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ON A DIOPHANTINE EQUATION ON TRIANGULAR NUMBERS

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Abstract. A number N is a triangular number if it can be written as $N = 0+1+\cdots+n$ for some natural number n. We study the problem of finding all nonnegative integer solutions of the Diophantine equation $(0+1+\cdots+X)+(0+1+\cdots+Y) = (0+1+\cdots+Z)$. Using this equation, some new and curious increasing integer sequences are built.

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1. Introduction

A triangular number is a number of the form $T_n = \sum_{k=0}^{n} k = n(n+1)/2$, where n is a natural number.

So the first few triangular numbers are $0,1,3,6,10,15,21,28,...$ (sequence A000217 in [6]). A well known fact about the triangular numbers is that X is a triangular number if and only if $8X+1$ is a perfect square. Triangular numbers can be thought of as the numbers of dots needed to make a triangle.

We are concerned by the Diophantine equation of the form

$$T_X + T_Y = T_Z \tag{1.1}$$

Papers [1–5] gave many interesting results concerning the problem of solvability of Diophantine equations related to triangular numbers. The aim of this paper is two-fold: on the one hand, it gives all solutions of the Diophantine equation (1.1), and on the other hand, it gives us a method of finding explicitly (and quickly) infinite families of solutions of (1.1) in where many new integer sequences are derived.

2. General solution

Let us start with some technical lemmas.

Lemma 1. *Let $a,b,c,d \in \mathbb{N}$. Then $ab = cd$ if and only if there exist $p,q,m,n \in \mathbb{N}$ such that $a = mn$, $b = pq$, $c = mp$ and $d = nq$.*

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Proof. Let $a = \prod_{i \in A \subset \mathbb{N}} p_i^{\alpha_i}$ and $b = \prod_{i \in B \subset \mathbb{N}} q_i^{\beta_i}$ where p_i, q_i are primes, $\alpha_i, \beta_i \in \mathbb{N}$ and A, B are finite sets. The fact $cd = \prod_{i \in A \subset \mathbb{N}} p_i^{\alpha_i} \prod_{i \in B \subset \mathbb{N}} q_i^{\beta_i}$ implies that $c = \prod_{i \in A' \subset A} p_i^{\alpha'_i} \prod_{i \in B' \subset B} q_i^{\beta'_i}$ and $d = \prod_{i \in A'' \subset A} p_i^{\alpha''_i} \prod_{i \in B'' \subset B} q_i^{\beta''_i}$, where $0 \le \alpha'_i, \alpha''_i \le \alpha_i$, $0 \le \beta'_i, \beta''_i \le \beta_i$ with $\alpha_i = \alpha'_i + \alpha''_i$ and $\beta_i = \beta'_i + \beta''_i$. If we put $m = \prod_{i \in A' \subset A} p_i^{\alpha'_i}$, $p = \prod_{i \in B' \subset B} q_i^{\beta'_i}$, $n = \prod_{i \in A'' \subset A} p_i^{\alpha''_i}$ and $q = \prod_{i \in B'' \subset B} q_i^{\beta''_i}$ then we obtain, $a = mn$, $b = pq$, $c = mp$ and $d = nq$. $\square$

Lemma 2. *Equation (1.1) is equivalent to* $X(X+1) = (Z-Y)(Z+Y+1)$.

Proof. Replacing T_X by $\frac{X(X+1)}{2}$, T_Y by $\frac{Y(Y+1)}{2}$ and T_Z by $\frac{Z(Z+1)}{2}$ equation (1.1) becomes

$$\frac{X(X+1)}{2} + \frac{Y(Y+1)}{2} = \frac{Z(Z+1)}{2},$$

i.e.,

$$X^2 + X + Y^2 + Y = Z^2 + Z. \tag{2.1}$$

Thus

$$X(X+1) = (Z-Y)(Z+Y+1).$$

$\square$

Now, we are able to establish the following theorem.

Theorem 1. *All nonnegative integer solutions of the equation* $T_X + T_Y = T_Z$ *are given by* $\begin{cases} X = mn \\ Y = \frac{1}{2}(nq - mp - 1) \\ Z = \frac{1}{2}(nq + mp - 1) \end{cases} : m, n, p, q \in \mathbb{N}$ *where* $pq - mn = 1$ *and* $nq - mp - 1 \in 2\mathbb{N}$.

Proof. According to Lemma 2, (1.1) is equivalent to $X(X+1) = (Z-Y)(Z+Y+1)$, thus thanks to Lemma 1, one has

$$\begin{aligned} X &= mn \\ X+1 &= pq \\ Z-Y &= mp \\ Z+Y+1 &= nq \quad \text{where } m, n, p, q \in \mathbb{N}. \end{aligned}$$

Therefore we get

$$\begin{aligned} X &= mn \\ Y &= \tfrac{1}{2}(nq - mp - 1) \end{aligned}$$

$$Z = \frac{1}{2}(nq + mp - 1) \text{ where } m, n, p, q \in \mathbb{N}$$

such that $pq - mn = 1$ and $nq - mp - 1 \in 2\mathbb{N}$. □

Example 1. $p = 4$, $q = 7$, $m = 3$ and $n = 9$. Clearly, $X = 27$, $Y = 25$ and $Z = 37$.

3. Solutions families

First of all, note that it is not easy to find integers m, n, p, q such that $pq - mn = 1$ and $nq - mp - 1 \in 2\mathbb{N}$.

The usage of matrices enables us to obtain a large number of solutions of (1.1). In this section, we will construct several infinite many families solutions of our triangular equation.

By completing squares in (2.1), we obtain

$$(X + \frac{1}{2})^2 + (Y + \frac{1}{2})^2 = (Z + \frac{1}{2})^2 + \frac{1}{4}.$$

Multiplying both sides by 4, we get

$$(2X + 1)^2 + (2Y + 1)^2 = (2Z + 1)^2 + 1.$$

By putting, $x = 2X + 1$, $y = 2Y + 1$ and $z = 2Z + 1$, we obtain

$$x^2 + y^2 = 1 + z^2. \tag{3.1}$$

Since the proof of the following proposition can be seen easily we omit it.

Proposition 1. *The two families* $\begin{pmatrix} 1 \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ a \end{pmatrix}$ *are solutions of (3.1) for all integer* a.

Theorem 2. *There exist some* 3×3 *matrices* $\mathfrak{M}$, *with elements in* $\{1, 2, 3\}$, *such that if* $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ *is a solution of (3.1) then* $\mathfrak{M}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ *is also a solution.*

Proof. Let $\mathfrak{M} = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix}$ and let $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ a solution of (3.1).

It is clear that $\begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} Ax_0 + By_0 + Cz_0 \\ Fy_0 + Gz_0 + Ex_0 \\ Lx_0 + My_0 + Nz_0 \end{pmatrix}$.

Now, if the following system holds

$$\begin{aligned} A^2 - L^2 + E^2 &= 1 \\ B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \end{aligned} \tag{3.2}$$

$$AB - LM + EF = 0$$
$$AC - LN + EG = 0$$
$$BC - MN + FG = 0$$

then

$$(Ax_0 + By_0 + Cz_0)^2 + (Fy_0 + Gz_0 + Ex_0)^2 - (Lx_0 + My_0 + Nz_0)^2 = 1.$$

It follows from this, $\begin{pmatrix} Ax_0 + By_0 + Cz_0 \\ Fy_0 + Gz_0 + Ex_0 \\ Lx_0 + My_0 + Nz_0 \end{pmatrix}$ is a solution of (3.1).

We will explore, in five steps, all solutions of the system (S) are in the space $\{1,2,3\}^9$.

Case 1: $A = L = E = 1$.

$$B^2 + F^2 - M^2 = 1 \tag{3.3}$$
$$C^2 + G^2 - N^2 = -1$$
$$B + F = M \tag{3.4}$$
$$C + G = N$$
$$BC - MN + FG = 0.$$

From (3.4), we obtain $F = 1$ or $B = 1$, then (3.3) implies $B = M$ or $F = M$ then $F = 0$ or $B = 0$ which is impossible.

Case 2 : $A = 1,\ L = E = 2$.

$$B^2 + F^2 - M^2 = 1$$
$$C^2 + G^2 - N^2 = -1$$
$$B - 2M + 2F = 0 \tag{3.5}$$
$$C - 2N + 2G = 0 \tag{3.6}$$
$$BC - MN + FG = 0.$$

Equations (3.5) and (3.6) imply that B and C are even.

Then from

$$M^2 - F^2 = 3 \tag{3.7}$$
$$N^2 - G^2 = 5 \tag{3.8}$$
$$M - F = 1$$
$$N - G = 1$$
$$MN - FG = 4.$$

Equation (3.7) gives $M = 2, F = 1$ and (3.8) gives $N = 3, G = 2$, then $\mathfrak{M}_1 = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

Case 3 : $A = 1, L = E = 3$.

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ B - 3M + 3F &= 0 \qquad (3.9) \\ C - 3N + 3G &= 0 \qquad (3.10) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.9) and (3.10) give us $B = C = 3$, then

$$\begin{aligned} F^2 - M^2 &= -8 \\ G^2 - N^2 &= -10 \qquad (3.11) \\ -M + F &= -1 \\ -N + G &= -1 \\ -MN + FG &= -9. \end{aligned}$$

It is easy to see that equation (3.11) is impossible in $\{1,2,3\}^2$

Case 4 : $A = 2$.

Equation 3.2 implies that $L = 2, E = 1$.

So, we get

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ 2B - 2M + F &= 0 \qquad (3.12) \\ 2C - 2N + G &= 0 \qquad (3.13) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.12) and (3.13) give us $F = G = 2$, thus

$$\begin{aligned} M^2 - B^2 &= 3 \qquad (3.14) \\ N^2 - C^2 &= 5 \qquad (3.15) \\ M - B &= 1 \\ N - C &= 1 \\ BC - MN + 4 &= 0. \end{aligned}$$

Equation (3.14) gives us $M = 2, B = 1$ and (3.15) gives $N = 3, C = 2$.

We obtain $\mathfrak{M}_2 = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

Case 5 : $A = 3$.

Equation (1_0) in (S) implies that : $L = 3, E = 1$. Now

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ 3B - 3M + F &= 0 \qquad (3.16) \\ 3C - 3N + G &= 0 \qquad (3.17) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.16) and (3.17) give us $F = G = 3$, thus

$$\begin{aligned} M^2 - B^2 &= 8 \\ N^2 - C^2 &= 10 \qquad (3.18) \\ B - M + 1 &= 0 \\ C - N + 1 &= 0 \\ BC - MN + 9 &= 0. \end{aligned}$$

It is easy to see that equation (3.18) is impossible in $\{1,2,3\}^2$. □

Corollary 1. *The equation (1.1) admits infinite many solutions.*

Proof. Let $\mathfrak{M} = \mathfrak{M}_1$ and $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ a solution of (3.1) then $\mathfrak{M}^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution of (3.1) for all integer $n \geq 0$. □

4. New integer sequences

Using the on-line Encyclopedia of integer sequences (see [6]), we can easily check that the following two integer sequences formed by solutions X of triangular equation (1.1) : $1, 5, 35, 203, 1179, 6929, 40391, 235415, 1372105, 7997213...$ and $0, 8, 54, 322, 1884, 10988, 64050, 373318, 2175864, 12681872, ...$ are new.

4.1. *Constructions*

1: We start with $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$, a solution of (3.1), thanks to theorem 2, by multiplying recursively by $\mathfrak{M}_1$, we obtain

$$\begin{pmatrix} 11 \\ 13 \\ 17 \end{pmatrix}, \begin{pmatrix} 71 \\ 69 \\ 99 \end{pmatrix}, \begin{pmatrix} 407 \\ 409 \\ 577 \end{pmatrix}, \begin{pmatrix} 2379 \\ 2377 \\ 3363 \end{pmatrix}, \begin{pmatrix} 13\,859 \\ 13\,861 \\ 19\,601 \end{pmatrix}, \begin{pmatrix} 80\,783 \\ 80\,781 \\ 114\,243 \end{pmatrix},$$
$$\begin{pmatrix} 470\,831 \\ 470\,833 \\ 665\,857 \end{pmatrix}, \begin{pmatrix} 2744\,211 \\ 2744\,209 \\ 3880\,899 \end{pmatrix}, \begin{pmatrix} 15\,994\,427 \\ 15\,994\,429 \\ 22\,619\,537 \end{pmatrix}, \ldots \text{ are also solutions of (3.1).}$$

We get, $x = 3, 11, 71, 407, 2379, 13859, 80783, 470831, 2744211, 15\,994\,427, \ldots$
But $x = 2X + 1$, so

$$X = 1, 5, 35, 203, 1179, 6929, 40391, 235415, 1372105, 7997213, \ldots$$

is a new sequence of odd integers formed by solutions X of our triangular equation (1.1).

2: We start with $\begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$ a solution of (3.1), thanks to theorem 2, by multiplying recursively by $\mathfrak{M}_2$, we obtain

$$\begin{pmatrix} 17 \\ 21 \\ 27 \end{pmatrix}, \begin{pmatrix} 109 \\ 113 \\ 157 \end{pmatrix}, \begin{pmatrix} 645 \\ 649 \\ 915 \end{pmatrix}, \begin{pmatrix} 3769 \\ 3773 \\ 5333 \end{pmatrix}, \begin{pmatrix} 21\,977 \\ 21\,981 \\ 31\,083 \end{pmatrix}, \begin{pmatrix} 128\,101 \\ 128\,105 \\ 181\,165 \end{pmatrix},$$
$$\begin{pmatrix} 746\,637 \\ 746\,641 \\ 1055\,907 \end{pmatrix}, \begin{pmatrix} 4351\,729 \\ 4351\,733 \\ 6154\,277 \end{pmatrix}, \begin{pmatrix} 25\,363\,745 \\ 25\,363\,749 \\ 35\,869\,755 \end{pmatrix}, \ldots \text{ are also solutions of (3.1).}$$

We get, $x = 1, 17, 109, 645, 3769, 21977, 128101, 746637, 4351729, 25\,363\,745, \ldots$
But $x = 2X + 1$, then

$$X = 0, 8, 54, 322, 1884, 10988, 64050, 373318, 2175864, 12681872, \ldots$$

is a new sequence of even integers formed by solutions X of our triangular equation (1.1).

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