mathbf{u}_h^-\right|\right).$

The form B_{cq} is the discretization of the operator $-\mathbf{u} \cdot \nabla c + q^I c$:

$$B_{cq}(c_h, w_h; \mathbf{u}_h) = \frac{1}{2}\left((\mathbf{u}_h\nabla c_h, w_h)_{\mathcal{E}_h} - (\mathbf{u}c_h, \nabla w_h)_{\mathcal{E}_h} + ((q^I + q^P)c_h, w_h)\right.$$

$$\left.+(c_h^{\mathrm{up}}\mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_{\Gamma_h} - (w_h^{\mathrm{down}}\mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h}\right) \quad (3.12)$$

with

$$\mathbf{u}_h \in \mathcal{P}_{\ell_1}[t_{n-1}, t_n; \mathbf{U}_h], p_h \in \mathcal{P}_{\ell_1}[t_{n-1}, t_n; P_h], c_h \in P_{\ell_2}[t_{n-1}, t_n; C_h]$$

Note, the problem is independent of the choice of normal vector $\mathbf{n}_e$ on the edges.

For the spacial discretization, we use the mixed finite element method and DG to maintain the mass-conservation. Also, the discretization enables us to obtain arbitrary order of approximation.

3.3 Consistency of the Numerical Scheme

We now give a more detailed analysis concerning the numerical scheme and its equivalence to the weak formulation of the problem.

3.3.1 Darcy's Law with Mixed Finite Element Method

We begin by examining the Darcy's Law:

$$\text{div}(\mathbf{u}) = q^I - q^P$$

$$\mathbf{u} = -\mathbb{K}(c)(\nabla p - \rho(c)\mathbf{g})$$

in which case we will rewrite as:

$$\text{div}(\mathbf{u}) = q^I - q^P$$

$$\mathbb{K}^{-1}(c)\mathbf{u} + \nabla p = \rho(c)\mathbf{g}$$

By integration over Ω, we have:

$$\int_\Omega \text{div}(\mathbf{u})q = \int_\Omega (q^I - q^P)q$$

$$\int_\Omega \mathbb{K}^{-1}(c)\mathbf{u}\cdot\mathbf{v} + \int_\Omega \nabla p\cdot\mathbf{v} = \int_\Omega \rho(c)\mathbf{g}\cdot\mathbf{v}$$

for all $\mathbf{v} \in H_0(\Omega, \text{div})$ and $q \in L_0^2(\Omega)$.

We note that according to Green's first identity:

$$\int_\Omega \nabla p\cdot\mathbf{v} = -\int_\Omega p\,\text{div}(\mathbf{v}) + \int_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n} = -\int_\Omega p\,\text{div}(\mathbf{v})$$

The desired weak form (3.5)-(3.6) is obtained

$$\int_\Omega \text{div}(\mathbf{u})q = \int_\Omega (q^I - q^P)q$$

$$\int_\Omega \mathbb{K}^{-1}(c)\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\,\text{div}(\mathbf{v}) = \int_\Omega \rho(c)\mathbf{g}\cdot\mathbf{v}$$

Once we pass in the piecewise polynomials from the finite element space we have the de-

sired spacial discretization for the Darcy's law. Next, we will show the consistency of the discretization of the transport equation.

3.3.2 Transport Equation with DG Method

For the left-hand side of the transport equation, we divide our analysis into two parts. The diffusion part $-\mathrm{div}(\mathbb{D}(\mathbf{u})\nabla c)$, and convection part $-\mathbf{u}\cdot\nabla c + q^I c$.

Diffusion Term

For all $w \in W^{1,4}(\Omega)$, we have according to the Green's theorem over each element E

$$-\int_E \mathrm{div}(\mathbb{D}(\mathbf{u})\nabla c)w = \int_E \mathbb{D}(\mathbf{u})\nabla c \cdot \nabla w - \int_{\partial E} \mathbb{D}(\mathbf{u})\nabla c \cdot \mathbf{n}_E w$$

Note, $\mathbf{n}_E$ is the outward normal vector of element E.

And if we sum up over all the elements we have,

$$-\sum_{E\in\mathcal{E}_h}\int_E \mathrm{div}(\mathbb{D}(\mathbf{u})\nabla c)w = \sum_{E\in\mathcal{E}_h}\int_E \mathbb{D}(\mathbf{u})\nabla c \cdot \nabla w - \sum_{E\in\mathcal{E}_h}\int_{\partial E} \mathbb{D}(\mathbf{u})\nabla c \cdot \mathbf{n}_E w$$

We can switch the last integral term to the sum over all interior edges with

$$\sum_{E\in\mathcal{E}_h}\int_{\partial E} \mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_E w = \sum_{e\in\Gamma_h}\int_e [\mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_e w] + \sum_{e\in\partial\Omega}\int_e \mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_e w = \sum_{e\in\Gamma_h}\int_e [\mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_e w]$$

given the boundary condition $\mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n} = 0$ on $\partial\Omega$. By the regularity of the solution c, we have $\mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_e = \{\mathbb{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_e\}$ a.e. over the edges. Therefore, the diffusion term can be rewritten as

$$\sum_{E\in\mathcal{E}_h}\int_E \mathbb{D}(\mathbf{u})\nabla c \cdot \nabla w - \sum_{e\in\Gamma_h}\int_e \{\mathbb{D}(\mathbf{u})\nabla c \cdot \mathbf{n}_e\}[w]$$

Also, note that the jump $[c] = 0$ a.e. on the interior edges. Therefore, we can add the terms

$$\epsilon \sum_{e \in \Gamma_h} \int_e \{\mathbb{D}(\mathbf{u}) \nabla w \cdot \mathbf{n}_e\}[c] \text{ and } \sum_{e \in \Gamma_h} \sigma h^{-1} \int_e (1 + \{|\mathbf{u}|\})[c][w]$$

So, we have the desired DG form for the diffusion operator

$$B_d(c, w; \mathbf{u}) = \sum_{E \in \mathcal{E}_h} \int_E \mathbb{D}(\mathbf{u}) \nabla c \cdot \nabla w - \sum_{e \in \Gamma_h} \int_e \{\mathbb{D}(\mathbf{u}) \nabla c \cdot \mathbf{n}_e\}[w]$$

$$+ \epsilon \sum_{e \in \Gamma_h} \int_e \{\mathbb{D}(\mathbf{u}) \nabla w \cdot \mathbf{n}_e\}[c] + \sum_{e \in \Gamma_h} \sigma h^{-1} \int_e (1 + \{|\mathbf{u}|\})[c][w]$$

$$= (\mathbb{D}(\mathbf{u}) \nabla c, \nabla w)_{\mathcal{E}_h} - (\{\mathbb{D}(\mathbf{u}) \nabla c \cdot \mathbf{n}_e\}, [w])_{\Gamma_h}$$

$$+ \epsilon(\{\mathbb{D}(\mathbf{u}) \nabla w \cdot \mathbf{n}_e\}, [c])_{\Gamma_h} + (\sigma h^{-1}(1 + \{|\mathbf{u}|\})[c], [w])_{\Gamma_h}$$

Hence, we can obtain the discretization of the diffusion term by passing in the piecewise polynomials from the finite element space. Next, we will show the consistency of the convection term.

Convection Term

We use a skew symmetric weak formulation for the convection in our numerical scheme. Nevertheless, the reader will soon notice this is nothing but upwind scheme with a stabilization term. Before we use the Green's theorem, consider to rewrite the convection term as

$$\mathbf{u} \cdot \nabla c = \tfrac{1}{2}\mathbf{u} \cdot \nabla c + \tfrac{1}{2}\mathbf{u} \cdot \nabla c$$

$$= \frac{1}{2}\mathbf{u} \cdot \nabla c + \frac{1}{2}\text{div}(\mathbf{u}c) - \frac{1}{2}\text{div}(\mathbf{u})c$$

$$= \frac{1}{2}(\mathbf{u} \cdot \nabla c + \text{div}(\mathbf{u}c) - (q^I - q^P)c)$$

Thus, we have

$$\mathbf{u} \cdot \nabla c + q^I c = \frac{1}{2}(\mathbf{u} \cdot \nabla c + \text{div}(\mathbf{u}c) + (q^I + q^P)c)$$

Now, we use the Green's theorem over each element

$$
\begin{aligned}
\int_E \left(\mathbf{u} \cdot \nabla c \, w + q^I c \, w \right) &= \frac{1}{2} \left(\int_E \mathbf{u} \cdot \nabla c \, w + \int_E \mathrm{div}(\mathbf{u}c) \, w + \int_E (q^I + q^P) c \, w \right) \\
&= \frac{1}{2} \left(\int_E \mathbf{u} \cdot \nabla c \, w - \int_E \mathbf{u}c \cdot \nabla w + \int_E (q^I + q^P) c \, w + \int_{\partial E} c\mathbf{u} \cdot \mathbf{n}_E w \right)
\end{aligned}
$$

where $\mathbf{n}_E$ is the outward normal vector of element E. We can sum up the term within the parentheses over all the elements from the equation above. Since $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we have

$$
\begin{aligned}
&\sum_{E \in \mathcal{E}_h} \int_E \mathbf{u} \cdot \nabla c \, w - \sum_{E \in \mathcal{E}_h} \int_E \mathbf{u}c \cdot \nabla w + \sum_{E \in \mathcal{E}_h} \int_E (q^I + q^P) c \, w + \sum_{e \in \Gamma_h} \int_e [c\mathbf{u} \cdot \mathbf{n}_e w] \\
&= \sum_{E \in \mathcal{E}_h} \int_E \mathbf{u} \cdot \nabla c \, w - \sum_{E \in \mathcal{E}_h} \int_E \mathbf{u}c \cdot \nabla w + \sum_{E \in \mathcal{E}_h} \int_E (q^I + q^P) c \, w + \sum_{e \in \Gamma_h} \int_e c^{\mathrm{up}} \mathbf{u} \cdot \mathbf{n}_e [w]
\end{aligned}
$$

with

$$
c^{\mathrm{up}} = \begin{cases} c|_{E_1^e} & \text{if } \mathbf{u} \cdot \mathbf{n}_e \geq 0 \\[2mm] c|_{E_2^e} & \text{if } \mathbf{u} \cdot \mathbf{n}_e < 0 \end{cases}
$$

and we define c^{down} as the opposite of c^{up} i.e.

$$
c^{\mathrm{down}} = \begin{cases} c|_{E_2^e} & \text{if } \mathbf{u} \cdot \mathbf{n}_e \geq 0 \\[2mm] c|_{E_1^e} & \text{if } \mathbf{u} \cdot \mathbf{n}_e < 0 \end{cases}
$$

Furthermore, since $[c] = 0$ a.e., we can add the stabilization term $\sum_{e \in \Gamma_h} \int_e w^{\mathrm{down}} \mathbf{u} \cdot \mathbf{n}_e [c]$. Therefore, for the convection term we have

$$
\begin{aligned}
B_{cq}(c, w; \mathbf{u}) = \frac{1}{2} \Big(&(\mathbf{u}\nabla c, w)_{\mathcal{E}_h} - (\mathbf{u}c, \nabla w)_{\mathcal{E}_h} + ((q^I + q^P)c, w) \\
&+ (c^{\mathrm{up}} \mathbf{u} \cdot \mathbf{n}_e, [w])_{\Gamma_h} - (w^{\mathrm{down}} \mathbf{u} \cdot \mathbf{n}_e, [c])_{\Gamma_h} \Big)
\end{aligned}
$$

So, we conclude that the spacial discretization is consistent.

3.4 DG Time Discretization

Since the pressure and velocity do not depend upon the time explicitly, we simply integrate over each time domain. Hence, we have

$$\int_{t_{n-1}}^{t_n} \left((\mathbb{K}^{-1}(c)\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}(\mathbf{v})) \right) = \int_{t_{n-1}}^{t_n} (\rho(c)\mathbf{g}, \mathbf{v})$$

$$\int_{t_{n-1}}^{t_n} (q, \mathrm{div}(\mathbf{u})) = \int_{t_{n-1}}^{t_n} ((q^I - q^P), q)$$

In the rest of the section we will focus on the discretization in time fpr the transport equation. First, observe that the transport equation can be viewed as follows

$$\phi c' + A(\mathbf{u})c = F(c)$$

where $A(\mathbf{u})$ is the convection-diffusion operator that is non-linearly depending upon $\mathbf{u}$. Let w be smooth in time and $w(t_N) = w^N = 0$ with $t_N = T$. We follow the standard procedure to obtain the weak form

$$\int_0^{t_N} (\phi c', w) + (A(\mathbf{u})c, w)dt = \int_0^{t_N} (F(c), w)$$

Use the integration by part,

$$\int_0^{t_N} (\phi c', w) = -\int_0^{t_N} (\phi c, w') + (\phi c, w)\big|_0^{t_N} = -\int_0^{t_N} (\phi c, w') - (\phi c_-^0, w_+^0)$$

where we set $c_-^0 = c_0$ as the initial condition. Hence, we have

$$-\int_0^{t_N} (\phi c, w') + (A(\mathbf{u})c, w)dt = (\phi c_-^0, w_+^0) + \int_0^{t_N} (F(c), w) \tag{3.13}$$

Next, we integrate the first term over each element

$$\int_0^{t_N} (\phi c, w')dt = -\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\phi c', w)dt + \sum_{n=1}^{N}(\phi c_-^n, w^n) - \sum_{n=1}^{N}(\phi c_+^{n-1}, w^{n-1})$$

$$= -\int_0^{t_N} (\phi c', w)dt - \sum_{n=1}^{N-1}([c^n]_t, \phi w_+^n) - (c_+^0, \phi w_+^0) \qquad (3.14)$$

By adding (3.13) and (3.14) we have,

$$\int_0^{t_N} (\phi c', w) + (A(\mathbf{u})c, w)dt + \sum_{n=0}^{N-1}([c^n]_t, \phi w_+^n) = \int_0^{t_N}(F(c), w)$$

We can now obtain the discretization

$$\int_0^{t_N} (\phi c_h', w_h) + (A(\mathbf{u}_h)c_h, w_h)dt + \sum_{n=0}^{N-1}([c_h^n]_t, \phi w_{h+}^n) = \int_0^{t_N}(F(c_h), w_h)$$

where $w_h \in \{v_h : v_h|_{[t_{n-1}, t_n]} \in \mathcal{P}_\ell[t_{n-1}, t_n; C_h]\}$. Choose $w_h \in P_\ell[t_{n-1}, t_n; C_h]$ such that it vanishes outside $[t_{n-1}, t_n]$ and we have

$$\int_{t_{n-1}}^{t_n} \left((\phi c_h', w_h) + a(c_h, w_h; \mathbf{u}_h)\right)dt + ([c_h^{n-1}]_t, \phi w_{h+}^{n-1}) = \int_{t_{n-1}}^{t_n}(F(c_h), w_h)dt$$

with $a(c_h, w_h; \mathbf{u}_h)$ is the spacial discretization of $(A(\mathbf{u})c, w)$, i.e. the spacial discretization of the convection and diffusion terms. Or for simplicity, one can regard the time discretization as integrating over each time domain while adding the stabilization term $([c_h^{n-1}]_t, \phi w_{h+}^{n-1})$. Therefore, we have the DG discretization in time,

$$\int_{t_{n-1}}^{t_n} \left((\phi \partial_t c_h, w_h) + B_d(c_h, w_h; \mathbf{u}_h) + B_{cq}(c_h, w_h; \mathbf{u}_h)\right) + \left([c_h^{n-1}]_t, \phi w_{h+}^{n-1}\right) = \int_{t_{n-1}}^{t_n}(\hat{c}q^I, w_h)$$

with $\mathbf{u}_h \in \mathcal{P}_\ell[t_{n-1}, t_n; \mathbf{U}_h], c_h \in P_\ell[t_{n-1}, t_n; C_h]$.

Chapter 4

Stability and Convergence of the Numerical Scheme

In this section, I will illustrate that the numerical scheme is stable and the numerical solutions converge under mesh refinement. I will begin the analysis of the numerical scheme with some preliminary results. With the help of the preliminary results, I will establish the stability of the numerical scheme. Before proving the convergence of the solution, I will present a more general compactness theorem for the functions that are discontinuous in space and time with some required assumptions. Finally, I will show the convergence of the numerical solutions by using the compactness theorem.

For the analysis of fluid pressure and velocity, I will refer to the analysis done by Walkington and Rivière [41] since the numerical methods for the pressure and velocity and the regularity of the functions in this case are identical to their analysis which have be studied in detail. Whereas, I will put a great emphasis on the transport equation concerning the solvent concentration.

4.1 Preliminary Results

4.1.1 Basic Inequalities

I will begin by stating several well-known inequalities that will be used to obtain some useful results in the setting concerning the numerical scheme. In following analysis, I require the mesh for the numerical method to be a regular mesh, i.e. there are positive constant $a_\circ$, a°,

$b_\circ$ and b° independent of h such that:

$$a_\circ \, |e|^{\frac{d}{d-1}} \leq |E| \leq a^\circ \, |e|^{\frac{d}{d-1}}$$

$$b_\circ \, |e|^{\frac{1}{d-1}} \leq h \leq b^\circ \, |e|^{\frac{1}{d-1}}$$

where E is a mesh element and its measure $|E|$, e is a face and its measure $|e|$. We use the notation "$\lesssim$" to denote the fact that the constant is independent of e, E and h. The properties above can be written as:

$$|e|^{\frac{d}{d-1}} \lesssim |E| \ \text{ and } \ |E| \lesssim |e|^{\frac{d}{d-1}}$$

$$|e|^{\frac{1}{d-1}} \lesssim h \text{ and } h \lesssim |e|^{\frac{1}{d-1}}$$

If it satisfies the properties as above, we use the notation "$\approx$" to describe the relationships. i.e.

$$|E| \approx |e|^{\frac{d}{d-1}} \, , h \approx |e|^{\frac{1}{d-1}} \ \text{ and } \ \frac{|E|}{|e|} \approx h \tag{4.1}$$

I shall now state the inverse inequality as follow.

Lemma 4.1.1 (Inverse Inequality [8])**.** *Let $\rho h \leq diam(E) \leq h$, where $0 < h \leq 1$, and $\mathcal{P}$ be finite dimensional subspace of $W_{\ell,p}(E) \cap W_{m,q}(E)$, where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $0 \leq m \leq \ell$. Then there exists $C = C(\hat{\mathcal{P}}, \hat{E}, \ell, p, q, \rho)$ then*

$$\forall v \in \mathcal{P}, \ \|v\|_{W_{\ell,p}(E)} \leq Ch^{m-\ell+\frac{d}{p}-\frac{d}{q}} \|v\|_{W_{m,q}(E)} \tag{4.2}$$

Another inequality that will be used frequently is a simplified version of Jensen's inequality, stated as

Lemma 4.1.2 (Jensen's Inequality [44])**.** *Let p, q, and n be positive integers. If $1 \leq q \leq$*

$p \leq \infty$, then

$$\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^q\right)^{1/q}, \forall a_i \in \mathbb{R} \tag{4.3}$$

Also, the trace inequality is extremely useful when one needs to translate the property of element from edge to the interior of the element.

Lemma 4.1.3 (Trace Inequality [43]). *If* $v \in \mathcal{P}$, *where* $\mathcal{P}$ *is a finite dimensional subspace, then*

$$\|v\|_{L^2(e)} \leq Ch^{-1/2}\|v\|_{L^2(E)} \tag{4.4}$$

$$\|v\|_{L^4(e)} \leq Ch^{-1/4}\|v\|_{L^4(E)} \tag{4.5}$$

where C *is positive and independent of* e *and* E.

4.1.2 Bounds for Stabilization Terms

The numerical scheme is as follows

$$\int_{t_{n-1}}^{t_n} \left((\mathbb{K}^{-1}(c_h)\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div}(\mathbf{v}_h))\right) = \int_{t_{n-1}}^{t_n} (\rho(c_h)g, \mathbf{v}_h)$$

$$\int_{t_{n-1}}^{t_n} (q_h, \operatorname{div}(\mathbf{u}_h)) = \int_{t_{n-1}}^{t_n} ((q^I - q^P), q_h),$$

$$\int_{t_{n-1}}^{t_n} \left((\phi\partial_t c_h, w_h) + B_d(c_h, w_h; \mathbf{u}_h) + B_{cq}(c_h, w_h; \mathbf{u}_h)\right) + \left(\left[c_h^{n-1}\right]_t, \phi w_{h+}^{n-1}\right) = \int_{t_{n-1}}^{t_n} (\hat{c}q^I, w_h)$$

where,

$$B_d(c_h, w_h; \mathbf{u}_h) = (\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) - ([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_{\Gamma_h}$$

$$+\epsilon([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} + (\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h}$$

$$B_{cq}(c_h, w_h; \mathbf{u}_h) = \frac{1}{2} \left((\mathbf{u}_h \nabla c_h, w_h) - (\mathbf{u}_h c_h, \nabla w_h) - (\mathbf{u}_h c_h, \nabla w_h) + ((q^I + q^P) c_h, w_h) \right.$$

$$\left. + (c^{\text{up}} \mathbf{u}_h \cdot \mathbf{n}_e, [w])_{\Gamma_h} - (w^{\text{down}} \mathbf{u}_h \cdot \mathbf{n}_e, [c])_{\Gamma_h}) \right)$$

for all $\mathbf{u}_h \in \mathcal{P}_\ell[t^{n-1}, t^n; \mathbf{U}_h], q_h \in \mathcal{P}_\ell[t^{n-1}, t^n; P_h], w_h \in P_\ell[t^{n-1}, t^n; C_h]$.

Now, let us look at the terms $([c_h], \{\mathbb{D}(\mathbf{u}_h) \nabla w_h \cdot \mathbf{n}\})_{\Gamma_h}$ and $([w_h], \{\mathbb{D}(\mathbf{u}_h) \nabla c_h \cdot \mathbf{n}\})_{\Gamma_h}$. The goal in this section is to establish the bound for the these terms as stated in Proposition 4.1.10. First, we obtain several inequalities that will prove to be useful for our analysis.

Lemma 4.1.4. *Let e be a given face of an arbitrary mesh element E. If $\mathbf{w} \in \mathcal{P}^d$ where $\mathbf{w}$ is a vector function and $\mathcal{P}$ is a finite dimensional subspace, then*

$$\|\mathbf{w}\|_{L^2(e)} \lesssim h^{-1/2} \|\mathbf{w}\|_{L^2(E)}$$

Proof. We write the definition of L^2 norm:

$$\|\mathbf{w}\|_{L^2(e)} = \left(\sum_{i=1}^d \int_e w_i^2 \right)^{1/2} = \left(\sum_{i=1}^d \|w_i\|_{L^2(e)}^2 \right)^{1/2}$$

Hence, applying the Trace Inequality in Lemma 4.1.3 we have

$$\|\mathbf{w}\|_{L^2(e)} \lesssim \left(\sum_{i=1}^d h^{-1} \|w_i\|_{L^2(E)}^2 \right)^{1/2} \lesssim h^{-1/2} \left(\sum_{i=1}^d \int_E w_i^2 \right)^{1/2} \lesssim h^{-1/2} \|\mathbf{w}\|_{L^2(E)}$$

$\square$

With the help of this inverse estimate, the following inequalities can be obtained.

Lemma 4.1.5. *Given $w_h \in \mathcal{P}$ and $\mathbf{u}_h \in \mathcal{P}^d$ then for a fixed element E and a face $e \in \partial E$,*

$$\|\nabla w_h\|_{L^2(e)} \lesssim h^{-1/2} \|\nabla w_h\|_{L^2(E)} \text{ and } \left\| |\mathbf{u}_h|_E|^{1/2} |\nabla w_h| \right\|_{L^2(e)} \lesssim h^{-1/2} \left\| |\mathbf{u}_h|^{1/2} |\nabla w_h| \right\|_{L^2(E)}.$$

Proof. The first inequality directly follows from Lemma 4.1.4.

For the second inequality,

$$\left\| |\mathbf{u}_h|^{1/2} \, |\nabla w_h| \right\|_{L^2(e)} \lesssim |e|^{1/4} \left(\int_e |\mathbf{u}_h|^2 \, |\nabla w_h|^4 \right)^{1/4} \lesssim |e|^{1/4} \left(\sum_{i,j=1}^{d} \int_e u_{h,i}^2 \left(\frac{\partial w_h}{\partial x_j} \right)^4 \right)^{1/4}$$

$$\lesssim |e|^{1/4} \left(\sum_{i,j=1}^{d} \left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^2(e)}^2 \right)^{1/4}$$

As the consequence of Trace Inequality from Lemma 4.1.3,

$$\left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^2(e)} \lesssim h^{-1/2} \left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^2(E)}$$

Hence, we related the face to the interior of the element E,

$$\left\| |\mathbf{u}_h|^{1/2} \, |\nabla w_h| \right\|_{L^2(e)} \lesssim |e|^{1/4} \left(h^{-1} \sum_{i,j=1}^{d} \left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^2(E)}^2 \right)^{1/4}$$

By the Inverse Inequality from Lemma 4.1.1,

$$\left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^2(E)} \lesssim h^{-d/2} \left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^1(E)}$$

Therefore, we can conclude

$$\left\| |\mathbf{u}_h|^{1/2} |\nabla w_h| \right\|_{L^2(e)} \lesssim |e|^{1/4} \left(h^{-1} h^{-d} \sum_{i,j=1}^{d} \left\| u_{h,i} \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right\|_{L^1(E)}^2 \right)^{1/4}$$

$$\lesssim h^{-1/2} \left(\sum_{i,j=1}^{d} \left(\fint_E |u_{h,i}| \left(\frac{\partial w_h}{\partial x_j} \right)^2 \right)^2 \right)^{1/4}$$

$$\lesssim h^{-1/2} \left(\fint_E \sum_{i=1}^{d} |u_{h,i}| \, |\nabla w_h|^2 \right)^{1/2}$$

$$\lesssim h^{-1/2} \left(\fint_E |\mathbf{u}_h| \, |\nabla w_h|^2 \right)^{1/2}$$

$$\lesssim h^{-1/2} \left\| |\mathbf{u}_h|^{1/2} |\nabla w_h| \right\|_{L^2(E)}$$

$\square$

Lemma 4.1.6. *If $w_h \in \mathcal{P}$, then we have*

$$\|\nabla w_h\|_{L^2(\mathcal{E}_h)} \lesssim \|\nabla w_h\|_{L^4(\mathcal{E}_h)}$$

Proof. We apply Cauchy-Schwarz inequality,

$$\left(\sum_{E \in \mathcal{E}_h} \|\nabla w_h\|_{L^2(E)}^2 \right)^{1/2} \leq \left(\sum_{E \in \mathcal{E}_h} |E|^{1/2} \left(\fint_E |\nabla w_h|^4 \right)^{1/2} \right)^{1/2}$$

$$\leq \left(\sum_{E \in \mathcal{E}_h} |E| \right)^{1/4} \left(\sum_{E \in \mathcal{E}_h} \fint_E |\nabla w_h|^4 \right)^{1/4} \lesssim \|\nabla w_h\|_{L^4(\mathcal{E}_h)}$$

$\square$

Lemma 4.1.7. *Let $\mathbf{u}_h \in \mathcal{P}^d$ and $c_h \in \mathcal{P}$, then for an element E and one of its face e,*

$$\left\| \mathbb{D}^{1/2}(\mathbf{u}_h|_E) \nabla c_h \right\|_{L^2(e)} \lesssim h^{-1/2} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(E)}$$

and

$$\left\|\mathbb{D}^{1/2}(\mathbf{u}_h|_E)\nabla c_h\right\|_{L^2(e)} \lesssim h^{-1/2}\left(\|\nabla c_h\|_{L^2(E)} + \|\mathbf{u}_h\|_{L^2(E)}^{1/2}\|\nabla c_h\|_{L^4(E)}\right)$$

Proof. Recall the property of diffusivity tensor in (3.4), we have

$$d_0(1 + |\mathbf{u}_h|)\,|\nabla c_h|^2 \leq \nabla c_h{}^T \mathbb{D}(\mathbf{u}_h)\nabla c_h \leq d_1(1 + |\mathbf{u}_h|)\,|\nabla c_h|^2$$

We therefore obtain the inequality,

$$\left(\int_e \mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \nabla c_h\right)^{1/2} \lesssim \left(\int_e (1 + |\mathbf{u}_h|)\,|\nabla c_h|^2\right)^{1/2}$$

$$\lesssim \left(\|\nabla c_h\|_{L^2(e)}^2 + \left\||\mathbf{u}_h|^{1/2}\,|\nabla c_h|\right\|_{L^2(e)}^2\right)^{1/2}$$

According to Lemma 4.1.5, we have

$$\left(\int_e \mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \nabla c_h\right)^{1/2} \lesssim h^{-1/2}\left(\|\nabla c_h\|_{L^2(E)}^2 + \left\||\mathbf{u}_h|^{1/2}\,|\nabla c_h|\right\|_{L^2(E)}^2\right)^{1/2}$$

$$\lesssim h^{-1/2}\left(\int_E (1 + |\mathbf{u}_h|)\,|\nabla c_h|^2\right)^{1/2}$$

Therefore, we obtain the first inequality using the property (3.4),

$$\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(e)} \lesssim h^{-1/2}\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(E)}$$

Also, by Lemma 4.1.2

$$\left(\int_E (1 + |\mathbf{u}_h|)\,|\nabla c_h|^2\right)^{1/2} = \left(\int_E |\nabla c_h|^2 + \int_E |\mathbf{u}_h|\,|\nabla c_h|^2\right)^{1/2}$$

$$\leq \left(\int_E |\nabla c_h|^2\right)^{1/2} + \left(\int_E |\mathbf{u}_h|\,|\nabla c_h|^2\right)^{1/2}$$

$$\leq \|\nabla c_h\|_{L^2(E)} + \|\mathbf{u}_h\|_{L^2(E)}^{1/2}\|\nabla c_h\|_{L^4(E)}$$

Therefore, we have

$$\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(e)} \lesssim h^{-1/2}\left(\|\nabla c_h\|_{L^2(E)} + \|\mathbf{u}_h\|_{L^2(E)}^{1/2}\|\nabla c_h\|_{L^4(E)}\right)$$

$\square$

With all the helpful inequalities attained so far, we can now bound the terms $([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_e$ and $([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_e$ in our scheme.

For the next result, let E_+^e and E_-^e be the mesh elements that share the face e. We define the average to be:

$$\{\|w\|_{L^p(E^e)}\} = \frac{1}{2}\left(\|w\|_{L^p(E_+^e)} + \|w\|_{L^p(E_-^e)}\right)$$

likewise,

$$\{\|w\|_{L^p(E^e)}\|v\|_{L^q(E^e)}\} = \frac{1}{2}\left(\|w\|_{L^p(E_+^e)}\|v\|_{L^q(E_+^e)} + \|w\|_{L^p(E_-^e)}\|v\|_{L^q(E_-^e)}\right)$$

we also use the notations

$$w^+ = w|_{E_+^e} \text{ and } w^- = w|_{E_-^e}$$

In the rest of the analysis, we will use the notations P_h, $\mathbf{U}_h$ and C_h corresponding to the finite element spaces for the numerical scheme. But, those results hold for all the piecewise polynomials.

Lemma 4.1.8. *Let e be a given face of an arbitrary mesh element E. Given $c_h, w_h \in C_h$, $\mathbf{u}_h \in \mathbf{U}_h$ and $\mathbb{D}$ the diffusion dispersion matrix satisfying the property (3.4), then we have*

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_e \lesssim \left(\int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h]^2\right)^{1/2}$$

$$\times \left\{\|\nabla w_h\|_{L^2(E^e)} + \|\mathbf{u}_h\|_{L^2(E^e)}^{1/2}\|\nabla w_h\|_{L^4(E^e)}\right\}$$

Proof. We begin by expanding and bounding the terms using Cauchy-Schwarz's inequality,

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_e \lesssim ([c_h], \mathbb{D}(\mathbf{u}_h^+)\nabla w_h^+ \cdot \mathbf{n}_e)_e + ([c_h], \mathbb{D}(\mathbf{u}_h^-)\nabla w_h^- \cdot \mathbf{n}_e)_e$$

$$\lesssim \{\int_e |\mathbb{D}^{1/2}(\mathbf{u}_h)\mathbf{n}_e| \, |[c_h]| \, |\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h|\}$$

$$\lesssim \{\left(\int_e |\mathbb{D}^{1/2}(\mathbf{u}_h)\mathbf{n}_e|^2 [c_h]^2\right)^{1/2} \left(\int_e |\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h|^2\right)^{1/2}\}$$

$$\lesssim \left(\int_e \{|\mathbb{D}^{1/2}(\mathbf{u}_h)\mathbf{n}_e|\}^2 [c_h]^2\right)^{1/2} \{\left(\int_e |\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h|^2\right)^{1/2}\}$$

By the property (3.4), we obtain

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_e \lesssim \left(\int_e (1 + \{|\mathbf{u}_h|\})[c_h]^2\right)^{1/2} \{\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h\right\|_{L^2(e)}\} \qquad (4.6)$$

By Lemma 4.1.7, therefore, we have

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_e \lesssim \left(\int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h]^2\right)^{1/2}$$

$$\times \left\{\|\nabla w_h\|_{L^2(E^e)} + \|\mathbf{u}_h\|_{L^2(E^e)}^{1/2} \|\nabla w_h\|_{L^4(E^e)}\right\}$$

$\square$

Lemma 4.1.9. *Given $c_h, w_h, \mathbf{u}_h$ and $\mathbb{D}$ as in Lemma 4.1.8, then*

$$([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_e \lesssim \left(\int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[w_h]^2\right)^{1/2} \{\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(E^e)}\}$$

Proof. From (4.6), we have

$$([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_e \lesssim \left(\int_e (1 + \{|\mathbf{u}_h|\})[w_h]^2\right)^{1/2} \{\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(e)}\}$$

And according to Lemma 4.1.7,

$$([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_e \lesssim \left(\int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[w_h]^2\right)^{1/2} \{\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(E^e)}\}$$

□

We now sum up the contributions over all the interior edge and establish the following proposition.

Proposition 4.1.10. *Let* c_h, w_h *be in* C_h *and* $\mathbf{u}_h$ *be in* $\mathbf{U}_h$. *We have*

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} \lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2}(\|\nabla w_h\|_{L^2(\mathcal{E}_h)} + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}\|\nabla w_h\|_{L^4(\mathcal{E}_h)}) \quad (4.7)$$

and

$$([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_{\Gamma_h} \lesssim R(w_h; \mathbf{u}_h)\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)} \quad (4.8)$$

with

$$J(c_h, c_h; \mathbf{u}_h) = \sum_{e \in \Gamma_h} h^{-1} \int_e (1 + \{|\mathbf{u}_h|\})[c_h]^2 \quad (4.9)$$

and

$$R(w_h; \mathbf{u}_h) = \left(1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}\right)\left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4\right)^{1/4} \quad (4.10)$$

Proof. To sum up over all the interior edges, by Lemma 4.1.8 one would have

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} = \sum_{e \in \Gamma_h} ([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_e$$

$$\lesssim \sum_{e \in \Gamma_h} \left(\int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h]^2\right)^{1/2} \{\|\nabla w_h\|_{L^2(E^e)} + \|\mathbf{u}_h\|_{L^2(E^e)}^{1/2}\|\nabla w_h\|_{L^4(E^e)}\}$$

$$\lesssim \sum_{e \in \Gamma_h} \left(\int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h]^2\right)^{1/2} \left(\{\|\nabla w_h\|_{L^2(E^e)}\} + \{\|\mathbf{u}_h\|_{L^2(E^e)}^{1/2}\|\nabla w_h\|_{L^4(E^e)}\}\right)$$

$$\lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2}\left(\left(\sum_{e \in \Gamma_h}\{\|\nabla w_h\|_{L^2(E^e)}\}^2\right)^{1/2} + \left(\sum_{e \in \Gamma_h}\{\|\mathbf{u}_h\|_{L^2(E^e)}^{1/2}\|\nabla w_h\|_{L^4(E^e)}\}^2\right)^{1/2}\right)$$

For the term,

$$\left(\sum_{e\in\Gamma_h}\{\|\nabla w_h\|_{L^2(E^e)}\}^2\right)^{1/2}$$

we have

$$\left(\sum_{e\in\Gamma_h}\{\|\nabla w_h\|_{L^2(E^e)}\}^2\right)^{1/2}\lesssim\left(\sum_{e\in\Gamma_h}\left(\|\nabla w_h\|_{L^2(E^e_+)}^2+\|\nabla w_h\|_{L^2(E^e_-)}^2\right)\right)^{1/2}\lesssim\|\nabla w_h\|_{L^2(\mathcal{E}_h)}$$

Likewise, we can obtain

$$\left(\sum_{e\in\Gamma_h}\{\|\mathbf{u}_h\|_{L^2(E^e)}^{1/2}\|\nabla w_h\|_{L^4(E^e)}\}^2\right)^{1/2}\lesssim\left(\sum_{E\in\mathcal{E}_h}\|\mathbf{u}_h\|_{L^2(E)}\|\nabla w_h\|_{L^4(E)}^2\right)^{1/2}$$

$$\lesssim\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}\|\nabla w_h\|_{L^4(\mathcal{E}_h)}$$

Therefore, for the term $([c_h],\{\mathbb{D}(\mathbf{u}_h)\nabla w_h\cdot\mathbf{n}_e\})_{\Gamma_h}$ we have

$$([c_h],\{\mathbb{D}(\mathbf{u}_h)\nabla w_h\cdot\mathbf{n}_e\})_{\Gamma_h}\lesssim J(c_h,c_h;\mathbf{u}_h)^{1/2}(\|\nabla w_h\|_{L^2(\mathcal{E}_h)}+\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}\|\nabla w_h\|_{L^4(\mathcal{E}_h)})$$

For the term $([w_h],\{\mathbb{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\})_{\Gamma_h}$ using Lemma 4.1.9 we have,

$$([w_h],\{\mathbb{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\})_{\Gamma_h}=\sum_{e\in\Gamma_h}([w_h],\{\mathbb{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\})_e$$

$$\lesssim\sum_{e\in\Gamma_h}\left(\int_e h^{-1}(1+\{|\mathbf{u}_h|\})[w_h]^2\right)^{1/2}\{\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2(E^e)}\}$$

$$\lesssim J(w_h,w_h;\mathbf{u}_h)^{1/2}\left(\sum_{e\in\Gamma_h}\{\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2(E^e)}\}^2\right)^{1/2}$$

$$\lesssim J(w_h,w_h;\mathbf{u}_h)^{1/2}\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2(\mathcal{E}_h)}$$

Thus,

$$([w_h],\{\mathbb{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\})_{\Gamma_h}\lesssim J(w_h,w_h;\mathbf{u}_h)^{1/2}\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2(\mathcal{E}_h)}\qquad(4.11)$$

For $J(w_h, w_h; \mathbf{u}_h)^{1/2}$, we can establish the inequality,

$$J(w_h, w_h; \mathbf{u}_h)^{1/2} = \left(\sum_{e \in \Gamma_h} \int_e h^{-1}(1 + \{|\mathbf{u}_h|\})[w_h]^2 \right)^{1/2}$$

$$\lesssim \left(\sum_{e \in \Gamma_h} \int_e h^{-1}[w_h]^2 \right)^{1/2} + \left(\sum_{e \in \Gamma_h} \int_e h^{-1}\{|\mathbf{u}_h|\}[w_h]^2 \right)^{1/2}$$

For the first term we have,

$$\left(\sum_{e \in \Gamma_h} \int_e h^{-1}[w_h]^2 \right)^{1/2} \lesssim \left(\sum_{e \in \Gamma_h} h^{-1} |e|^{1/2} \left(\int_e [w_h]^4 \right)^{1/2} \right)^{1/2}$$

$$\lesssim \left(\sum_{e \in \Gamma_h} |E| \right)^{1/4} \left(\sum_{e \in \Gamma_h} h^{-2} \frac{|e|}{|E|} \int_e [w_h]^4 \right)^{1/4}$$

Using the property of regular mesh in (4.1), we have

$$\left(\sum_{e \in \Gamma_h} h^{-2} \frac{|e|}{|E|} \int_e [w_h]^4 \right)^{1/4} \lesssim \left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4 \right)^{1/4}$$

For the second term we notice,

$$\left(\sum_{e \in \Gamma_h} \int_e h^{-1} |\mathbf{u}_h^+| [w_h]^2 \right)^{1/2} \lesssim \left(\sum_{e \in \Gamma_h} h^{-1} \left(\int_e |\mathbf{u}_h^+|^2 \right)^{1/2} \left(\int_e [w_h]^4 \right)^{1/2} \right)^{1/2}$$

$$\lesssim \left(\sum_{e \in \Gamma_h} \|\mathbf{u}_h^+\|_{L^2(e)}^2 \right)^{1/4} \left(\sum_{e \in \Gamma_h} h^{-2} \int_e [w_h]^4 \right)^{1/4}$$

$$\lesssim \left(\sum_{e \in \Gamma_h} h^{-1} \|\mathbf{u}_h\|_{L^2(E_+^e)}^2 \right)^{1/4} \left(\sum_{e \in \Gamma_h} h^{-2} \int_e [w_h]^4 \right)^{1/4}$$

$$\lesssim \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4 \right)^{1/4}$$

In the same way we can establish,

$$\left(\sum_{e\in\Gamma_h}\int_e h^{-1}\left|\mathbf{u}_h^-\right|[w_h]^2\right)^{1/2}\lesssim\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}\left(\sum_{e\in\Gamma_h}h^{-3}\int_e[w_h]^4\right)^{1/4}\tag{4.12}$$

To summarize we have,

$$\left(\sum_{e\in\Gamma_h}h^{-1}\int_e(1+\{|\mathbf{u}_h|\})[w_h]^2\right)^{1/2}\lesssim\left(1+\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}\right)\left(\sum_{e\in\Gamma_h}h^{-3}\int_e[w_h]^4\right)^{1/4}\tag{4.13}$$

Therefore, we conclude

$$([w_h],\{\mathbb{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\})_{\Gamma_h}\lesssim R(w_h;\mathbf{u}_h)\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)}$$

$\square$

These results will be used extensively in the analysis to come concerning the stability and compactness theorem. In our analysis, we use a rather unconventional jump term to bypass the difficulty of the low regularity condition.

4.2   Stability Analysis

4.2.1   Stability of Pressure and Velocity

The stability of the fluid pressure and velocity follows the same argument as the result in Walkington and Rivière [41]. For completeness, this section recalls the existing results.

Lemma 4.2.1. *There exists a constant $m > 0$ depending only upon Ω such that*

$$\sup_{\mathbf{u}_h\in\mathbf{U}_h}\frac{\int_\Omega p_h\,div(\mathbf{u}_h)}{\|\mathbf{u}_h\|_{H(\Omega;div)}}\geq m\,\|p_h\|_{L^2(\Omega)}\,,\ p_h\in P_h$$

In particular, if $Z_h=\{\mathbf{u}_h\in\mathbf{U}_h\mid div(\mathbf{u}_h)=0\}$ and $\mathbf{U}_h=Z_h\oplus Z_h^\perp$ is the orthogonal decomposition, then there exists a linear operator $L_h:\ P_h\to Z_h^\perp$ with $\|L_h\|_{\mathcal{L}(P_h,\mathbf{U}_h)}\leq 1$ such

that

$$m \left\| p_h \right\|_{L^2(\Omega)}^2 \leq \int_\Omega p_h \, div(L_h(p_h)), \ \ p_h \in P_h$$

and if $\mathbf{u}_h \in Z_h^\perp$ *then* $m \left\| \mathbf{u}_h \right\|_{H(\Omega;div)} \leq \left\| div(\mathbf{u}_h) \right\|_{L^2(\Omega)}$.

Lemma 4.2.2. *Let V be a linear space and $(.,.)_V$ be a (semi) inner product on V; $w \geq 0$ be a non-zero element of $L^1(0,1)$; and $0 < a < b$. Then there exists a constant $M_\ell > 0$, depending only upon ℓ and w, such that for all $u \in \mathcal{P}_\ell[a,b;V]$*

$$\left\| u \right\|_{L^p[a,b;V]} \leq (b-a)^{1/p-1/2} \left(M_\ell \int_a^b w((t-a)/(b-a)) \left\| u(t) \right\|_V^2 \, dt \right)^{1/2}, \ \ 1 \leq p \leq \infty$$

In particular, if $1/p + 1/p' = 1$ then

$$\left\| u \right\|_{L^p[a,b;V]} \left\| u \right\|_{L^{p'}[a,b;V]} \leq M_\ell \int_a^b w((t-a)/(b-a)) \left\| w(t) \right\|_V^2$$

Now, we state and prove the stability for the pressure and velocity.

Theorem 4.2.3. *There exists a constant $M > 0$ independent of h and Δt such that solutions of the numerical scheme satisfy the following bounds.*

- *If $1 \leq p, q \leq \infty$ and q^I, $q^P \in L^p[0,T;L^q(\Omega)]$, then*

$$\left\| div(\mathbf{u}_h) \right\|_{L^p[0,T;L^q(\Omega)]} \leq M \left(\left\| q^I \right\|_{L^p[0,T;L^q(\Omega)]} + \left\| q^P \right\|_{L^p[0,T;L^q(\Omega)]} \right)$$

- *If $1 \leq p \leq \infty$, q^I, $q^P \in L^p[0,T;L^2(\Omega)]$, then*

$$\left\| \mathbf{u}_h \right\|_{L^p[0,T;H(\Omega,div)]} + \left\| p_h \right\|_{L^p[0,T;L^2(\Omega)]} \leq M \left(\left\| q^I \right\|_{L^p[0,T;L^2(\Omega)]} + \left\| q^P \right\|_{L^p[0,T;L^2(\Omega)]} \right.$$
$$\left. + \left\| \rho_1 \mathbf{g} \right\|_{L^p[0,T;L^2(\Omega)]} \right)$$

Proof. For each $E \in \mathcal{E}_h$, let $\Pi_h : L^2(t_{n-1}, t_n; E) \to \mathcal{P}_\ell[t_{n-1}, t_n, \mathcal{P}_k(E)]$ denote the L^2 projec-

tion. A parent element calculation shows that there exists a constant $M >)$ depending only on the parent element such that

$$\left\|\Pi_h(q^I - q^P)\right\|_{L^p[t_{n-1},t_n,L^q(E)]} \leq M \left\|q^I - q^P\right\|_{L^p[t_{n-1},t_n,L^q(E)]}, \ 1 \leq p, q \leq \infty$$

Since $\text{div}(\mathbf{u}_h) \in P_h$ it follows from (3.9) that

$$\text{div}(\mathbf{u}_h) = \Pi_h(q^I - q^P)$$

Next, we introduce the orthogonal decomposition $\mathbf{U}_h = Z_h \oplus Z_h^{\perp}$, thus we can let $\mathbf{u}_h = \mathbf{z}_h + \mathbf{u}_h^{\perp}$ be the decomposition of $\mathbf{u}_h$. From Lemma 4.2.1 we find

$$M \left\|\mathbf{u}_h^{\perp}\right\|_{H(\Omega;\text{div})} \leq \left\|\text{div}(\mathbf{u}_h^{\perp})\right\|_{L^2(\Omega)} = \left\|\text{div}(\mathbf{u}_h)\right\|_{L^2(\Omega)}$$

and since $\text{div}(\mathbf{u}_h) = \Pi_h(q^I - q^P)$ it follows that

$$\left\|\mathbf{u}_h^{\perp}\right\|_{L^p[t_{n-1},t_n;H(\Omega;\text{div})]} \leq M \left\|\text{div}(\mathbf{u}_h)\right\|_{L^p[t_{n-1},t_n;L^2(\Omega)]}$$
$$\leq M(\left\|q^I\right\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} + \left\|q^P\right\|_{L^p[t_{n-1},t_n,L^2(\Omega)]})$$

To estimate $\mathbf{z}_h$ select it to be the test function in (3.8) and we have

$$\int_{t_{n-1}}^{t_n} (\mathbb{K}^{-1}(c_h)(\mathbf{z}_h + \mathbf{u}_h^{\perp}), \mathbf{v}_h) = \int_{t_{n-1}}^{t_n} (\mathbb{K}^{-1}(c_h)\mathbf{u}_h, \mathbf{v}_h) = \int_{t_{n-1}}^{t_n} (\rho(c_h)g, \mathbf{v}_h)$$

Upon rescaling that $\|\mathbf{z}_h\|_{H(\Omega;\text{div})} = \|\mathbf{z}_h\|_{L^2(\Omega)}$ and the assumption on $\mathbb{K}$, it follows that

$$
\begin{aligned}
\|\mathbf{z}_h\|^2_{L^2[t_{n-1},t_n;\text{div}(\Omega,\text{div})]} &\leq M \int_{t_{n-1}}^{t_n} (\mathbb{K}^{-1}(c_h)\mathbf{z}_h, \mathbf{z}_h) \\
&\leq M \left(\left| \int_{t_{n-1}}^{t_n} (\rho(c_h)g, \mathbf{v}_h) \right| + \left| \int_{t_{n-1}}^{t_n} (\mathbb{K}^{-1}(c_h)\mathbf{u}_h^{\perp}, \mathbf{v}_h) \right| \right) \\
&\leq M \|\mathbf{z}_h\|_{L^{p'}[t_{n-1},t_n;H(\Omega;\text{div})]} \left(\|\rho_1\mathbf{g}\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} + \left\|\mathbf{u}_h^{\perp}\right\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} \right)
\end{aligned}
$$

And since $1/p + 1/p' = 1$ we can use Hölder's inequality

$$
\|\mathbf{z}_h\|_{L^p[t_{n-1},t_n;\text{div}(\Omega,\text{div})]} \leq M \left(\|\rho_1\mathbf{g}\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} + \left\|\mathbf{u}_h^{\perp}\right\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} \right)
$$

Therefore, we can construct the bound

$$
\|\mathbf{z}_h\|_{L^p[t_{n-1},t_n;\text{div}(\Omega,\text{div})]} \leq M \left(\|\rho_1\mathbf{g}\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} + \left\|q^I\right\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} + \left\|q^P\right\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} \right)
$$

from which we can find the bound for $\|\mathbf{z}_h\|_{L^p[t_{n-1},t_n;\text{div}(\Omega,\text{div})]}$. Since the operator $L_h : P_h \to Z_h^{\perp}$ in Lemma 4.2.1 is independent of time, it follows that $L_h(p_h) \in \mathcal{P}_\ell[t_{n-1}, t_n, \mathbf{U}_h]$. We may then set $\mathbf{v}_h = L_h(p_h)$ in (3.8) to find

$$
M \int_{t_{n-1}}^{t_n} \|p_h\|^2_{L^2(\Omega)} \leq \int_{t_{n-1}}^{t_n} (p_h, \text{div}(L_h(p_h))) = \int_{t_{n-1}}^{t_n} ((\mathbb{K}^{-1}(c_h)\mathbf{u}_h, L_h(p_h)) - (\rho(c_h)g, L_h(p_h))
$$

By Lemma 4.2.2 we have

$$
\begin{aligned}
\|p_h\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} &\leq M \left(\|\mathbf{u}_h\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} + \|\rho_1\mathbf{g}\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} \right) \\
&\leq M \left(\|\rho_1\mathbf{g}\|_{L^p[t_{n-1},t_n;L^2(\Omega)]} + \left\|q^I\right\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} + \left\|q^P\right\|_{L^p[t_{n-1},t_n,L^2(\Omega)]} \right)
\end{aligned}
$$

$\square$

4.2.2 Stability of Concentration

In this subsection, I will show that the scheme is stable for the concentration.

Define the energy semi-norm $\|\cdot\|_{X_h}$ in following way:

$$\|v\|_{X_h} = \left(\sum_{E \in \mathcal{E}_h} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla v \right\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h} h^{-1} \left\| (1 + \{|\mathbf{u}_h|\})^{1/2} [v] \right\|_{L^2(e)}^2 \right)^{1/2} \tag{4.14}$$

I will first show the coercivity of the diffusion term:

Lemma 4.2.4. *There always exists penalty parameter $\sigma > 0$ such that*

$$B_d(w_h, w_h; \mathbf{u}_h) \geq \frac{1}{2} \|w_h\|_{X_h}^2, \quad \forall w_h \in C_h$$

Proof. From our numerical scheme, we have

$$B_d(w_h, w_h; \mathbf{u}_h) = (\mathbb{D}(\mathbf{u}_h) \nabla w_h, \nabla w_h) + (\epsilon - 1)([w_h], \{\mathbb{D}(\mathbf{u}_h) \nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h}$$

$$+ (\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[w_h], [w_h])_{\Gamma_h}$$

According to results attained previously in (4.11)

$$([w_h], \{\mathbb{D}(\mathbf{u}_h) \nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} \leq M \left(\sum_{e \in \Gamma_h} h^{-1} \left\| (1 + \{|\mathbf{u}_h|\})^{1/2} [w_h] \right\|_{L^2(e)}^2 \right)^{1/2} \left\| D^{1/2}(\mathbf{u}_h) \nabla w_h \right\|_{L^2(\mathcal{E}_h)}$$

for a constant M independent upon h.

We use Young's inequality to obtain,

$$([w_h], \{\mathbb{D}(\mathbf{u}_h) \nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} \leq \frac{\delta \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla w_h \right\|_{L^2(\mathcal{E}_h)}^2}{2} + \frac{M^2}{2\delta} \sum_{e \in \Gamma_h} h^{-1} \left\| (1 + \{|\mathbf{u}_h|\})^{1/2} [w_h] \right\|_{L^2(e)}^2$$

for all $\delta > 0$.

Thus,

$$
\begin{aligned}
B_d(w_h, w_h; \mathbf{u}_h) \geq & (1 + \frac{\delta}{2}(\epsilon - 1)) \left\| \mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h \right\|_{L^2(\mathcal{E}_h)}^2 \\
& + \sum_{e \in \Gamma_h} \left(\sigma + \frac{\epsilon - 1}{2\delta} M^2 \right) h^{-1} \left\| (1 + \{|\mathbf{u}_h|\})^{1/2}[w_h] \right\|_{L^2(e)}^2
\end{aligned}
$$

When $\epsilon = 1$, immediately one obtains $B_d(w_h, w_h; \mathbf{u}_h) = \|w_h\|_{X_h}^2$; (since $\epsilon = 1$ in this case)

When $\epsilon = 0$, choose $\delta = 1$ and $\sigma \geq \frac{1}{2}(1 + M^2)$;

When $\epsilon = -1$, choose $\delta = \frac{1}{2}$ and $\sigma \geq \frac{1}{2} + 2M^2$.

These criteria will guarantee $B_d(w_h, w_h; \mathbf{u}_h) \geq \frac{1}{2} \|w_h\|_{X_h}^2$. $\qquad\square$

We just showed the coercivity of the diffusion term. Now, with the help of this property, we will proceed by proving the stability of the concentration solution.

Theorem 4.2.5. *The numerical scheme is stable with respect to the fluid concentration, so that* $\|c_h\|_{\ell^\infty[L^2(\Omega)]}$, $\|c_h\|_{L^2[0,T;X_h]}$ *and* $\|c_h\|_{L^2[0,T;H^1(\mathcal{E}_h)]}$ *are bounded independent of h and Δt. In particular, we have:*

$$
\begin{aligned}
\max_{1 \leq n \leq N} \left\| \phi^{1/2} c_{h-}^n \right\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \left\| [\phi^{1/2} c_h^{n-1}] \right\|_{L^2(\Omega)}^2 + \int_0^T & \left(\|c_h\|_{X_h}^2 + \left\| \sqrt{q^P} c_h \right\|_{L^2(\Omega)}^2 \right. \\
& \left. + (|\mathbf{u}_h \cdot \mathbf{n}_e| [c_h], [c_h])_{\Gamma_h} \right) \leq \left\| \phi^{1/2} c_{h-}^0 \right\|_{L^2(\Omega)}^2 + \int_0^T \left\| \sqrt{q^I} \hat{c} \right\|_{L^2(\Omega)}^2
\end{aligned}
$$

Proof. According to result in Lemma 4.2.4, we have

$$
B_d(c_h, c_h; \mathbf{u}_h) \geq \frac{1}{2} \|c_h\|_{X_h}^2
$$

Also according to our numerical scheme,

$$
\begin{aligned}
B_{cq}(c_h, w_h; \mathbf{u}_h) = \frac{1}{2} \big(& (\mathbf{u}_h \nabla c_h, w_h) - (\mathbf{u}_h c_h, \nabla w_h) + ((q^I + q^P)c_h, w_h) \\
& + (c_h^{\text{up}} \mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_{\Gamma_h} - (w_h^{\text{down}} \mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h} \big)
\end{aligned}
$$

then we have

$$B_{cq}(c_h, c_h; \mathbf{u}_h) = \frac{1}{2}\left((\mathbf{u}\nabla c_h, c_h)_{\mathcal{E}_h} - (\mathbf{u}_h c_h, \nabla c_h)_{\mathcal{E}_h} + ((q^I + q^P)c_h, c_h)\right.$$

$$\left. + (c_h^{\text{up}}\mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h} - (c_h^{\text{down}}\mathbf{u}_h \cdot \mathbf{n}_e, [c])_{\Gamma_h}\right)$$

$$= \frac{1}{2}\left(((q^I + q^P)c_h, c_h) + (|\mathbf{u}_h \cdot \mathbf{n}_e| [c_h], [c_h])_{\Gamma_h}\right)$$

And we conclude

$$B_{cq}(c_h, c_h; \mathbf{u}_h) = \frac{1}{2}\left(((q^I + q^P)c_h, c_h) + (|\mathbf{u}_h \cdot \mathbf{n}_e| [c_h], [c_h])_{\Gamma_h}\right) \tag{4.15}$$

Now, we expand the numerical scheme:

$$\int_{t_{n-1}}^{t_n} \left((\phi\partial_t c_h, c_h) + B_d(c_h, c_h; \mathbf{u}_h) + B_{cq}(c_h, c_h; \mathbf{u}_h)\right) + (c_{h+}^{n-1}, \phi c_{h+}^{n-1})$$

$$= (c_{h-}^{n-1}, \phi c_{h+}^{n-1}) + \int_{t_{n-1}}^{t_n} (\hat{c}q^I, c_h)$$

Notice,

$$\int_{t_{n-1}}^{t_n} (\phi\partial_t c_h, c_h) = \int_{t_{n-1}}^{t_n} \frac{1}{2}\partial_t(\phi c_h, c_h) = \frac{1}{2}(\phi c_{h-}^n, c_{h-}^n) - \frac{1}{2}(\phi c_{h+}^{n-1}, c_{h+}^{n-1})$$

Thus, we have

$$\int_{t_{n-1}}^{t_n} (\phi\partial_t c_h, c_h) + (c_{h+}^{n-1}, \phi c_{h+}^{n-1}) = \frac{1}{2}(\phi c_{h-}^n, c_{h-}^n) + \frac{1}{2}(\phi c_{h+}^{n-1}, c_{h+}^{n-1})$$

$$= \frac{1}{2}\left\|\phi^{1/2}c_{h-}^n\right\|^2 + \frac{1}{2}(\phi[c_h^{n-1}]_t, [c_h^{n-1}]_t) + (\phi c_{h+}^{n-1}, c_{h-}^{n-1}) - \frac{1}{2}(\phi c_{h-}^{n-1}, c_{h-}^{n-1})$$

$$= \frac{1}{2}\left\|\phi^{1/2}c_{h-}^n\right\|^2 + \frac{1}{2}\left\|\phi^{1/2}[c_h^{n-1}]_t\right\|^2 + (\phi c_{h+}^{n-1}, c_{h-}^{n-1}) - \frac{1}{2}\left\|\phi^{1/2}c_{h-}^{n-1}\right\|^2$$

Therefore,

$$
\int_{t_{n-1}}^{t_n} \left(B_d(c_h, c_h; \mathbf{u}_h) + B_{cq}(c_h, c_h; \mathbf{u}_h) \right) + \frac{1}{2} \left\| \phi^{1/2} c_{h-}^n \right\|^2 + \frac{1}{2} \left\| \phi^{1/2} [c_h^{n-1}]_t \right\|^2
$$

$$
+ (\phi c_{h+}^{n-1}, c_{h-}^{n-1}) - \frac{1}{2} \left\| \phi^{1/2} c_{h-}^{n-1} \right\|^2 = (c_{h-}^{n-1}, \phi c_{h+}^{n-1}) + \int_{t_{n-1}}^{t_n} (\hat{c} q^I, c_h)
$$

Hence, we obtain

$$
\frac{1}{2} \left\| \phi^{1/2} c_{h-}^n \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| [\phi^{1/2} c_h^{n-1}]_t \right\|_{L^2(\Omega)}^2 + \int_{t_{n-1}}^{t_n} \left(B_d(c_h, c_h; \mathbf{u}_h) + B_{cq}(c_h, c_h; \mathbf{u}_h) \right)
$$

$$
= \frac{1}{2} \left\| \phi^{1/2} c_{h-}^{n-1} \right\|_{L^2(\Omega)}^2 + \int_{t_{n-1}}^{t_n} (\hat{c} q^I, c_h)
$$

The equation above can be simplified into by Lemma 4.2.4 and 4.15.

$$
\frac{1}{2} \left\| \phi^{1/2} c_{h-}^n \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| [\phi^{1/2} c_h^{n-1}]_t \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{t_{n-1}}^{t_n} \left(\|c_h\|_{X_h}^2 + ((q^I + q^P) c_h, c_h) \right.
$$

$$
+ (|\mathbf{u}_h \cdot \mathbf{n}_e| [c_h], [c_h])_{\Gamma_h}) \le \frac{1}{2} \left\| \phi^{1/2} c_{h-}^{n-1} \right\|_{L^2(\Omega)}^2 + \int_{t_{n-1}}^{t_n} (\hat{c} q^I, c_h) \qquad (4.16)
$$

Now, again use Cauchy-Schwarz's inequality and Young's inequality to obtain

$$
(\hat{c} q^I, c_h) \le \left\| \hat{c} \sqrt{q^I} \right\|_{L^2(\Omega)} \left\| \sqrt{q^I} c_h \right\|_{L^2(\Omega)} \le \frac{\left\| \sqrt{q^I} c_h \right\|_{L^2(\Omega)}^2}{2} + \frac{\left\| \hat{c} \sqrt{q^I} \right\|_{L^2(\Omega)}^2}{2}
$$

Thus, substitute this term into (4.16) and have

$$
\frac{1}{2} \left\| \phi^{1/2} c_{h-}^n \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| [\phi^{1/2} c_h^{n-1}]_t \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{t_{n-1}}^{t_n} \left(\|c_h\|_{X_h}^2 + ((q^I + q^P) c_h, c_h) \right.
$$

$$
+ (|\mathbf{u}_h \cdot \mathbf{n}_e| [c_h], [c_h])_{\Gamma_h}) \le \frac{1}{2} \left\| \phi^{1/2} c_{h-}^{n-1} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{t_{n-1}}^{t_n} \left\| \sqrt{q^I} c_h \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{t_{n-1}}^{t_n} \left\| \hat{c} \sqrt{q^I} \right\|_{L^2(\Omega)}^2
$$

Therefore,

$$\left\|\phi^{1/2}c_{h-}^{n}\right\|_{L^2(\Omega)}^2 + \left\|[\phi^{1/2}c_h^{n-1}]_t\right\|_{L^2(\Omega)}^2 + \int_{t_{n-1}}^{t_n}\left(\|c_h\|_{X_h}^2 + \left\|\sqrt{q^P}c_h\right\|_{L^2(\Omega)}^2 + (|\mathbf{u}_h\cdot\mathbf{n}_e|\,[c_h],[c_h])_{\Gamma_h}\right)$$

$$\leq \left\|\phi^{1/2}c_{h-}^{n-1}\right\|_{L^2(\Omega)}^2 + \int_{t_{n-1}}^{t_n}\left\|\sqrt{q^I}\hat{c}\right\|_{L^2(\Omega)}^2.$$

We sum up overall the time interval and obtain:

$$\max_{1\leq n\leq N}\left\|\phi^{1/2}c_{h-}^{n}\right\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N}\left\|[\phi^{1/2}c_h^{n-1}]_t\right\|_{L^2(\Omega)}^2 + \int_0^T\left(\|c_h\|_{X_h}^2 + \left\|\sqrt{q^P}c_h\right\|_{L^2(\Omega)}^2\right.$$

$$\left.+ (|\mathbf{u}_h\cdot\mathbf{n}_e|\,[c_h],[c_h])_{\Gamma_h}\right) \leq \left\|\phi^{1/2}c_{h-}^{0}\right\|_{L^2(\Omega)}^2 + \int_0^T\left\|\sqrt{q^I}\hat{c}\right\|_{L^2(\Omega)}^2$$

Therefore, the scheme is stable for the concentration. Now, we show that $\|c_h\|_{L^2[0,T;H^1(\mathcal{E}_h)]}$ is bounded. Define the semi-norm for $H^1(\mathcal{E}_h)$ to be

$$|v|_{H^1(\mathcal{E}_h)} = \left(\sum_{E\in\mathcal{E}_h}\|\nabla v\|_{L^2(E)}^2 + \sum_{e\in\Gamma_h}h^{-1}\|[v]\|_{L^2(e)}^2\right)^{1/2}$$

and the $H^1(\mathcal{E}_h)$ norm to be

$$\|v\|_{H^1(\mathcal{E}_h)} = \left(\|v\|_{L^2(\Omega)}^2 + |v|_{H^1(\mathcal{E}_h)}^2\right)^{1/2}$$

We exclude the case when $\int_\Omega q^P = 0$, since this implies $q^P = 0$ and $q^I = 0$ which implies $c = 0$ according to (3.1)-(3.3).

Consider $\int_\Omega q^P > 0$, then we apply the Poincaré's inequality for the broken Sobolev space from [7].

$$\|c_h\|_{L^2(\Omega)}^2 \leq C_p^2\left(|c_h|_{H^1(\mathcal{E}_h)}^2 + \left(\int_\Omega \sqrt{q^P}c_h\right)^2\right)^{1/2}$$

where C_p is the Poincaré constant independent of h on a regular mesh. Hence, we use

Cauchy-Schwarz's inequality and obtain

$$\|c_h\|_{L^2(\Omega)}^2 \leq C \left(|c_h|_{H^1(\mathcal{E}_h)}^2 + \left\|\sqrt{q^P}c_h\right\|_{L^2(\Omega)}^2 \right)^{1/2}$$

Therefore,

$$\|c_h\|_{H^1(\mathcal{E}_h)} \lesssim \left(|c_h|_{H^1(\mathcal{E}_h)}^2 + \left\|\sqrt{q^P}c_h\right\|_{L^2(\Omega)}^2 \right)^{1/2} \lesssim \left(\|c_h\|_{X_h}^2 + \left\|\sqrt{q^P}c_h\right\|_{L^2(\Omega)}^2 \right)^{1/2}$$

Therefore, $\|c_h\|_{L^2[0,T;H^1(\mathcal{E}_h)]}$ is bounded as well. $\qquad\square$

This completes the stability analysis which we will find it essential for us to establish the convergence.

4.3 Compactness Theorem for the Concentration

In this section I will lay down the foundation for proving the convergence of the concentration term by establishing a compactness theorem for the concentration.

4.3.1 Generalized Compactness Theorem

First, we state and prove a general compactness theorem that can be applied to broken Sobolev spaces. The proof of the theorem relies on the existing results stated and proved in the Appendix A.

Theorem 4.3.1. *Let H be a Hilbert space with inner-product $(\cdot,\cdot)_H$ and V and W be Banach spaces equipped with norms $\|\cdot\|_V$ and $\|\cdot\|_W$. Assume that $W \subset H$ is dense and*

$$W \hookrightarrow V \hookrightarrow\!\!\!\!\rightarrow H \hookrightarrow W'$$

are embeddings with V compactly embedded in H. Let $h \in (0,\infty)$ be a (mesh) parameter and for each $h > 0$ let $W(\mathcal{E}_h)$ be a Banach space with $W \hookrightarrow W(\mathcal{E}_h) \hookrightarrow V$ where the embedding

constants are independent of h.

For each h, let $W_h \subset W(\mathcal{E}_h)$ be a closed subspace and let $\{t_h^n\}_{n=0}^{N_h}$ be a quasi-uniform family of partitions of $[0,T]$. Let $\Pi_h : H \to W_h$ denote the orthogonal projection, and assume that its restriction to $W(\mathcal{E}_h)$ is stable in the sense that there exists a constant $M > 0$ independent of h such that $\|\Pi_h w\|_{W(\mathcal{E}_h)} \leq M \|w\|_{W(\mathcal{E}_h)}$ for $w \in W(\mathcal{E}_h)$.

Fix $\ell \geq 0$ an integer and $1 < p < \infty$, $1 \leq q < \infty$, with $1/p + 1/q \geq 1$, and assume that

1. For each $h > 0$, $w_h \in \{w_h \in L^p[0,T;W_h] \mid w_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; W_h]\}$ and on each interval satisfies

$$\forall z_h \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; W_h], \quad \int_{t_h^{n-1}}^{t_h^n} (w_{ht}, z_h)_H + (w_{h+}^{n-1} - w_{h-}^{n-1}, z_{h+}^{n-1})_H = \int_{t_h^{n-1}}^{t_h^n} F_h(z_h).$$

2. The sequence $\{w_h\}_{h>0}$ is bounded in $L^p[0,T;V]$.

3. For each $h > 0$, $F_h \in L^q[0,T;W_h']$ and $\{\|F_h\|_{L^q[0,T;W_h']}\}_{h>0} \subset \mathbb{R}$ is bounded.

Then the set $\{w_h\}_{h>0}$ is precompact in $L^p[0,T;H] \cap L^r[0,T;W']$ for each $1 \leq r < \infty$.

Proof. We fix $h > 0$, consider the space $L^p[\delta, T; W(\mathcal{E}_h)]$ with $\sigma > 0$.

The dual space of $L^p[\delta, T; W(\mathcal{E}_h)]$ is $L^{p'}[\delta, T; W(\mathcal{E}_h)']$ with $1/p + 1/p' = 1$.

Since $W(\mathcal{E}_h)$ is a Banach space with $W(\mathcal{E}_h) \hookrightarrow H$, then $W(\mathcal{E}_h)$ is a Hilbert space equipped with the inner-product $(\cdot, \cdot)_H$. Consider an element in the dual space $z \in L^{p'}[\delta, T; W(\mathcal{E}_h)']$, it is identified to an element in $L^p[\delta, T; W(\mathcal{E}_h)]$. Hence, the dual norm for such element is

$$\left(\int_\delta^T \|z(t)\|_{W_h'}^{p'} \, dt \right)^{1/p'} = \sup_{v \in L^p[\delta,T;W(\mathcal{E}_h)]} \frac{\int_\delta^T (z(t), v)_H dt}{\|v\|_{L^p[\delta,T;W(\mathcal{E}_h)]}}$$

We apply this to the function $w_h(t) - w_h(t - \delta)$

$$\left(\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W_h'}^{p'} \, dt \right)^{1/p'} = \sup_{v \in L^p[\delta,T;W(\mathcal{E}_h)]} \frac{\int_\delta^T (w_h(t) - w_h(t - \delta), v)_H dt}{\|v\|_{L^p[\delta,T;W(\mathcal{E}_h)]}}$$

Since the function $t \to w_h(t) - w_h(t - \delta)$ belongs to W_h, we use the definition of the projection Π_h onto W_h to have:

$$\sup_{v \in L^p[\delta, T; W(\mathcal{E}_h)]} \frac{\int_\delta^T (w_h(t) - w_h(t - \delta), v)_H dt}{\|v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}} = \sup_{v \in L^p[\delta, T; W(\mathcal{E}_h)]} \frac{\int_\delta^T (w_h(t) - w_h(t - \delta), \Pi_h v)_H dt}{\|v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}}$$

So we have

$$\left(\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W_h'}^{p'} dt \right)^{1/p'}$$

$$= \sup_{v \in L^p[\delta, T; W(\mathcal{E}_h)]} \frac{\int_\delta^T (w_h(t) - w_h(t - \delta), \Pi_h v)_H dt}{\|\Pi_h v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}} \frac{\|\Pi_h v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}}{\|v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}}$$

Next we use the assumption that $\|\Pi_h v\|_{W(\mathcal{E}_h)} \le M \|v\|_{W(\mathcal{E}_h)}$, this yields:

$$\|\Pi_h v\|_{L^p[\delta, T; W(\mathcal{E}_h)]} \le M \|v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}, \quad \forall v \in L^p[\delta, T; W(\mathcal{E}_h)]$$

So,

$$\left(\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W_h'}^{p'} dt \right)^{1/p'} \le M \sup_{v \in L^p[\delta, T; W(\mathcal{E}_h)]} \frac{\int_\delta^T (w_h(t) - w_h(t - \delta), \Pi_h v)_H dt}{\|\Pi_h v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}}$$

This implies

$$\left(\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W_h'}^{p'} dt \right)^{1/p'} \le M \sup_{v \in L^p[\delta, T; W_h]} \frac{\int_\delta^T (w_h(t) - w_h(t - \delta), v)_H dt}{\|v\|_{L^p[\delta, T; W(\mathcal{E}_h)]}} \quad (4.17)$$

At this point, we want to apply Lemma .0.6 with the spaces $W(\mathcal{E}_h), W, H$ in the lemma to be the spaces $W_h, W(\mathcal{E}_h), H$ of the theorem. First we check the assumptions of the lemma. By the assumptions of the theorem, we have

$$W(\mathcal{E}_h) \hookrightarrow H.$$

Since H is a Hilbert space, this implies

$$H \hookrightarrow W(\mathcal{E}_h)'$$

In addition, since $W \hookrightarrow W(\mathcal{E}_h) \hookrightarrow H$, and W is dense in H, we have that $W(\mathcal{E}_h)$ is dense in H. This implies that H is dense in $W(\mathcal{E}_h)'$ by Lemma .0.2.

Lemma .0.6 then gives that

$$\sup_{v_h \in L^p[\delta,T;W_h]} \frac{\int_\delta^T (w_h(t) - w_h(t-\delta), v_h)_H \, dt}{\|v_h\|_{L^p[\delta,T;W(\mathcal{E}_h)]}} \leq M(\ell, \vartheta) \, \|F_h\|_{L^q[0,T;W_h']} \max(\Delta t, \delta)^{1/q'} \delta^{1/p'}.$$

Thus equation (4.17) becomes (with a different constant M that depends on $\|\Pi_h\|_{\mathcal{L}(W(\mathcal{E}_h),W_h)}$)

$$\left(\int_\delta^T \|w_h(t) - w_h(t-\delta)\|_{W_h'}^{p'} \, dt \right)^{1/p'} \leq M(\ell, \vartheta) \, \|F_h\|_{L^q[0,T;W_h']} \max(\Delta t, \delta)^{1/q'} \delta^{1/p'}.$$

Next, since $W \hookrightarrow W(\mathcal{E}_h)$, we have $W(\mathcal{E}_h)' \hookrightarrow W'$ by Lemma .0.1 so we have for a constant M

$$\|w_h(t) - w_h(t-\delta)\|_{W'} \leq M \, \|w_h(t) - w_h(t-\delta)\|_{W_h'}$$

Therefore

$$\left(\int_\delta^T \|w_h(t) - w_h(t-\delta)\|_{W'}^{p'} \, dt \right)^{1/p'} \leq M(\ell, \vartheta) \, \|F_h\|_{L^q[0,T;W_h']} \max(\Delta t, \delta)^{1/q'} \delta^{1/p'}. \quad (4.18)$$

By assumption, $\|F_h\|_{L^q[0,T;W_h']}$ is uniformly bounded. We now show $(w_h)_{h>0}$ is equicontinuous in $L^{p'}[0,T;W']$:

Fix $\epsilon > 0$. We want to show there is $\delta_0 > 0$ such that

$$\left(\int_\delta^T \|w_h(t) - w_h(t-\delta)\|_{W'}^{p'} \, dt \right)^{1/p'} \leq \epsilon, \quad \forall h > 0, \forall \delta < \delta_0 \quad (4.19)$$

Since $p > 1$, we have $p' < \infty$. Consider the case $q = 1$ first, then $q' = \infty$, and $1/q' = 0$. The

bound (4.18) above becomes for some constant M:

$$\left(\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W'}^{p'} \, dt \right)^{1/p'} \leq M \delta^{1/p'}.$$

Choose δ_0 such that $M \delta_0^{1/p'} < \epsilon$ and we get (4.19).

Consider now the case $q > 1$, then $q' < \infty$. It suffices to find δ_0 such that

$$M \max(\Delta t, \delta_0)^{1/q'} \delta_0^{1/p'} < \epsilon$$

We can assume that $\delta_0 < \Delta t$ and take

$$\delta_0 = \min \left(\frac{1}{2} (\frac{\epsilon}{M \Delta t^{1/q'}})^{p'}, \Delta t \right)$$

We apply now Theorem .0.7 with the spaces $B_0 = V$, $B = W'$. The theorem is recalled below. We first check the assumptions that are required in Theorem .0.7. V and W' are Banach spaces. One can easily show that $V \hookrightarrow W'$ is a compact embedding by lemma .0.4.

By assumption $(w_h)_h$ is bounded in $L^p[0, T; V]$ with $p > 1$. This implies that $(w_h)_h$ is bounded in $L^1[0, T; V]$. In addition, we showed that $(w_h)_h$ is equicontinuous in $L^{p'}[0, T; W']$ for $1 < p' < \infty$. Then, Theorem .0.7 says that for all $0 < \theta < T/2$, the set $(w_h|_{(\theta, T-\theta)})_h$ is precompact in $L^{p'}[\theta, T - \theta; W']$.

Equation (4.18) gives:

$$\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W'}^{p'} \, dt \leq M(p') \max(\Delta t, \delta)^{p'/q'} \delta.$$

Assume now that $0 < \delta < T$, then we have for a constant M independent of δ:

$$\int_\delta^T \|w_h(t) - w_h(t - \delta)\|_{W'}^{p'} \, dt \leq M \delta.$$

We then apply Lemma .0.8 with W and p in the lemma taken equal to W' and p'. We conclude that $w_h \in L^r[0, T; W']$ for any $1 \le r < \infty$.

Remark following Theorem .0.7 also says that if $(w_h)_h$ is bounded in $L^r[0, T; W']$ for some $r > p'$, then we have uniform integrability, and this gives us the precompactness result in $L^{p'}[0, T; W']$. Therefore, we conclude that $(w_h)_h$ is precompact in $L^{p'}[0, T; W']$.

Now, the fact that $(w_h)_h$ is bounded in $L^r[0, T; W']$ for any $1 \le r < \infty$ and that $(w_h)_h$ is precompact in $L^{p'}[0, T; W']$, implies that $(w_h)_h$ is precompact in $L^r[0, T; W']$ for any $1 \le r < \infty$ by Lemma .0.3.

Finally it remains to show that $(w_h)_h$ is precompact in $L^p[0, T; H]$. From a result in [47], we have for all $\epsilon > 0$ there exists $M(\epsilon) > 0$ such that

$$\|w_h(t)\|_H \le \epsilon \|w_h(t)\|_V + M(\epsilon)\|w_h(t)\|_{W'}$$

So,

$$\|w_h\|_{L^p[0,T;H]} \le \epsilon \|w_h\|_{L^p[0,T;V]} + M(\epsilon)\|w_h\|_{L^p[0,T;W']}$$

Since $(w_h)_h$ is bounded in $L^p[0, T; V]$ and precompact in $L^p[0, T; W']$ it follows it is also precompact in $L^p[0, T; H]$ by Lemma .0.5. $\quad\square$

The only thing that remains is to put our numerical scheme in the context of this theorem in order to show the compactness of the concentration $\{c_h\}_{h>0}$.

Before we begin, we shall introduce several function spaces and related concepts. First, we introduce the bounded variation functions or simply BV functions. Define the total variation to be

$$V(u, \Omega) = \sup\{\int_\Omega u \, \text{div}\phi \ : \ \phi \in C^1(\bar{\Omega})^d, \ \|\phi\|_{L^\infty(\Omega)} \le 1\}$$

then the space of the BV functions is defined as

$$BV(\Omega) = \{u \in L^1(\Omega) : V(u, \Omega) < \infty\}$$

It worth to note that $BV(\Omega)$ is a Banach space, but is not separable nor reflexive.

Now, assume X_0 and X_1 are Banach spaces,

$$|u(x)| \le \lambda u_0^{1-\theta}(x) u_1^\theta(x), \text{ with } u_0 \in X_0 \text{ and } u_1 \in X_1, \|u_0\|_{X_0} = \|u_1\|_{X_1} = 1, \text{ and } \lambda, u_0, u_1 \ge 0$$

then we define the fractional space $[X_0, X_1]_\theta$ to be

$$[X_0, X_1]_\theta = \{u : \inf \lambda < \infty\}$$

For this definition we refer to [1]. Now, we set $W = W^{1,4}(\Omega)$, $W(\mathcal{E}_h) = W^{1,4}(\mathcal{E}_h)$, $V = [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2}$ and $H = L^2(\Omega)$, with norm

$$\|w\|_{W(\mathcal{E}_h)} = \left(\|w\|_{L^4(\Omega)}^4 + \sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 + \sum_{e \in \Gamma_h} h^{-3} \|[w]\|_{L^4(e)}^4 \right)^{1/4} \tag{4.20}$$

It is clear that $\|\cdot\|_{W(\mathcal{E}_h)}$ is a norm.

Using Brenner's Poincaré's inequality for Broken Sobolev space [7] and the embedding property from [3] we have:

$$\|w\|_{L^2(\Omega)} \le C \|w\|_V, \quad \|w\|_{L^4(\Omega)} \le C \|w\|_V, \quad \|w\|_{L^4(\Omega)} \le C \|w\|_{W^{1,4}(\Omega)}$$

where the constant C is independent of the mesh size.

Without loss of generality, I will use the regular inner product on $L^2(\Omega)$, rather than the weighted inner product with the weight ϕ since $\phi \in L^\infty$ the inners products are equivalent.

Let Π_h be the L^2 projection to the finite element space. Now, I will verify the properties of

the spaces $W, V, H,$ and W' so that they satisfy the requirements in Theorem 4.3.1.

Lemma 4.3.2. *Let* $W = W^{1,4}(\Omega),$ $W(\mathcal{E}_h) = W^{1,4}(\mathcal{E}_h),$ $V = [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2}$ *and* $H = L^2(\Omega),$ *then* $V,$ $W(\mathcal{E}_h)$ *and* W *are Banach spaces with the norm* $\|\cdot\|_W,$ $\|\cdot\|_{W(\mathcal{E}_h)}$ *and* $\|\cdot\|_V$ *and*

$$W \hookrightarrow V \hookrightarrow\!\!\!\!\rightarrow H \hookrightarrow W'$$

$W \subset H$ *is dense embedding with* V *compactly embedded in* $H,$ *and* $W \hookrightarrow W(\mathcal{E}_h) \hookrightarrow V$ *with the embedding constant independent of* $h.$

Proof. It is clear that W is a Banach space. The spaces $BV(\Omega)$ and $L^4(\Omega)$ are Banach spaces which implies $BV(\Omega) \cap L^4(\Omega)$ is a Banach space. We know that the interpolating space of two Banach spaces is still a Banach space. We can conclude V is a Banach space.

Now, let us verify $W(\mathcal{E}_h)$ is a Banach space. Let the sequence $\{w_n\} \subset W(\mathcal{E}_h)$ be a Cauchy sequence. Thus for any $\epsilon > 0,$ there exists N such that $m, n > N$ implies $\|w_n - w_m\|_{W(\mathcal{E}_h)} < \epsilon.$ Since

$$\|w\|_{W(\mathcal{E}_h)} = \left(\|w\|_{L^4(\Omega)}^4 + \sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 + \sum_{e \in \Gamma_h} h^{-3} \|[w]\|_{L^4(e)}^4 \right)^{1/4}$$

Fix E in $\mathcal{E}_h,$ then the sequence is $\nabla w_n|_E$ is a Cauchy sequence in $L^4(E),$ then this implies

$$\lim_{n \to \infty} \|\nabla w_n|_E - v_E\|_{L^4(E)} = 0 \quad \forall E \in \mathcal{E}_h$$

Also because $\|w\|_{L^4(\Omega)} \leq C \|w\|_{W(\mathcal{E}_h)},$ we have

$$\lim_{n \to n} \|w_n - w_E\|_{L^4(E)} = 0 \quad \forall E \in \mathcal{E}_h$$

Hence, we have

$$\int_E w_E \nabla \cdot \phi = \lim_{n \to \infty} \int_E w_n \nabla \cdot \phi = -\lim_{n \to \infty} \int_E \nabla w_n \cdot \phi = -\int_E v_E \cdot \phi \quad \forall \phi \in C^\infty(\Omega)^d$$

which implies $v_E = \nabla w_E$.

Let w be the function s.t. $w|_E = w_E \quad \forall E \in \mathcal{E}_h$ and by trace theorem,

$$\|w_n - w\|_{L^4(e)} < C_1 \|w_n - w\|_{L^4(E)} + C_2 \|\nabla w_n - \nabla w\|_{L^4(E)}$$

where C_1, C_2 are independent of n.

Therefore,

$$\lim_{n \to \infty} \|w_n - w\|_{W(\mathcal{E}_h)} = 0$$

hence $W(\mathcal{E}_h)$ is a Banach space.

From [3] have the following embeddings.

$$BV(\Omega) \cap L^4(\Omega) \hookrightarrow [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2} \hookrightarrow\hspace{-1.2em}\rightarrow L^2(\Omega)$$

Also, $BV(\Omega) \cap W^{1,4}(\Omega) \hookrightarrow L^4(\Omega)$. Hence, combine the embedding results

$$W^{1,4}(\Omega) \hookrightarrow [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2} \hookrightarrow\hspace{-1.2em}\rightarrow L^2(\Omega)$$

with $[BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2}$ compactly embedded in $L^2(\Omega)$. And from [41] we have $L^2(\Omega) \hookrightarrow W^{1,4}(\Omega)'$. Hence, we can establish the following:

$$W^{1,4}(\Omega) \hookrightarrow [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2} \hookrightarrow\hspace{-1.2em}\rightarrow L^2(\Omega) \hookrightarrow W^{1,4}(\Omega)'$$

We also know that $W^{1,4}(\Omega) \subset L^2(\Omega)$ is dense because $C^\infty(\Omega) \subset W^{1,4}(\Omega)$ is dense in $L^2(\Omega)$. What remains is to show the embedding

$$W \hookrightarrow W(\mathcal{E}_h) \hookrightarrow V$$

with embedding constants independent of h.

First, we observe

$$W^{1,4}(\Omega) \subset W^{1,4}(\mathcal{E}_h), \text{ then } W^{1,4}(\Omega) \hookrightarrow W^{1,4}(\mathcal{E}_h)$$

Second, we notice

$$W^{1,4}(\mathcal{E}_h) \hookrightarrow BV(\Omega) \cap L^4(\Omega) \hookrightarrow [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2}$$

Therefore, we conclude

$$W \hookrightarrow W(\mathcal{E}_h) \hookrightarrow V$$

$\square$

What remains is to show the stability of L^2 projection in the context of the broken Sobolev space.

Lemma 4.3.3. *The L^2 projection*

$$\Pi_h : H \to C_h$$

is stable in $W(\mathcal{E}_h) = W^{1,4}(\mathcal{E}_h)$, i.e. there is a constant $M > 0$ independent of h such that

$$\|\Pi_h w\|_{W(\mathcal{E}_h)} \le M \|w\|_{W(\mathcal{E}_h)} \quad \forall w \in W(\mathcal{E}_h)$$

Proof. Define the semi-norm:

$$|w|_{W^{1,4}(\mathcal{E}_h)} = \left(\sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 + \sum_{e \in \Gamma_h} h^{-3} \|[w]\|_{L^4(e)}^4 \right)^{1/4}$$

Then,

$$\|w\|_{W(\mathcal{E}_h)} = \left(\|w\|_{L^4(\Omega)}^4 + |w|_{W^{1,4}(\mathcal{E}_h)}^4 \right)^{1/4}$$

So,

$$\|\Pi_h w\|_{W(\mathcal{E}_h)} = \left(\|\Pi_h w\|_{L^4(\Omega)}^4 + |\Pi_h w|_{W^{1,4}(\mathcal{E}_h)}^4\right)^{1/4}$$

For the first term we can construct a bound using inverse inequality from Lemma 4.1.1,

$$\|\Pi_h w\|_{L^4(\Omega)} = \left(\sum_{E\in\mathcal{E}_h} \|\Pi_h w\|_{L^4(E)}^4\right)^{1/4} \lesssim \left(\sum_{E\in\mathcal{E}_h} h^{-d}\|\Pi_h w\|_{L^2(E)}^4\right)^{1/4}$$

We use the property

$$\|\Pi_h w\|_{L^2(E)} \le \|w\|_{L^2(E)}$$

and Cauchy-Schwarz's inequality to obtain

$$\|\Pi_h w\|_{L^4(\Omega)} \lesssim \left(\sum_{E\in\mathcal{E}_h} h^{-d}\|w\|_{L^2(E)}^4\right)^{1/4} \lesssim \left(\sum_{E\in\mathcal{E}_h} \|w\|_{L^4(E)}^4\right)^{1/4} = \|w\|_{L^4(\Omega)}$$

For the second term in the $W(\mathcal{E}_h)$ norm, let $\bar{w}$ be the average of w on each element, i.e.

$$\bar{w}|_E = \frac{1}{|E|}\int_E w$$

Thus,

$$|\Pi_h w|_{W^{1,4}(\mathcal{E}_h)} \le |\Pi_h(w-\bar{w})|_{W^{1,4}(\mathcal{E}_h)} + |\Pi_h\bar{w}|_{W^{1,4}(\mathcal{E}_h)}$$

$$\le \left(\sum_{E\in\mathcal{E}_h} \|\nabla\Pi_h(w-\bar{w})\|_{L^4(E)}^4 + \sum_{e\in\Gamma_h} h^{-3}\|[\Pi_h(w-\bar{w})]\|_{L^4(e)}^4\right)^{1/4} + \left(\sum_{e\in\Gamma_h} h^{-3}\|[\Pi_h\bar{w}]\|_{L^4(e)}^4\right)^{1/4}$$

Next, we apply inverse inequality

$$\|\nabla\Pi_h(w-\bar{w})\|_{L^4(E)} \le Mh_E^{-d/4}\|\nabla\Pi_h(w-\bar{w})\|_{L^2(E)} \le Mh_E^{-d/4}h_E^{-1}\|\Pi_h(w-\bar{w})\|_{L^2(E)}$$

$$\le Mh_E^{-d/4}h_E^{-1}\|w-\bar{w}\|_{L^2(E)}$$

We now use Poincaré's inequality,

$$\|\nabla\Pi_h(w-\bar{w})\|_{L^4(E)} \le Mh_E^{-d/4}\|\nabla w\|_{L^2(E)} \le M\|\nabla w\|_{L^4(E)}$$

This implies $\displaystyle\sum_{E\in\mathcal{E}_h}\|\nabla\Pi_h(w-\bar{w})\|_{L^4(E)}^4 \le M\sum_{E\in\mathcal{E}_h}\|\nabla w\|_{L^4(E)}^4$

Furthermore, by trace and inverse inequality we obtain

$$\|\Pi_h(w-\bar{w})\|_{L^4(e)} \le Mh_E^{-1/4}\|\Pi_h(w-\bar{w})\|_{L^4(E)} \le Mh_E^{-1/4}h_E^{-d/4}\|\Pi_h(w-\bar{w})\|_{L^2(E)}$$

$$\le Mh_E^{-1/4}h_E^{-d/4}\|w-\bar{w}\|_{L^2(E)} \le Mh_E^{1/4}h_E^{-d/4}h_E^{-1/2}\|w-\bar{w}\|_{L^2(E)}$$

$$\le Mh_E^{1/4}h_E^{-d/4}h_E^{1/2}\|\nabla w\|_{L^2(E)} \le Mh_E^{1/4}h_E^{-d/4}h_E^{1/2}h_E^{d/4}\|\nabla w\|_{L^4(E)}$$

$$\le Mh^{3/4}\|\nabla w\|_{L^4(E)}$$

Hence, $\displaystyle\sum_{e\in\Gamma_h}h^{-3}\|[\Pi_h(w-\bar{w})]\|_{L^4(e)}^4 \le M\sum_{E\in\mathcal{E}_h}\|\nabla w\|_{L^4(E)}^4.$

For the last term, we have

$$\|[\Pi_h\bar{w}]\|_{L^4(e)} = \|[\bar{w}]\|_{L^4(e)} \le \|[w-\bar{w}]\|_{L^4(e)} + \|[w]\|_{L^4(e)}$$

From [9], we have

$$\sum_{e\subset\Gamma_h}h^{-3}\|[w-\bar{w}]\|_{L^4(e)}^4 \le M\sum_{E\in\mathcal{E}_h}\|\nabla w\|_{L^4(E)}^4$$

Hence, $\sum_{e \subset \mathcal{E}_h^0} h^{-3} \|[\Pi_h \bar{w}]\|_{L^4(e)}^4 \leq M \left(\sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 + \sum_{e \subset \Gamma_h} h^{-3} \|[w]\|_{L^4(e)}^4 \right)$

So, we can conclude

$$|\Pi_h w|_{W^{1,4}(\mathcal{E}_h)} \leq M \left(\sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 \right)^{1/4} + M \left(\sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 + \sum_{e \subset \Gamma_h} h^{-3} \|[w]\|_{L^4(e)}^4 \right)^{1/4}$$

$$\leq M \left(\sum_{E \in \mathcal{E}_h} \|\nabla w\|_{L^4(E)}^4 + \sum_{e \subset \Gamma_h} h^{-3} \|[w]\|_{L^4(e)}^4 \right)^{1/4} \leq M |w|_{W^{1,4}(\mathcal{E}_h)}$$

Thus,

$$\|\Pi_h w\|_{W^{1,4}(\mathcal{E}_h)} \leq M \|w\|_{W^{1,4}(\mathcal{E}_h)}$$

with M independent of the mesh size. Therefore, the L^2 projection is stable. $\qquad \square$

Following the format in Theorem 4.3.1, the scheme can be rewritten as:

$$\int_{t_{n-1}}^{t_n} (c_{ht}, w_h)_H + ([c_h^{n-1}], w_{h+}^{n-1})_H = \int_{t_{n-1}}^{t_n} F_h(w_h) \qquad (4.21)$$

$$F_h(w_h) = (\hat{c}q^I, w_h) - B_d(c_h, w_h; \mathbf{u}_h) - B_{cq}(c_h, w_h; \mathbf{u}_h) \qquad (4.22)$$

where we recall:

$$B_d(c_h, w_h; \mathbf{u}_h) = (\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) - ([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_{\Gamma_h}$$

$$+ \epsilon([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} + (\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h}$$

$$B_{cq}(c_h, w_h; \mathbf{u}_h) = \frac{1}{2} \left((\mathbf{u}_h \nabla c_h, w_h) - (\mathbf{u}_h c_h, \nabla w_h) + ((q^I + q^P)c_h, w_h) \right)$$

$$+ (c_h^{\text{up}} \mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_{\Gamma_h} - (w_h^{\text{down}} \mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h})$$

One still need to show that $F_h \in L^1[0, T; W_h']$ where $W_h = C_h$ and it is bounded. First, I

will show the diffusion term $B_d(c_h, w_h; \mathbf{u}_h)$ is bounded.

4.3.2 Upper Bound for Diffusion

In this subsection, we will obtain an upper bound for the discretization of diffusion.

Lemma 4.3.4. *Given $\mathbf{u}_h \in \mathbf{U}_h$ and c_h, $w_h \in C_h$, then we have*

$$(\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) \lesssim \|c_h\|_{X_h} (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|w_h\|_{W^{1,4}(\mathcal{E}_h)} \tag{4.23}$$

Proof.

$$(\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) \leq \sum_{E \in \mathcal{E}_h} \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(E)} \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h\right\|_{L^2(E)}$$

Notice that by (3.4),

$$\left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla w_h\right\|_{L^2(E)} \lesssim (\int_E (1 + |\mathbf{u}_h|)\,|\nabla w_h|^2)^{1/2} \lesssim \|\nabla w_h\|_{L^2(E)} + (\int_E |\mathbf{u}_h|\,|\nabla w_h|^2)^{1/2}$$

$$\lesssim \|\nabla w_h\|_{L^2(E)} + \|\mathbf{u}_h\|_{L^2(E)}^{1/2} \|\nabla w_h\|_{L^4(E)}$$

So, we have

$$(\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) \lesssim \sum_{E \in \mathcal{E}_h} \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(E)} (\|\nabla w_h\|_{L^2(E)} + \|\mathbf{u}_h\|_{L^2(E)}^{1/2} \|\nabla w_h\|_{L^4(E)})$$

$$\lesssim \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)} (\|\nabla w_h\|_{L^2(\mathcal{E}_h)} + (\sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\|_{L^2(E)} \|\nabla w_h\|_{L^4(E)}^2)^{1/2})$$

$$\lesssim \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)} (\|\nabla w_h\|_{L^2(\mathcal{E}_h)} + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|\nabla w_h\|_{L^4(\mathcal{E}_h)})$$

And consequently using Lemma 4.1.6 we have,

$$(\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) \lesssim \|c_h\|_{X_h} (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

□

Lemma 4.3.5. *Given* $\mathbf{u}_h \in \mathbf{U}_h$ *and* $c_h, w_h \in C_h$, *we have*

$$(\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h} \lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2} R(w_h; \mathbf{u}_h) \tag{4.24}$$

where J *and* R *are defined in (4.9) and (4.10) respectively.*

Proof. We recall from numerical scheme,

$$(\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h} = \sum_{e \in \Gamma_h} \sigma h^{-1} \int_e (1 + \{|\mathbf{u}_h|\})[c_h][w_h]$$

By Cauchy-Schwarz's inequality,

$$(\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h} \lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2} J(w_h, w_h; \mathbf{u}_h)^{1/2}$$

Furthermore, according to (4.13) we have

$$(\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h} \lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2} R(w_h; \mathbf{u}_h)$$

□

Consequently, we can obtain the bound for the diffusion term as follows.

Proposition 4.3.6. *For* $\mathbf{u}_h \in \mathbf{U}_h$ *and* $c_h, w_h \in C_h$, *we have*

$$|B_d(c_h, w_h; \mathbf{u}_h)| \lesssim (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|c_h\|_{X_h} \|w_h\|_{W^{1,4}(\mathcal{E}_h)} \tag{4.25}$$

Proof. We summarize from (4.7), (4.8), (4.23) and (4.24)

$$([c_h], \{\mathbb{D}(\mathbf{u}_h)\nabla w_h \cdot \mathbf{n}_e\})_{\Gamma_h} \lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2}(\|\nabla w_h\|_{L^2(\mathcal{E}_h)} + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|\nabla w_h\|_{L^4(\mathcal{E}_h)})$$

$$([w_h], \{\mathbb{D}(\mathbf{u}_h)\nabla c_h \cdot \mathbf{n}_e\})_{\Gamma_h} \lesssim R(w_h; \mathbf{u}_h) \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)}$$

$$(\mathbb{D}(\mathbf{u}_h)\nabla c_h, \nabla w_h) \lesssim \|c_h\|_{X_h} (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

$$(\sigma h^{-1}(1 + \{|\mathbf{u}_h|\})[c_h], [w_h])_{\Gamma_h} \lesssim J(c_h, c_h; \mathbf{u}_h)^{1/2}R(w_h; \mathbf{u}_h)$$

To sum up what we have,

$$\begin{aligned}
|B_d(c_h, w_h; \mathbf{u}_h)| \lesssim\ & J(c_h, c_h; \mathbf{u}_h)^{1/2}(\|\nabla w_h\|_{L^2(\mathcal{E}_h)} + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|\nabla w_h\|_{L^4(\mathcal{E}_h)}) \\
& + R(w_h; \mathbf{u}_h) \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)} \\
& + \|c_h\|_{X_h} (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|w_h\|_{W^{1,4}(\mathcal{E}_h)} \\
& + J(c_h, c_h; \mathbf{u}_h)^{1/2}R(w_h; \mathbf{u}_h)
\end{aligned}$$

Note that according to the definitions of the norms $\|\cdot\|_{X_h}$ and $\|\cdot\|_{W^{1,4}(\mathcal{E}_h)}$, we have

$$J(c_h, c_h; \mathbf{u}_h)^{1/2} \lesssim \|c_h\|_{X_h} \text{ and } R(w_h; \mathbf{u}_h) \lesssim (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

Therefore,

$$|B_d(c_h, w_h; \mathbf{u}_h)| \lesssim (1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|c_h\|_{X_h} \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

$\square$

Notice that the constant does not depend on the mesh size. So, the diffusion term is bounded. Now, let us bound the convection term.

4.3.3 Upper bound for Convection

For the convection:

$$\begin{aligned}
B_{cq}(c_h, w_h; \mathbf{u}_h) = \frac{1}{2} \Big(& (\mathbf{u}_h\nabla c_h, w_h) - (\mathbf{u}_h c_h, \nabla w_h) + ((q^I + q^P)c_h, w_h) \\
& + (c_h^{\text{up}}\mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h} - (c_h^{\text{down}}\mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h} \Big)
\end{aligned}$$

we can derive the bound as follows.

Proposition 4.3.7. *For* $\mathbf{u}_h \in \mathbf{U}_h$ *and* $c_h, w_h \in C_h$*, we have*

$$|B_{cq}(c_h, w_h; \mathbf{u}_h)|$$
$$\lesssim \left(\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|c_h\|_{X_h} + \|q^I + q^P\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \right) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

$$(4.26)$$

Proof. For the first term we have:

$$(\mathbf{u}_h \nabla c_h, w_h)_E = \int_E \mathbf{u}_h \cdot \nabla c_h w_h \le \int_E |\mathbf{u}_h| \, |\nabla c_h| \, w_h \le \left(\int_E |\mathbf{u}_h| \, |\nabla c_h|^2 \right)^{1/2} \left(\int_E |\mathbf{u}_h| \, w_h^2 \right)^{1/2}$$

$$= \frac{1}{\sqrt{d_\circ}} \left(\int_E d_\circ |\mathbf{u}_h| \, |\nabla c_h|^2 \right)^{1/2} \left(\int_E |\mathbf{u}_h| \, w_h^2 \right)^{1/2}$$

$$\le \frac{1}{\sqrt{d_\circ}} \left(\int_E \mathbb{D}(\mathbf{u}_h) \nabla c_h \cdot \nabla c_h \right)^{1/2} \left(\int_E |\mathbf{u}_h|^2 \right)^{1/4} \left(\int_E w_h^4 \right)^{1/4}$$

$$= \frac{1}{\sqrt{d_\circ}} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(E)} \|\mathbf{u}_h\|_{L^2(E)}^{1/2} \|w_h\|_{L^4(E)}$$

Hence,

$$(\mathbf{u}_h \nabla c_h, w_h) = \sum_{E \in \mathcal{E}_h} (\mathbf{u}_h \nabla c_h, w_h)_E \le \frac{1}{\sqrt{d_\circ}} \sum_{E \in \mathcal{E}_h} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(E)} \|\mathbf{u}_h\|_{L^2(E)}^{1/2} \|w_h\|_{L^4(E)}$$

$$\le \frac{1}{\sqrt{d_\circ}} \left(\sum_{E \in \mathcal{E}_h} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(E)}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\|_{L^2(E)} \|w_h\|_{L^4(E)}^2 \right)^{1/2}$$

$$\le \frac{1}{\sqrt{d_\circ}} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(\mathcal{E}_h)} \left(\sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\|_{L^2(E)}^2 \right)^{1/4} \left(\sum_{E \in \mathcal{E}_h} \|w_h\|_{L^4(E)}^4 \right)^{1/4}$$

$$\le \frac{1}{\sqrt{d_\circ}} \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(\mathcal{E}_h)} \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|w_h\|_{L^4(\Omega)}$$

Therefore,

$$(\mathbf{u}_h \nabla c_h, w_h) \lesssim \left\| \mathbb{D}^{1/2}(\mathbf{u}_h) \nabla c_h \right\|_{L^2(\mathcal{E}_h)} \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|w_h\|_{L^4(\Omega)} \qquad (4.27)$$

For the second term we have:

$$(\mathbf{u}_h c_h, \nabla w_h)_E = \int_E \mathbf{u}_h \cdot \nabla w_h c_h \le \int_E |\mathbf{u}_h| \, |\nabla w_h| \, |c_h| \le \|\mathbf{u}_h\|_{L^2(E)} \|c_h\|_{L^4(E)} \|\nabla w_h\|_{L^4(E)}$$

And then we have:

$$(\mathbf{u}_h c_h, \nabla w_h) = \sum_{E \in \mathcal{E}_h} (\mathbf{u}_h c_h, \nabla w_h)_E \le M \sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\|_{L^2(E)} \|c_h\|_{L^4(E)} \|\nabla w_h\|_{L^4(E)}$$

$$\le M \left(\sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\| \, \|\nabla w_h\|_{L^2(E)}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\|_{L^2(E)} \|c_h\|_{L^2(E)}^2 \right)^{1/2}$$

$$\le M \left(\sum_{E \in \mathcal{E}_h} \|\nabla w_h\|_{L^4(E)}^4 \right)^{1/4} \left(\sum_{E \in \mathcal{E}_h} \|\mathbf{u}_h\|_{L^2(E)}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} \|c_h\|_{L^4(E)}^4 \right)^{1/4}$$

$$\le M \|\nabla w_h\|_{L^4(\mathcal{E}_h)} \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)}$$

Therefore,

$$(\mathbf{u}_h c_h, \nabla w_h) \lesssim \|\nabla w_h\|_{L^4(\mathcal{E}_h)} \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \tag{4.28}$$

We apply the same technique to the term $((q^I + q^p)c_h, w_h)$, then we have:

$$((q^I + q^p)c_h, w_h) \lesssim \|q^I + q^p\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \|w_h\|_{L^4(\Omega)} \tag{4.29}$$

Now, for $(c_h^{\text{up}} \mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_{\Gamma_h}$ we have as follows.

$$(c_h^{\text{up}} \mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_e \le \int_e |c_h^{\text{up}}| \, |\mathbf{u}_h| \, |[w_h]|$$

Notice that,

$$|c_h^{\text{up}}| \le \max\{|c_h^+|, |c_h^-|\} \le |c_h^+| + |c_h^-|$$

Consequently, according to the property of Raviart-Thomas space $\mathbf{u}_h^+ \cdot \mathbf{n}_e = \mathbf{u}_h^- \cdot \mathbf{n}_e$,

$$
\begin{aligned}
(c_h^{\text{up}}\mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_e &\leq \int_e |c_h^+|\,|\mathbf{u}_h^+ \cdot \mathbf{n}_e|\,|[w_h]| + \int_e |c_h^-|\,|\mathbf{u}_h^- \cdot \mathbf{n}_e|\,|[w_h]| \\
&\leq \int_e |c_h^+|\,|\mathbf{u}_h^+|\,|[w_h]| + \int_e |c_h^-|\,|\mathbf{u}_h^-|\,|[w_h]|
\end{aligned}
$$

By Cauchy-Schwarz's inequality, inverse inequality and trace inequality in Lemma 4.1.3, we have

$$
\begin{aligned}
\int_e |c_h^+|\,|\mathbf{u}_h^+|\,|[w_h]| &\leq \left(\int_e |\mathbf{u}_h^+|\,|c_h^+|^2 \right)^{1/2} \left(\int_e |\mathbf{u}_h^+|\,[w_h]^2 \right)^{1/2} \\
&\lesssim \|\mathbf{u}_h\|_{L^2(E_+^e)}^{1/2} \|c_h\|_{L^4(E_+^e)} \left(h^{-1} \int_e \{|\mathbf{u}_h|\}[w_h]^2 \right)^{1/2}
\end{aligned}
$$

Next, we sum up over all interior faces while applying the inverse inequality,

$$
\begin{aligned}
\sum_{e \in \Gamma_h} \int_e |c_h^+|\,|\mathbf{u}_h^+|\,|[w_h]| &\lesssim \sum_{e \in \Gamma_h} \|\mathbf{u}_h\|_{L^2(E_+^e)}^{1/2} \|c_h\|_{L^4(E_+^e)} \left(h^{-1} \int_e \{|\mathbf{u}_h|\}[w_h]^2 \right)^{1/2} \\
&\lesssim \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|c_h\|_{L^4(\Omega)} \left(\sum_{e \in \Gamma_h} h^{-1} \int_e \{|\mathbf{u}_h|\}[w_h]^2 \right)^{1/2}
\end{aligned}
\tag{4.30}
$$

Using (4.12) we have,

$$
\begin{aligned}
\sum_{e \in \Gamma_h} \int_e |c_h^+|\,|\mathbf{u}_h^+|\,|[w_h]| &\lesssim \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|c_h\|_{L^4(\Omega)} \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4 \right)^{1/4} \\
&\lesssim \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4 \right)^{1/4}
\end{aligned}
$$

Therefore, we have

$$
(c_h^{\text{up}}\mathbf{u}_h \cdot \mathbf{n}_e, [w_h])_{\Gamma_h} \lesssim \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4 \right)^{1/4}
\tag{4.31}
$$

We apply the same idea as in (4.30) to the last term and have:

$$(w_h^{\text{down}} \mathbf{u}_h \cdot \mathbf{n}_e, [c_h])_{\Gamma_h} \lesssim \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|w_h\|_{L^4(\Omega)} J(c_h, c_h; \mathbf{u}_h)^{1/2} \qquad (4.32)$$

Therefore, according to (4.27), (4.28), (4.29), (4.31) and (4.32)

$$|B_{cq}(c_h, w_h; \mathbf{u}_h)|$$

$$\lesssim \left\|\mathbb{D}^{1/2}(\mathbf{u}_h)\nabla c_h\right\|_{L^2(\mathcal{E}_h)} \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|w_h\|_{L^4(\Omega)} + \|\nabla w_h\|_{L^4(\mathcal{E}_h)} \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)}$$

$$+ \left\|q^I + q^p\right\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \|w_h\|_{L^4(\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \left(\sum_{e \in \Gamma_h} h^{-3} \int_e [w_h]^4\right)^{1/4}$$

$$+ \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|w_h\|_{L^4(\Omega)} J(c_h, c_h; \mathbf{u}_h)^{1/2}$$

$$\lesssim \left(\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|c_h\|_{X_h} + \left\|q^I + q^P\right\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)}\right) \|w_h\|_{L^4(\Omega)}$$

$$+ \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

Since we have the embedding according to the definition of $W^{1,4}(\mathcal{E}_h)$,

$$W^{1,4}(\mathcal{E}_h) \hookrightarrow L^4(\Omega)$$

the convection term is bounded by

$$|B_{cq}(c_h, w_h; \mathbf{u}_h)|$$

$$\lesssim \left(\|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|c_h\|_{X_h} + \left\|q^I + q^P\right\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)}\right) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

$$\square$$

Theorem 4.3.8. $\{\|F_h\|_{L^1[0,T,W_h']}\}_{h>0}$ *is bounded with* $W_h = C_h$.

Proof. Recall from (4.22),

$$F_h(w_h) = (\hat{c}q^I, w_h) - B_d(c_h, w_h; \mathbf{u}_h) - B_{cq}(c_h, w_h; \mathbf{u}_h)$$

One can easily obtain

$$(\hat{c}q^I, w_h) \leq |\Omega|^{1/4} \left\|q^I\right\|_{L^2(\Omega)} \left\|w_h\right\|_{L^4(\Omega)}$$

Therefore, by (4.25) and (4.26) we have,

$$|F_h(w_h)| \leq M \left((1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|c_h\|_{X_h} + \left\|q^I + q^P\right\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} \right.$$
$$\left. + \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{L^4(\Omega)} + \left\|q^I\right\|_{L^2(\Omega)}\right) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$

with the constant M independent of the mesh.

From [3], we know that

$$\|c_h\|_{L^4(\Omega)} \lesssim \|c_h\|_{H^1(\mathcal{E}_h)}$$

Hence, using Cauchy-Schwarz's inequality

$$\int_0^T |F_h(w_h)| \leq M \int_0^T \left((1 + \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2}) \|c_h\|_{X_h} + \left\|q^I + q^P\right\|_{L^2(\Omega)} \|c_h\|_{H^1(\mathcal{E}_h)} \right.$$
$$\left. + \|\mathbf{u}_h\|_{L^2(\Omega)} \|c_h\|_{H^1(\mathcal{E}_h)} + \left\|q^I\right\|_{L^2(\Omega)}\right) \|w_h\|_{W^{1,4}(\mathcal{E}_h)}$$
$$\leq M \left(\|c_h\|_{L^2[0,T;X_h]} + \left\|q^I\right\|_{L^\infty[0,T;L^2(\Omega)]} + \|c_h\|_{L^2[0,T;X_h]} \|\mathbf{u}_h\|_{L^\infty[0,T;L^2(\Omega)]} \right.$$
$$\left. + \|c_h\|_{L^2[0,T;X_h]} \left\|q^I + q^P\right\|_{L^\infty[0,T;L^2(\Omega)]} + \|\mathbf{u}_h\|_{L^\infty[0,T;L^2(\Omega)]}^{1/2} \|c_h\|_{L^2[0,T;X_h]}\right) \|w_h\|_{L^4[0,T;W^{1,4}(\mathcal{E}_h)]}$$

Therefore,

$$\|F_h(w_h)\|_{L^1[0,T;W_h']} \leq M \left(\|c_h\|_{L^2[0,T;X_h]} + \left\|q^I\right\|_{L^\infty[0,T;L^2(\Omega)]} + \|c_h\|_{L^2[0,T;X_h]} \|\mathbf{u}_h\|_{L^\infty[0,T;L^2(\Omega)]} \right.$$
$$\left. + \|c_h\|_{L^2[0,T;X_h]} \left\|q^I + q^P\right\|_{L^\infty[0,T;L^2(\Omega)]} + \|\mathbf{u}_h\|_{L^\infty[0,T;L^2(\Omega)]}^{1/2} \|c_h\|_{L^2[0,T;X_h]}\right) \|w_h\|_{L^4[0,T;W^{1,4}(\mathcal{E}_h)]}$$

Furthermore, according to the stability analysis in Theorem 4.2.3 and Theorem 4.2.5, we know that $\|\mathbf{u}_h\|_{L^\infty[0,T;L^2(\Omega)]}$, $\|c_h\|_{L^2[0,T;X_h]}$ and $\|c_h\|_{L^2[0,T;X_h]}$ are bounded by a constant determined by the source terms. Therefore, $\{\|F_h\|_{L^1[0,T;W_h']}\}_{h>0}$ is bounded. $\qquad\square$

4.3.4 Compactness of the Concentration

Finally, with all the preliminary results being established and the requirements in the statement of Theorem 4.3.1 being satisfied I will state and prove the compactness theorem.

Theorem 4.3.9. *Suppose the maximal time step Δt tends to zero with the mesh parameter. Then the concentration $\{c_h\}_{h>0}$ computed using our numerical scheme are precompact in $L^2[0,T;L^2(\Omega)] \cap L^r[0,T;W^{1,4}(\Omega)']$ for all $1 \leq r < \infty$.*

Proof. In Lemma 4.3.3, we have shown the stability of L^2 projection in $W(\mathcal{E}_h) = W^{1,4}(\mathcal{E}_h)$. Assumption (1) in Theorem 4.3.1 is immediately satisfied since this the numerical scheme can be rewritten as (4.21). The sequence $\{c_h\}_{h>0}$ is bounded in $H^1(\mathcal{E}_h)$ and $H^1(\mathcal{E}_h) \hookrightarrow BV(\Omega) \cap L^4(\Omega) \hookrightarrow [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2}$, thus, it is bounded in $L^2[0,T;V]$ which shows that assumption (2) is satisfied. Assumption (3) is satisfied by Theorem 4.3.8. Therefore, one can conclude that $\{c_h\}_{h>0}$ is precompact in $L^2[0,T;L^2(\Omega)] \cap L^r[0,T;W^{1,4}(\Omega)']$ for all $1 \leq r < \infty$ by Theorem 4.3.1. $\qquad\qquad\square$

Remark 4.3.10. This result is significant because it allows us to construct a convergence subsequence of the sequence $\{c_h\}_{h>0}$ which will be essential when proving the convergence of the solvent concentration.

4.4 Convergence of the Numerical Solutions

Using the machinery we established from previous section, we are able to use the compactness theorem to construct a convergent subsequence. This result will allow us to establish the convergence of pressure and velocity and eventually the convergence of the concentration to the true solution as we will illustrate next.

4.4.1 Convergence of the Velocity and Pressure

Now, we show the convergence of velocity and pressure using exact argument from [41].

Theorem 4.4.1. *Given the data, parameters and numerical scheme, and suppose the maximal time step Δt tends to zero with the mesh parameter. Suppose that the sequence $\{c_h\}_{h>0} \subset L^2[0, T; L^2(\Omega)]$ converges to c in $L^2[0, T; L^2(\Omega)]$, then the velocity and pressure computed using the scheme (3.8)-(3.10) over the regular family of meshes convergence strongly to the solutions of the weak forms (3.5) and (3.6).*

Proof. For completeness, we repeat the proof given in [41]. Let $\mathbb{U} = L^2[0, T; U]$ and $\mathbb{P} = L^2[0, T; L^2(\Omega)]$ and denote the finite element subspaces to be

$$\mathbb{U}_h = \{\mathbf{u}_h \in \mathbb{U} \mid \mathbf{u}_h|_{(t_{n-1}, t_n)} \in \mathcal{P}_\ell[t_{n-1}, t_n; \mathbf{U}_h]\}, \text{ and}$$

$$\mathbb{P}_h = \{p_h \in \mathbb{P} \mid p_h|_{(t_{n-1}, t_n)} \in \mathcal{P}_\ell[t_{n-1}, t_n; P_h]\}$$

by Lemma 4.2.3 we know the numerical approximation $\{(\mathbf{u}_h, p_h)\}_{h>0}$ are bounded in $\mathbb{U} \times \mathbb{P}$, so we may pass to a subsequence for which $(\mathbf{u}_h, p_h)$ converges weakly to a pair $(\mathbf{u}, p)$ in $\mathbb{U} \times \mathbb{P}$. Also, we can use dominate convergence theorem to show $\mu(c_h) \to \mu(c)$ in $L^r[0, T; L^r(\Omega)]$ for each $1 \leq r < \infty$.

To show $(\mathbf{u}, p)$ is the weak solution of the mixed problem, we fix $(\mathbf{v}, q) \in C^\infty([0, T] \times \bar{\Omega}) \cap (\mathbb{U} \times \mathbb{P})$. Approximation theory tells us that there exists a sequence $((\mathbf{v}_h, q_h))_h \subset \mathbb{U}_h \times \mathbb{P}_h$ such that $(\mathbf{v}_h, q_h) \to (\mathbf{v}, q)$ in $W^{1,\infty}((0, T) \times \Omega)$. Hence, we can pass the limit term-by-term in equation (3.8) and (3.9) to show that

$$\int_0^T (\mathbb{K}^{-1}(c)\mathbf{u}, v) - (p, \mathrm{div}(v)) = \int_0^T (\rho(c)g, v)$$

$$\int_0^T (q, \mathrm{div}(\mathbf{u})) = \int_0^T (q^I - q^P, q)$$

Since $C^\infty([0, T] \times \bar{\Omega}) \cap (\mathbb{U} \times \mathbb{P})$ is dense in $\mathbb{U} \times \mathbb{P}$, it follows that $(\mathbf{u}, p)$ is a weak solution of the mixed problem.

In order to show strong convergence we introduce the notation $b(\cdot, \cdot; c)$ such that for a

fixed $c \in L^2[0, T; L^2(\Omega)]$ we have $b(\cdot, \cdot; c) : (\mathbb{U} \times \mathbb{P})^2 \to \mathbb{R}$ where

$$b((\mathbf{u}, p), (\mathbf{v}, q); c) = \int_0^T \left((\mathbb{K}^{-1}(c)\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}(\mathbf{v})) + (q, \operatorname{div}(\mathbf{u})) \right)$$

Lemma 4.2.1 shows that $b(\cdot, \cdot; c)$ is coercive on $\mathbb{U}_h \times \mathbb{P}_h$. Cleary, $b(\cdot, \cdot; c)$ is continuous. Hence, we can use the Strang's Lemma

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathbb{U} \times \mathbb{P}} \leq \inf_{(\mathbf{v}_h, q_h) \in \mathbb{U}_h \times \mathbb{P}_h} \|(\mathbf{u} - \mathbf{v}_h, p - q_h)\|_{\mathbb{U} \times \mathbb{P}}$$
$$+ \sup_{(\mathbf{v}_h, q_h) \in \mathbb{U}_h \times \mathbb{P}_h} \frac{|b((\mathbf{u}, p), (\mathbf{v}_h, q_h); c) - b((\mathbf{u}, p), (\mathbf{v}_h, q_h); c_h)|}{\|(\mathbf{v}_h, q_h)\|_{\mathbb{U} \times \mathbb{P}}}$$

Since we have

$$b((\mathbf{u}, p), (\mathbf{v}_h, q_h); c) - b((\mathbf{u}, p), (\mathbf{v}_h, q_h); c_h) = \int_0^T (\mathbb{K}^{-1}(c) - \mathbb{K}^{-1}(c_h))\mathbf{u}, \mathbf{v}_h)$$

so

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathbb{U} \times \mathbb{P}}$$
$$\leq \inf_{(\mathbf{v}_h, q_h) \in \mathbb{U}_h \times \mathbb{P}_h} \|(\mathbf{u} - \mathbf{v}_h, p - q_h)\|_{\mathbb{U} \times \mathbb{P}} + \left\| (\mathbb{K}^{-1}(c) - \mathbb{K}^{-1}(c_h))\mathbf{u} \right\|_{L^2[0, T; L^2(\Omega)]}$$

The assumptions on $\mathbb{K}$ guarantee that $\left|\mathbb{K}^{-1}(c_h)\mathbf{u}\right|^2$ converges pointwise to $\left|\mathbb{K}^{-1}(c)\mathbf{u}\right|^2$, and since $\mathbb{K}^{-1}$ takes values in a compact set it follows that $\left|\mathbb{K}^{-1}(c_h)\mathbf{u}\right|^2 \leq M \left|\mathbf{u}\right|^2$. Apply the dominated convergence theorem shows $\mathbb{K}^{-1}(c_h)\mathbf{u} \to \mathbb{K}^{-1}(c)\mathbf{u}$ in $L^2[0, T; L^2(\Omega)]$, and strong convergence of the velocity and pressure follows. $\qquad\square$

4.4.2 Convergence of the Concentration

To prove the convergence of concentration, we first state a result related to the approximation spaces from [3]. The result concerns the convergence of sequence from DG approximation

spaces.

Lemma 4.4.2. *Consider a sequence* $\{v_h\} \in \prod_{n=1}^{N} \mathcal{P}_\ell[t_{n-1}, t_n, C_h]$, *such that*

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|v_h\|_{H^1(\mathcal{E}_h)}^2 \, dt < M \tag{4.33}$$

for some $M > 0$. *Then there exists a subsequence* $\{v_h\}_h$ *which converges weakly to* $v \in L^2[(0, T) \times \Omega]$. *As* $h \to 0$ *every weak accumulation point in* $L^2[(0, T) \times \Omega]$ *belongs to* $L^2[0, T, H^1(\Omega)]$. *Moreover,* $\|v\|_{L^2[0,T,H^1(\Omega)]} \lesssim M$, *and the gradients* $\{\nabla v_h\}_h$ *converges weakly in* $L^2[0, T; H^{-1}(\Omega)]$ *to* ∇v.

We now show the convergence of the concentration.

Theorem 4.4.3. *Suppose that the maximal time step* Δt *and* h *tend to zero with mesh parameter. Then upon passage to a subsequence, the concentrations* $\{c_h\}_h$ *computed using the scheme (3.8)-(3.10) over a regular family of meshes converge strongly in* $L^2[(0, T) \times \Omega]$ *to* $c \in L^2[0, T; H^1(\Omega)]$ *and* $\{\nabla c_h\}_h$ *converges weakly in* $L^2[0, T; H^{-1}(\Omega)]$ *to* ∇c.

Proof. From Theorem 4.3.9 we know $\{c_h\}_{h>0}$ is precompact in $L^2[0, T; L^2(\Omega)] \cap L^r[0, T; W^{1,4}(\Omega)']$ for all $1 \leq r < \infty$ by Theorem 4.3.1. There exists a subsequence $\{c_h\}_h$ that converges to $c \in L^2[0, T; L^2(\Omega)]$ strongly in $L^2[0, T; L^2(\Omega)]$. The condition (4.33) in Lemma 4.4.2 is satisfied since from boundedness of the concentration from Theorem 4.2.5, there exists $M > 0$ such that $\|c_h\|_{L^2[0,T;H^1(\mathcal{E}_h)]} < M$. Therefore, there exists a subsequence $\{\nabla c_h\}_h$ that converges weakly in $L^2[0, T; H^{-1}(\Omega)]$ to ∇c. $\qquad\square$

Remark 4.4.4. The analysis has showed the convergence of the concentration $\{c_h\}_{h>0}$ to a solution. Further analysis is required for us to show that the solution satisfies the weak form (3.7). However, if we use an L^2-projection of the diffusion-dispersion tensor

$$\Pi_h : \mathbb{D} \to \mathbb{D}_h$$

in our numerical scheme as in [3], then one can prove the convergence of the concentration to weak solution in (3.7) using SIPG.

Chapter 5

Numerical Examples

In this chapter, numerical simulations of the miscible displacement problem are given in two and three dimensions for analytical and physical problems. Convergence rates of the numerical solutions with respect to time and space will be presented.

I shall begin by offering some more detail information about the numerical implementation.

5.1   Implementation Outline

For the numerical implementation, due to the coupling nature and nonlinearity of the PDE system, it would require us to use Newton's method to solve the coupled equations. But, for simplicity we use the decoupling concept and numerical quadrature for time integration, hence on each sequential update we only need to solve two separate linear PDE systems.

However, there is a challenge concerning using the sequential update. At time t_{n-1} we only know the value of c_h from 0 to t_{n-1}. But, if we use the numerical quadrature for integration over the time $[t_{n-1}, t_n]$, the time integration for the Darcy's Law requires us to know the value of c_h at each quadrature point or at least some accurate approximations over the interval. If $\mathbf{u}_h$ is not very sensitive to the time fluctuation we can simply use $c_h(\cdot, t_{n-1})$ which is nothing but a 1st-order approximation of the numerical integral in time, although this most likely will cause deterioration of the convergent rate. One can also use extrapolation to approximate c_h at the quadrature points using the previous computed c_h, or approximate $\mathbf{u}_h$ at the quadrature points using the previous computed $\mathbf{u}_h$ see [33], with the assumption that the function is continuous in time. What we will do is to introduce a class

of diagonalizable DG time updating which we will discuss in further detail next section. We consider a time stepping method to be diagonalizable if the upper triangular entries Butcher's tableau are all zero. So, instead of solving the Darcy's Law over the entire time domain we approximate the velocity at first quadrature point call it $\mathbf{u}_h^{(1)}$ using $c_h(\cdot, t_{n-1})$. Since the time updating is diagonalizable we can use $\mathbf{u}_h^{(1)}$ to solve for $c_h^{(1)}$. And with $c_h^{(1)}$ we can have a better approximation of $\mathbf{u}_h^{(2)}$. Therefore, instead solving $\mathbf{u}_h^{(1)}, \mathbf{u}_h^{(2)}, \cdots, \mathbf{u}_h^{(s)}$ over $[t_{n-1}, t_n]$ once for all in each update in the Darcy's Law where s is the number of quadrature points, we bootstrap them according to our need while updating the transport equation.
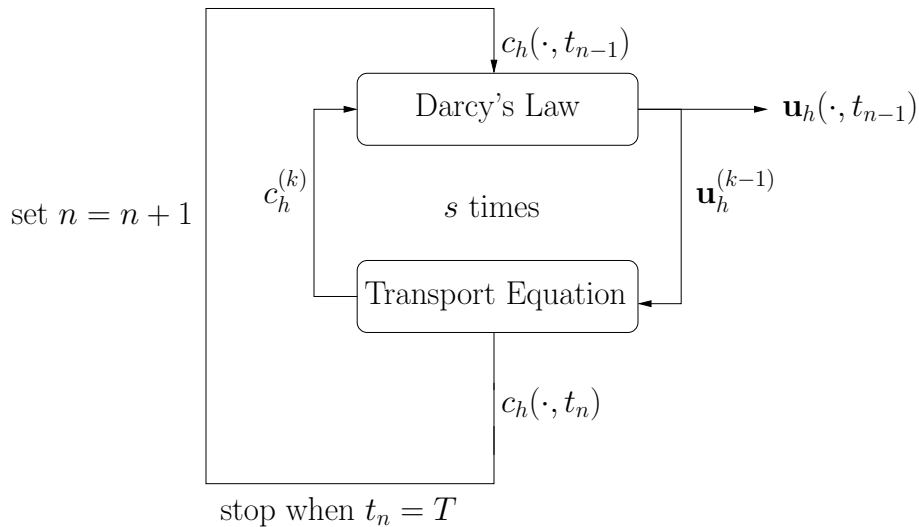
Figure 5.1.1 : Diagram for the numerical algorithm

Figure 5.1.1 above illustrates the concept of the decoupling algorithm. We give the algorithm as follows.

Algorithm 5.1.1. *For $n \leq N$ with $t_N = T$, set $c_h^{(0)} = c_h(\cdot, t_{n-1})$. Let i go from 1 to s where s is the number of quadrature points.*

Find $(\mathbf{u}_h^{(i)}, p_h^{(i)}) \in (\mathbf{U}_h, P_h)$ *such that*

$$\left((\mathbb{K}^{-1}(c_h^{(i-1)})\mathbf{u}_h, \mathbf{v}_h) - (p_h^{(i)}, div(\mathbf{v}_h)) \right) = (\rho(c_h^{(i-1)})\mathbf{g}, \mathbf{v}_h)$$

$$(q_h, div(\mathbf{u}_h^{(i)})) = ((q^I - q^P)^{(i)}, q_h)$$

Find $c_h^{(i)} \in C_h$ *using DG time updating with previously computed* $c_h^{(0)}, \cdots, c_h^{(i-1)}$ *and* $\mathbf{u}_h^{(1)}, \cdots,$ $\mathbf{u}_h^{(i)}$.

Update $c_h(\cdot, t_n)$ *and set* $n = n + 1$.

The implementation of this decoupling scheme rests on the diagonalizability of DG time updating we use. In the next section, we will give a detail description of the diagonalizable DG time updating.

5.2 Implementation Details

5.2.1 Implementing DG in Time

One of the biggest challenges in the implementation is to translate rather abstract DG time-steppings into practice. The most practical way for the implementation is to use Butcher tableaux. Here, I will introduce a unified approach to accomplish the task. I will illustrate how to obtain Butcher Tableaux for 1st-order DG in time (DG0) up to 4th-order DG in time (DG3).

Recall the discretization of the transport equation

$$\int_{t_{n-1}}^{t_n} \left((\phi\partial_t c_h, w_h) + B_d(c_h, w_h; \mathbf{u}_h) + B_{cq}(c_h, w_h; \mathbf{u}_h) \right) + \left(\left[c_h^{n-1} \right]_t, \phi w_{h+}^{n-1} \right) = \int_{t_{n-1}}^{t_n} (\hat{c}q^I, w_h)$$

with $\mathbf{u}_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h], c_h \in P_\ell[t^{n-1}, t^n; C_h]$.

We can regard the discretization as

$$\int_{t_{n-1}}^{t_n} (\partial_t c, w)_H + ([c^{n-1}]_t, w_+^{n-1})_H = \int_{t_{n-1}}^{t_n} (f(c), w)$$

with $c, w \in P_\ell[t^{n-1}, t^n; C_h]$ for simplicity, where $(\cdot, \cdot)_H$ is the weighted inner product with the weight ϕ.

We use the integration by part for the first term,

$$\int_{t_{n-1}}^{t_n} (\partial_t c, w)_H = -\int_{t_{n-1}}^{t_n} (c, \partial_t w)_H + (c_-^n, w_-^n) - (c_+^{n-1}, w_+^{n-1})_H$$

Therefore, the scheme becomes

$$(c_-^n, w_-^n)_H = (c_-^{n-1}, w_+^{n-1})_H + \int_{t_{n-1}}^{t_n} (c, \partial_t w)_H + \int_{t_{n-1}}^{t_n} (f(c), w) \tag{5.1}$$

DG0

We select the basis functions on the reference time interval $[0, 1]$ to be the piecewise constant function. So, the scheme (5.1) becomes

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \int_{t_{n-1}}^{t_n} (f(c), w) \; \forall w \in \mathcal{P}_0[t_{n-1}, t_n, \mathcal{P}_k(\mathcal{E}_h)]$$

Since we use piecewise constant approximation in time, the integral can be approximated as

$$\int_{t_{n-1}}^{t_n} (f(c), w) \approx \Delta t (f(c^n), w)$$

to get the first order accuracy in our implementation.

Hence, we have

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t(f(c^n), w)$$

Therefore, we can construct the Butcher tableau for DG0 time stepping

$$
\begin{array}{c|c}
1 & \\
\hline
 & 1
\end{array}
$$

Notice, the time-stepping is Backward Euler which is a first-order method.

DG1

In this case we use the Gauss I quadrature with quadrature points and weights over the interval $[0, 1]$

$$Q = \left\{ \frac{1}{2} \right\} \text{ and } W = \{1\}$$

Define

$$c^{(1)} = c(t_{n-1} + \Delta t Q_1)$$

We pick the basis functions on the reference time interval $[0, 1]$ to be

$$\{1, 2x - 1\}$$

So, the basis on $I_n = [t_{n-1}, t_n]$ is

$$\left\{ 1, \frac{2}{\Delta t}(t - t_{n-1}) - 1 \right\} = \{p_0, p_1\}$$

For $p_0 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t(f(c^{(1)}), w) \quad \forall w \in \mathcal{P}_k(\mathcal{E}_h) \tag{5.2}$$

For $p_1 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = -(c_-^{n-1}, w)_H + 2(c^{(1)}, w)_H \quad \forall w \in \mathcal{P}_k(\mathcal{E}_h) \tag{5.3}$$

Equations (5.2) and (5.3) imply

$$2(c_-^{n-1}, w)_H + \Delta t(f(c^{(1)}), w) = 2(c^{(1)}, w)_H \quad \forall w \in \mathcal{P}_k(\mathcal{E}_h)$$

Hence, we have for all $w \in \mathcal{P}_k(\mathcal{E}_h)$

$$(c^{(1)}, w)_H = (c_-^{n-1}, w)_H + \frac{1}{2}\Delta t(f(c^{(1)}), w)$$

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t(f(c^{(1)}), w)$$

Therefore, we can construct the Butcher tableau for DG1 time stepping

$$
\begin{array}{c|c}
1/2 & 1/2 \\
\hline
 & 1
\end{array}
$$

This is a second-order method [11].

DG2

In this case we use the Radau II quadrature with quadrature points and weights over the interval $[0, 1]$

$$Q = \left\{ \frac{1}{3}, 1 \right\} \quad \text{and} \quad W = \left\{ \frac{3}{4}, \frac{1}{4} \right\}$$

Define

$$c^{(i)} = c(t_{n-1} + \Delta t Q_i)$$

We pick the basis functions on the reference time interval $[0, 1]$ to be

$$\left\{ 1, x, \frac{1}{2}(3x^2 - 1) \right\}$$

So, the basis on $I_n = [t_{n-1}, t_n]$ is

$$\left\{ 1, \frac{1}{\Delta t}(t - t_{n-1}), \frac{3}{2\Delta t^2}(t - t_{n-1})^2 - \frac{1}{2} \right\} = \{p_0, p_1, p_2\}$$

For $p_0 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{3}{4}(f(c^{(1)}), w) + \Delta t \frac{1}{4}(f(c^{(2)}), w) \tag{5.4}$$

For $p_1 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = \frac{3}{4}(c^{(1)}, w)_H + \frac{1}{4}(c^{(2)}, w)_H + \Delta t \frac{1}{4}(f(c^{(1)}), w) + \Delta t \frac{1}{4}(f(c^{(2)}), w) \tag{5.5}$$

For $p_2 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = -\frac{1}{2}(c_-^{n-1}, w)_H + \frac{3}{4}(c^{(1)}, w)_H + \frac{3}{4}(c^{(2)}, w)_H - \Delta t \frac{1}{4}(f(c^{(1)}), w) + \Delta t \frac{1}{4}(f(c^{(2)}), w) \tag{5.6}$$

with $w \in \mathcal{P}_k(\mathcal{E}_h)$. Thus, (5.4)-(5.5) implies

$$0 = (c_-^{n-1}, w)_H - \frac{3}{4}(c^{(1)}, w)_H - \frac{1}{4}(c^{(2)}, w)_H + \Delta t \frac{1}{2}(f(c^{(1)}), w) \tag{5.7}$$

(5.5)-(5.6) implies

$$0 = \frac{1}{2}(c_-^{n-1}, w)_H - \frac{1}{2}(c^{(2)}, w)_H + \Delta t \frac{1}{2}(f(c^{(1)}), w) \tag{5.8}$$

(5.7)-$\frac{1}{2}$(5.8) implies

$$(c^{(1)}, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{1}{3}(f(c^{(1)}), w) \tag{5.9}$$

From (5.8) we have

$$(c^{(2)}, w)_H = (c_-^{n-1}, w)_H + \Delta t(f(c^{(1)}), w) \tag{5.10}$$

Hence, from (5.4),(5.9) and (5.10) we have for all $w \in \mathcal{P}_k(\mathcal{E}_h)$

$$(c^{(1)}, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{1}{3}(f(c^{(1)}), w)$$

$$(c^{(2)}, w)_H = (c_-^{n-1}, w)_H + \Delta t(f(c^{(1)}), w)$$

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{3}{4}(f(c^{(1)}), w) + \Delta t \frac{1}{4}(f(c^{(2)}), w)$$

Therefore, we can construct the Butcher tableau for DG2 time stepping

$$
\begin{array}{c|cc}
1/3 & 1/3 & 0 \\
1 & 1 & 0 \\
\hline
& 3/4 & 1/4
\end{array}
$$

This is a third-order method [11].

DG3

In this case we use the Lobatto III quadrature with quadrature points and weights over the interval $[0, 1]$

$$Q = \left\{ 0, \frac{1}{2}, 1 \right\} \text{ and } W = \left\{ \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \right\}$$

We pick the basis functions on the reference time interval $[0, 1]$ to be

$$\left\{ 1, x, \frac{1}{2}(3x^2 - 1) \right\}$$

and we require an additional constraint for the polynomial on the interval $I_n = [t_{n-1}, t_n]$ such that

$$c(t_{n-1}) = c^{n-1}$$

So, the basis on I_n is

$$\left\{ 1, \frac{1}{\Delta t}(t - t_{n-1}), \frac{3}{2\Delta t^2}(t - t_{n-1})^2 - \frac{1}{2} \right\} = \{p_0, p_1, p_2\}$$

For $p_0 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{1}{6}(f(c^{(1)}), w) + \Delta t \frac{2}{3}(f(c^{(2)}), w) + \Delta t \frac{1}{6}(f(c^{(3)}), w) \tag{5.11}$$

For $p_1 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = \frac{1}{6}(c^{(1)}, w)_H + \frac{2}{3}(c^{(2)}, w)_H + \frac{1}{6}(c^{(3)}, w)_H + \Delta t \frac{1}{3}(f(c^{(2)}), w) + \Delta t \frac{1}{6}(f(c^{(3)}), w) \tag{5.12}$$

For $p_2 w$, the scheme (5.1) becomes

$$(c_-^n, w)_H = -\frac{1}{2}(c_-^{n-1}, w)_H + (c^{(2)}, w)_H + \frac{1}{2}(c^{(3)}, w)_H - \Delta t \frac{1}{12}(f(c^{(1)}), w) - \Delta t \frac{1}{12}(f(c^{(2)}), w)$$

$$+ \Delta t \frac{1}{6}(f(c^{(3)}), w)$$

$$(5.13)$$

With the additional constraint $c^{(1)} = c(t_{n-1})$ on the polynomial basis, we have

$$(c^{(1)}, w)_H = (c_-^{n-1}, w)_H \tag{5.14}$$

We combine the equations (5.11), (5.12), (5.13) and (5.14) and have for all $w \in \mathcal{P}_k(\mathcal{E}_h)$

$$(c^{(1)}, w)_H = (c_-^{n-1}, w)_H$$

$$(c^{(2)}, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{1}{4}(f(c^{(1)}), w) + \Delta t \frac{1}{4}(f(c^{(2)}), w)$$

$$(c^{(3)}, w)_H = (c_-^{n-1}, w)_H \qquad\qquad\qquad + \Delta t \frac{1}{4}(f(c^{(2)}), w)$$

$$(c_-^n, w)_H = (c_-^{n-1}, w)_H + \Delta t \frac{1}{6}(f(c^{(1)}), w) \;\; + \Delta t \frac{2}{3}(f(c^{(2)}), w) + \Delta t \frac{1}{6}(f(c^{(3)}), w)$$

Therefore, we can construct the Butcher tableau for DG3 time stepping

0	0	0	0
1/2	1/4	1/4	0
1	0	1	0
	1/6	2/3	1/6

This is a fourth-order method [11].

One should notice that we have a particular way of choosing the basis functions to guarantee the upper-triangular entries of Butcher's table to be zeros. Only these types of time-stepping schemes can be incorporated into the Algorithm 5.1.1 since we need to have

the previous intermediate value of the concentration to approximate the velocity at next intermediate point over each time interval. For this approach of deriving the time-updating scheme, we refer to the survey done by Gottlieb et al [30].

5.2.2   DUNE and DUNE-PDELab Software

For the numerical experiment, I decided to use DUNE and DUNE-PDELab for 2D and 3D implementation. DUNE is an open source C++ library for solving partial differential equations which has undergone active development since 2002 by several universities [4]. The main purpose is to take advantage of the object oriented programming, whereby to enhance the flexibility and the productivity of the numerical implementation. DUNE consists of several modules: dune-common, dune-grid, dune-localfunction, dune-istl.(see: Fig(5.2.2)) The dune-geometry is added in the 2.2 release.

The basic classes such as vector, matrices and parallel computing tools are included in dune-common. Dune-grid is used for abstract grid and mesh interface. The iterative solver library is contained in dune-istl. The interface for finite element shape functions is provided by dune-localfunction. Also, DUNE can be linked with several external libraries, such as SuperLU, ALUGrid, Alberta, METIS, ParMETIS.
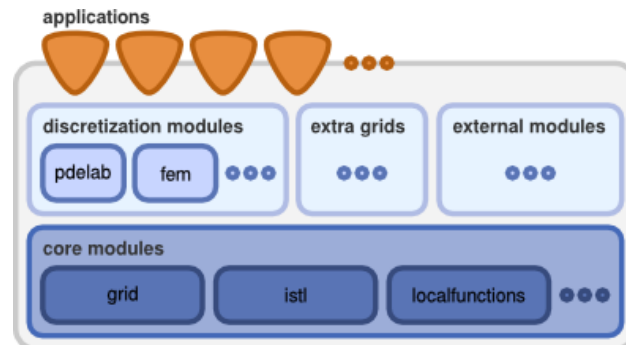

Figure 5.2.2 : DUNE design*

DUNE-PDELab is a discretization module based on DUNE, that allows rapid prototyping the numerical scheme. The development of DUNE-PDELab started in 2009. A large number

of finite element spaces were added into this module for solving different types of problems both stationary and time dependent. The other attraction for me to use this package is that it provides several easily modifiable time stepping schemes, so that DG time stepping can be incorporated by modifying the existing source code for the time stepping method.

Here are the details of contributions of my application using DUNE-PDELab to solve miscible displacement equations. I used the internal mesh generator `YaspGrid` in DUNE to create and refine rectangular meshes for the computational domain. I selected monomial and Raviart-Thomas finite element basis in DUNE-PDELab for space discretization to solve the Darcy's law using the mixed finite element method. Also, I have created subroutine to handle the pure Neumann boundary condition for the Darcy's law. For simplicity, the monomial basis is used in space discretization for the transport equation using DG scheme. Because those two equations are dependent, it has required me considerable effort to modify the time stepping method in DUNE-PDELab for DG time stepping. For the implementation of the Raviart-Thomas method, the discretization has been modified to handle pure Nuemann boundary condition. I used external package `SuperLU` to solve the assembled systems. The numerical results will be presented in the next section.

---

*www.dune-project.org/

5.3 Numerical Results

In this section we present both numerical results for analytical and physical problems. We will observe the advantages of using high order methods for solving the miscible displacement problems. We begin by defining some notation. If we use $\mathbf{U}_h = RT_i(\mathcal{E}_h)$ for the Raviart-Thomas element for the Darcy's Law and $C_h = \{c_h \in H^1(\mathcal{E}_h) : c_h|_E \in \mathcal{P}_j(E), E \in \mathcal{E}_h\}$ for the transport equation, then we express the space discretization as RT_i-$NIPG_j$, RT_i-$SIPG_j$ or RT_i-$IIPG_j$ depending on the DG discretization.

5.3.1 Analytical Problem and Convergence Rate

Consider an problem with analytical solutions,

$$p(x, y, t) = \left(2 - e^{-x}\left(1 + x + x^2\right) - e^{-y}\left(1 + y + y^2\right)\right) e^{\frac{\pi t}{2}}$$
$$c(x, y, t) = \frac{1}{2}\left(\sin(2\pi x)^2 + \cos(2\pi y)^2\right) \sin\left(\frac{\pi t}{2}\right)$$

given the parameters,

$$\phi = 0.2 \ , \ \mathbb{K}(c) = \frac{9.44 \times 10^{-3}}{1 + (0.0524c)^{4.74}} \ , \ g = 0 \ , \ q^I = 1$$
$$\mathbb{D}(\mathbf{u}) = \frac{\mathbf{u}\mathbf{u}^T}{|\mathbf{u}|}\left(1.8 \times 10^{-5} - 1.8 \times 10^{-6}\right) + \left(1.8 \times 10^{-7} + 1.8 \times 10^{-6}|\mathbf{u}|\right)\mathbf{I}$$

The profiles of the function take form at t=1.0 as follows in Figure 5.3.3, 5.3.4 and 5.3.5.
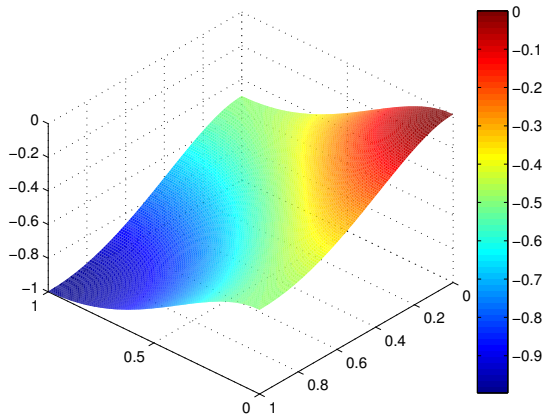
Figure 5.3.3 : exact p at time $t = 1$
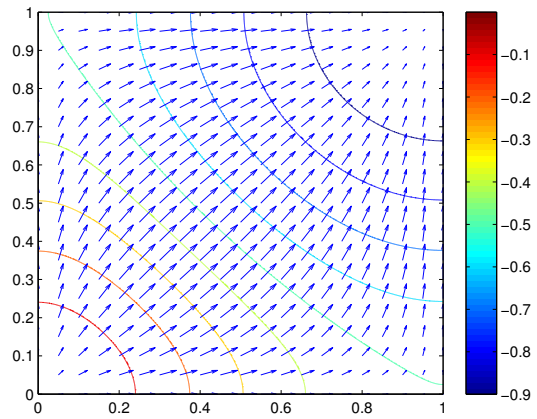
Figure 5.3.4 : exact $\mathbf{u}$ at time $t = 1$
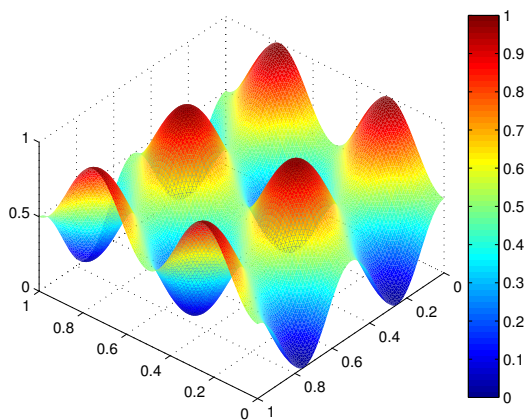
Figure 5.3.5 : exact c at time $t = 1$

For the discretization in space we use $RT_0\text{-}NIPG_1$, $RT_1\text{-}NIPG_2$, and $RT_2\text{-}NIPG_3$, with DG3 in time to obtain high accuracy in time with time step $\Delta t = 0.01$ and obtain the convergence rate at time $t = 0.5$ as follows.
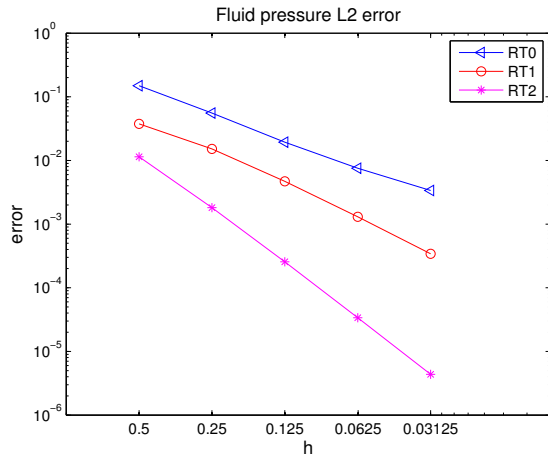
Figure 5.3.6 : cvg. rate for p in L^2
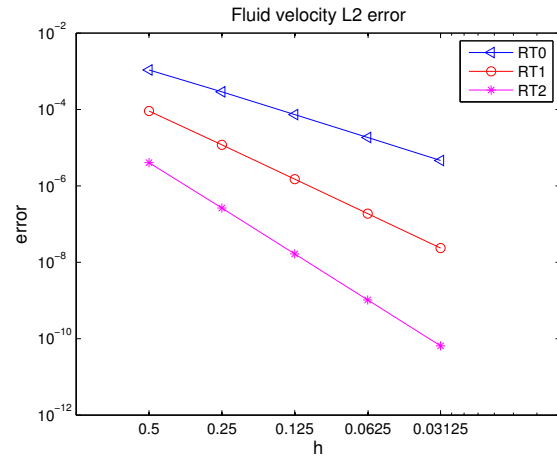
Figure 5.3.7 : cvg. rate for $\mathbf{u}$ in L^2
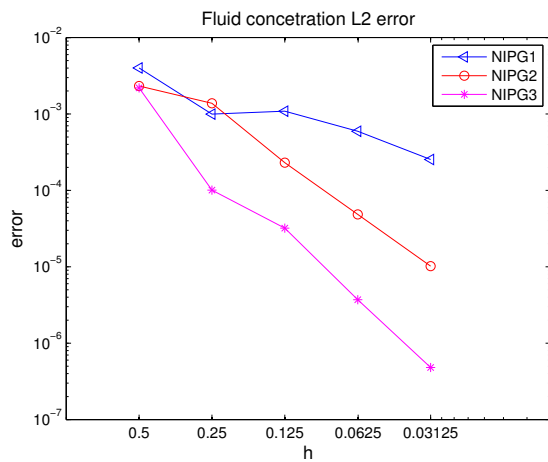
Figure 5.3.8 : cvg. rate for c in L^2
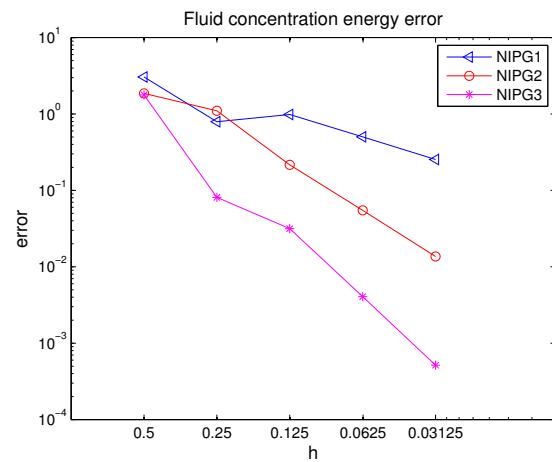
Figure 5.3.9 : cvg. rate for c in energy norm

Indeed, we have observed the increases of the convergence rates as the order of approximations increase as in Figure 5.3.6, 5.3.7, 5.3.8 and 5.3.9. We also present the error.

Pressure				
h	$\|p - p_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
2^{-1}	1.49467723e-1	–	1.08683259e-3	–
2^{-2}	5.55713388e-2	1.427	2.90026132e-4	1.906
2^{-3}	1.94075368e-2	1.518	7.37036372e-5	1.976
2^{-4}	7.53977315e-3	1.364	1.85015781e-5	1.994
2^{-5}	3.37151442e-3	1.161	4.63013722e-6	1.999
Concentration				
h	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
2^{-1}	4.00554402e-3	–	3.04940173	–
2^{-2}	1.08790458e-3	2.003	7.92174173e-1	1.945
2^{-3}	1.08790458e-3	-0.123	9.86513989e-1	-0.317
2^{-4}	5.93557729e-4	0.874	5.02300845e-1	0.974
2^{-5}	2.76874914e-4	1.100	2.53417337e-1	0.987

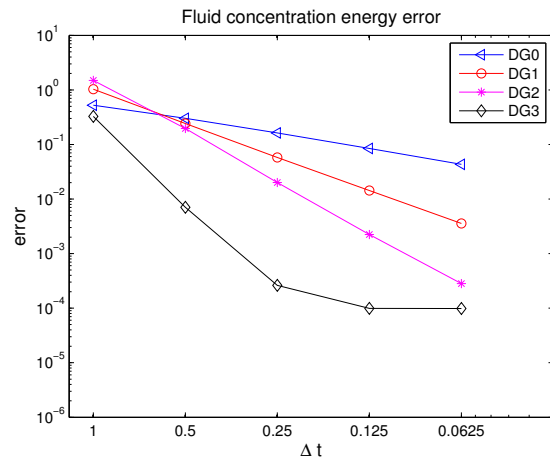Table 5.3.1 : error and rate for pressure and concentration with RT_0-$NIPG_1$

Pressure				
h	$\|p - p_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
2^{-1}	3.75612627e-2	–	9.12292627e-5	–
2^{-2}	1.51988594e-2	1.305	1.18909883e-2	2.940
2^{-3}	4.70278178e-3	1.692	1.50219946e-6	2.985
2^{-4}	1.30205703e-3	1.853	1.88274337e-7	2.996
2^{-5}	3.42238663e-4	1.928	2.35499363e-8	2.999
Concentration				
h	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
2^{-1}	2.32838162e-3	–	1.86209926	–
2^{-2}	1.37922246e-3	0.755	1.10375937	0.755
2^{-3}	2.30891708e-4	2.579	2.16460653e-1	2.350
2^{-4}	4.84970125e-5	2.251	5.47554909e-2	1.983
2^{-5}	1.01955089e-5	2.250	1.36293565e-2	2.006

Table 5.3.2 : error and rate for pressure and concentration with RT_1-$NIPG_2$

Pressure				
h	$\|p - p_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
2^{-1}	1.14265447e-2	–	4.08259591e-6	–
2^{-2}	1.81358816e-3	2.655	2.62942571e-7	3.957
2^{-3}	2.55346692e-4	2.828	1.65601216e-8	3.989
2^{-4}	3.38735338e-5	2.914	1.03699484e-9	3.997
2^{-5}	4.36191826e-6	2.957	6.48427172e-11	3.999
Concentration				
h	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
2^{-1}	2.19480438e-3	–	1.78168909	–
2^{-2}	1.00751042e-4	4.445	8.09128327e-2	4.461
2^{-3}	3.20635143e-5	1.652	3.15839214e-2	1.357
2^{-4}	3.69915500e-6	3.116	4.05515779e-3	2.961
2^{-5}	4.80717242e-7	2.944	5.14981839e-4	2.977

Table 5.3.3 : error and rate for pressure and concentration with RT_2-$NIPG_3$

For the DG time stepping we use space discretization RT_2-$NIPG_3$ with 64×64 grid to obtain high accuracy in space and plot the errors at time $t = 1.0$ as follows.
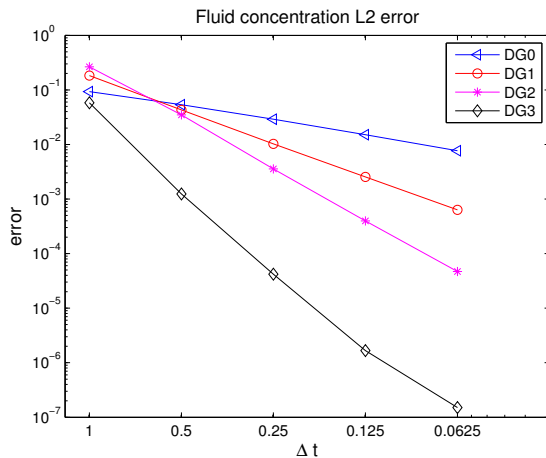
Figure 5.3.10 : cvg. rate for c in L^2 Figure 5.3.11 : cvg. rate for c in energy norm

Again, we observe in Figure 5.3.10 and 5.3.11 the high accuracy as well as increase of the convergence rates obtained by using high order DG in time. Errors and rates are presented below.

Concentration				
Δt	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
1	9.31436128e-2	–	5.22918703e-1	–
0.5	5.36939808e-2	0.795	3.01516836e-1	0.794
0.25	2.89774349e-2	0.890	1.62751631e-1	0.890
0.125	1.50834032e-2	0.942	8.47244201e-2	0.942
0.0625	7.69898626e-3	0.970	4.32471913e-2	0.970

Table 5.3.4 : error and rate of concentration with DG0

Concentration				
Δt	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
1	1.83083992e-1	–	1.02901530	–
0.5	4.30672057e-2	2.088	2.42081307e-1	2.088
0.25	1.02830466e-2	2.066	5.78029857e-2	2.066
0.125	2.54485428e-3	2.015	1.43067080e-2	2.014
0.0625	6.34705845e-4	2.003	3.57076445e-3	2.002

Table 5.3.5 : error and rate of concentration with DG1

Concentration				
Δt	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
1	2.65020659e-1	–	1.49304362	–
0.5	3.47976408e-2	2.929	1.96155266e-1	2.928
0.25	3.56332588e-3	3.287	2.00777497e-2	3.288
0.125	3.97630912e-4	3.145	2.24174160e-3	3.163
0.0625	4.69802381e-5	3.081	2.82088129e-4	2.990

Table 5.3.6 : error and rate of concentration with DG2

Concentration				
Δt	$\|c - c_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla c - \nabla c_h\|_{L^2(\mathcal{E}_h)}$	Cvg. rate
1	5.79618138e-2	–	3.27028209e-1	–
0.5	1.23576954e-3	5.552	7.04188045e-3	5.537
0.25	4.19441867e-5	4.881	2.60932509e-4	4.754
0.125	1.66442910e-6	4.655	9.88695054e-5	1.400
0.0625	1.50408449e-7	3.468	9.84599519e-5	0.006

Table 5.3.7 : error and rate of concentration with DG3

Table 5.3.4, 5.3.5, 5.3.6, and 5.3.7 verifies the improvement of the convergence rate in time as we increase the order of approximation in time.

5.3.2 Physical Problem

Homogeneous grain size

Now, we turn our attention to a physical problem over the space domain $[0,1] \times [0,1]$. For the diffusion/dispersion tensor we use the semi-empirical relation:

$$\mathbb{D}(\mathbf{u}) = d_m \mathbf{I} + |\mathbf{u}| \left(\alpha_l E(\mathbf{u}) + \alpha_t (\mathbf{I} - E(\mathbf{u})) \right)$$

where $E(\mathbf{u}) = \dfrac{\mathbf{u}\mathbf{u}^T}{|\mathbf{u}|^2}$ and we set,

$$d_m = 1.8 \times 10^{-7} \; , \; \alpha_l = 1, 8 \times 10^{-5} \text{ and } \alpha_t = 1.8 \times 10^{-6}$$

We neglect the gravity by setting $g = 0$. We set porosity $\phi = 0.2$. We set the permeability to be $\mathcal{K}(x) = 9.44 \times 10^{-3}$ throughout the domain and fluid viscosity $\mu(c) = 1 + (0.0524c)^{4.74}$. Thus, we have

$$\mathbb{K}(c) = \frac{9.44 \times 10^{-3}}{1 + (0.0524c)^{4.74}}$$

We fix the injection concentration to be $\hat{c} = 1$ and initial concentration $c_0 = 0$. For the injection source and production sink we have

$$\int_\Omega q^I = \int_\Omega q^P = 0.018$$

where q^I is piecewise constant on $[0, 0.1] \times [0, 0.1]$ and $q^I = 0$ elsewhere and q^P is piecewise constant on $[0.9, 1] \times [0.9, 1]$ and $q^P = 0$ elsewhere. We use the solver to simulate the fluid flow and plot the fluid profile from $t = 0$ to $t = 10$ as follows with $RT_0\text{-}NIPG_1$, $RT_1\text{-}NIPG_2$

and RT_2-$NIPG3$ with 1024 elements and $\Delta t = 0.05$ using DG0 up to DG3 in time.

In this case it appears that fluid pressure and velocity remain constant. We plot the pressure and velocity together, with pressure in the background and velocity streamlines in the foreground.
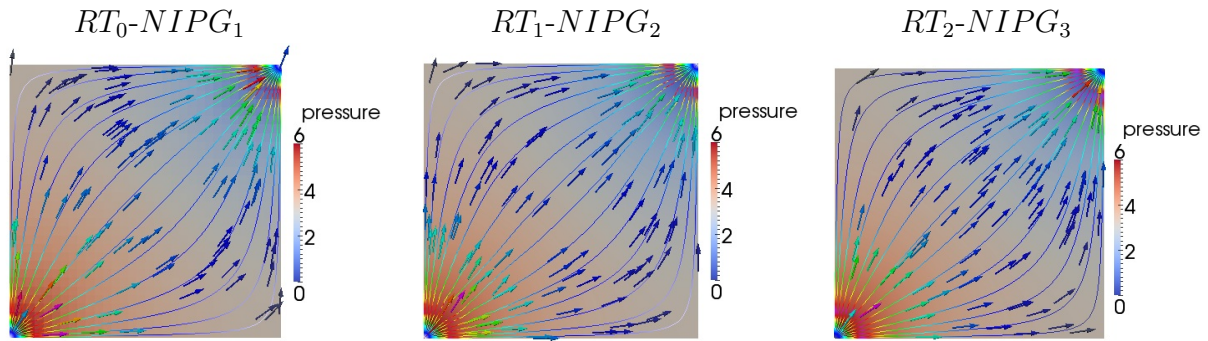

Table 5.3.8 : Fluid pressure and velocity streamlines at $t = 5$ with DG_0 in time
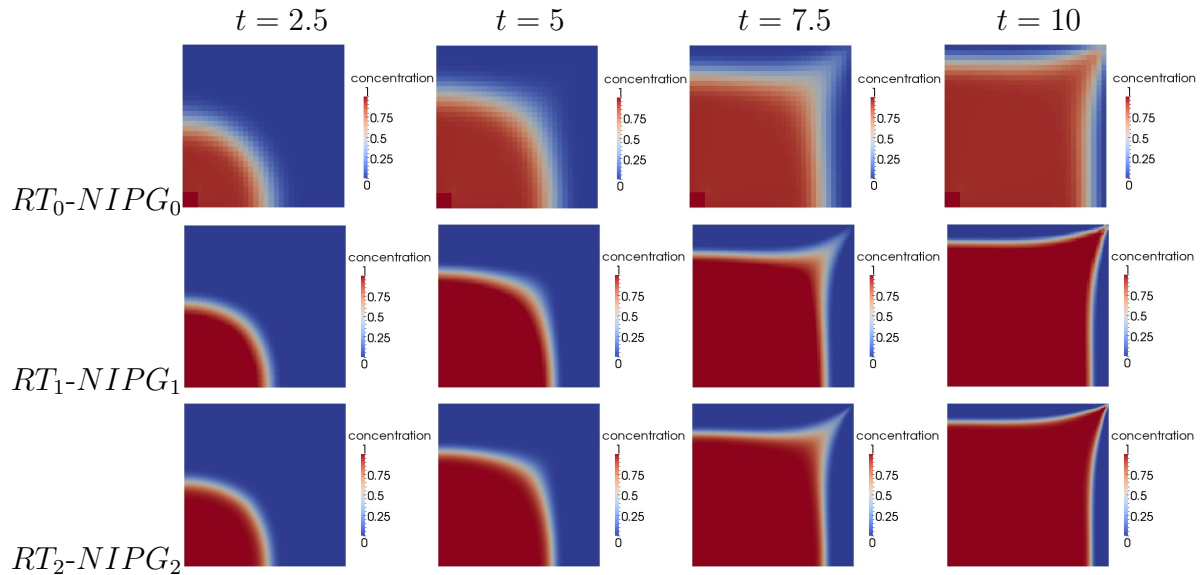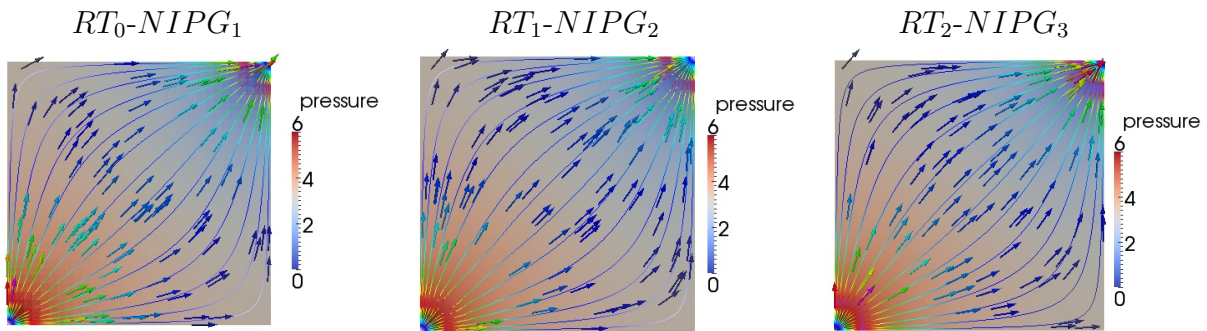
For the concentration,

Table 5.3.9 : Simulations of the fluid concentration with DG_0 in time

One can observe the increase of the quality of the simulations as we use higher order approximations in Table 5.3.9.

$RT_0\text{-}NIPG_1$ $RT_1\text{-}NIPG_2$ $RT_2\text{-}NIPG_3$

Table 5.3.10 : Fluid pressure and velocity streamlines at $t = 5$ with DG_1 in time
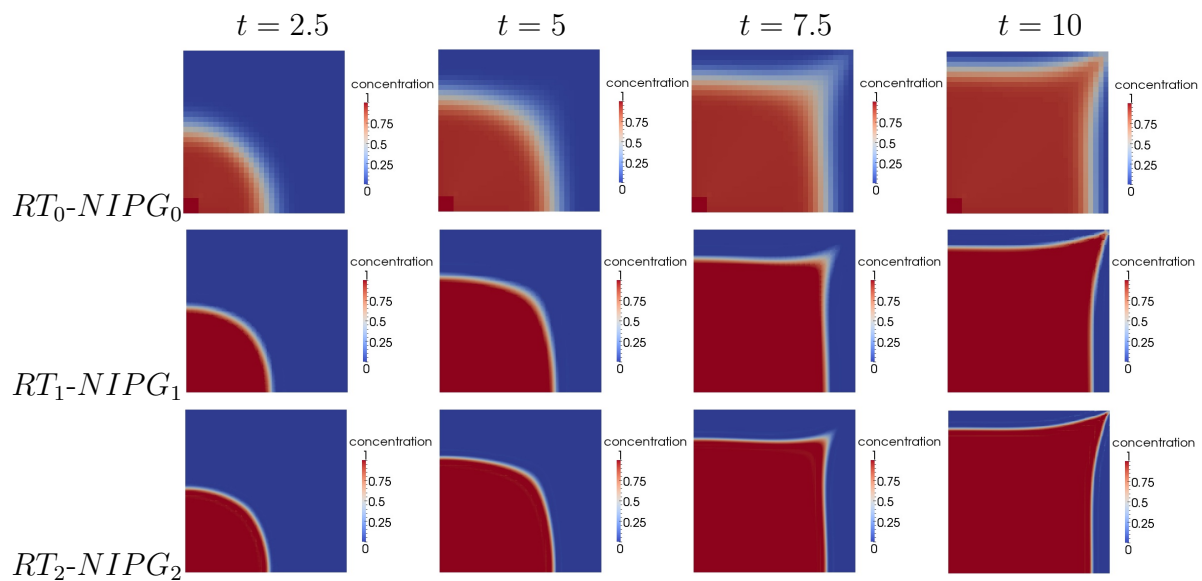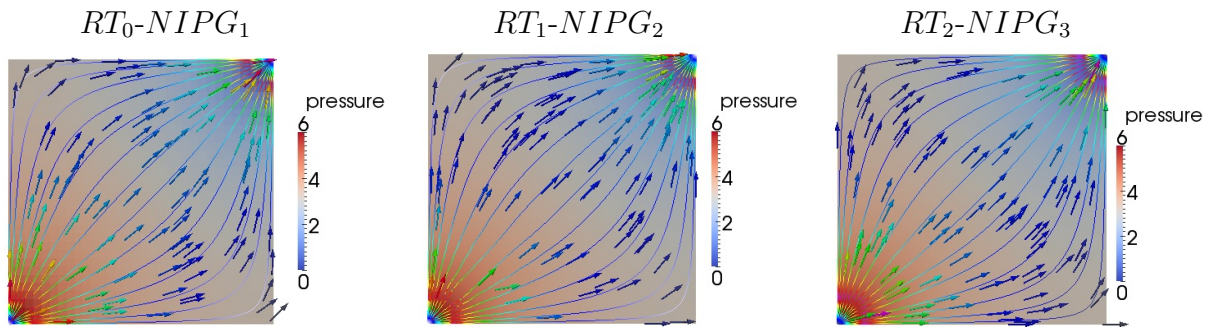
For the concentration,

Table 5.3.11 : Simulations of the fluid concentration with DG_1 in time

In Tables 5.3.10 and 5.3.11, we obverse the solution remains stable as we increase the order of approximation in time which is consistent with our theoretical analysis.

Table 5.3.12 : Fluid pressure and velocity streamlines at $t = 5$ with DG_2 in time
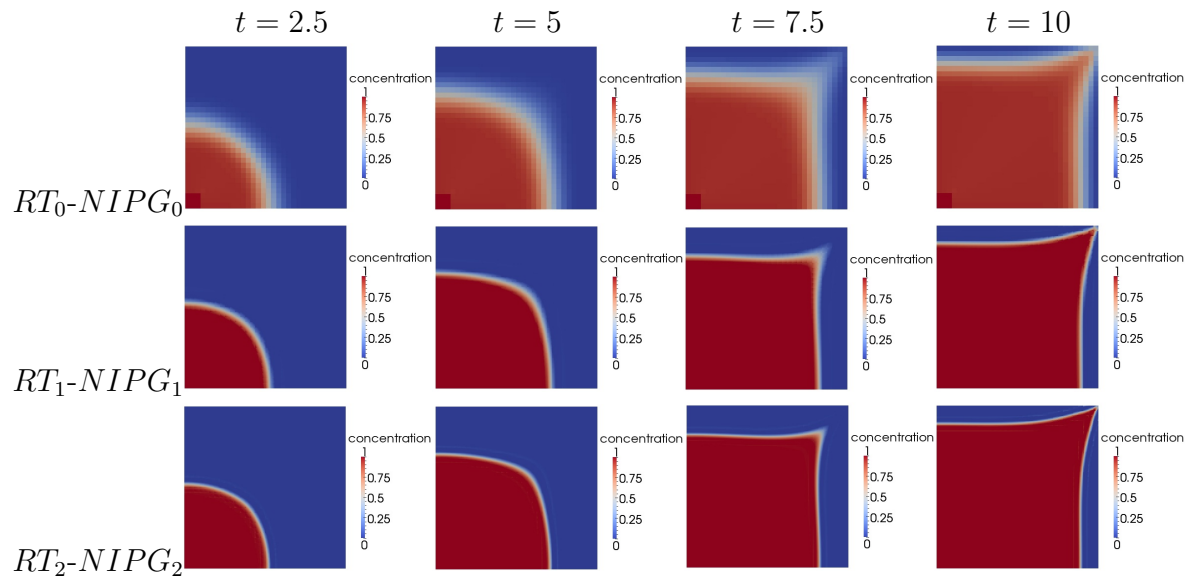
For the concentration,

Table 5.3.13 : Simulations of the fluid concentration with DG_2 in time

The fluid profiles remain the same as we varying the orders of approximations in space and time. Hence, it offers us strong evidence of the convergence of the numerical solutions.
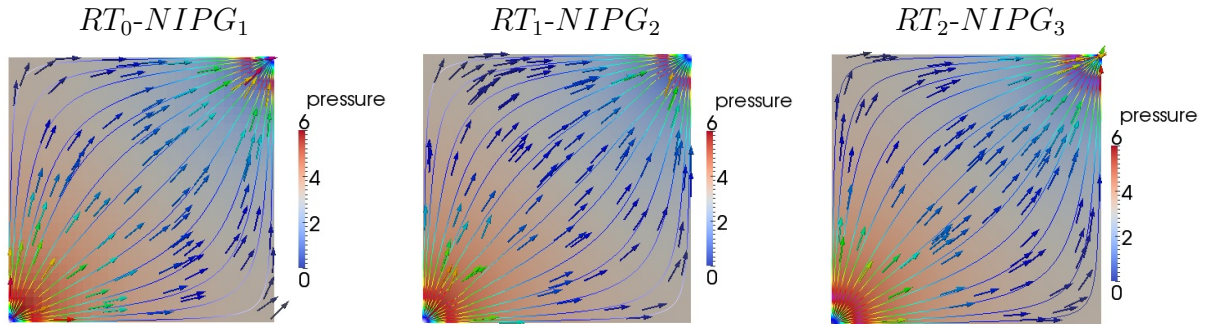
RT_0-$NIPG_1$ RT_1-$NIPG_2$ RT_2-$NIPG_3$

Table 5.3.14 : Fluid pressure and velocity streamlines at $t = 5$ with DG_3 in time
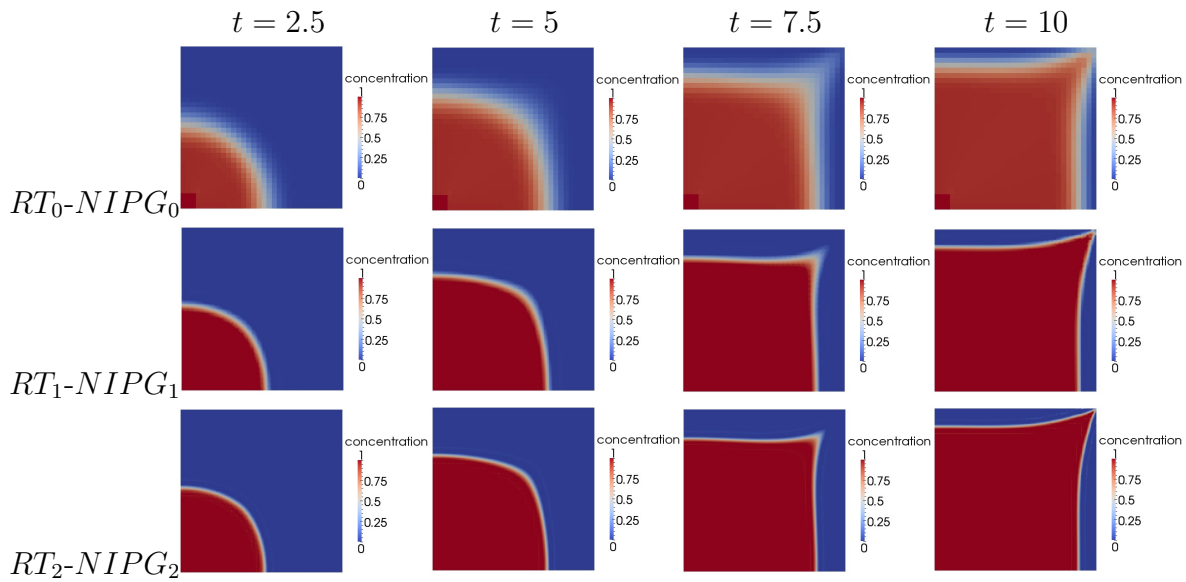
For the concentration,

Table 5.3.15 : Simulations of the fluid concentration with DG_3 in time

Perhaps it is not very clear to see the differences between the effect of different DG time updating schemes. So, we plot the intersection curves of the concentration alone the line $x = y$ with RT_2-$NIPG_2$ in space.
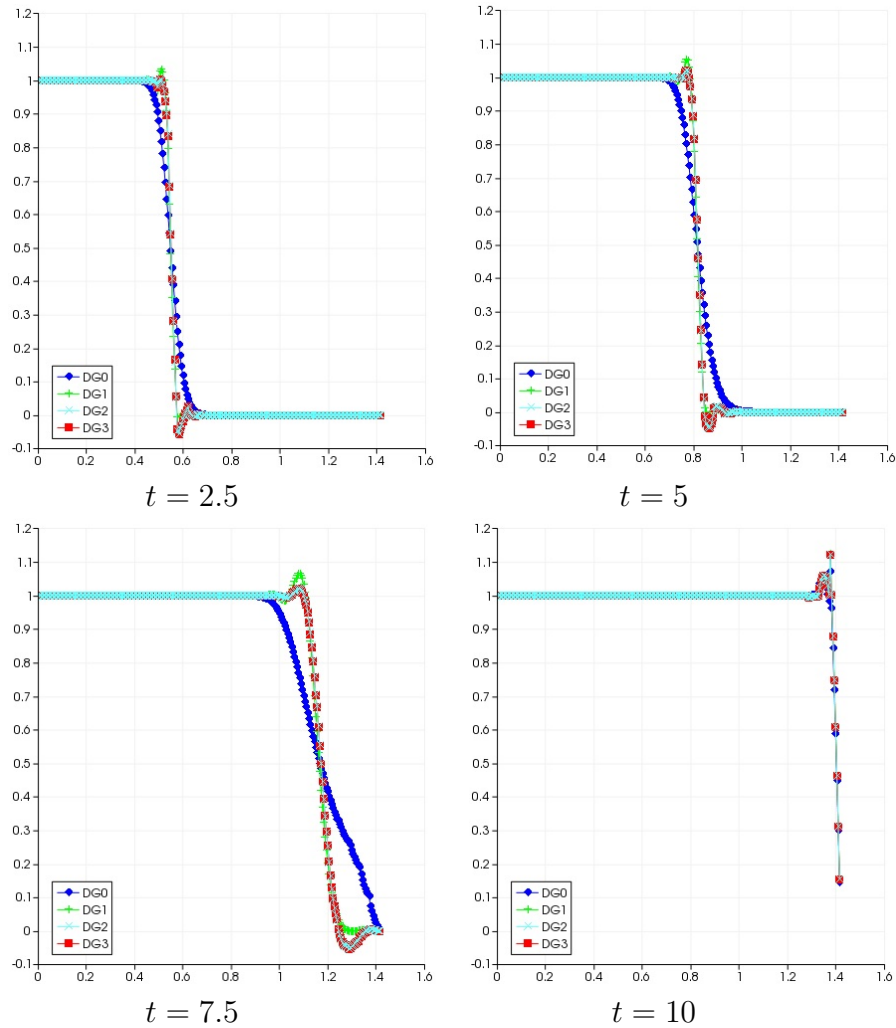
Table 5.3.16 : Concentration curve intersection with RT_2-$NIPG_2$

Table 5.3.16 illustrates the effect of using higher order approximations in time. On one hand, we observe the localized overshoot and undershoot phenomena using higher order time approximations. On the other hand, we have gained considerable accuracy globally using the high order approximations in time and the reduction numerical diffusion effect, despite the overshoot and undershoot.

Homogeneous grain size with a discontinuous lens

Now, let us study the case with discontinuous permeability which is always the case for the permeability in the real world problems. We use all the parameters from previous problem, except the permeability.
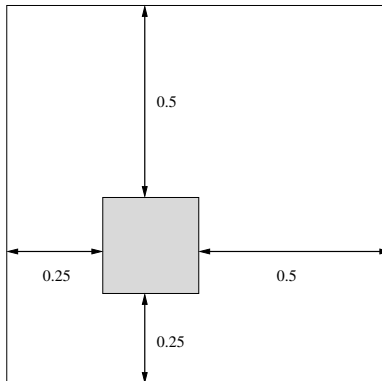

Figure 5.3.12 : domain with discontinuous permeability

In Figure 5.3.12, we set the permeability of the shaded area to be $\mathcal{K}(x) = 9.44 \times 10^{-6}$ and in the rest of the domain the permeability remains the same as before i.e.$\mathcal{K}(x) = 9.44 \times 10^{-3}$. We simulate the flow problem using spacial discretization $RT_2 - NIPG_3$ with 4096 elements. For the time discretization, we use DG_0 and DG_1 in time with $\Delta t = 0.05$ from $t = 0$ to $t = 10.0$. We compare the pressure and velocity as follows.
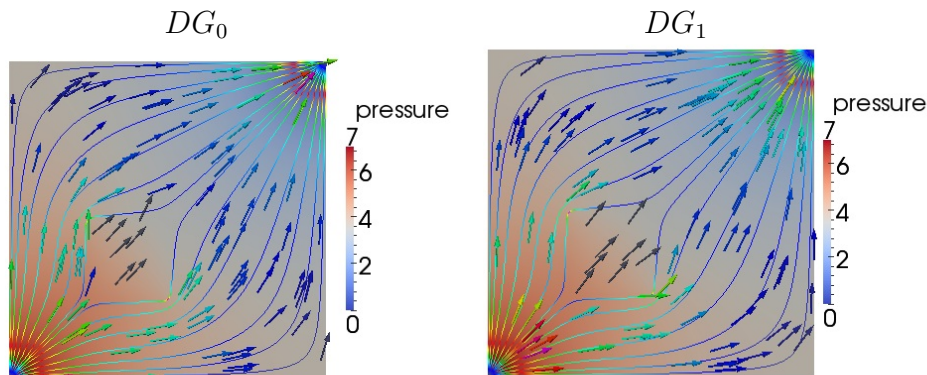

Table 5.3.17 : Fluid pressure and velocity streamlines at $t = 5$ with RT_2-$NIPG_2$ in space

We observe in Table 5.3.17 the streamlines of the fluid avoid the region with low permeability

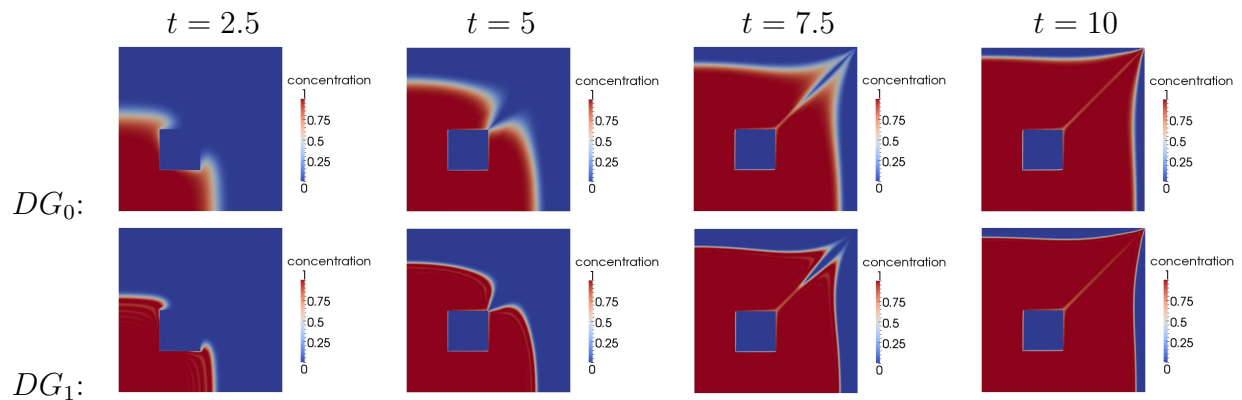which is consistent with physical phenomenon. For the concentration, we have

| $t = 2.5$ | $t = 5$ | $t = 7.5$ | $t = 10$ |

DG_0:

DG_1:

Table 5.3.18 : Simulations of the fluid concentration with RT_2-$NIPG_2$ in space

In Table 5.3.18, we have observe the robustness of the numerical scheme that it is capable of capturing the lens in the domain.

SPE10 problem

In addition, we test our solver on the snapshot of the SPE10 problem with given permeability fields. We shall conduct two tests with Tarbert and Upper Ness permeability field. For the Tarbert, we have as follows.
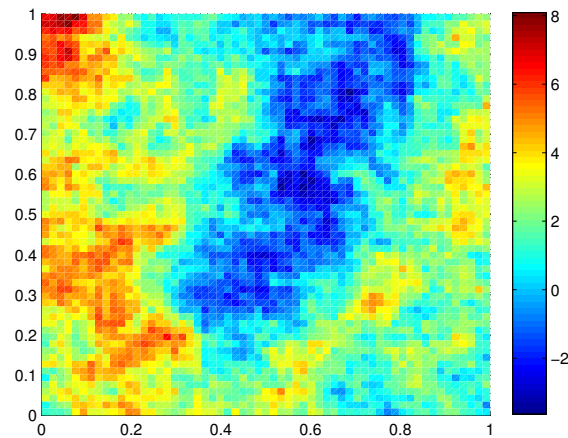

Figure 5.3.13 : SPE10 permeability field layer 30 in log scale

This permeability field consists of wide range of permeability. We present our approximation of the solution with DG_1 in time as follows.

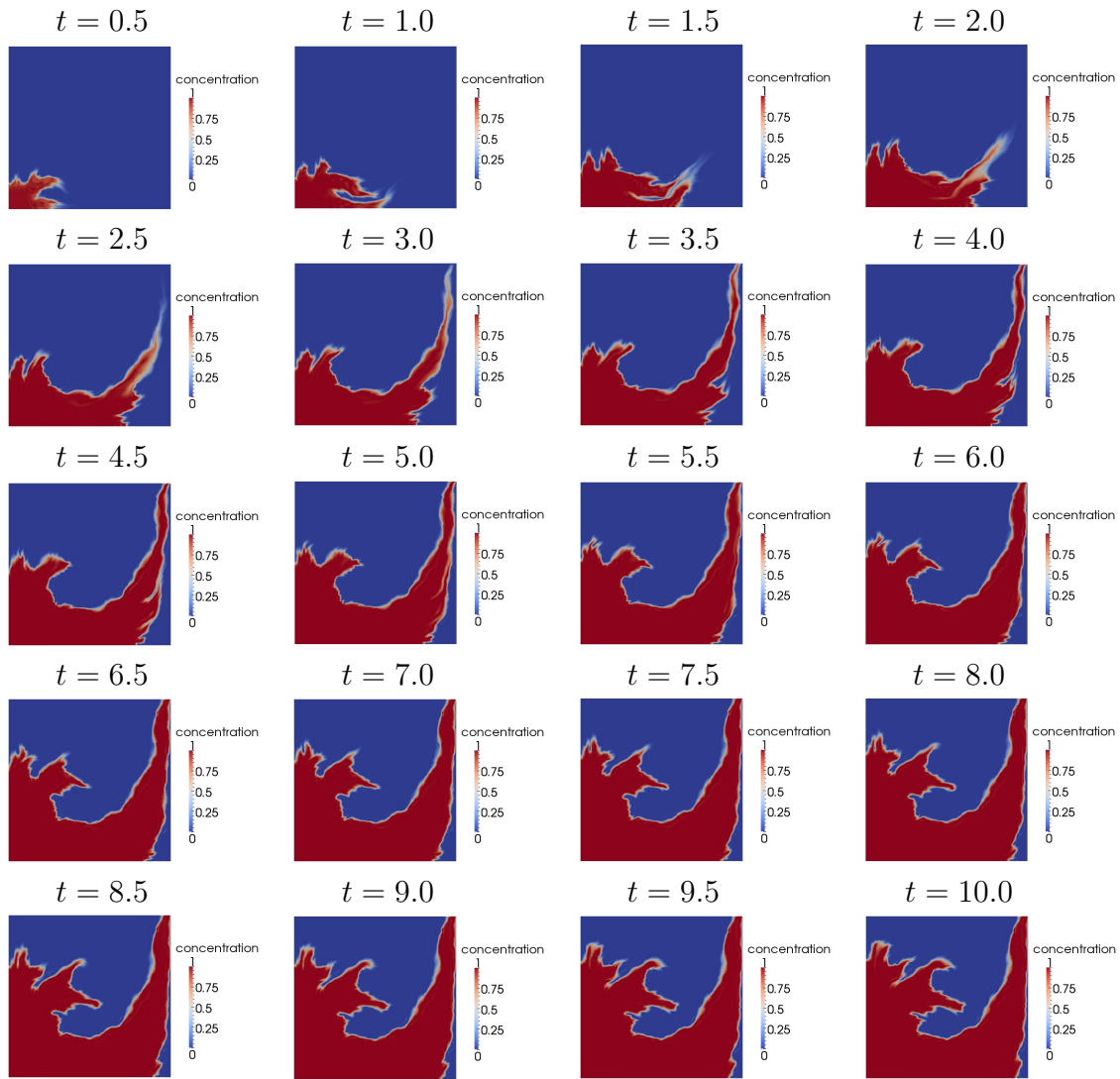| $t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
$t = 2.5$	$t = 3.0$	$t = 3.5$	$t = 4.0$
$t = 4.5$	$t = 5.0$	$t = 5.5$	$t = 6.0$
$t = 6.5$	$t = 7.0$	$t = 7.5$	$t = 8.0$
$t = 8.5$	$t = 9.0$	$t = 9.5$	$t = 10.0$

Table 5.3.19 : Simulations of the fluid concentration with RT_2-$NIPG_2$ in space

Observe in Table 5.3.19, our numerical solution remains stable despite the wide range of scales and discontinuity in the permeability field. Also, the fluid flow is clearly resembling the permeability field, as we observe that it avoids the region of low permeability.

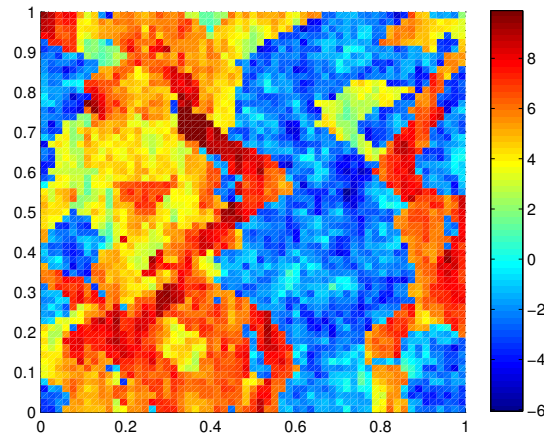Now, for the Upper Ness we have the permeability field taken from SPE10 layer 60.

Figure 5.3.14 : SPE10 permeability field layer 60 in log scale

This permeability field consists of even wider range of permeability than the Tarbert. With the fractures in the in the field, it is much harder to simulate the fluid flow. We present our approximation of the solution with DG_1 in time as follows.

$t = 6.5$	$t = 7.0$	$t = 7.5$	$t = 8.0$
$t = 8.5$	$t = 9.0$	$t = 9.5$	$t = 10.0$
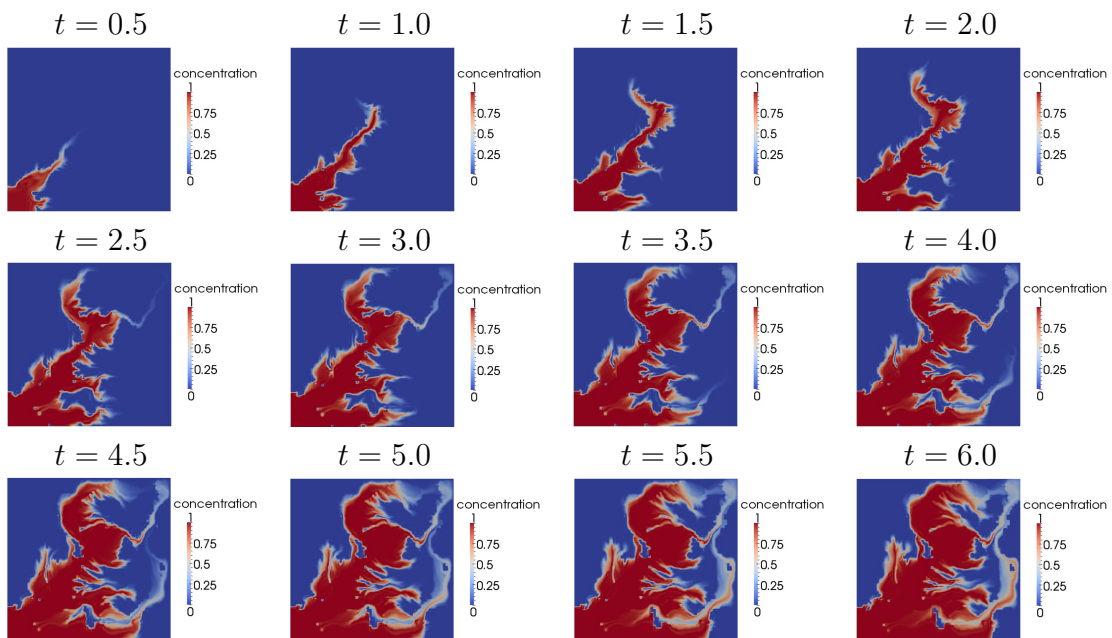
Table 5.3.20 : Simulations of the fluid concentration with RT_2-$NIPG_2$ in space

Again we observe the fluid flow is consistent with the distribution of different permeabilities. We also notice that the simulation produced by the numerical algorithm was able to capture the areas with low permeability.

Chapter 6

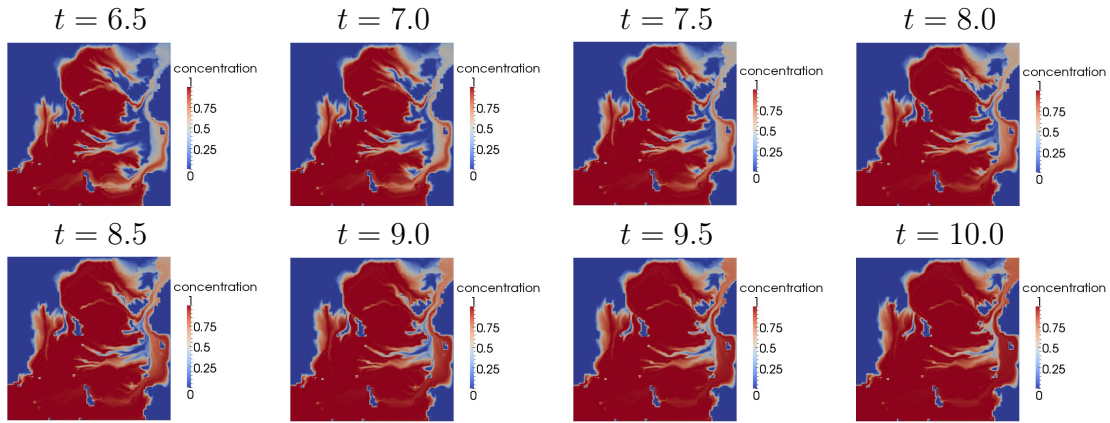# Conclusions and Future Work

In this chapter, I will begin by summarizing the results obtained so far. Also, I would like to present the possibility for future work concerning the miscible displacement simulation under low regularity condition.

6.1   Summary

In the thesis, I developed a numerical method for solving the miscible displacement equations under low regularity which is an important mathematical model in enhanced oil recovery .

For the numerical discretization of the PDE system, there are three different aspects concerning the concept of the discretization. First, there is the discretization in space for the Darcy's law, for which I used the locally mass-conservative mixed finite element. Then there is the discretization of the transport equation in space. Due to the difficulty posed by the low regularity of the solution from Darcy's law, I introduced a modified discontinuous Galerkin spacial discretization for the transport equation. The last essential aspect of the discretization concept is the use of the discontinuous Galerkin method for time updating which allows arbitrary degree of approximation in time.

After establishing the spacial and time discretizations, I analyzed the numerical scheme. I began by showing the stability of the numerical method. Using the results from stability analysis, I then proved compactness of the concentration through a much more general compactness theorem. For the analysis of the convergence of the numerical solutions, the convergence of the pressure and velocity is verified by standard technique used for the analysis of the mixed finite element method. With the help of the compactness theorem for the

concentration, I also discussed the convergence of concentration.

The chapter following the stability and convergence analysis provides the implementation concept of the numerical scheme. Because of the concern of the efficiency, I decided to use the decoupling sequential approach for the implementation. I introduce an implementation strategy to maintain the order of the approximation in time for the sequential updating. I developed the software for 2D miscible displacement problems according to the numerical algorithm I proposed and I tested on various different problems. For the case of when the analytical solutions are known, the numerical experiments suggested the improvement of the accuracy and convergence rate as I increased the order of approximation in space and time. Numerical experiments were also conducted for the physical problem with unknown solutions. I tested two cases of porous media: one with homogeneous grain size and one with homogeneous grain size yet with a lens of different permeability inside the domain. Numerous comparison studies have been done to compare the differences caused by using the different order of approximations. Finally, I used the permeability values from SPE10 data to test the numerical method. I tested the solver on layers with Tarbert structure and layer with Upper Ness structure. The solutions remained stable and the fluid flows also corresponded to the distribution of the permeability inside the domain. The numerical experiments suggested the improvement of the quality of the simulations and the robustness of the numerical method using higher order approximation.

In conclusion, the combination of mixed finite element and discontinuous Galerkin method in space and time provides an alternative for solving miscible displacement problem while handling the low regularity condition. This thesis provides rigorous analyses of the method. Apart from the numerical method I proposed, the compactness theorem has theoretical significance when Aubin-Lions theorem is no longer applied for the analysis of the convergence of solution of PDE. In the light of what I have done in this thesis, there are still many open questions concerning the numerical methods for solving miscible displacement equa-

tions under low regularity as well as the implementations of the methods when handling the discontinuous parameters.

6.2 Future Work

In the near future, I would like to carry on further studies on the numerical method I proposed in the thesis. The convergence of the concentration solution to the weak solution requires further investigation. Currently, the numerical experiments are restricted to 2D test case. I will begin to parallelize numerical implementation for the scheme on the cluster machines and extend the numerical experiments to 3D domains while including gravity effects. Also, I would like to conduct test with varying porosity. Slope limiters will be introduced in the numerical implementation as well. I also plan to observe the effect of different mesh structures have on the numerical solutions.

Furthermore, I want to increase the flexibility of the numerical implementation for higher order methods. Mixed finite element method itself poses considerable difficulties when it comes to the implementation of higher order approximations, imposing the boundary conditions, and using adaptive mesh refinement. Hence, I propose to extend the DG discretization to the Darcy's law as well. Therefore, most naturally for the same problem we would have a new discretization as follows.

$$\int_{t_{n-1}}^{t_n} B_{d,p}(p_h, q_h; c_h) = \int_{t_{n-1}}^{t_n} \left((q^I - q^P, q_h) + (\mathbb{K}(c_h)\rho(c_h)\mathbf{g}, \nabla q_h)_{\mathcal{E}_h} \right.$$

$$\left. -(\{\mathbb{K}(c_h)\rho(c_h)\mathbf{g} \cdot \mathbf{n}_e\}, [q_h])_{\Gamma_h} \right)$$

$$\mathbf{u}_h = -\mathbb{K}(c_h)\nabla p_h$$

$$\int_{t_{n-1}}^{t_n} \left((\phi \partial_t c_h, w_h) + B_d(c_h, w_h; \mathbf{u}_h) + B_{cq}(c_h, w_h; \mathbf{u}_h) \right) + \left(\left[c_h^{n-1} \right]_t, \phi w_{h+}^{n-1} \right) = \int_{t_{n-1}}^{t_n} (\hat{c}q^I, w_h)$$

for all $q_h \in \mathcal{P}_\ell[t^{n-1}, t^n; P_h], w_h \in P_\ell[t^{n-1}, t^n; C_h]$.

where,

$$B_{d,p}(p, q; c) = (\mathbb{K}(c)\nabla p, \nabla q)_{\mathcal{E}_h} - ([q], \{\mathbb{K}(c)\nabla p \cdot \mathbf{n}_e\})_{\Gamma_h}$$

$$+ \epsilon([p], \{\mathbb{K}(c)\nabla q \cdot \mathbf{n}_e\})_{\Gamma_h} + (\sigma h^{-1}[p], [q])_{\Gamma_h}$$

$$B_d(c, w; \mathbf{u}) = (\mathbb{D}(\mathbf{u})\nabla c, \nabla w)_{\mathcal{E}_h} - ([w], \{\mathbb{D}(\mathbf{u})\nabla c \cdot \mathbf{n}_e\})_{\Gamma_h}$$

$$+ \epsilon([c], \{\mathbb{D}(\mathbf{u})\nabla w \cdot \mathbf{n}_e\})_{\Gamma_h} + (\sigma h^{-1}(1 + \{|\mathbf{u}|\})[c], [w])_{\Gamma_h}$$

and

$$B_{cq}(c, w; \mathbf{u}) = \frac{1}{2}\left((\mathbf{u}\nabla c, w)_{\mathcal{E}_h} - (\mathbf{u}c, \nabla w)_{\mathcal{E}_h} + ((q^I + q^P)c, w)\right.$$

$$\left. + (c^{\mathrm{up}}\mathbf{u} \cdot \mathbf{n}_e, [w])_{\Gamma_h} - (w^{\mathrm{down}}\mathbf{u} \cdot \mathbf{n}_e, [c])_{\Gamma_h}\right)$$

Implementation wise, it becomes much easier to achieve arbitrary order of approximations in space for the Darcy's law. Hence, one would expect for it to provide much more accurate approximation for the pressure and velocity. Another advantage of this scheme formulation is that all boundary conditions can be imposed weakly. Furthermore, one can use hanging nodes for the adaptive mesh refinement in both equations.

Nevertheless, the new scheme poses new challenges in terms of theoretical analysis. The traditional stability analysis for the mixed finite element method is no longer applied when analyzing the stability of the pressure and velocity. The low regularity condition makes the analysis even more sophisticated. Whether the compactness theorem will still be applicable for studying the compactness of the concentration solutions is unknown. There is even a possibility that one has to modify the new DG scheme to handle the difficulties posed by the challenges I listed above.

Furthermore, numerically how does the new scheme performs in comparison to the scheme I previously proposed in my thesis is worth investigating.

With all the advantages and challenges, I would like to devote more time and effort in

the hope to shed some light on numerical methods for modeling the miscible displacement processes and its related problems.

Appendix A

In this appendix, I will state and prove several results concerning functional analysis which are extremely useful for the analysis on the broken Sobolev spaces.

Lemma .0.1. *For X and Y Banach spaces, let $X \hookrightarrow Y$: we say that X is embedded into Y. Then we have*

$$Y' \hookrightarrow X'$$

Proof. Indeed, denote by $i : X \to Y$ the identity operator. Pick $F \in Y'$ and define

$$G(x) = F(ix), \quad \forall x \in X$$

Then

$$|G(x)| \le \|F\|_{Y'} \|ix\|_Y \le \|F\|_{Y'} C \|x\|_X$$

By linearity of F and i, the map G is linear. Since it is also continuous, $G \in X'$.

In addition, we have

$$\|G\|_{X'} \le C \|F\|_{Y'}$$

$\square$

Lemma .0.2. *Let $X \hookrightarrow H$, where H is Hilbert space, and X is Banach space. Then X is an inner product space with $(\cdot, \cdot)_H$. Assume that X is dense in H. Then $H \hookrightarrow X'$ and H is dense in X'.*

Proof. Pick $F \in X'$. By Riesz representation theorem, $F(x) = (y, x)_H$ for some $y \in X \subset H$.

Then, since X is dense in H, there is $(x_n)_n \in X$ such that x_n tends to y. Define

$$F_n(v) = (x_n, v)_H \forall v \in H$$

Then $F_n \in H'$, and in fact F_n is identified to x_n. Claim: $\|F_n - F\|_{X'}$ tends to zero. Indeed:

$$F_n(x) - F(x) = (x_n - y, x)_H \leq \|x_n - y\| \|x\|$$

$\square$

Lemma .0.3. *Assume $(w_h)_h$ is bounded in $L^r[0,T;W']$ for any $1 \leq r < \infty$ and that $(w_h)_h$ is precompact in $L^{p'}[0,T;W']$. Then $(w_h)_h$ is precompact in $L^r[0,T;W']$ for any $1 \leq r < \infty$.*

Proof. Pick $1 \leq r < \infty$. If $r \leq p'$, then Hölder's inequality yields

$$\|w\|_{L^r[0,T;W']} \leq T^{1-r/p'} \|w\|^r_{L^{p'}[0,T;W']}, \quad \forall w \in L^{p'}[0,T;W']$$

Since $(w_h)_h$ is precompact in $L^{p'}[0,T;W']$, for any $\epsilon > 0$, there is an $\epsilon-$ net for the closure of $(w_h)_h$. Denote $\mathcal{F}$ the closure of $(w_h)_h$. Then, we have

$$\mathcal{F} \subset \cup_{i=1}^N B_\epsilon(w_i)$$

for some $w_i \in L^{p'}[0,T;W']$. The inequality above says that w_i belongs to $L^r[0,T;W']$. Pick $\epsilon > 0$ and $v \in \mathcal{F}$. There is a $w_i \in L^{p'}[0,T;W']$ (and thus also in $L^r[0,T;W']$) such that

$$\|v - w_i\|_{L^{p'}[0,T;W']} \leq \left(\frac{\epsilon}{T^{1-r/p'}}\right)^{1/r}$$

Therefore we have from the inequality above:

$$\|v - w_i\|_{L^r[0,T;W']} \leq \epsilon$$

So we prove that $\mathcal{F}$ is totally bounded in $L^r[0,T;W']$.

If $r > p'$, then we use the fact that

$$\|u\|_{L^r[0,T;W']} \leq \|u\|_{L^{p'}[0,T;W']}^{\theta} \|u\|_{L^q[0,T;W']}^{1-\theta}$$

with

$$\frac{1}{r} = \frac{\theta}{p'} + \frac{1-\theta}{q}$$

and $0 < \theta < 1$. Since $(w_h)_h$ is bounded in $L^q[0,T;W']$, its closure is also bounded and this gives for a positive constant M:

$$\|v\|_{L^r[0,T;W']} \leq M\|v\|_{L^{p'}[0,T;W']}^{\theta}, \quad \forall v \in \mathcal{F}$$

Let $\{w_h\}$ be a sequence in $\mathcal{F}$, then since $\mathcal{F}$ is precompact in $L^{p'}[0,T;W']$ there is a subsequence $\{w_{h_k}\}$ such that $w_{h_k} \to v$ in $L^{p'}[0,T;W']$. Hence,

$$\|w_{h_k} - v\|_{L^r[0,T;W']} \leq M\|w_{h_k} - v\|_{L^{p'}[0,T;W']}^{\theta}, \quad \forall v \in \mathcal{F}$$

So, $w_{h_k} \to v$ in $L^r[0,T;W']$. Therefore, $(w_h)_h$ is precompact in $L^r[0,T;W']$. $\qquad\square$

Lemma .0.4. *Given the dense embedding:*

$$W \hookrightarrow V \overset{c}{\hookrightarrow} H \hookrightarrow W'$$

And V is compactly embedded in H, then V is compactly embedded in W'.

Proof. Let S is a bounded subset of V, if $\{x_n\} \in S$, then there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ in H. Hence, we have $\|x_{n_k} - x\|_{W'} \leq C \|x_{n_k} - x\|_H$. So, $x_{n_k} \to x$ in W', which implies S is precompact in W'. Therefore, V is compactly embedded in W'. $\quad\square$

Lemma .0.5. *Given $(w_h)_h$ is bounded in $L^p[0,T;V]$ and precompact in $L^p[0,T;W']$ it follows it is also precompact in $L^p[0,T;H]$.*

Proof. Since $(w_h)_h$ is precompact in $L^p[0,T;W']$, then there exists a subsequence $\{w_{h_k}\}$ such that $w_{h_k} \to w$ in $L^p[0,T;W']$. Hence, $\{w_{h_k}\}$ is a Cauchy sequence in $L^p[0,T;W']$. i.e. there exits $N > 0$ such that for all $m,n > N$ we have $\|w_{h_n} - w_{h_m}\|_{L^p[0,T;W']} \leq \epsilon/M(\epsilon)$. Therefore, $\|w_{h_n} - w_{h_m}\|_{L^p[0,T;H]} \leq \epsilon \|w_{h_n} - w_{h_m}\|_{L^p[0,T;V]} + M(\epsilon) \|w_{h_n} - w_{h_m}\|_{L^p[0,T;W']}$. We know that $(w_h)_h$ is bounded in $L^p[0,T;V]$, hence, $\|w_{h_n} - w_{h_m}\|_{L^p[0,T;H]} \leq \epsilon M + \epsilon \leq (M+1)\epsilon$. So, $\{w_{h_k}\}$ is Cauchy in $L^p[0,T;H]$. Hence, $\{w_{h_k}\}$ converges in $L^p[0,T;H]$ since H is a Hilbert space. Therefore, $\{w_h\}$ is precompact in $L^p[0,T;H]$. $\quad\square$

Lemma .0.6. *Let H be a Hilbert space with inner-product $(\cdot,\cdot)_H$, let W be a Banach space, and let $W \hookrightarrow H \hookrightarrow W'$ be dense embeddings. Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of $[0,T]$, let $W(\mathcal{E}_h) \subset W$ be a subspace, and $\ell \geq 0$. Fix $1 \leq p, q < \infty$ with $1/p + 1/q \geq 1$ and assume that $w_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;W(\mathcal{E}_h)]$ and*

$$\int_{t^{n-1}}^{t^n} (w_{ht}, v_h)_H + (w_{h+}^{n-1} - w_{h-}^{n-1}, v_{h+}^{n-1})_H = \int_{t^{n-1}}^{t^n} F_h(v_h)$$

for all $v_h \in \mathcal{P}_\ell(t^{n-1},t^n;W(\mathcal{E}_h))$, where $F_h \in L^q[0,T;W(\mathcal{E}_h)']$.

Then for all $0 \leq \delta \leq T$ there exists a constant $M(\ell,\vartheta) > 0$ such that

$$\sup_{v_h \in L^p[\delta,T;W_h]} \frac{\int_\delta^T (w_h(t) - w_h(t-\delta), v_h)_H \, dt}{\|v_h\|_{L^p[\delta,T;W]}} \leq M(\ell,\vartheta) \|F\|_{L^q[0,T;W(\mathcal{E}_h)']} \max(\Delta t, \delta)^{1/q'} \delta^{1/p'}.$$

With $1/p' = 1 - 1/p$ and $1/q' = 1 - 1/q$. The parameter ϑ is:

$$\vartheta = \min_{1 \leq n \leq N} (t^n - t^{n-1})/\tau, \quad \tau = \max_{1 \leq n \leq N} (t^n - t^{n-1})$$

Proof. This result is taken from lemma 3.9 in [41] (which comes from Lemma 3.3. in [54]). $\quad \square$

Theorem .0.7. *Let B_0 and B be Banach spaces, and let B_0 be compactly embedded into B. Let $\mathcal{F} \subset L^1[0, T; B_0]$ be bounded, and suppose for some $1 \leq p < \infty$ that $\mathcal{F}$ is equicontinuous in $L^p[0, T; B]$ in the sense that for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{\delta'}^{T} \|u(t) - u(t - \delta')\|_B^p dt \leq \epsilon, \quad u \in \mathcal{F}, \quad \delta' < \delta$$

Then for all $0 < \theta < T/2$ the set $\mathcal{F}|_{(\theta, T-\theta)}$ is precompact in $L^p[\theta, T - \theta; B]$.

Proof. This result is taken from theorem 3.2 in [54]. $\quad \square$

The next lemma is a slight modification of Lemma 3.4 in [54].

Lemma .0.8. *Let W be a Banach space and let $u \in L^p[0, T; W]$ for some $1 \leq p < \infty$. Assume*

$$\int_{\delta}^{T} \|u(t) - u(t - \delta)\|_W^p dt \leq C\delta, \quad 0 < \delta < T$$

then $u \in L^q[0, T; W]$ for any $1 \leq q < \infty$.

Proof. If $\delta < 1$, then we have

$$\int_{\delta}^{T} \|u(t) - u(t - \delta)\|_W^p dt \leq C\delta \leq C\delta^{\alpha}, \ \forall 0 \leq \alpha < 1$$

Hence, by lemma 3.4 in [54] we have the $u \in L^q[0, T; W]$ for any $1 \leq q < p/(1 - \alpha)$. Since it holds for all $0 \leq \alpha < 1$, then it is true for $1 \leq q < \infty$.

If $\delta \geq 1$, then we have

$$\int_{\delta}^{T} \|u(t) - u(t-\delta)\|_W^p dt \leq C\delta \leq CT^{1-\alpha}\delta^{\alpha} \leq CT\delta^{\alpha}, \ \forall 0 \leq \alpha < 1$$

Hence, again using lemma 3.4 in [54], we have the $u \in L^q[0, T; W]$ for any $1 \leq q < p/(1-\alpha)$. Since it holds for all $0 \leq \alpha < 1$, then it is true for $1 \leq q < \infty$. Therefore, $u \in L^q[0, T; W]$ for any $1 \leq q < \infty$ $\qquad \square$

Bibliography

[1] R. A. ADAMS AND J. J. FOURNIER, *Sobolev spaces*, vol. 140, Academic press, 2003.

[2] V. AMBATI AND O. BOKHOVE, *Space–time discontinuous Galerkin finite element method for shallow water flows*, Journal of computational and applied mathematics, 204 (2007), pp. 452–462.

[3] S. BARTELS, M.JENSEN, AND R.MÜLLER, *Discontinuous Galerkin finite element convergence for incompressible miscible displacement problems of low regularity*, SIAM J. Num. Anal., 47 (2009), pp. 3720–3743.

[4] P. BASTIAN, M. BLATT, C. ENGWER, A. DEDNER, R. KLÖFKORN, S. KUTTANIKKAD, M. OHLBERGER, AND O. SANDER, *The distributed and unified numerics environment (dune)*, in Proc. of the 19th Symposium on Simulation Technique in Hannover, 2006.

[5] J. BEAR, *Dynamics of fluids in porous media*, Dover publications, 1988.

[6] P. BEDIENT, H. RIFAI, AND C. NEWELL, *Ground water contamination: transport and remediation*, (1994).

[7] S. BRENNER, K. WANG, AND J. ZHAO, *Poincaré–friedrichs inequalities for piecewise h 2 functions*, Numerical Functional Analysis and Optimization, 25 (2004), pp. 463–478.

[8] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, vol. 15, Springer, 2008.

[9] S. C. Brenner and R. Scott, *The mathematical theory of finite element methods*, vol. 15, Springer, 2007.

[10] M. Buès and C. Oltean, *Numerical simulations for saltwater intrusion by the mixed hybrid finite element method and discontinuous finite element method*, Transport in Porous Media, 40 (2000), pp. 171–200.

[11] J. Butcher, *Numerical methods for ordinary differential equations*, Wiley, 2008.

[12] G. Chavent, G. Chavent, G. Cohen, J. Jaffre, R. Eyard, D. Guerillot, and L. Weill, *Discontinuous and mixed finite elements for two-phase incompressible flow*, SPE Reservoir Engineering, 5 (1990), pp. 567–575.

[13] Z. Chen and R. Ewing, *Mathematical analysis for reservoir models*, SIAM Journal on Mathematical Analysis, 30 (1999), pp. 431–453.

[14] Z. Chen and R. Ewing, *Mathematical analysis of reservoir models*, SIAM J. Math. Anal., 30 (1999), pp. 431–543.

[15] A. Coutinho and J. Alves, *37399-parallel finite element simulation of miscible displacements in porous media*, SPE Journal, 1 (1996), pp. 487–500.

[16] M. Cui, *A combined mixed and discontinuous galerkin method for compressible miscible displacement problem in porous media*, Journal of computational and applied mathematics, 198 (2007), pp. 19–34.

[17] B. Darlow, R. E. Ewing, and M. Wheeler, *Mixed finite element methods for miscible displacement problems in porous media*, tech. rep., Exxon Prod Res Co, USA, 1982.

[18] J. Douglas, Jr, *Superconvergence in the pressure in the simulation of miscible displacement*, SIAM journal on numerical analysis, 22 (1985), pp. 962–969.

[19] J. Douglas Jr, *Simulation of miscible displacement in porous media by a modified method of characteristic procedure*, in Numerical analysis, Springer, 1982, pp. 64–70.

[20] J. Douglas Jr, *Numerical methods for the flow of miscible fluids in porous media*, Numerical methods in coupled systems, (1984), pp. 405–439.

[21] R. G. Durán, *On the approximation of miscible displacement in porous media by a method of characteristics combined with a mixed method*, SIAM journal on numerical analysis, 25 (1988), pp. 989–1001.

[22] L. J. Durlofsky, *Accuracy of mixed and control volume finite element approximations to darcy velocity and related quantities*, Water Resources Research, 30 (1994), pp. 965–973.

[23] Y. Epshteyn and B. Rivière, *Convergence of high order methods for miscible displacement*, International Journal of Numerical Analysis and Modeling, 5 (2008), pp. 47–63.

[24] R. Ewing, T. Russell, and M. Wheeler, *Simulation of miscible displacement using mixed methods and a modified method of characteristics*, in SPE Reservoir Simulation Symposium, 1983.

[25] R. E. Ewing, T. F. Russell, and M. F. Wheeler, *Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics*, Computer Methods in Applied Mechanics and Engineering, 47 (1984), pp. 73–92.

[26] R. E. Ewing and M. F. Wheeler, *Galerkin methods for miscible displacement problems in porous media*, SIAM Journal on Numerical Analysis, 17 (1980), pp. pp. 351–365.

[27] X. Fang, Q. Wang, and P. Shuler, *Bio-engineering high performance micro-bial strains for meor by directed protein evolution technology*, Tech. Rep. DE-FC26-06NT15525, California Institute of Technology, 1201 E. California Blvd,Pasadena, CA 91125, September 2007.

[28] X. Feng, *On existence and uniqueness result for a coupled system modeling miscible displacement in porous medias*, J. Math. Anal. Appl., 194 (1995), pp. 883–910.

[29] A. Garder Jr., D. Peaceman, and A. Pozzi Jr., *Numerical calculation of mul-tidimensional miscible displacement by the method of characteristics*, Old SPE Journal, 4 (1964), pp. 26–36.

[30] S. Gottlieb, G. Wei, and S. Zhao, *A unified discontinuous galerkin framework for time integration*, preprint, (2010).

[31] H. Hoteit and A. Firoozabadi, *Multicomponent fluid flow by discontinuous galerkin and mixed methods in unfractured and fractured media*, Water Resources Research, 41 (2005).

[32] H. Hoteit and A. Firoozabadi, *Compositional modeling by the combined discon-tinuous galerkin and mixed methods*, SPE Journal, 11 (2006), pp. 19–34.

[33] M. Jensen and R. Müller, *Stable CrankNicolson discretisation for incompressible miscible displacement problems of low regularity*, Numerical Mathematics and Advanced Applications 2009, (2010), pp. 469–477. 10.1007/978-3-642-11795-4_50.

[34] C. Klaij, J. Van der Vegt, and H. Van der Ven, *Space–time discontinuous Galerkin method for the compressible Navier–Stokes equations*, Journal of Computa-tional Physics, 217 (2006), pp. 589–611.

[35] P. Lin and D. Yang, *An iterative perturbation method for the pressure equation in*

the simulation of miscible displacement in porous media, SIAM Journal on Scientific Computing, 19 (1998), pp. 893–911.

[36] A. F. LOULA, E. L. GARCIA, AND A. L. COUTINHO, *Miscible displacement simulation by finite element methods in distributed memory machines*, Computer methods in applied mechanics and engineering, 174 (1999), pp. 339–354.

[37] R. MOSÉ, P. SIEGEL, P. ACKERER, AND G. CHAVENT, *Application of the mixed hybrid finite element approximation in a groundwater flow model: Luxury or necessity?*, Water resources research, 30 (1994), pp. 3001–3012.

[38] D. PEACEMAN, *Fundamentals of numerical reservoir simulation*, vol. 6, Elsevier, 1977.

[39] D. PEACEMAN AND R. HH, *Numerical calculation of multidimensional miscible displacement*, Old SPE Journal, 2 (1962), pp. 327–339.

[40] B. RIVIÈRE, *Discontinuous Galerkin methods for solving the miscible displacement problem in porous media*, PhD thesis, University of Texas at Austin, 2000.

[41] B. RIVIÈRE AND N. WALKINGTON, *Convergence of a discontinuous Galerkin method for the miscible displacement under low regularity*, SIAM J. Num. Anal., 49 (2011), pp. 1085–1110.

[42] B. RIVIÈRE AND M. WHEELER, *Discontinuous Galerkin methods for flow and transport problems in porous media*, Communications in numerical methods in engineering, 18 (2002), pp. 63–68.

[43] B. M. RIVIÈRE, *Discontinuous Galerkin methods for solving elliptic and parabolic equations: theory and implementation*, vol. 35, Cambridge University Press, 2008.

[44] W. RUDIN, *Real and complex analysis*, Tata McGraw-Hill Education, 2006.

[45] T. Russell and M. Wheeler, *Finite element and finite difference methods for continuous flows in porous media*, The mathematics of reservoir simulation, 1 (1983), pp. 35–106.

[46] T. F. Russell, *Time stepping along characteristics with incomplete iteration for a galerkin approximation of miscible displacement in porous media*, SIAM journal on numerical analysis, 22 (1985), pp. 970–1013.

[47] R. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, 1996.

[48] P. Siegel, R. Mosé, P. Ackerer, and J. Jaffré, *Solution of the advection–diffusion equation using a combination of discontinuous and mixed finite elements*, International journal for numerical methods in fluids, 24 (1997), pp. 595–613.

[49] J. Sudirham, J. Van der Vegt, and R. Van Damme, *Space–time discontinuous Galerkin method for advection–diffusion problems on time-dependent domains*, Applied numerical mathematics, 56 (2006), pp. 1491–1518.

[50] S. Sun, B. Rivière, and M. Wheeler, *A combined mixed finite element and discontinuous Galerkin method for miscible displacement problem in porous media*, in Recent progress in computational and applied PDEs, conference proceedings for the international conference held in Zhangjiajie in July, 2001, pp. 321–348.

[51] L. Tongchao, *A collocation method for incompressible miscible displacement in porous media*, Numerical Mathematics A Journal of Chinese Universities, 3 (1992), p. 003.

[52] J. Van der Vegt and H. Van der Ven, *Space–time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flows: I. general formulation*, Journal of Computational Physics, 182 (2002), pp. 546–585.

[53] H. VAN DER VEN AND J. VAN DER VEGT, *Space–time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flows: II. Efficient flux quadrature*, Computer methods in applied mechanics and engineering, 191 (2002), pp. 4747–4780.

[54] N. WALKINGTON, *Compactness properties of the dg and cg time stepping schemes for parabolic equations*, SIAM Journal on Numerical Analysis, 47 (2010), pp. 4680–4710.

[55] J. WARREN, H. PRICE, F. SKIBA, AND R. VARGA, *Miscible displacement: The liquid-liquid case*, in Fall Meeting of the Society of Petroleum Engineers of AIME, 1961.

[56] Y. WEI, *Stabilized finite element methods for miscible displacement in porous media*, ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique, 28 (1994), pp. 611–665.

[57] D. YANG, *Mixed methods with dynamic finite-element spaces for miscible displacement in porous media*, Journal of computational and applied mathematics, 30 (1990), pp. 313–328.

[58] D. YANG, *Simulation of miscible displacement in porous media by a modified uzawa's algorithm combined with a characteristic method*, Computer methods in applied mechanics and engineering, 162 (1998), pp. 359–368.