Re-Reforming the Bostonian System: A Novel Approach to the Schooling Problem☆

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Abstract

This paper proposes the notion of ε-stability to conciliate Pareto efficiency and fairness. We propose the use of a centralized procedure, the *Exchanging Places Mechanism*. It endows students a position according with the Gale and Shapley students optimal stable matching as tentative allocation and allows the student to trade their positions. We show that the final allocation is ε-stable, i.e. efficient, fair and immune to any justifiable objection that students can formulate.

Keywords: School allocation problem, Pareto efficient matching.

JEL: C78, D63, I28

1. Introduction

In many countries local authorities coordinate the admission process to allocate the children, entering public education, a place in a primary school. Different cities use distinct procedures to do such a task. Almost all of then take into account the parents' interests for the schools and build priority

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as preference for, otherwise indifferent, centers. When this priority list is not enough to break all relevant ties the municipalities use a sort of random process to break the ties.

The usual way to deal with the problem of how students would be prioritized is solved by building a function that associates each student a score for each school. The scoring functions, or the priority orderings they induce, are constructed to reduce the well-known efficiency-equity trade-off (Roth, 1982). First, to reach a (Rawlsian) efficiency, when allocating places to students, the Public Administration tries to reduce the cost in which households incur, due to the children's attendance to the school by accurately describing school areas and other relevant criteria, as the presence of siblings. Second, and once efficient equivalence classes has been established, a fair lottery is used to break ties inside each such an efficiency category. The fact that all the students have the same (ex-ante) probability of being ranked at any position, inside their class, induces a kind of ex-ante internal equity relative to each efficiency category.

Therefore, we can think that students' priorities are decided following an efficient intra-equitable criterion. Nevertheless, it is hard to argue that this criterion is fully equitable since some agents are ex-ante prioritized, relative to the final ranking. Furthermore, the particular procedure use to match students and schools' places, could induce inefficient allocations.

The literature on this issue concentrates on three allocation mechanisms: the Boston mechanism, the Student-Optimal Stable mechanism and the Top Trading Cycle.

The first one or "Boston mechanism" coincides with the "Now-or-Never mechanism" introduced by Alcalde (1996) to implement the set of stable allocations in undominated Nash equilibria. The main properties that this mechanism exhibits are:

(1) When agents do not play strategically, i.e. when the preferences that they declare are their true preferences, the allocation it suggests is Pareto efficient.

(2) When agents act strategically,[3] the expected allocations are stable. Furthermore, any stable allocation can be decentralized by a Nash equilibrium.

The second mechanism that has been explored is the Student-Optimal Stable mechanism, SOSM hence for, which coincides with the realization of the Deferred Acceptance algorithm in which students send proposals to the schools. This mechanism, introduced by Gale and Shapley (1962), always selects a stable allocation. Moreover, all the students (weakly) prefer this allocation to any other stable allocation. Furthermore, when this mechanism is used, students have no interest on playing strategically.

The third mechanism was proposed, under the name of "Top Trading Cycles" mechanism, TTCM henceforth, by Abdulkadiroğlu and Sönmez (2003). It is inspired in the homonym algorithm used by Shapley and Scarf (1974) to prove the existence of stable allocations in their "Housing Markets" model. It

[3]The result by Alcalde (1996) is established for undominated strategies. This is because in the framework analyzed in this paper, agents in both sides of the market are allowed to select strategically their actions. Nevertheless, in the problem modeled in the present paper, since schools are not allowed to act strategically, there is no need to assume that agents use undominated strategies to reach the result.

is remarkable that this mechanism always selects Pareto efficient allocations and that students have no interest on misrepresent their true characteristics. Nevertheless, the way in which it has been described yields to consider the TTCM as hardly used in practice. In our opinion, it is not easy to convince the students (or their relatives) about the intuition behind such an allocation procedure.

The redesign that has been introduced in several markets has proposed the SOSM as the best option for school markets, i.e. for almost one sided-markets with indifferences. However, as Erdil and Ergin (2008), Kesten (2010) and Abdulkadiroğlu et al. (2009) agrees, the welfare lost due to select the SOSM can be troubledly large. The alternative is to move to a Pareto efficient mechanism such the TTCM. However, it can be argued against the TTCM that some priorities as, for example, the priority obtaining for having siblings attending the same school, should not be exchangeable. It can also be said that the TTCM dilutes the impact of priorities that do not belongs to parents but to the district. Kesten (2010) address this problem proposing an intermediate approach: an efficiency adjusted SOSM, the Efficiency Adjusted Deferred Acceptance Mechanism or EADAM, that allows a student to consent to waive a certain priority that has no effect on his assignment.

In this paper we present an alternative way to deal with the problem. We propose a new solution concept, to be called ε-stability. Our solution concept is inspired in the usual way in which *Public Decision-Makers* act. In fact, what it seems to be the main objective for the Public School Administrators is to allocate students optimally. This can be one of the main reasons supporting systems like the (former) mechanism used in the Boston area. The idea behind ε-stability aims to improve the inefficient solution obtained if we impose the equity criterion to be fulfilled.

The concept of ε-stability adapts the idea of exchange-proofness of an allocation, introduced by Alcalde (1995), to the context of School Allocation Problems. The purpose of such a notion is to guarantee the efficiency of allocations. To introduce ε-stability, let us consider that each student has been allocated a place in some school. And let us assume that there is no student *justifiably envying* some other student's place. Under such a consideration, one might well assume that students owns or have some *rights* on their places and that they are free to exchange their rights if they wished to. A solution is ε-stable if no group of students have interest on exchanging their school places. The way to reach our objective is simple. We start by allocating each student at the best school she can reach, provided that the allocation must be stable. This job can be done by applying the students-propose deferred algorithm designed by Gale and Shapley (1962). Then, let us consider the Walrasian market where agents are the students, commodities are the schools' places; and each agent's initial endowment is her best stable allocation, as previously stated. This market has the formal structure of a "Housing Market", as modeled by Shapley and Scarf (1974). Therefore, let us allow the students to exchange *their places*. As a consequence of this interaction an efficient allocation is reached, in which all the students will be, at least, as well as in any stable allocation and where there will no be additional incentives to trade. This allocation would be ε-stable.

When analyzind the related literature, the paper by Balinski and Sönmez (1999) can be viewed as the starting point on a large, recent, literature on solutions for the so called *student allocation problem.*[4] Balinski and Sönmez (1999) formalized the way in which students should be allocated to the different available positions in the education system. What it is important in their model, and imposes a new way to explore how to study the problem, related to the Matching Theory introduced by Gale and Shapley (1962), is that agents in a side of the market (schools, colleges, institutions, etc.) have no preferences on which their mates are. These agents just declare some lists describing students' priorities when the allocation of some place is contested. These priorities are obtained to reach allocation efficiency. For instance, at primary school level, when students can be considered, from an academic point of view, as undistinguished, the only variables that are considered lie on family background (related or not to the schools) as walking distance, sibling, per-capita household income, etc.; for higher education, since students' characteristics are differentiated, students' academic skills are used to decide priorities. Note that, from a *Social* perspective, the use of academic skills as a relevant variable lies on allocative efficiency. This is because the success of students' effort is more likely if they exhibit the appropriate skills to follow some particular studies.

Given theses primitives, Balinski and Sönmez (1999) provided a way to associate a solution concept in this family of problems to the classical notion of stability introduced by Gale and Shapley (1962). This idea is nice, original

[4]Some authors made a formal distinction between the model by Balinski and Sönmez (1999) and the school allocation problem presented in Ergin and Sönmez (2006). Nevertheless, for our purposes, when the schools do not act strategically, and their only role is providing educational services, we can treat both literatures as coincident.

and has been useful to introduce some important reforms in the admission systems used at the Elementary School in some areas in the US.[5] The key notion introduced by Balinski and Sönmez (1999) is their *fairness criterion*.[6] It implicitly assumes that some students are the *usufructuary* of certain places. Nevertheless, in their model, students are not free to exchange their *rights on using the places*. However there is evidence that agents are often allow to improve the initial allocation by exchanging their rights. This evidence is abundant both in the ranges of civil servants and army. For instance, civil servants are allowed to exchange their places in Spain,[7] and a similar exchange can be done in the US Army under the so-called *Enlisted Assignment Exchanges* (SWAPS).[8] And, following a similar spirit, i.e. allowing agents to exchange goods they are not the owners, but they *retain some rights*, it can be found some socially accepted systems like several *international students exchange programs*, or the recent *kidney exchange*[9] programs.

The rest of the paper is organized as follows. Section 2 introduces the basic framework and provides some definitions which are classical in the lit-

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[5]See, for instance the papers by Abdulkadiroğlu et al. (2009), relative to the New York City High School Match, or Abdulkadiroğlu et al. (2006) for the Boston Public School Match.

[6]This criterion is also called *non justified envy* by Haeringer and Klijn (2009). We consider both expressions as equivalent throughout the paper.

[7]Art. 62 in the Spanish law that rules civil servants or Law 315/1964, B.O.E 15.02.1964. This regulation can be obtained from http://www.ua.es/oia/es/legisla/funcion.htm.

[8]The reader is gathered to http://usmilitary.about.com/od/armyassign/a/swap for further information on this matter.

[9]Transplant services in Ronald Reagan UCLA Medical Center provides some information via the web page http://transplants.ucla.edu/body.cfm?id=112

erature. Section 3 proposes how to modify the allocation system, provided that colleges are free to exhibit any way to prioritize students. Section 4 introduces a discussion on the equilibrium concept. It also proposes a procedure selecting, for each School Allocation Problem, a φ-stable allocation. The way in which this mechanism is described points out how the actual Boston system might be *re-reformed*. Section 5 studies the strategic properties of the mechanism. Conclusions are gathered to Section 6. Finally, all the proofs are relegated to the Appendix.

2. The School Allocation Problem

This section is devoted to introduce some formalisms related to the School Allocation Problem. This family of problems faces two set of non-empty disjoint agents to be called Students and Schools. The set of Students is denoted by $\mathcal{S}$, and has n individuals, i.e. $\mathcal{S} = \{s_1, \ldots, s_i, \ldots, s_n\}$. The set of Schools is denoted by $\mathcal{C}$, and has m elements, i.e. $\mathcal{C} = \{c_1, \ldots, c_j, \ldots, c_m\}$.

Each school has a number of seats (or places), to be distributed among the students, that will be called its capacity. Let $q_{c_j} \geq 1$ denote school c_j's capacity; and let $Q = \left\{q_{c_1}, \ldots, q_{c_j}, \ldots, q_{c_m}\right\}$ the vector summarizing schools' capacities. Schools are also endowed a priorities linear ordering over the set of students. Let $\pi_{c_j} \in \mathbb{R}^n$ be the students' ordering for school c_j and Π the $(m \times n)$-matrix summarizing these priorities. Formally, π_{c_j} is described as a n-dimensional vector such that for each $k \in \{1, \ldots, n\}$ there is a unique student s_i for which $\pi_{c_j s_i} = k$; given this description, the j-th row for matrix Π coincides with vector π_{c_j}.

Note that, under our description, no school would consider a student to be inadmissible. Notice that most schools systems impose such a restriction in the way that the schools rank their potential students. Nevertheless, our model might capture the possibility of a student to be inadmissible at a low cost: just by introducing a new variable for each school defining the priority level of the last admissible student.

On the other side, each student has linear preferences over the set of schools, so that no student will consider two different schools as equivalent (or indifferent), and no school is neither considered as inadmissible by any student. Let ρ_{s_i} denote the schools' ranking induced by student s_i's preferences,[10] and Φ the $(n \times m)$-matrix summarizing these rankings. Note that our model assumes that each student considers all the school as admissible.[11] Nevertheless, we can also reformulate this model by assuming that each student might consider some schools as unacceptable. The essence of this paper is the same in both frameworks.

Therefore, a School Allocation Problem can be described by listing the elements above: $\mathcal{SAP} = \{\mathcal{S}, \mathcal{C}; \Phi, \Pi, Q\}$. We will say that a School Allocation Problem is non-scarce whenever there is enough places to allocate all the students

$$\sum_{c_j \in \mathcal{C}} q_{c_j} \geq n.$$

Given a School Allocation Problem, $\mathcal{SAP}$, a solution for it is an application μ that matches students and schools' places. Such a correspondence is

[10] I.e., $\rho_{s_i c_j} = 3$ indicates that student s_i considers that c_j is her third-best school.

[11] Here, we can also invoke legislative regulations establishing that school attendance is compulsory for the children of certain ages.

called a matching. Formally,

Definition 1. A *matching* for $\mathcal{SAP}$, a School Allocation Problem, is a correspondence μ, applying $\mathcal{S} \cup \mathcal{C}$ into itself, such that:

1. For each s_i in $\mathcal{S}$, if $\mu(s_i) \neq s_i$, then $\mu(s_i) \in \mathcal{C}$;
2. For each c_j in $\mathcal{C}$, $\mu(c_j) \subseteq \mathcal{S}$, and $|\mu(c_j)| \leq q_{c_j}$;[12] and
3. For each s_i in $\mathcal{S}$, and any c_j in $\mathcal{C}$, $\mu(s_i) = c_j$ if, and only if, $s_i \in \mu(c_j)$.

The central solution concept used through the literature is stability, as defined by Balinski and Sönmez (1999). This stability notion coincides with the pair-wise stability introduced by Gale and Shapley (1962). Under our considerations (i.e., each school is acceptable for any student and vice versa), stability is defined as follows.

Definition 2. A matching for $\mathcal{SAP}$, say μ, is said *stable* if there is no student-school pair (s_i, c_j) such that

1. $\mu(s_i) = s_i$, or $\rho_{s_i c_j} < \rho_{s_i \mu(s_i)}$; and
2. $|\mu(c_j)| < q_{c_j}$, or $\pi_{c_j s_i} < \pi_{c_j s_h}$ for some $s_h \in \mu(c_j)$.

Throughout this paper, we adopt the convention that $\rho_{s_i \mu(s_i)} = m + 1$ whenever $\mu(s_i) = s_i$.

The idea of instability comes basically from the notion of justified envy. (See Haeringer and Klijn, 2009). Let us consider a matching μ, and let us assume that student s_i prefers to study at school c_j rather that developing her educative formation at her actual school $\mu(s_i)$. If s_i has a priority higher

[12] Throughout this paper $|T|$ will denote the cardinality of set T.

than some of the actual students attending school c_j, or this school is still having some vacant, she might claim that the allocation process has been *unfair*.

A second notion that has also been analyzed in this framework is that of efficiency. To introduce appropriately this concept, let us remember that the only role for the schools is to provide educational services needed by the students. Therefore the natural notion of efficiency, as proposed by Balinski and Sönmez (1999) for this framework, is Pareto efficiency (from the students' point of view).

Definition 3. Given a School Allocation Problem, $\mathcal{SAP}$, we say that matching μ is *Pareto efficient* if for any other matching μ' there is a student, say s_i, such that

$$\rho_{s_i \mu(s_i)} < \rho_{s_i \mu'(s_i)}.$$

Note that, for any non-scarce School Allocation Problem, stability and/or efficiency of a matching μ implies that, for each student s_i, $\mu(s_i) \in \mathcal{C}$.

A matching mechanism is a regular procedure that associates to each School Allocation Problem a matching for such a problem. A matching mechanism $\mathcal{M}$ is said to be stable if, for any given problem, it always selects a stable matching. Similarly, we say that a matching mechanism is Pareto efficient whenever its outcome is always Pareto efficient, related to its input. It is easy to see that there are stable matching mechanisms. In fact, any of the versions of the deferred-acceptance algorithms proposed by Gale and Shapley (1962) associates a stable matching for the related School Allocation Problem. On the other hand, the *now-or-never mechanism* introduced by

Alcalde (1996) always selects a Pareto efficient matching when the proposals are made by the students.[13]

The first question that we deal with is the possibility of designing matching mechanisms that always select stable and Pareto efficient allocations. As Proposition 1 states it is a well known result that it might be an impossible task to conciliate the “fairness” notion involving stability and Pareto efficiency.

Proposition 1. There is no matching mechanism selecting a stable and Pareto efficient allocation for each School Allocation Problem.

What Proposition 1 suggests is the need of proposing a new solution concept that accurately combines the notions of fairness, reflected by stability, and Pareto efficiency.

3. ε-Stability: A New Solution Concept

In this section we propose a new solution concept for the School Allocation Problem. It tries to reduce the trade-off between equity (in terms of stability) and efficiency. The central idea to reach our objective is just to restrict which are the statements, made by some student, that are considered “admissible” to induce instability of an allocation.

Following a large tradition on cooperative games, it is fairly important to be precise when defining which objections (made by a set of agents) are admissible and which are not. This is the essence of the concept of Bargaining

[13] The now-or-never mechanism is also known as the Boston mechanism because it was used in the Boston school district.

Set introduced by Aumann and Maschler (1964), and the solutions concepts that appeared following that paper. The idea behind the Bargaining Set is that any agent is free to formulate an objection against an allocation. What she should do is to propose an alternative allocation fitting some properties. Then, if an agent formally presents an objection against an allocation, any other agent might formulate an objection against this *new proposal* in the same fashion that previously did the former agent. That is, any other agent might *counter-object*. What φ-stability of an allocation imposes is that:

1. no agent will object against this allocation, or
2. any objection presented by an agent will be counter-objected.

In this paper we capture the idea behind stability considering as valid only objections against an allocation that cannot be counter-objected. To illustrate our proposal, let us analyze the following example.

Example 1. Let us consider the following School Allocation Problem. $\mathcal{S} = \{1,2,3\}$; $\mathcal{C} = \{a,b,c\}$; $Q = (1,1,1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Note that matching μ, with $\mu(1) = a$; $\mu(2) = b$; and $\mu(3) = c$ is not stable. This is because student 2 claims that she has priority, related to student 1, for studying at school a. Now, let us propose to student 2 the following deal:

> "If you are able to propose a matching, preferred by you to μ, and no other student would claim that the new proposal is

unfair (as you did when μ was proposed), the new matching will be implemented."

The conclusion will be that student 2 will not be able to propose an alternative matching.

Therefore, and adapting the arguments above relative the Bargaining Set, the process that φ-stability captures can be informally described as follows. Let us consider a matching μ. Then, any student is free to claim that such an allocation is, from her point of view, unfair. Nevertheless, her criticism has to be supported by proposing an alternative matching. The new proposal is accepted only if no student is able to show, using identical arguments, that the new matching unfair too.

Definition 4. [**Fair Objection**]
Let $\mathcal{SAP}$ be a School Allocation Problem, and let μ be a matching for such a problem. A *fair objection* from student $s_i \in \mathcal{S}$ against μ is a pair (s_i, μ') such that

1. $\rho_{s_i\mu'(s_i)} < \rho_{s_i\mu(s_i)}$, and
2. $|\mu'(\mu'(s_i))| < q_{\mu'(\mu'(s_i))}$, or $\pi_{\mu'(s_i)s_i} < \pi_{\mu'(s_i)s_h}$ for some $s_h \in \mu'(\mu'(s_i))$.

Definition 5. [**Counter-Objection**]
Let (s_i, μ') be a fair objection against matching μ. A *counter-objection* from student s_h against (s_i, μ') is a pair (s_h, μ'') that constitutes a fair objection against matching μ'.

We say that (s_i, μ') is a *justified fair objection* against μ if it cannot be counter-objected

Definition 6. [**φ-Stability**]
Let $\mathcal{SAP}$ be a School Allocation Problem. We say that matching μ is *φ-stable* if any objection against it can be counter-objected.

Therefore, the idea of φ-stability for μ is that when some student might claim that such a matching is unfair, she is unable to propose an alternative solution that no student would consider an unfair matching.

Note that, for any School Allocation Problem, $\mathcal{SAP}$, the set of φ-stable matchings is a super-set of the set of stable matchings. Therefore, the next statement follows.

Proposition 2. Let SAP be a School Allocation Problem. Then, it has a φ-stable matching.

What it is also relevant is that, in general, there are School Allocation Problems having φ-stable matchings that are not stable. Notice that matching μ, proposed in Example 1, is unstable, but it is φ-stable for the related problem.

The central solution concept that we propose in this section, ε-stability, comes from the confluence of two solution ideas, namely Pareto efficiency and φ-stability.

Definition 7. [**ε-Stability**]
Let $\mathcal{SAP}$ be a School Allocation Problem. We say that matching μ is *ε-stable* if it is Pareto efficient and φ-stable for $\mathcal{SAP}$.

The next question that we deal with is the existence of ε-stable allocations. Even though that the sets of Stable and Pareto efficient matchings might not intersect (Proposition 1), when we concentrate on φ-stable allocations, instead on stable ones, such an intersection is always non-empty.

Theorem 1. Let $\mathcal{SAP}$ be a School Allocation Problem. Then, it has a matching μ which is ε-stable.

4. The Exchanging Places Mechanism and ε-Stability

This section proposes an algorithm that always selects a ε-stable matching. Therefore it can be seen as a constructive proof for Theorem 1.

The mechanism that we propose can be introduced as a combination of two algorithms. The first one is the classic students-proposing deferred acceptance algorithm. The second one follows the idea reflected in the Gale's Top Trading Cycle, introduced by Shapley and Scarf (1974).

Since the deferred acceptance algorithm is well-known in the literature, we concentrate on a description of how the second algorithm works. First, we need some additional definitions.

Definition 8. Let $\mathcal{SAP}$ be a School Allocation Problem, and μ a matching. For s_i given, let δ_{s_i} a ranking of the set of students.[14] We say that δ_{s_i} μ-agrees ρ_{s_i} if, and only if,

1. $\delta_{s_i s_i} < \delta_{s_i s_h}$, for each $s_h \neq s_i$ such that $\mu(s_h) \notin \mathcal{C}$,
2. $\delta_{s_i s_i} < \delta_{s_i s_h}$, for each $s_h \neq s_i$ such that $\mu(s_h) = \mu(s_i)$, and

[14] This is, for each $k \in \{1, \ldots, n\}$ there is one, and only one, student s_h such that $\delta_{s_i s_h} = k$.

3. for each two students s_h and s_l, with $\mu(s_h), \mu(s_l) \in \mathcal{C} \setminus \mu(s_i)$,

$$\delta_{s_i s_h} < \delta_{s_i s_l} \text{whenever } \rho_{s_i \mu(s_h)} < \rho_{s_i \mu(s_l)}.$$

By extension, if we denote by Σ the matrix whose i-th row is δ_{s_i}, we say that Σ μ-agrees Φ whenever for each student s_i, δ_{s_i} μ-agrees ρ_{s_i}.

Definition 9. [**Students' Incidence Matrix**]
Let $\mathcal{S}$ be the set of students, and let Σ be a matrix of students' rankings, whose i-th row represents s_i's ranking. For each subset of students $\mathcal{S}' \subseteq \mathcal{S}$ we define its incidence matrix as the $(|\mathcal{S}'| \times |\mathcal{S}'|)$-matrix that associates, to each s_i and s_h in $\mathcal{S}'$, the value

$$I^{\Sigma}_{\mathcal{S}'}(s_i, s_h) = \begin{cases} 1 & \text{if } \delta_{s_i s_h} < \delta_{s_i s_l} \text{for each } s_l \in \mathcal{S}' \setminus \{s_h\} \\ 0 & \text{otherwise} \end{cases}$$

Definition 10. Let Σ be a matrix of students' rankings, and $\mathcal{S}' \subseteq \mathcal{S}$ be a subset of students. A Cycle for the incidence matrix $I^{\Sigma}_{\mathcal{S}'}$ is a (non-empty) ordered set of students in $\mathcal{S}'$, $\{s^1, \ldots, s^i, \ldots, s^t\}$ such that, for each $i \leq t-1$,

$$I^{\Sigma}_{\mathcal{S}'}\left(s^i, s^{i+1}\right) = I^{\Sigma}_{\mathcal{S}'}\left(s^t, s^1\right) = 1.$$

Note that, since for each subset of students $\mathcal{S}'$ each row of its incidence matrix has a unique element whose value is 1, it is easy to see that

1. $I^{\Sigma}_{\mathcal{S}'}$ has, at least one cycle, and
2. each student $s_i \in \mathcal{S}'$ is involved in, at most, one cycle.

We are now ready to introduce the workings for the *Top Trading Cycle algorithm* in our framework.

Definition 11. [**The μ-Σ-Top Trading Cycle Algorithm**]
Let $\mathcal{SAP}$ be a School Allocation Problem, and μ a matching. Let Σ be a matrix μ-agreeing the students' ranking matrix Φ. The μ-Σ-Top Trading Cycle algorithm, μ-Σ-TTCA henceforth, works as follows:

(Step 1) Let us consider the students' incidence matrix $I_{\mathcal{S}}^{\Sigma}$, and let $\mathcal{S}^1$ be the students belonging to a cycle for $I_{\mathcal{S}}^{\Sigma}$. Then associate each student $s_i \in \mathcal{S}^1$ her mate

$$\mu^{TTC}(s_i) = \mu(s_h), \text{ where } s_h \text{ satisfies } I_{\mathcal{S}}^{\Sigma}(s_i, s_h) = 1.$$

Let $\mathcal{S}_1 = \mathcal{S} \setminus \mathcal{S}^1$. If $\mathcal{S}_1 = \emptyset$ the algorithm ends, and matching μ^{TTC} is implemented. Otherwise, go to Step 2.

(Step k) Let us consider the students' incidence matrix $I_{\mathcal{S}_{k-1}}^{\Sigma}$, and let $\mathcal{S}^k \subseteq \mathcal{S}_{k-1}$ be the students belonging to a cycle for $I_{\mathcal{S}_{k-1}}^{\Sigma}$. Then associate each student $s_i \in \mathcal{S}^k$ her mate

$$\mu^{TTC}(s_i) = \mu(s_h) \text{ where } s_h \text{ satisfies } I_{\mathcal{S}_{k-1}}^{\Sigma}(s_i, s_h) = 1.$$

Let $\mathcal{S}_k = \mathcal{S}_{k-1} \setminus \mathcal{S}^k$. If $\mathcal{S}_k = \emptyset$ the algorithm ends, and matching μ^{TTC}, as described throughout steps 1 to k, is implemented. Otherwise, go to Step $k+1$.

The algorithm ends at the step t for which $\mathcal{S}_t = \emptyset$.

Note that, since the set of students is finite and, for each step k, $|\mathcal{S}_{k+1}| < |\mathcal{S}_k|$, this algorithm always ends in a finite number of steps.

Relative to the output of the μ-Σ-TTCA, for any given μ and each matrix Σ, we can guarantee that the following properties are satisfied:

1. μ^{TTC} is Pareto efficient;
2. for each student s_i,

$$\rho_{s_i \mu^{TTC}(s_i)} \leq \rho_{s_i \mu(s_i)}.$$

3. $\mu = \mu^{TTC}$ if, and only if, the former matching is Pareto efficient; and
4. if students are asked to reveal their rankings to implement matching μ^{TTC}, they will obtain no advantage from misrepresenting their rankings.

Note that the above properties can be seen as a conclusion derived from theorems 4 and 5 in Alcalde-Unzu and Molis (2009).

We can now establish the following result.

Theorem 2. Let $\mathcal{SAP}$ be a School Allocation Problem, and μ^{SO} its student optimal stable matching. Let Σ^{SO} be a matrix μ^{SO}-agreeing the students' rankings matrix Φ. Then the matching μ^{TTC} obtained by applying the μ^{SO}-Σ^{SO}-TTCA is ε-stable.

Summarizing the process that we have introduced in this section, let us propose a formal description for the *Exchanging Places Mechanism*. To describe how this procedure operates, let fix the set of students $\mathcal{S}$, and define, for each student s_i, an exchanging-priorities vector ω_{s_i} that will be understood as a rule for prioritizing exchanges.[15] In other words, let us imagine

[15] ω_{s_i} can be determined by the local school committee when establishing a students ordering to be used for breaking ties in schools' scores. For instance, and in order to make up for the draw effect, each student's vector ω_{s_i} might represent the ordering reversing the draw result.

that student s_i is located a place at school c_j, and she would like to exchange her place to some student attending to school c_t. What ω_{s_i} describes is how s_i orders the students having a place at c_t to (sequentially) propose them such an exchange. Given this vector for each student, we summarize this information by defining a matrix of rankings Ω.

The "Exchanging Places Mechanism" operates as follows. Given a School Allocation Problem, $\mathcal{SAP}$, and matrix Ω, let μ^{SO} be the students optimal stable matching for $\mathcal{SAP}$, and Σ^{SO} a matrix μ^{SO}-agreeing students' rankings matrix Φ, which is obtained by preserving the priorities established in Ω. Then, apply the μ-Σ-TTCA for $\mu = \mu^{SO}$ and $\Sigma = \Sigma^{SO}$. The result of this procedure is the outcome for the Exchanging Places Mechanism.

Definition 12. [**The Exchanging Places Mechanism**]
We define the Exchanging Places Mechanism, EPM henceforth, as the function that associates to each School Allocation Problem, $\mathcal{SAP}$, and $(n \times n)$-ranking matrix, Ω,[16] the matching μ^* which is obtained by applying the μ-Σ-TTCA, where

1. μ is the student optimal stable matching for $\mathcal{SAP}$, and
2. Σ is the matrix μ-agreeing the students' ranking matrix Φ that, for each s_i and any two students s_h and s_ℓ in $\mathcal{S} \setminus \{s_i\}$ such that $\mu(s_h) = \mu(s_\ell)$, $\Sigma_{ih} < \Sigma_{i\ell}$ if, and only if $\Omega_{ih} < \Omega_{i\ell}$.

To conclude this section, let us propose the following example to show how the EPM works.

[16] By "ranking matrix" we mean that it satisfies that for each row i, and any two different columns j, and h, $\Omega_{ij} \neq \Omega_{ih}$.

Example 2. Let us consider the following Schools Allocation Problem. $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{C} = \{a, b, c, d\}$, the capacity for each school is 2; and the Rankings and Priorities matrices are

$$\Phi = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 2 & 3 & 1 \\ 1 & 4 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix}; \text{ and } \qquad \Pi = \begin{bmatrix} 2 & 3 & 8 & 1 & 7 & 4 & 6 & 5 \\ 6 & 2 & 1 & 5 & 8 & 4 & 3 & 7 \\ 7 & 5 & 6 & 8 & 2 & 3 & 1 & 4 \\ 8 & 5 & 3 & 4 & 1 & 2 & 7 & 6 \end{bmatrix}$$

Let us assume that, for each student, the vector of exchanging priorities is $\omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The EPM proceeds as follows.

1. Let us calculate the student optimal stable matching, μ^{SO}. The application of the students-proposing deferred acceptance algorithm is summarized in the following table[17]

[17] A row in the table indicates the applications that each school receives at such a step. The students marked in red are the ones whose application is refused, whereas the students marked in green are tentatively accepted by the school.

Step	a		b		c		d
1	5, 6, 7		1, 8		2, 3		4
2	6, 7		1, 8		2, 3, 5		4
3	6, 7		1, 3, 8		2, 5		4
4	6, 7, 8		1, 3		2, 5		4
5	6, 8		1, 3, 7		2, 5		4
6	1, 6, 8		3, 7		2, 5		4
7	1, 6		3, 7		2, 5, 8		4
8	1, 2, 6		3, 7		5, 8		4
9	1, 2		3, 6, 7		5, 8		4
10	1, 2		3, 7		5, 6, 8		4
11	1, 2		3, 7		5, 6		4, 8
μ^{SO} :=	1, 2		3, 7		5, 6		4, 8

2. Matrix Σ^{MO} is the following

$$\Sigma^{MO} = \begin{bmatrix} 3 & 4 & 1 & 7 & 5 & 6 & 2 & 8 \\ 4 & 3 & 7 & 5 & 1 & 2 & 8 & 6 \\ 5 & 6 & 3 & 7 & 1 & 2 & 4 & 8 \\ 7 & 8 & 3 & 1 & 5 & 6 & 4 & 2 \\ 1 & 2 & 7 & 5 & 3 & 4 & 8 & 6 \\ 1 & 2 & 3 & 7 & 6 & 5 & 4 & 8 \\ 1 & 2 & 4 & 5 & 7 & 8 & 3 & 6 \\ 3 & 4 & 1 & 8 & 5 & 6 & 2 & 7 \end{bmatrix};$$

3. and the successive incidence matrices are[18]

$$I_{\mathcal{S}}^{\Sigma^{SO}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{; and thus } \mathcal{S}_1 = \{2,\, 6,\, 7,\, 8\}\,;$$

$$I_{\mathcal{S}_1}^{\Sigma^{SO}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{; and thus } \mathcal{S}_2 = \{7,\, 8\}\,;$$

$$I_{\mathcal{S}_2}^{\Sigma^{SO}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{; and thus } \mathcal{S}_3 = \{8\}\,.$$

Since $\mathcal{S}_3$ is a singleton, matrix $I_{\mathcal{S}_3}^{\Sigma^{SO}}$ will have a cycle containing the unique element in such a set.

Therefore, and following the order in which cycles has been reached, we have that

(1.1) student 1 will get the place that μ^{MO} assigns to student 3; the latter will take the one assigned to student 5, which will obtain the place that

[18]Each cycle in a matrix is marked by using one color (red or blue) for all the elements belonging to that cycle.

1 got;

(1.2) student 4 will keep the place that μ^{MO} assigned to her;

(2.1) students 2 and 6 will exchange the seats that μ^{MO} assigned to them;

(3.1) student 7 will retain her place; and

(4.1) since no student prefers 8's place to her own, this agent will remain in her place.

To conclude this example, and with the aim of presenting a comparative for the application of some allocation procedures, relative to the data proposed in the present example, let us consider the following table. It associates each student two items: the school in which she gets a place (in blue), and the position of such school in the student's ranking (in red).

	Comparing Systems															
	1		2		3		4		5		6		7		8	
Boston Mechanism	b	1	c	1	c	1	d	1	d	3	a	1	a	1	b	1
Top Trading Cycles	b	1	c	1	c	1	d	1	a	1	b	2	a	1	d	4
Student Optimal	a	2	a	2	b	2	d	1	c	2	c	3	b	2	d	4
Exchange Places	b	1	c	1	c	1	d	1	a	1	a	1	b	2	d	4

Let us note that,

(1) The solution proposed by the *Boston mechanism* is not ε-stable. In fact, student 5 might *fairly object* this solution by proposing the student optimal stable matching. Since the last allocation is stable, no student will be able to counter-object;

(2) relative to the proposal by the TTCM, and comparing it to the one suggested by the *EPM*, let us observe that both allocations are efficient. The main difference can be founded in a *fairness criterion*. Let us concentrate on schools a, and b that are the ones in which both allocations differ. Note that, in the former allocation, student 6 envies students 5 and 7; and it is justifiable; for the latter allocation, 7 is the only student justifiable envying another student. She envies 5's allocation. Therefore, form a cardinal point of view, the outcome for the EPM is 'fairer' than the allocation proposed by the TTCM;

(3) finally, when comparing the *SOSM* and the *EPM*, it is easy to see that the latter Pareto-dominates the former. Moreover, both mechanisms induce φ-stable allocations.

Notice that the EPM does not coincides with the EADAM proposed by Kesten (2010) as we can see in the following example.

Example 3. Let us consider the following School Allocation Problem. $\mathcal{S} = \{1,2,3,4\}$; $\mathcal{C} = \{a,b,c\}$; $Q = (1,2,1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Notice that if student 1 trades with student 2, the allocation is the one provide by EDAM however if she trades with 3 the allocation is different an both are possible allocations of the EPM.

5. Strategy-Proofness

As expected being efficient the EPM proposed for the School Allocation Problem is not strategy proof.

Example 4. Let us consider the following School Allocation Problem. $\mathcal{S} = \{1, 2, 3, 4, 5\}$; $\mathcal{C} = \{a, b, c, d, e\}$; $Q = (1, 1, 1, 1, 1)$; and the ranking and priorities matrices are[19]

$$\Phi = \begin{bmatrix} 1 & 2 & - & - & 3 \\ - & 1 & 2 & - & 3 \\ 2 & - & 1 & - & 3 \\ 1 & 5 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 3 & - & 1 & - & 2 \\ 1 & 3 & - & - & 2 \\ 5 & 2 & 4 & 1 & 3 \\ - & - & - & - & - \\ - & - & - & 1 & 2 \end{bmatrix}$$

The student optimal stable matching is $\mu^{SO}(1) = b$, $\mu^{SO}(2) = \mu^{SO}(3) = a$, $\mu^{SO}(4) = e$, and $\mu^{SO}(5) = d$. When we apply the TTC we obtain $\mu^{SO-TTC}(1) = a$, $\mu^{SO-TTC}(2) = b$, $\mu^{SO-TTC}(3) = c$, $\mu^{SO-TTC}(4) = e$, and $\mu^{SO-TTC}(5) = d$. If student 4 misreport her preferences by setting $\rho'_4 = (1,\ 5,\ 2,\ 4,\ 3)$, then when applying the EPM we reach that

1. $\mu^{SO}(1) = b$, $\mu^{SO}(2) = e$, $\mu^{SO}(3) = a$, $\mu^{SO}(4) = c$, and $\mu^{SO}(5) = d$;
2. and when we apply the TTC the allocation becomes $\mu^{SO-TTC}(1) = b$, $\mu^{SO-TTC}(2) = e$, $\mu^{SO-TTC}(3) = c$, $\mu^{SO-TTC}(4) = a$, and $\mu^{SO-TTC}(5) = d$.

[19] In the next tables, when the ranking and/or priority is not completely determined, it is understood that any ranking and/or priority agreeing our description are likely valid to show our result.

Therefore the mechanism is not strategy proof.

This fact is closely related with some previous results. First, Erdil and Ergin (2008) show that no student-optimal stable mechanism is strategy proof. Kesten (2010) shows that when preferences are strict, there is no strategy-proof and Pareto efficient mechanism that Pareto dominates the deferred acceptance mechanism. Abdulkadiroğlu et al. (2009) provided a tighter bound, i.e., even when the efficiency requirement is relaxed, no strategy-proof mechanism can Pareto improve upon deferred acceptance with some tie-breaking, were the preferences strict or not. We find then that there is a trade-off between efficiency and strategy-proofness.

6. Concluding Remarks

Let us start this section by referring the reform hold in Boston (Citing Abdulkadiroğlu et al., 2005, Section IV).

> A memorandum from Superintendent Payzant in December 2004 states that BPS plans to change the computerized process used to assign students to schools. Although the task-force report recommended that BPS adopt the TTC assignment algorithm, the School Committee is interested in simulations of both mechanisms and in understanding the extent of preference manipulation under the Boston mechanism. They are also thinking through their philosophical position on the trade-off between stability and efficiency.

This interest for defining a philosophical position on the trade-off between stability and efficiency is at the origin of a modification in the mechanism used in the Boston Area, decided by July 2005. (See Abdulkadiroğlu et al., 2006). As these authors mention, the solution was to adopt a deferred acceptance mechanism because it is strategy-proof. Nevertheless, as we have pointed out in the present paper, this solution is far from solving the trade-off which is at the origin of this reform. In fact the use of such a solution can be justified because it considers that stability is the central issue. If, moreover, the best that agents can do is to reveal their true characteristics, it is straightforward to conclude, as the Boston School Committee did, that the SOSM would be adopted.

The main contribution of this paper is to provide a way for avoiding the efficiency-equity dilemma. Our approach to escape to this trade-off between stability and efficiency comes from a reinterpretation of the instability notion. In this sense, the additional reform that would be introduced in systems focusing on stability is allowing students to exchange the places that were allocated to them in the actual system. Thus, when the Public authority does not want to restrict the schools freedom when prioritizing students, the EPM can be introduced to minimize the (inevitable) trade-off between efficiency and equity.

A further aspect that can be used to promote the use of the EPM comes from the arguments given in Abdulkadiroğlu et al. (2006). What these authors report, concerning the recent changes introduced by the Boston School Committee, is:

> As far as we know, it is the first time that "strategyproof-

> ness," a central concept in the game theory literature on mechanism design, has been adopted as a public policy concern related to transparency, fairness, and equal access to public facilities. (Abdulkadiroğlu et al., 2006, pg. 2)

Nevertheless, strategy-proofness was not considered as a "sufficient condition" justifying a modification in the mechanism used to allocate schools places among students. In particular, the TTCM, introduced by Abdulkadiroğlu and Sönmez (2003) was not considered by the Boston School Committee as a satisfactory proposal to modify the former system. This mechanism is strategy-proof and selects efficient allocations. When comparing the TTCM and the SOSM, which is also strategy-proof, Abdulkadiroğlu et al. (2006) conclude the following:

> While TTC is a Pareto efficient mechanism when only students are considered, and the student-proposing deferred acceptance mechanism is not, the former does not Pareto dominate the latter. One implication is, based on a stronger efficiency notion (such as a cardinal efficiency notion relying on the rank order of schools) the student-proposing deferred acceptance mechanism may perform better than the TTC for some problems. For example, the student-proposing deferred acceptance mechanism may assign more students to their first choices than TTC. Moreover while each Nash equilibrium outcome of the complete information preference revelation game induced by the Boston mechanism is weakly Pareto dominated by each dominant-strategy equilibrium

outcome of the student-proposing deferred acceptance mechanism (Ergin and Sönmez, 2006), equilibrium outcomes induced by the Boston mechanism and TTC are not Pareto ranked. (Abdulkadiroğlu et al., 2006, pg. 10)

Given the above comparison, we can conclude that our EPM performs better than all the three mechanisms considered in the literature, namely the TTC, the SOSM and the Boston mechanisms. The reason is that, using the arguments by Abdulkadiroğlu et al. (2006), and taking into account that the EPM Pareto dominates the SOSM, we get

1. The EPM never assigns less students to their first choices than the SOSM do, and
2. when students do not act strategically, the EPM weakly Pareto dominates the SOSM.

APPENDIX

I. A Proof for Proposition 1

To prove Proposition 1, let us consider the following School Allocation Problem.

$\mathcal{S} = \{1, 2, 3\}$; $\mathcal{C} = \{a, b, c\}$; $Q = (1, 1, 1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \text{ and } \qquad \Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Note that in such a problem there is only one stable matching, μ, such that $\mu(1) = c$; $\mu(2) = b$; and $\mu(3) = a$. Nevertheless, μ fails to be Pareto efficient since μ' defined as $\mu'(1) = a$; $\mu'(2) = b$; and $\mu'(3) = c$, Pareto dominates μ. $\boxdot$

II. A Proof for Theorem 2

To prove Theorem 2, let us consider a School Allocation Problem, $\mathcal{SAP}$, and let μ^{SO} its student optimal stable matching. By Martínez et al. (2001), we know that for any matching μ', if $\rho_{s_i\mu'(s_i)} < \rho_{s_i\mu^{SO}(s_i)}$ for some student c_i, then μ' is unstable.

Now, let Σ^{SO} be a matrix μ^{SO}-agreeing Φ, and μ^{TTC} the matching obtained by applying the μ^{SO}-Σ^{SO}-TTC algorithm. By Alcalde-Unzu and Molis (2009), Theorem 4, we have that μ^{TTC} is efficient. Moreover, Corollary 3 in Alcalde-Unzu and Molis (2009) also establishes that, for each student s_i,

$$\rho_{s_i\mu^{TTC}(s_i)} \leq \rho_{s_i\mu^{MO}(s_i)}.$$

Now, let us assume that μ^{TTC} is not ε-stable. Since it is efficient, it should fail to be φ-stable. Therefore, there should be a student, s_i, that can fairly object μ^{MO} via some matching, say μ'. Then, μ' must satisfy that

$$\rho_{s_i\mu'(s_i)} < \rho_{s_i\mu^{TTC}(s_i)} \leq \rho_{s_i\mu^{MO}(s_i)},$$

which implies that μ' is unstable. Note that this instability implies that there should be a student, say s_h, and matching μ'' such that s_h can counter-object μ' via μ''. A contradiction. $\boxdot$

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