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# Binary Phase Detector Gain in Bang-Bang Phase-Locked Loops with DCO Jitter

Stefan Tertinek, James P. Gleeson, and Orla Feely, *Fellow, IEEE*

**Abstract**—Bang-bang phase-locked loops (BBPLLs) are hard nonlinear systems due to the nonlinearity introduced by the binary phase detector (BPD). In the presence of jitter, the nonlinear loop is typically analyzed by linearizing the BPD and applying linear transfer functions in the analysis. In contrast to a linear PD, the linearized gain of a BPD depends on the rms jitter and the type of jitter (either non-accumulative or accumulative). Previous works considered the case of non-accumulative reference clock jitter and showed that the BPD gain is inversely proportional to the rms jitter when the latter is small or large. In this brief we consider the case of accumulative DCO jitter and derive an asymptotic closed-form expression for the BPD gain which becomes exact in the limit of small and large jitter. Contrary to the reference clock jitter case, the BPD gain is constant for small DCO jitter and is inversely proportional to the square of jitter for large DCO jitter; in the latter case, the timing jitter has a normal-Laplace distribution.

**Index Terms**—Bang-bang phase-locked loop, timing jitter, binary phase detector gain, asymptotic analysis.

## I. INTRODUCTION

Bang-bang phase-locked loops (BBPLLs) are widely used for clock and data recovery in communication systems, mainly because of their high-frequency capabilities [1]. While they are typically implemented based on the charge-pump PLL architecture [2], [3], recent progress in the development of low-noise digitally-controlled oscillators (DCOs) has resulted in several digital BBPLL (DBBPLL) implementations suitable for high-bandwidth digital frequency synthesis [4], [5].

The distinct feature of BBPLLs is the binary phase detector (BPD) which binary-quantizes the timing jitter seen at its input. The hard nonlinearity introduced by the BPD makes the loop behavior inevitably nonlinear and thus complicates the analysis. In the absence of noise and jitter, the loop exhibits periodic (limit cycle) or quasiperiodic behavior, and a nonlinear analysis is required to investigate the loop stability [6]. In practice, phase noise on the clock sources causes jitter on the clock edges, mainly in the form of non-accumulative

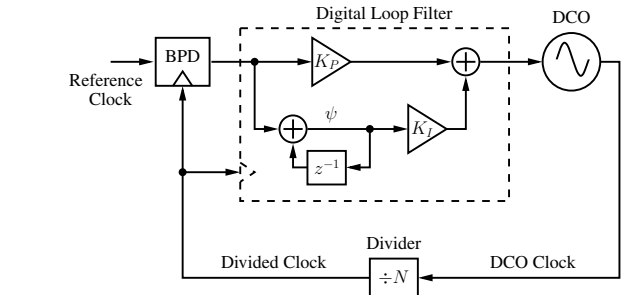


Fig. 1. Second-order DBBPLL architecture [6].

jitter (white phase noise) and accumulative jitter (random-walk and flicker phase noise); a discussion on their physical origin can be found in [7]. Although the effect of jitter on the loop behavior can be accurately analyzed using Markov models [8]–[12], the more common approach is to linearize the BPD nonlinearity and apply linear transfer functions in the analysis [13]–[15]. Unlike a linear PD [16], the linearized gain of a BPD depends on the rms jitter and the type of jitter (either non-accumulative or accumulative). The gain expression known in the literature has been obtained for the case of non-accumulative reference clock jitter [8], [9], [16] and used in the linear analysis of a second-order loop with dominant reference clock jitter [14]. A recent paper [15] extended this linear analysis to the case of dominant accumulative DCO jitter due to random-walk phase noise, which typically occurs in frequency synthesis; however, since a gain expression for the accumulative DCO jitter case has not appeared previously, the authors of [15] used a gain expression based on a hypothesis, lacking any analytical investigation.

The goal of this brief is to analytically study the BPD gain in BBPLLs with accumulative DCO jitter<sup>1</sup>, thereby improving on the gain expression used in [15]. As in [8], [9] we confine ourselves to a first-order loop or to a second-order loop where the gain of the proportional path in the digital loop filter (DLF) is much bigger than the gain of the integral path. Building on our recent work in this area [12] we derive an asymptotic closed-form expression for the BPD gain which becomes exact in the limit of small and large jitter. Compared to the non-accumulative reference clock jitter case, the gain shows a fundamentally different dependence on accumulative DCO jitter. In particular, the gain is constant for small jitter and is inversely proportional to the square of jitter for large jitter—

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<sup>1</sup>Although accumulative jitter of both the reference clock and the DCO leads to the same gain expression, we emphasize the DCO jitter case to be consistent with [15]. Furthermore, we neglect jitter due to flicker phase noise.

this stands in marked contrast to the inversely proportional dependence on reference clock jitter in these regimes [8]. The analysis also shows that for large DCO jitter, the timing jitter (and hence the output jitter) is normal-Laplacian distributed.

## II. DBBPLL ARCHITECTURE AND MODEL

A block diagram of a second-order DBBPLL architecture is shown in Fig. 1 [6]. The BPD compares the rising edges of the reference clock with those of the divided clock and produces binary phase-error information at its output, with a logical value high if the reference clock leads the divided clock, and a logical value low if the reference clock lags the divided clock. The BPD output is fed into the DLF, which consists of a proportional branch with gain coefficient  $K_P$  and an integral branch with gain coefficient  $K_I$ . The DLF output changes the frequency of the DCO clock, which is divided by  $N$  and is fed back as divided clock to the BPD and the DLF.

The BPD gain depends on the timing jitter between the reference clock and the divided clock, also called untracked jitter [8]. Because updating the BPD output and clocking the DLF occurs only every divided clock cycle (see Fig. 1), the loop behavior can be described at discrete time instants  $n = 0, 1, \dots$ . Hence it suffices to consider the time instants of the rising reference clock and divided clock edges, denoted by  $t_{r,n}$  and  $t_{d,n}$ , respectively. Defining the timing jitter by  $\Delta t_n = t_{r,n} - t_{d,n}$  and denoting the integrator state in the DLF by  $\psi_n$ , the second-order DBBPLL is described by the following first-order difference equations [6]:

$$\Delta t_{n+1} = \Delta t_n + T_{r,n} - NT_{v0} - NK_I K_T \psi_n - NK_P K_T \operatorname{sgn} \Delta t_n \quad (1)$$

$$\psi_{n+1} = \psi_n + \operatorname{sgn} \Delta t_{n+1} \quad (2)$$

where  $T_{r,n}$  is the  $n$ th reference clock period. The BPD is modeled as the signum function, which is defined as  $\operatorname{sgn} x = 1$  for  $x \geq 0$ , and  $\operatorname{sgn} x = -1$  for  $x < 0$ . The DCO is considered as a linear block, with free-running clock period  $T_{v0}$  and period gain constant  $K_T$ .

To study the effect of DCO jitter on the BPD gain, we assume noise-free PLL blocks and a jitter-free reference clock, with nominal clock period  $T_{r0}$  equal to  $NT_{v0}$  (locked loop). The accumulative DCO jitter is modeled by replacing  $T_{v0}$  in (1) by  $T_{v0} + \xi_n$ , where the random variable (RV)  $\xi_n$  accounts for the jitter on the  $n$ th DCO period, and the jitter sequence  $\{\xi_n\}$  is assumed to be a sequence of independent, identically distributed Gaussian RVs with zero mean and variance  $\sigma_v^2$  [7]. Under these assumptions, the statistical time evolution of the timing jitter is described by

$$\Delta t_{n+1} = \Delta t_n - NK_I K_T \psi_n - NK_P K_T \operatorname{sgn} \Delta t_n + \xi_n \quad (3)$$

$$\psi_{n+1} = \psi_n + \operatorname{sgn} \Delta t_{n+1} \quad (4)$$

where the DCO jitter RV  $\xi_n$  now has the scaled variance  $\sigma^2 = N^2 \sigma_v^2$ . As in [8], [9] we confine our study to a first-order loop or to a second-order loop satisfying  $K_P \gg K_I$ ; the latter assumption corresponds to an overdamped loop and is usually satisfied in practice [2], [17], [18]. Thus we approximate (3)–(4) by the first-order stochastic difference equation

$$\Delta t_{n+1} = \Delta t_n - K \operatorname{sgn} \Delta t_n + \xi_n \quad (5)$$

where  $K = NK_P K_T$  is the bang-bang step size. In our recent work [12] we studied the statistical timing jitter properties of a first-order BBPLL when the reference clock is subject to accumulative jitter, assuming the  $n$ th reference clock period to be  $T_{r,n} = T_{r0} + \xi_n$ . It can be seen that both accumulative jitter cases lead to the stochastic difference equation (5) and thus to the same gain expression, with the difference that the variance of  $\xi_n$  is scaled by  $N^2$  in case of DCO jitter. This observation is crucial in that it allows us to apply our results from [12] to the analysis of the BPD gain.

## III. BPD GAIN EXPRESSION FOR DCO JITTER

In this section we analytically study the BPD gain  $K_{\text{bpd}}$  in the presence of accumulative DCO jitter. Based on our recent work [12] we derive an exact integral representation of  $K_{\text{bpd}}$ , from which we obtain an asymptotic closed-form expression and the asymptotes for small and large jitter. Since linearizing the nonlinear loop has already been considered elsewhere [8], [14], [15], we provide here only the definition of  $K_{\text{bpd}}$ , which is given by [8]

$$K_{\text{bpd}} = 2p_{\Delta t}(0) \quad (6)$$

where  $p_{\Delta t}(t)$  is the steady-state timing-jitter probability density function (pdf).

In [12] we used a sign-dependent random-walk model to analyze the limiting behavior of (5). A main result of that work is that the steady-state timing jitter  $\Delta t$  can be decomposed into the sum of statistically independent RVs

$$\Delta t = \eta + \xi + o \quad (7)$$

where the RV  $\xi$  models the DCO jitter in steady-state. The hunting jitter RV  $\eta$  and the overload jitter RV  $o$  model, respectively, static and dynamic jitter components caused by the binary phase-error quantization. More specifically, hunting jitter is introduced by the coarseness of the binary PD characteristic, and accounts for the DCO phase hunting randomly around the reference clock phase. Overload jitter is introduced by DCO phase updates that are larger than the bang-bang step size  $K$ . As we shall see, these jitter components reflect the dependence of the BPD gain on the DCO jitter.

### A. Exact Integral Representation of the BPD Gain

Let  $z$  be a real variable, and let  $\phi(z)$  denote the characteristic function (CF) of a RV with pdf  $p(t)$ . It follows from [12, (27)] that the CF of  $\Delta t$  is given by the product

$$\phi_{\Delta t}(z) = \phi_{\eta}(z)\phi_{\xi}(z)\phi_o(z) \quad (8)$$

where

$$\phi_{\eta}(z) = \frac{\sin(Kz)}{Kz} \quad (9)$$

is the CF of  $\eta$  which is uniformly distributed on  $[-K, K]$ , and

$$\phi_{\xi}(z) = e^{-\sigma^2 z^2 / 2} \quad (10)$$

is the CF of  $\xi$  which is Gaussian distributed with variance  $\sigma^2$ . The CF [12]

$$\phi_o(z) = \exp\left(2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} (\cos(zx) - 1) f_n(x; -K, \sigma^2) dx\right) \quad (11)$$

where

$$f_n(x; \mu, \sigma^2) \equiv f(x; n\mu, n\sigma^2) \quad (12)$$

and  $f(x; \mu, \sigma^2)$  denotes the Gaussian pdf with mean  $\mu$  and variance  $\sigma^2$ , corresponds to the overload jitter RV  $o$ ; its distribution is not known in closed form, but we will shortly obtain a closed-form expression valid in the large- $\sigma$  regime.

Since  $p(t)$  is related to  $\phi(z)$  by the inverse Fourier transform formula [19, p. 482]

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt} \phi(z) dz \quad (13)$$

we obtain, after substituting (8)–(11) into (13) and the result into (6), the expression

$$K_{\text{bpd}} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(Kz)}{Kz} \exp(-\sigma^2 z^2/2 - I(z)) dz \quad (14)$$

with

$$I(z) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} (1 - \cos(zx)) f_n(x; -K, \sigma^2) dx. \quad (15)$$

Equation (14) is an exact integral representation of the BPD gain as a function of the DCO jitter  $\sigma$  and the bang-bang step size  $K$ . To simplify this integral, it is natural to perform an asymptotic analysis by allowing the parameter of interest,  $\sigma$ , to become small and large. As will be shown next, the result is an asymptotic closed-form expression for  $K_{\text{bpd}}$ .

### B. Asymptotic Closed-Form Expression for the BPD Gain

To begin the asymptotic analysis, let us simplify the integral in (15) by first letting  $\sigma \rightarrow 0$  and then letting  $\sigma \rightarrow \infty$ . Since the integrand is non-negative for all  $z$ , we may change the order of summation and integration, and consider the sum term  $\sum_{n=1}^{\infty} (1/n) f_n(x; -K, \sigma^2)$ . Substitution of (12) and the Gaussian pdf gives

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma n^{3/2}} \exp\left(-\frac{(x + Kn)^2}{2\sigma^2 n}\right). \quad (16)$$

By partitioning the interval  $(0, \infty)$  as  $0 < \sigma < 2\sigma < \dots$  and making the change of variable  $\sigma^2 n = y$ , we can consider (16), in the asymptotic limit  $\sigma \rightarrow 0$ , as a Riemann sum for the integral

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi}y^{3/2}} \exp\left(-\frac{(x + Ky/\sigma^2)^2}{2y}\right) dy. \quad (17)$$

Similarly, by choosing the partition  $0 < 1/\sigma < 2/\sigma < \dots$  and making the change of variable  $n/\sigma^2 = y$ , we can consider (16), in the asymptotic limit  $\sigma \rightarrow \infty$ , as a Riemann sum for the integral

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2 y^{3/2}} \exp\left(-\frac{(x + K\sigma^2 y)^2}{2\sigma^4 y}\right) dy. \quad (18)$$

Using formula 3.471.12 in [20, p. 368], both the integrals (17) and (18) can be evaluated explicitly to yield

$$\frac{1}{x} \exp\left(-\frac{2Kx}{\sigma^2}\right). \quad (19)$$

Plugging (19) into (15) and employing formula 3.943 in [20, p. 497] gives the asymptotic expression<sup>2</sup>

$$\begin{aligned} I(z) &\sim 2 \int_0^{\infty} (1 - \cos(zx)) \frac{1}{x} \exp\left(-\frac{2Kx}{\sigma^2}\right) dx \\ &= \log\left(1 + \frac{\sigma^4}{4K^2} z^2\right) \end{aligned} \quad (20)$$

as  $\sigma \rightarrow 0$  or  $\sigma \rightarrow \infty$ . In words, (15) is asymptotically equivalent to the (simple) expression (20) as  $\sigma$  tends to zero or infinity.

Consequently, after substitution of (20) into (14), the BPD gain  $K_{\text{bpd}}$  may be written as the single integral

$$K_{\text{bpd}} \sim \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(Kz)}{Kz} e^{-\sigma^2 z^2/2} \frac{1}{1 + \frac{\sigma^4}{4K^2} z^2} dz \quad (21)$$

as  $\sigma$  tends to the above limits. This integral can be simplified further. With the help of the partial fraction expansion  $1/(z(1 + cz^2)) = 1/z - cz/(1 + cz^2)$ , where  $c = \sigma^4/(4K^2)$ , we may rewrite (21) as

$$\begin{aligned} K_{\text{bpd}} &\sim \frac{1}{K\pi} \int_{-\infty}^{\infty} \frac{\sin(Kz)}{z} e^{-\sigma^2 z^2/2} dz \\ &\quad - \frac{1}{K\pi} \int_{-\infty}^{\infty} \sin(Kz) e^{-\sigma^2 z^2/2} \frac{z}{\frac{4K^2}{\sigma^4} + z^2} dz. \end{aligned} \quad (22)$$

Both these integrals can be evaluated in closed form by using formula 3.954.1 in [20, p. 504]. After integration and some algebra we get the final expression

$$K_{\text{bpd}} \sim \frac{1}{K} + \frac{1}{2K} e^{4K^2/\sigma^2} \operatorname{erfc}\left(\frac{3K}{\sqrt{2}\sigma}\right) - \frac{3}{2K} \operatorname{erfc}\left(\frac{K}{\sqrt{2}\sigma}\right) \quad (23)$$

where  $\operatorname{erfc}$  denotes the complementary error function. Equation (23) is an asymptotic closed-form expression for  $K_{\text{bpd}}$  which becomes exact in the limits  $\sigma \rightarrow 0$  or  $\sigma \rightarrow \infty$ . A comparison of this expression with simulation results will be presented in Sec. IV.

### C. Small- $\sigma$ and Large- $\sigma$ Asymptotes

Let us now determine the asymptotes for the BPD gain. In the limit  $\sigma \rightarrow 0$ , the second and third term in (23) tend to zero, and we immediately obtain the asymptote

$$K_{\text{bpd}} \rightarrow \frac{1}{K} \quad \text{as } \sigma \rightarrow 0. \quad (24)$$

Note that this result may also be deduced from the decomposition (7). Clearly, because the overload jitter  $o$  depends on  $\sigma$ , whereas the hunting jitter  $\eta$  is independent of it, the distribution of  $\Delta t$  in the small- $\sigma$  regime will be dominated by  $\eta$ ; the latter has a uniform distribution on  $[-K, K]$  from which (24) follows.

The asymptote in the limit  $\sigma \rightarrow \infty$  requires a more careful analysis. However, by expanding the exponential term in (23) into a Taylor series about zero and retaining only the first two terms, we have

$$K_{\text{bpd}} \sim \frac{2K}{\sigma^2} \operatorname{erfc}\left(\frac{3K}{\sqrt{2}\sigma}\right). \quad (25)$$

<sup>2</sup>Two functions  $f(x)$  and  $g(x)$  are asymptotically equivalent, denoted by  $f(x) \sim g(x)$ , as  $x \rightarrow x_0$  if  $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$  [21, p. 78].

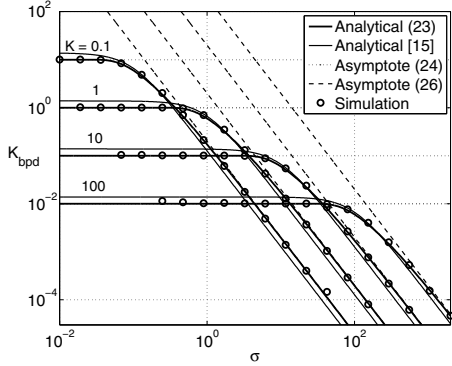


Fig. 2. BPD gain  $K_{\text{bpd}}$  vs. rms DCO jitter  $\sigma$  for varying  $K$ .

The leading-order asymptotic behavior of this expression gives the asymptote

$$K_{\text{bpd}} \rightarrow \frac{2K}{\sigma^2} \quad \text{as } \sigma \rightarrow \infty. \quad (26)$$

As a corollary to this asymptotic analysis we obtain the distribution law of  $\Delta t$  for large  $\sigma$ . Due to (13), the integrand in (21) is a CF consisting of a product of three terms: The first two terms are the CFs of a uniform and a Gaussian distribution; the third term is the CF of a Laplace distribution with mean zero and variance  $\sigma^4/(2K^2)$  [22, p. 930], corresponding to an asymptotic approximation for the CF  $\phi_o$ . For sufficiently large  $\sigma$ , the term  $\sin(Kz)/(Kz)$  in (21) can be replaced by its value at  $z = 0$ , which yields the limiting distribution of  $\Delta t$  to be the convolution of a Gaussian distribution with variance  $\sigma^2$  and a Laplace distribution with variance  $\sigma^4/(2K^2)$ , both having zero mean. Put differently, the timing jitter is distributed as the sum of two independent RVs, one Gaussian and the other Laplacian. The distribution law of this sum, called the normal-Laplace distribution, has only recently been introduced [23]. It follows from [23, (4.2)] that the pdf of  $\Delta t$  for large  $\sigma$  is given by

$$p_{\Delta t}(t) = \frac{K}{2\sigma^2} e^{2K^2/\sigma^2} \left[ e^{-2Kt/\sigma^2} \operatorname{erfc} \left( \frac{2K-t}{\sqrt{2}\sigma} \right) + e^{2Kt/\sigma^2} \operatorname{erfc} \left( \frac{2K+t}{\sqrt{2}\sigma} \right) \right]. \quad (27)$$

The importance of this result is that the pdf is clearly non-Gaussian, a result that contradicts the intuitive idea of large jitter linearizing the BPD and leading to a linear system behavior, in which case the pdf would be Gaussian. The same conclusion was drawn in [12] by computing the kurtosis, but we now have an explicit expression for the pdf in this regime. Note that by evaluating  $2p_{\Delta t}(0)$  and expanding the first exponential term into a Taylor series about zero, we recover the large- $\sigma$  asymptote in (26).

#### IV. SIMULATION RESULTS AND DISCUSSION

The BPD gain expression will now be compared against Monte Carlo simulation results and the expression used in the linear analysis in [15]. For each simulation we generated a realization of the stochastic difference equation (5) of length  $10^7$ , discarding the first 100 values. To obtain  $p_{\Delta t}(0)$  we

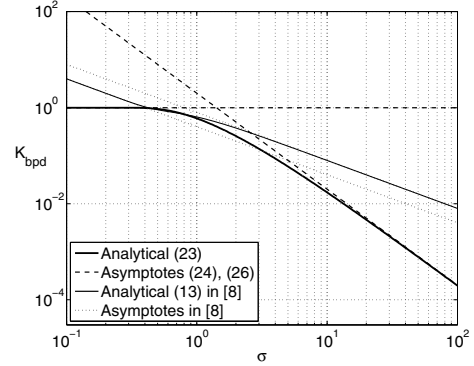


Fig. 3. Comparison of  $K_{\text{bpd}}$  and its asymptotes for accumulative DCO jitter and non-accumulative reference clock jitter [8] ( $K = 1$ ).

estimated the timing-jitter pdf by computing the histogram of this realization, and took the value of the bin at zero. Although this method gives a good approximation for the value of  $p_{\Delta t}(0)$ , a more accurate value can be obtained by the numerical method described in [10].

Figure 2 plots the BPD gain  $K_{\text{bpd}}$  as a function of the rms DCO jitter  $\sigma$ , the parameter being the bang-bang step size  $K$ . The agreement between the closed-form analytical expression (23), the asymptotes (24) and (26), and the simulation results confirms our analysis. For comparison, the figure also plots the BPD gain expression used by Zanuso et. al. [15, (1)]. Based on the hypothesis of Gaussian distributed timing jitter  $\Delta t$ , the BPD gain was defined as  $K_{\text{bpd}} = \sqrt{2/\pi}/\sigma_{\Delta t}$ , where the rms timing jitter  $\sigma_{\Delta t}$  is replaced by the approximate expression given in [15, (13)]. The deviation between simulation and theory seen in the figure is due to the incorrect assumption of Gaussian distributed timing jitter. In fact, it follows from the decomposition (7) that there is only one value of  $\sigma$  for which the timing jitter is Gaussian, as discussed in [12]. Clearly, for small jitter, the dominance of the hunting jitter implies that  $\Delta t$  is uniformly distributed; for large jitter, the increasing contribution of the overload jitter implies that  $\Delta t$  is normal-Laplacian distributed with the pdf given in (27).

The peculiarity of the BPD is that the linearized gain depends not only on the rms jitter but also on the type of jitter, as the following comparison with the non-accumulative reference clock jitter case illustrates [8]. Figure 3 plots  $K_{\text{bpd}}$  for both jitter cases and a normalized loop ( $K = 1$ ), where  $\sigma$  denotes the rms value of both the DCO jitter and the reference clock jitter. The asymptotes shown for the reference clock jitter case are given by  $1/(\sqrt{2\pi}\sigma)$  for small  $\sigma$  and by  $2/(\sqrt{2\pi}\sigma)$  for large  $\sigma$  [8].

In the small- $\sigma$  regime<sup>3</sup>, the BPD gain is inversely proportional to reference clock jitter, but it shows no dependence on DCO jitter. This different behavior has practical significance. For PLL design, an important parameter is the loop bandwidth, which depends on the loop gain and thus, when linearizing the BPD, on  $K_{\text{bpd}}$ . Since  $K_{\text{bpd}}$  depends on the reference clock

<sup>3</sup>Strictly speaking, for sufficiently small DCO jitter, the loop dynamics will be nonlinear, and a linear analysis (BPD gain, loop bandwidth etc.) cannot be applied [14], [15]. The BPD gain comparison in this regime is included to be consistent with previous work [8], [9].

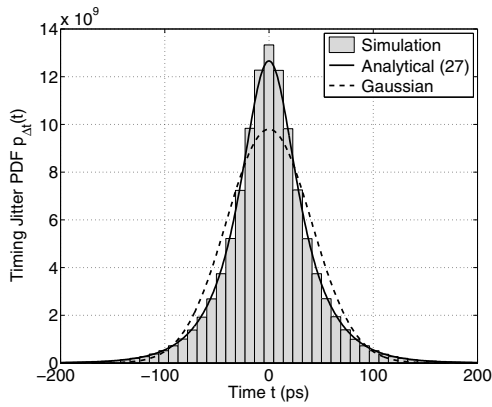


Fig. 4. Steady-state timing-jitter pdf of a second-order DBBPLL with an rms accumulative DCO jitter of  $\sigma_v = 1.74$  ps: Comparison between simulated pdf (histogram), analytical pdf (27) and a Gaussian pdf of equal variance.

jitter, so does the loop bandwidth, so that variations of the clock jitter will affect the loop characteristic. By contrast, the constant  $K_{\text{bpd}}$  in case of DCO jitter may suggest that the loop bandwidth is not affected by jitter in this regime.

A strikingly different behavior is observed in the large- $\sigma$  regime. The BPD gain is again inversely proportional to reference clock jitter, but now it is inversely proportional to the square of DCO jitter. Thus, variations of the clock jitter will affect the loop bandwidth in either jitter case, but the effect is more pronounced for DCO jitter.

We conclude this section with a practical example. Consider a second-order DBBPLL with parameters as in [15]. The DCO has a free-running clock period of  $T_{v0} = 1/(320 \text{ MHz})$  and an rms accumulative jitter of  $\sigma_v = 1.74$  ps, corresponding to a  $-100 \text{ dBc/Hz}$  (random-walk) phase noise at 1 MHz offset from the carrier. The divider value is  $N = 8$  which gives  $\sigma = N\sigma_v = 13.92$  ps. The gain coefficients of the proportional path and integral path in the DLF are given by  $K_P = 8$  and  $K_I = 0.5$ , respectively, and the DCO gain is taken to be  $K_T = 55$  fs. The steady-state timing-jitter pdf obtained from Monte Carlo simulation of (3)–(4) is shown in Fig. 4, along with the analytical prediction (27) and a Gaussian pdf of equal variance. The agreement between simulation and theory confirms the validity of the normal-Laplace distribution law also for an overdamped loop. The BPD gain for this example is  $K_{\text{bpd}} = 2.52 \cdot 10^{10}$ . Note that because the reference clock is assumed to be jitter free, the timing jitter equals the DBBPLL output jitter (scaled by  $N$ ), and so the pdf in (27) is also that of the output jitter.

## V. CONCLUSION

We have studied the BPD gain in BBPLLs subject to accumulative DCO jitter, which has not been considered previously. Our main result is (23), an asymptotic closed-form expression for the gain, yielding the exact asymptotes (24) and (26) for small and large jitter, respectively. Unlike the gain expression known for the non-accumulative reference clock jitter case [8], the gain for the DCO jitter case shows a different dependence on jitter: It is constant for small jitter and is inversely proportional to the square of jitter for large

jitter. The closed-form expression (27) for the timing-jitter pdf corresponds to a normal-Laplace distribution law, proving the non-Gaussianity of the timing jitter for large jitter.

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