Developing the use of diagrammatic representations in primary mathematics through professional development

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Background: The research on diagrammatic representations (e.g. Leinhardt et al 1991; Brophy 1991; Hall 1998) highlights their importance for the teaching and learning of mathematics. However, the empirical evidence to support their use in the classroom is mixed and somewhat lacking.

Purpose: The aim of this study was to develop the use of diagrammatic representations of mathematical concepts in primary classrooms, through introducing primary teachers to the research literature in this area, and researching the subsequent impact on children and teachers. A professional development programme was designed, involving three one-day training sessions for mathematics co-ordinators. They were asked to implement the ideas from the training back in their schools.

Sample: Mathematics co-ordinators from 8 primary schools attended the professional development programme. The study focussed on Year 3 pupils (aged 7-8) and Year 5 pupils (aged 9 – 10).

Design and methods: In this paper, we report the qualitative findings from the larger project looking at the overall impact of the professional development programme. The paper focuses on semi-structured interviews carried out with the mathematics co-ordinators attending the professional development sessions, and the Year 3 and Year 5 class teachers who subsequently worked with the co-ordinators on their use of diagrammatic representations in their teaching of mathematics. Lesson observations involving the class teachers were also carried out in order to further explore the possible impact of the project on classroom practice.

Results: The qualitative results identified the impact of the project on mathematics co-ordinators and class teachers' knowledge and practice. However, the nature of this impact was complex, with a variety of facilitating and hindering factors identified for the transfer of the professional development ideas on the use of diagrammatic representations. In addition, different levels of sophistication of class teachers' use of diagrammatic representations were identified.

Conclusions: Implications for the development of professional development programmes to facilitate the transfer of research into practice were identified. Recommendations for the use of diagrammatic representations are also put forward.

Keywords: mathematics instruction, diagrammatic representations, teacher professional development, primary education

Introduction

The aim of the reported study was to develop the use of diagrammatic representations of mathematical concepts in primary classrooms. More specifically, this research project aimed to bring about this development by introducing primary teachers to the research literature on the importance of and issues concerning the use of diagrammatic representations in the teaching and learning of mathematics. In this context, diagrammatic representations are examples of external representations which "serve to denote or to exemplify" mathematical concepts (Perkins and Unger 1994, 2). Kaput (1991, 56) describes external representations as "materially instantiated" entities. Examples of external representations are physical apparatus, marks on paper, pictures, symbols, sounds, spoken words and computerised objects (Kaput 1991), or numerals, algebraic equations, graphs, tables, diagrams, and charts (Pape and Tchoshanov 2001). Focussing specifically on the example of pictures or diagrams, Larkin and Simon (1987) use the term *diagrammatic representation* to mean "a data structure in which information is indexed by two-dimensional location" (p. 68). Examples of these diagrammatic representations would include the number line and the array (see Figures 1 to 7 for further examples for multiplication and fractions). The distinction is made between diagrammatic representations and *visual representations*, where visual representations or visual imagery are more often associated in the research literature with specific types of internal or mental representations of mathematical concepts held by subjects (Lean and Clements 1981; Presmeg 1986; Goldin 1998). Although there is a relationship (albeit complex) between the external representations experienced by subjects and their internal representations of mathematical concepts (Hiebert and Wearne 1992), this study specifically focussed on external diagrammatic representations.

Literature Review

The use of diagrammatic representations in the teaching of mathematics

The research literature highlights the importance of diagrammatic representations both for teachers and children in their teaching and learning of mathematics. The use of multiple representations in general is an important part of teachers' knowledge of mathematics and they can play an important role in the explanation of mathematical ideas (Leinhardt et al 1991).

> "Skilled teachers have a repertoire of such representations available for use when needed to elaborate their instruction in response to student comments or questions or to provide alternative explanations for students who were unable to follow the initial instruction" (Brophy 1991, 352)

Hall (1998, 35) argues that "…diagrammatic and concrete representation systems have the pedagogical advantage of being easier to talk about, to describe and to analyse than language-based solutions". Also, external representations can highlight specific aspects of a mathematical concept thereby supporting this explanatory process (Kaput 1991; Ainsworth 1999). For example, the array representation (Figure 1) can be used to explain the commutative property of multiplication where we can swap round the two numbers involved – for example why 2×6 gives the same answer as 6×2.

In addition, the ability to draw on multiple representations is an important aspect of children's mathematical understanding (Hiebert and Carpenter 1992; Greeno and Hall 1997). Diagrammatic representations enable children to make connections between their existing experience and mathematical concepts (Post and Cramer 1989), and to gain insight into more abstract mathematical ideas (Duval 1999; Flevares and Perry 2001). Other benefits of external representations include their role as memory aids through the externalisation of calculation processes (Paivio 1969; Perkins and Unger 1994), and also their role in supporting problem solving processes (Lesh, Landau and Hamilton 1983; Webb, Gold and Qi 1990).

However, in addition to recognising the benefits of using external representations, the possible difficulties involved in using external representations such as pictures and diagrams must also be acknowledged. In particular, teachers cannot assume that students recognise representations in the manner expected (Hall 1998); the meaning that particular representations have for the teacher may be quite different from the meaning they have for the student (Cobb, Yackel and Wood 1992). Therefore, if particular representations are to be used in the classroom, then teachers need to support students in learning to interpret representations (Flevares and Perry 2001), through providing "effective transitional experiences" (Boulton-Lewis 1998, 222) to support students' progression onto using these different representations.

Perhaps due to these possible drawbacks, despite the theoretical importance of diagrammatic representations in the teaching and learning of mathematics, the empirical evidence to support the use of these representations in the classroom is mixed and somewhat lacking. Sowell (1989) carried out a meta-analysis looking at the effectiveness of external representations in mathematics classrooms and concluded that there were no significant benefits associated with using diagrammatic representations in comparison to abstract representations. A more recent meta-analysis by Gersten et al. (2009) found more positive results. This meta-analysis looked specifically at mathematics instruction for students with learning disabilities and found that the use of diagrammatic representations by students during problem solving, or by teachers during initial instruction or demonstration, significantly benefitted the students. Due to the mixed nature of these results and the lack of empirical studies specifically looking at the use of diagrammatic representations in the primary/elementary school context, this study contributes to the research on this pedagogical approach to teaching mathematics.

This paper, in addition to looking at the broader research on diagrammatic representations given above, focuses specifically on theoretical aspects related to the two topics of multiplication and fractions. The paper outlines the design of a professional development programme which aimed to introduce teachers to the use of diagrammatic representations in the teaching and learning of these topics within mathematics. The qualitative methods approach and design utilised for the project are described, followed by the results of the project in terms of the impact on the children and teachers involved. Finally, implications for pedagogical practice in mathematics classrooms are considered, and the findings from the study are used to suggest how research ideas can be transferred into classroom practice through teacher professional development.

A focus on multiplication and fractions

In trying to exemplify the theoretical issues highlighted above, we chose two areas of primary/elementary school mathematics to focus upon, namely multiplication and fractions. In coming to understand multiplication, Greer (1992) highlighted the importance of a range of different 'classes of situations' associated with the concept. These included equal groups, equal measures, rate, multiplicative comparison, multiplicative change and Cartesian product situations. Examples of each of these situations are given below:

- Equal groups - Six children are holding two books each. Altogether they will have 2 × 6 = 12 books.

- Equal measures – Each book is 2cm wide. 6 books stacked together will be 2 × 6 = 12cm.

- Rate – A boat moves at 2 metres per second. In 6 seconds, it will move 2 × 6 = 12m.

- Multiplicative comparison - Emily swims six times further than James. If James swims two lengths of the pool, Emily will swim 2 × 6 = 12 lengths.

- Multiplicative change - When Sarah's guinea pig was born, it was only two centimetres long. It grew up to be six times this size. It grew to 2 × 6 = 12cm.

- Cartesian product – A sandwich shop has two types of bread and six different fillings. Altogether, it sells 2 × 6 = 12 different types of sandwiches.

These different situations illustrate the range of possible ways of thinking about multiplication that can constitute our understanding of the concept (Hiebert and

Carpenter 1992). Closely linked to these different situations, Greer (1992) also highlighted the range of different external diagrammatic representations associated with different contexts of multiplication. For example, equal groups can be represented by diagrams of equal groups of objects and/or arrays respectively (an example is given in Figure 1).

(Insert Figure 1 here)

Cartesian product situations can also be represented by arrays (Figure 2).

(Insert Figure 2 here)

The number line can be used to represent multiplication, particularly in relation to multiplication as repeated addition (Figure 3).

(Insert Figure 3 here)

Alongside the representations highlighted above, Outhred and Mitchelmore (2004) stated that the rectangular array model (or area representation) is an important model for multiplication (Figure 4).

(Insert Figure 4 here)

Also, Lampert (1986) highlighted the importance of computational knowledge, manipulating numerical symbols often according to procedural rules. This can be done through traditional methods of multiplication or through methods such as the grid method (Figure 5).

(Insert Figure 5 here)

The different diagrammatic representations of multiplication also facilitate the explanation of particular properties of multiplication. For example, Skemp (1986) highlights the usefulness of the array representation in showing the commutative and distributive laws for multiplication. For example, in Figure 1, the array can show why 6 × 2 is the same as 2 × 6 by reorienting the array, or why 6 × 2 can be redistributed as (4 × 2) + (2 × 2). Likewise, the grid method makes more explicit the use of the distributive law. For example, in Figure 5, 18 × 16 is split into (10 × 10) + (8 × 10) + (10 × 6) + (8 × 6). However, students can also face difficulties in accessing meaning from these diagrammatic representations. For example, Barmby et al. (2009b) identified children's difficulties in associating the array representation with multiplication calculations. They therefore suggested a progressive teaching sequence of diagrammatic representations, moving from equal groups to the array and on to the grid method, and therefore providing students with ‘transitional experiences’ as highlighted by Boulton-Lewis (1998).

In terms of fractions, Behr et al. (1983, 92) argue that an understanding of any rational number “… involves a rich set of integrated subconstructs and processes”. Basing their ideas on Kieren’s (1976) earlier work, Behr et al. (1983) details five sub-constructs for the rational number concept: the part-whole construct (both for continuous and discrete quantities) and the closely-associated measure construct; the ratio construct; rational numbers as indicated division or quotient; and rational number as an operator (e.g. as a transforming function). As for multiplication, this range of sub-constructs for fractions can constitute our understanding of this topic. Related to these sub-constructs are the ways in which these constructs can be represented. For example, diagrammatically, the part-whole construct can be represented by a part of a continuous whole (e.g. a part of a shape) or by a group of discrete objects (Figure 6). The measure construct can be represented by the number line (Figure 7).

(Insert Figures 6 and 7 here)

Again, the different diagrammatic representations can facilitate the explanation of particular properties of fractions. For example, the rectangular part-whole representation in Figure 6 is particularly useful for explaining the concept of equivalent fractions (Barmby et al. 2009a). However, students also face difficulties in identifying different representations of fractions. For example, although it is recognised that the ‘part-whole’ construct for fractions is fundamental to introducing fractions (Charalambous and Pitta-

Pantazi 2007), children become overly reliant on these part-whole representations (Kerslake 1986; Mack 1990; Pitkethley and Hunting 1996), particularly the circular or ‘pizza’ representation for fractions given in Figure 6 (Peck and Jencks 1981). This limitation in children’s use of representations results in difficulties in relating fractions to other representations such as locating them on a number line such as shown in Figure 7 (Baturo and Cooper 1999; Hannula 2003). In relation to the above point, this reliance also results in difficulties in understanding the ordering and equivalence of fractions (Behr et al. 1984; Hart 1981; Kamii and Clark 1995; Kerslake 1986; Ni 2001).

The design of professional development programme for teachers

Based on the above literature on the use of diagrammatic representations, the research project aimed to develop primary teachers’ use of these representations through professional development. Examining the research on the professional development of teachers, the literature, particularly for teachers of mathematics, suggests a number of key characteristics for successful professional development programmes. Firstly, a programme should specify and highlight subject matter and knowledge required by the teachers (Garet et al. 2001; Borko 2004; Hill and Ball 2004). Secondly, a programme should highlight how children learn that subject matter (Wilson and Berne 1999; Franke et al. 2001; Garet et al. 2001; Borko 2004). Thirdly, during the programme, teachers should be actively engaged in their learning (Desimone et al. 2002), providing opportunities for teachers to work together (Franke et al. 2001; Hill and Ball 2004) with meaningful discussion and planning (Garet et al. 2001), with a “privileging of teachers’ interaction with one another” (Wilson and Berne 1999, 195). Fourthly, the programme should strive for a ‘community of practice’ that is sustained over time (Garet et al. 2001).

Looking critically at the above research, one possible drawback we could highlight is that this research on characteristics of successful professional development has largely been carried out in the American context. Viewing this issue from the UK context, a recent report (Back et al. 2009) published by the National Centre for Excellence in the Teaching of Mathematics (NCETM) into effective professional development in mathematics education, echoed these recommendations highlighting the following characteristics of successful professional development:

- Led by facilitators with good understanding of current practice;
- Practical and stimulating approach;
- Opportunities to network with other teachers;
- Focussing on ways of teaching the subject and students’ conceptions, and getting teachers to use their subject knowledge;
- Encouraging reflection;
- Encouraging change and supporting the embedding of change

However, in addition to identifying the characteristics that support successful professional development of teachers, the factors that hinder this professional development must be recognised as well. With regards to teacher professional development generally, Rhodes and Houghton-Hill (2000, 430) again within the UK context identified ‘time’ as an issue, where “lack of time was identified as the major detractor from the potential of staff development to impact in classrooms”. Related to this issue, responding to other priorities such as the administrative workload and external inspections by OfSTED were seen to influence the impact of professional development. Resistance by teachers and difficulties in cascading information obtained on professional development programmes within schools were also identified as factors. Within mathematics education, Back et al. (2009) also identified time as an issue for participants on professional development programmes, and that teachers therefore appreciated the opportunities in terms of time to examine and reflect on their practice during professional development opportunities.

The possible barriers facing providers of continuing professional development programmes, particular in the UK context, are outlined by the above pieces of research. However, despite these, there is still little research on the factors facilitating or hindering professional development programmes, particularly in the UK context and focussing specifically on mathematics at the primary level. Therefore, although the main focus of the present paper is to examine whether research-informed professional development in a particular area of mathematics education was successful, the study will also contribute more broadly to the considerations that need to be made when designing professional development opportunities in this UK context.

Aims

The aim of the study reported here was to use the research on diagrammatic representations to develop teachers' use of these representations through a professional development programme, and as a result of this programme, develop children's understanding in mathematics. Operationalising these aims (Cohen, Manion and Morrison 2000), the study sought to answer the following research questions:

- How did the professional development programme impact on mathematics co-ordinators' and classroom teachers' practice in terms of using diagrammatic representations?
- Did the programme of professional development result in subsequent development of children's understanding of mathematics?
- What factors facilitated or hindered the development of co-ordinators' and classroom teachers' practice in terms of using diagrammatic representations and the development of children's understanding?

In order to answer these questions, a study was carried out which used a mixed methods approach, "combining qualitative and quantitative approaches within different phases of the research process" (Tashakkori and Teddlie 1998, 19). Quantitative measures of pupils' understanding of multiplication and fractions were collected alongside qualitative data obtained through interviews and observations. In this paper, we specifically report on the qualitative data obtained as part of the larger study.

Methods

From the research questions, the focus of the study was not only on the potential impact of the project, but also on the 'mechanism' for any potential impact on mathematics co-ordinators attending the professional development sessions and the classroom teachers that they were working with. In examining this mechanism, a *realist* methodological perspective was taken for the study (Robson 2002, 32) where the focus was on "explanation … concerned with how mechanisms produce events". In seeking to understand more the mechanism behind developing teachers' use of diagrammatic representations, this question warranted a qualitative approach (Punch 2009).

Setting

The study involved 8 volunteer primary schools (A to J) in the North East of England. In recruiting these schools, Head Teachers from 13 schools were approached and information on the project provided to them, and 10 schools agreed to take part in the project, with 8 schools subsequently being involved to the professional development sessions (the other two schools were control schools in the quantitative part of the study which will be reported elsewhere). The 8 participating schools varied in size and location, two being small (one class per year of less than 15 children, with mixed age classes), four being medium (one class per year of around 25 children), and two being large (two classes per year of around 25 children each). The location of the schools also varied, ranging from urban settings for the larger primary schools and three of the medium sized schools, to rural settings for the very small primary schools and one of the medium sized schools.

Participants

Having gained agreement from the 8 participating schools, the mathematics co-ordinator from each school was invited to the professional development programme. The role of the co-ordinator, alternatively referred to as the subject leader, is to secure high quality teaching, effective use of resources and improved learning for pupils through their management and leadership of the subject within the school (Teacher

Training Agency 1998). Therefore the input through the professional development programme was provided to the mathematics co-ordinators rather than directly to the class teachers who were subsequently interviewed and observed, since the co-ordinators had the specific remit to improve the teaching of the subject in their establishments. This 'dissemination' approach to professional development is commonly used with schools, particularly in primary schools which have smaller establishments than say secondary schools, and where there are difficulties around releasing larger numbers of teachers for professional development purposes. The possible drawback of this 'indirect' approach to the transfer of research ideas was recognised. However, advantages were also identified through the hoped-for discussions and reflections taking place between co-ordinators and class teachers. The study will consider further the possible advantages and disadvantages of this professional development design in subsequent sections.

Design and structure of the professional development programme

The training was designed so that the mathematics co-ordinators were provided with an understanding of the research ideas on the use of diagrammatic representations in the teaching of mathematics (as outlined above) so that they could subsequently make complex decisions in the classroom, and were not simply 'passive practitioners' (Kaestle 1993). The emphasis was on an "on-going process of experiencing practical teaching and learning situations, reflecting on them under the guidance of an expert, and developing one's own insights into teaching" (Korthagen and Kessels 1999, 6). This emphasis was in line with the characteristics of successful professional development identified earlier.

Therefore, based on the above literature, the professional development programme for the mathematics co-ordinators involved in the project was structured as shown in Table 1.

(Insert Table 1 here)

In addition to introducing the co-ordinators to the research on the use of diagrammatic representations, an emphasis during the training days was to encourage discussions between co-ordinators by providing necessary opportunities. This included pairing up co-ordinators as partners for discussion during the training days themselves, and also requesting them to meet and discuss their progress in the project outside of the training sessions. In providing the training over three days across the school year, the project tried to achieve the balance between providing sustained opportunities for professional development, and minimising the impact of taking the co-ordinators out of schools.

Also, the design of the training was wary of 'feed-forward' issues (Korthagen and Kessels 1999), where barriers for teachers in implementing ideas may occur due to a lack of personal concerns from the teachers about the issues at hand. For example, teachers not responsible for developing the teaching of mathematics throughout the school may be less engaged in the training compared to mathematics co-ordinators with this role.

Study design

A number of different considerations informed the design of the research methods used in the study. Firstly, the requirements of the professional development programme were influential in terms of incorporating sustained involvement from co-ordinators and schools over time, and also opportunities for collaboration between co-ordinators. Secondly, the qualitative data collected for this project was part of a larger study a involving a clustered pre-test/post-test experimental design (with randomisation at the school level) with two treatment groups and one control group (Muijs 2004). The two treatment groups consisted of the co-ordinators receiving the professional training, but only half of the co-ordinators (i.e. one of the treatment groups) in each term were asked to implement the research ideas from the particular professional development day in their schools. With this experimental design, the treatment groups therefore provided a comparison group to each other. A smaller control group that did not receive the professional development was also included as a further comparison group. The design of the study, including the quantitative and qualitative parts of the research, is summarised in Table 2 in order to provide an overall context for the study. Included in the table are the points at which the qualitative data described in this paper were obtained. This design was used in each of the Autumn and Spring terms in which the study was carried out.

(Insert Table 2 here)

As highlighted above, in each of the first two terms of the study, only half of the mathematics co-ordinators were asked to implement the ideas from the professional development training. Therefore, qualitative data was only collected from the schools implementing the ideas in a given term. Stratified random sampling was carried out to allocate four schools to each of the treatment groups (1 small school, 2 medium schools and 1 large school) to ensure comparability of the treatment groups in terms of the numbers of children and the characteristics of the schools. After the mathematics co-ordinators from the eight treatment schools attended the first professional development day with the focus on multiplication, half of the co-ordinators (from the schools in the first treatment group) were then asked to implement the ideas from the training back in their schools, with a particular focus on Year 3, over the course of this Autumn term. The other co-ordinators (from schools in the other treatment group), although having attended the professional development day, were asked not to implement any of the ideas in their schools. Then in the Spring term, the same design as in the autumn term was employed, except the focus was on fractions and Year 5. The other half of the co-ordinators (the second treatment group) were this time involved in implementing the ideas with their Year 5 teachers.

The reason for focussing on Year 3 and Year 5 in the first and second terms respectively was largely determined by the focus of the mathematical topics, i.e. multiplication and fractions. Previous research suggests that Year 3 children lack an understanding of multiplication representations (Barmby et al. 2009b). Year 5 was chosen because we wanted children to have had at least some experience of fractions but we did not want to disrupt schools or children during the busy Year 6 (final year of primary school).

Data collection

As summarised in Table 2, the following methods of qualitative data collection were included in the study:

- Interviews with class teachers implementing the research ideas in a given term, at the start and end of that term;
- Interviews with mathematics co-ordinators implementing the research ideas in their school in a given term, at middle and end of that term;
- Lesson observations of the class teachers implementing the research ideas in a given term, at the start and end of that term.

The purposes of the interviews was to gather data on the mathematics co-ordinators' and classroom teachers' views on their use of diagrammatic representations in the mathematics classroom, whether the project had an impact on their classroom practice and identifying issues which supported or hindered this impact on practice. The interviews with class teachers at the start of the term were designed to find out how they used diagrammatic representations before the implementation of the project ideas. The interviews with co-ordinators in the middle of the term were designed to elicit their reflections on the professional development training, and also their initial views on implementing the ideas. The interviews with both co-ordinators and class teachers at the end of the term was to evaluate the impact of the project both on the co-ordinators and class teachers themselves, and also on the pupils being taught by the class teachers. In examining these issues, semi-structured interviews were used to gain descriptions from the teachers of their interpretation of the phenomenon at hand (Kvale 2007). The questions around which the interviews were structured are given in Tables 3 and 4. The interviews lasted in the region of 15 to 25 minutes each time.

(Insert Table 3 and 4 here)

These interviews were recorded and transcribed for further analysis. The interviews were carried out on a one-to-one basis except for two occasions where due to constraints within the school, the co-ordinator and the class teacher or two class teachers in the case of a larger school, were interviewed together. In the first case however, the views of the co-ordinator and the class teacher were differentiated in the subsequent analysis of the interview data. .

The purpose of the lesson observations was to gain further insight into whether and to what extent the class teachers' pedagogical practices had changed as a result of the professional development programme. In meeting this aim, descriptive observations focussing on the use of diagrammatic representations by the class teacher and children in their classroom were carried out (Robson 2002). These observations were carried out during the same visit for the interviews with the class teachers, and carried out by the same researcher that carried out the interview. In the case of the larger schools, two observations were carried out for each of the class teachers in the given year group. Although focussed, the observations remained unstructured, where predetermined categories and classifications were not used, rather observations were carried out in a "more natural open-ended way" (Punch 2009, 154), with categories for describing the observational data emerging later in the analysis. Observations were recorded as field notes, starting as jottings and descriptions of the classroom setting, the class teacher's and children's actions and observations from scanning children's books. These were then assembled and written out more fully following the lesson to form a more comprehensive account of the lesson, ready for further analysis (Cohen, Manion and

Morrison 2000). The role of the researcher within this observation process was that of observer-as-participant (Flick 2009), where the researcher had been introduced to the class as a researcher, but otherwise has little input to the lesson observed (Cohen, Manion and Morrison 2000).

Data analyses

Data obtained from teacher interviews and lesson observations were coded to reduce the information to themes/categories (Creswell 1994). In doing so, the analysis was guided by Tesch's (1990) (cited in Creswell 1994, 154-155) systematic steps to analysing qualitative data. Specifically, the categories used for initial coding emerged from an initial examination of the transcripts/accounts, but these codes were subsequently modified to form new categories and codes, used for the final coding. Due to the semi-structured nature of the questions used for the interviews, the emerging codes from the interviews were closely related to the focus of the questions, namely co-ordinators' and class teachers' views on their use of diagrammatic representations, the impact of the project, and issues which supported or hindered this impact. The codes emerging from the analysis of the accounts from the lessons were quite different, reflecting the observational nature of these accounts (e.g. a focus on the observed use of diagrammatic representations by the class teacher and the pupils, and any observed changes in the use of these representations between the two observations). Therefore, the interview transcripts and the accounts of the lesson observations were analysed separately, with the main qualitative findings emerging from the more detailed information obtained from the interviews. However, the findings of the lesson observations were used to gain further insight into particular ways in which diagrammatic representations were being used in the classroom.

Findings

The final categories that emerged from the analysis of the interviews with co-ordinators and class teachers are given in Figure 8, relating the different categories emerging in a causal network (Robson 2002), and illustrating the perceived relationships between the different categories identified in the qualitative analysis of the interviews. We discuss each of these final categories in turn below, with selected quotations from class teachers and co-ordinators to illustrate the issues at hand.

(Insert Figure 8 here)

Findings from the interview data

<u>Class teachers' and co-ordinators' views on the impact on the children</u>

The consensus from the class teachers and co-ordinators interviewed, particularly the class teachers, was that the use of diagrammatic representations had a positive impact on the children concerned. Specifically, most teachers identified the beneficial impact of the use of these representations on children's understanding in mathematics, as these quotations illustrate:

> They've got a better understanding of what a fraction actually is. If we ask them what a fraction is they'll get the vocabulary and they'll say it has to be equal, its starts with a whole ... And they know the different ways of how to explain the numerator and the denominator and things like that. (Year 5 class teacher)

> They could explain things. Some were still struggling but they could explain things and they could draw pictures for me and they were wanting to draw little diagrams for me. (Mathematics co-ordinator)

In addition, this positive impact may have been in a more general sense, with some teachers highlighting the impact of the use of diagrammatic representations on the confidence of the children in mathematics as shown below:

> I think it's given them more confidence if I'm honest. I mean there's one little girl, I don't know if you saw her ... She's a lower ability child who came with no maths confidence at all, and I've tried to do everything very visually for her. Even addition, subtraction, not just multiplication. And because she sees it visually, it's easier for me to talk it through where she's gone wrong and correct it. (Year 3 class teacher)

This latter quotation also raises the issue raised by some of the teachers of the possible differential impact of the use of the diagrammatic representations, depending on the ability of the child. Most class teachers and co-ordinators who raised this issue felt that the representations had a greater impact on the lower ability children, although one teacher felt that the impact was greater for the higher ability children, as shown in their quote:

> And for the enquiring brighter minds amongst them, it's definitely given them a lot more food for thought. (Mathematics co-ordinator)

Class teachers' views on the impact on their teaching

In terms of whether the project led to a direct impact on the class teachers involved, a small number of teachers, particularly in the second term of the project stated that they were 'doing it anyway' and little or no change had occurred in their teaching. This quotation exemplifies this view:

> I don't think it's made any, I think because I'm a visual learner anyway. If I'd been asked to teach fractions I would have started it using pictures and visual representations of it to start introducing them to it. (Year 5 class teacher)

The issue of changing practice could also have been related to the background of the teacher, as will be highlighted below. For many other teachers however, they did identify ways in which their practice had changed, the following quote being one example:

> I mean the way I've taught it has been definitely different. I've not taught it using arrays before so I think that's had an impact on the children but also on me. (Year 3 class teacher)

As identified above, this could have also been through a greater focus on teaching for understanding with the children, getting children to explain their thinking and thereby highlighting misconceptions as shown below:

> I'm more aware of the misconceptions children have of multiplication through doing it this way. Because you're not just saying to them "how have you got the answer?" They're actually showing you, "well, this is how I got the answer". (Year 3 class teacher)

In addition to changes in practice, many teachers (even those who stated that their practice had not changed), identified the opportunities provided by the project of thinking more about their teaching and developing their knowledge and awareness of the issues around teaching the particular mathematical topics. The following quotes illustrate some examples of these changes:

> It has made me think more about how I would teach it and the different ways to teach it. (Year 5 class teacher)

In terms of this awareness of using diagrammatic representations, a particular issue identified by the teachers was the progression involved in the use of different representations as shown below:

> Seeing the simplest visual method because sometimes it's easy to maybe kind of jump a step and you won't perform the next step. So I suppose it's having to think "right, what's the first stage", and then working through in kind of relevant and logical steps, and progress the steps. (Year 3 class teacher)

<u>Co-ordinators' views on the impact of the professional development</u>

In addition to the class teachers themselves, the mathematics co-ordinators identified the impact of the project on themselves and also wider benefits. All of the co-ordinators identified the impact of the project on their knowledge and practice, for example in the quote below on the co-ordinator's knowledge about teaching multiplication:

> For me, yeah, knowledge about the teaching of multiplication … I think we've thought about it in a bit of a deeper kind of way, that perhaps some children do understand but they can't tell you what they're doing. (Mathematics co-ordinator)

The impact more broadly throughout the school was also identified in some cases:

> We'll do a staff meeting and kind of go through the kind of visualisations, the methods and things we've used and talked about. (Mathematics co-ordinator)

> We have looked at visual representations throughout school and we've kind of looked at our resources and looked at all the different ways, and decided on instead of using lots and lots of different things, using a kind of more refined set of visual representations. We've also particularly looked at the number line and our understanding of how to use it, and I think that has helped because I think before, we were perhaps not using it incorrectly, but we weren't using it to its fullest advantage. (Mathematics co-ordinator)

The resources provided during the training day supported this impact as well in some cases as shown below:

> Loved the training day because of the ideas that I brought away, with the practical ideas, we didn't have those previously … Those hands on practical activities were really fantastic. Brought them back into school, I've used them in my class. (Mathematics co-ordinator)

With regards to the training days, the opportunity to discuss ideas with colleagues from other schools was highlighted by all of the co-ordinators as particularly positive as highlighted below:

> To talk about what works and it was good to get ideas and hear about the research and some of the academic research side of it. But then there was plenty of chance for us to talk about how it's really going to work in the classroom, what's going to fit together ... It's always a nice chance to get together with another group of teachers and have a talk about how things work. (Mathematics co-ordinator)

In addition to the impact on the co-ordinators themselves, some wider benefits were also identified (examples are given below).

> It's made us I think as a school talk a bit more about maths and how we teach it and getting more of the sort of continuity. (Mathematics co-ordinator)

Class teachers' and co-ordinators' views on difficulties and issues of implementation

The 'mechanisms' by which the project impacted on the children and the class teachers were related to the different backgrounds of the teachers involved. For instance:

> I'm a young teacher who's recently come out of [removed] University and it's a [removed] University maths project, I think I've been aware of the principles behind it. (Year 3 class teacher)

> But that's because I've got quite good subject knowledge in maths, I've probably had the appropriate training and I've pretty much picked up most of the techniques that have been impacted through the project anyway, prior to the project. (Year 3 class teacher)

> I suppose initially it was to become familiar with the range of different representations … I suppose obviously being my NQT (newly qualified teacher) year like some of it's trial and error … I think maybe that's my main difficulty. (Year 3 class teacher)

One teacher felt that the fact of also being the mathematics co-ordinator and attending the training helped the impact in her school. In addition to the background of the teachers, some of the co-ordinators highlighted that the context of the class or the school had an impact in some cases, in particular the pressure of external inspections (by OFSTED) or statutory assessments (the National Curriculum Tests, referred to informally as 'SATS' by the interviewees):

> So she's going to use some arrays and some visualisations to help them with the division side of that. The only thing she said is sometimes the work is a bit messy in the books and thinking about the school side of it, especially because we're due OFSTED this year, it's kind of make sure we've got lots of evidence in books and putting it in. Sometimes it's not the neatest work … (Mathematics co-ordinator)

> With it being a Year 5/6 class you can guess what I'm about to say. She's on the way to SATs so she's, it’s very much teaching toward the Year 6s at the moment. Which is not the best for your research, I understand that. (Mathematics co-ordinator)

> I mean in terms of fractions, it’s such a small element of a large, you know of maths as a whole ... If you think of the hundred marks in a SATs paper, the maximum will be four questions on fractions. (Mathematics co-ordinator)

In two schools, the background of the class teacher also supported the ability for discussions to take place between the teacher and the mathematics co-ordinator about the project, as shown below:

> Well lucky for me because (the teacher) was an NQT so it meant I was down with him quite a lot. Initially I fed back what happened with the first training day to (the teacher) and then I went down for two lessons with him and worked with him in the class. (Mathematics co-ordinator)

This issue of whether class teachers were able to discuss the project ideas with the co-ordinator was raised by many of the class teachers and the co-ordinators.

> When I came back from the training day, I shared all my information with the Year 5 teacher. She looked at my notes that I’d made from talking with the other teachers and from the sessions that we’d had. We talked about what might be useful for her class and then I know she’s implemented a lot of those ideas. (Mathematics co-ordinator)

For some, the issue of time and unforeseen circumstances had a negative impact on discussions and implementations of ideas in the classroom, as illustrated by the following quote:

> Not sitting down and discussing it all, just in little bits ... But that's just the reality as it was. You know (the co-ordinator) went to the thing, came back and said this is what we’re going to do. Gave us the power point and information, the different

> resources that we got. Some of the online stuff and then get on with it ... I mean to be fair it was probably the timing that its happened as well … When do you have time to get together and sit down and talk when you're not running off to football matches and various different things? (Year 5 class teacher)

This issue of time also possibly impacted on the discussions between mathematics co-ordinators outside of the training sessions, as shown below:

> My partner teacher ... I haven't really heard anything from him. You know, we haven't done this sort of meet up. (Mathematics co-ordinator)

For the second term of the project, the timing of the project was also identified as a negative factor by some of the class teachers, as illustrated by the following quote:

> I mean part of the problem that I've had would be really that this would have been better if we'd had this in the autumn, because it was in the autumn term that Year 5 did this kind of fractions work. (Year 5 class teacher)

<u>Class teachers' and co-ordinators suggestions for improving the project</u>

A number of suggestions for improving the project were identified by teachers. Firstly, as identified above, some of the class teachers highlighted that the timing of parts of the project could have been improved, as illustrated by this quote:

> It would be much better if we knew at the beginning of the year for example or we could get together at the beginning of the year and look at the structure of what we're going to be teaching, when we're going to be teaching and what you've got to offer. (Year 5 class teacher)

Related to this was the request from many co-ordinators for more advanced information to be provided to the schools before the start of the project. Following on from this, the opportunity for the class teachers themselves to be involved in the training directly was particularly highlighted by most co-ordinators and class teachers:

> The best improvement you could make … I'm the delivery teacher, allowing us to go on the training. If I had received the training, even though my co-ordinator did disseminate the training she got and passed me all the notes and that, I still think that learning that first hand and being there would have been more beneficial for me, to know what was going on rather than being disseminated. (Year 5 class teacher)

A request for more resources and examples of teaching was made by some of the co-ordinators involved in the project:

> I would have liked a bit more guidance on what resource-wise or different ways of approaching it. Maybe some guidelines of saying "well you could do it in this sort of way". Just to get an idea of what exactly, to make sure I was doing it right and not just my interpretation of what the project wanted if that makes sense. (Mathematics co-ordinator)

> Maybe if we kind of saw demonstrations of different things being used, I suppose kind of if there were more examples from that. I suppose a lot of mine may have been trial and error and I might not know all the resources even now. (Mathematics co-ordinator)

The short-term nature of the project (in terms of implementation only being over one term) was highlighted as an issue by some of the co-ordinators, so developing the project over a more sustained period of time was suggested as shown below:

> The time that you've given yourselves. It's such a short time to see the progress. (Year 3 class teacher)

Findings from the Lesson observations

As previously stated, alongside the interviews with mathematics co-ordinators and class teachers, lesson observations of the class teachers at the start and end of each term were carried out to gain further insight into the extent to which the class teachers' pedagogical practices had changed as a result of the professional development programme to co-ordinators. Analysing the themes emerging from these lesson observations, in line with the interviews, the observations suggested that the impact of the project varied across different classrooms. In some of the classrooms observed, it was evident that the project had had little impact on the use of diagrammatic representations for teaching mathematics. An excerpt from the account of one of the lesson observations illustrates this.

> The next activity involved further SATs practice, with all the types of Fractions, Decimals, Percentages questions that would appear in the SATs. It was a 10 minute activity. During this task, there were a lot of explanations, but they were all verbal – no recourse to any other types of representations. Looking around at the children trying to do the questions, they were all using very procedural methods. For example, when converting 6 7/11 – drawing a triangle through the mixed fraction was recommended. Again, to find 4/5 of 60, a completely rote method of calculation was suggested. The teacher did suggest later that some of these rote methods were put in place for the less able members of the class, but it seemed that the pupils had very little else to fall back on. So, based on this observation, it seemed that the project had had very little impact on the teaching of this teacher.
> (Year 5 lesson observation – end of term)

In some classrooms, it was observed that teachers already used diagrammatic representations in the initial observations of their lessons. However, in some cases, this was due to awareness of the nature of the research project.

> The learning objective was stated as ‘Know by heart facts of 2x and 5x tables’. The lesson started with the children on the carpet and the teacher showed, on the IWB, a picture of a sheet of paper with a smiley faced sticker on it. She told the children that this could be seen as 1 sticker on 1 sheet or 1x1 ... The teacher returned to the IWB and presented 2 sheets with 2 stickers on, then 3 sheets with 2 stickers on, each time asking the children to suggest multiplication sentences. Most did so. (Year 3 lesson observation – start of term)

> The teacher then brought up a programme on the Interactive White Board showing six ladybirds, each showing two black dots on their back. She showed the children how the total number of dots could be arrived at by adding (i.e. repeated addition) but that it is more efficient to use multiplication. The teacher offered another screen via the Interactive White Board showing stools, each with three legs ... The teacher admitted afterwards that the lesson was planned with the aims of the research project in mind. I think this showed in that the teacher tried to include as many visual representations as she could without a great deal of thought about how the children might make the links between the different ones she used. (Year 3 lesson observation – start of term)

This latter issue suggested that it was problematic to gauge from the observations the extent to which class teachers’ pedagogical practices had changed. However, comparing the observations of lessons at the start and end of each term, some clear developments in the use of diagrammatic representations emerged. For example, relating to the excerpts give above, a particular finding from the observations was the development in progression in terms of teachers’ introducing diagrammatic representations to the pupils. As the above excerpt show, even if teachers were initially using diagrammatic representations, they also needed to consider how pupils can make the necessary connections between different representations. The excerpts below, from observations with one of the class teachers at the start and end of term, illustrate their developing use of diagrammatic representations.

> The teacher started off with the children on the carpet and began by trying to introduce the quotitive/grouping model of division to the children by showing (via the Interactive White Board) a number of visual representations. For instance, she first modelled 15 ÷ 5 by showing 15 bananas. She then asked the children how many groups of 5 fit into 15 and then showed them, dynamically by moving the bananas, how if they were grouped into 5s then there would be 3 groups, so 15 ÷ 5 = 3. The teacher moved the bananas into an array-type arrangement ... It was obvious to me that she was trying to make links with the children's previous learning and in particular their experiences with the array. (Year 3 lesson observation – end of term)

For this teacher, although they had been using diagrammatic representations prior to the start of the project, they developed their use in terms of making connections to other mathematical concepts using the diagrammatic representations. In this case, in discussions after the lesson, the class teacher highlighted that they had not previously used diagrammatic representations to make the connection between multiplication and division, and the project specifically encouraged them to do this.

The other development in pedagogical practice highlighted by the lesson observations was in the range of diagrammatic representations use by the class teachers in their lessons. For example, two of the Year 5 class teachers started off using the circular 'pizza' representation in their teaching of fractions, but by the end of the term, they had moved on to the use of the fractions of discrete objects and fractions on the number line, the latter being identified by the research literature as an area of difficulty for pupils.

> The teacher explained that children would today be learning how to put a fraction on a number line. Teacher: 'What is a fraction?'Children replied using language such as 'equally divided'. Teacher showed unlabelled number line on the board divided into eighths. Children were encouraged to work out what fraction the line was divided into. Children successfully described the denominator. Children were

> invited to the board to mark 1/8 and 4/8 on the number line. Higher ability pupils simplified 4/8 to 1/2. A discussion took place about how to find 1/4 on the number line. The teacher showed another unlabelled number line divided into 16ths with an arrow pointing to 1/16. Children asked to identify the fraction marked and to work out where 5/16 would be. (Year 3 lesson observation – end of term)

Therefore, what was surmised from the lesson observations was that although it was difficult to observe a 'quantitative' change in terms of the amount of use of diagrammatic representations by the class teachers as a result of the project, there was evidence that there had been a 'qualitative' change in terms of the considerations made when using these representations and the types of representations being used. These also link back to the issues highlighted in the interviews with regards to the impact of the professional development programme.

Discussion

The qualitative data obtained suggest that the professional development training programme, which introduced mathematics co-ordinators to the research on using diagrammatic representations, had a positive impact on both the children in the respective schools, and the classroom teaching experienced by those children. The findings are in disagreement with Sowell (1989) who found little evidence for the impact of diagrammatic representations on children's learning. The findings suggest that diagrammatic representations can benefit children's understanding, in line with Greeno and Hall's (1997) view of mathematical understanding, and also the confidence of children in mathematics. Interestingly, and returning to the focus of the previous study by Gersten et al. (2009), the qualitative data from teachers also suggested a possible differential impact of the use of these representations on lower and higher ability children, although there was no clear consensus on which group of children would benefit more.

In addition to the impact of the use of diagrammatic representations on children, the project also identified the possible impact of the project on teachers' knowledge and practice. Although not all of the teachers identified changes in their actual classroom practice, there was evidence both from interviews and the lesson observations that a greater range of diagrammatic representations were being used by class teachers. More importantly perhaps, most of the classroom teachers interviewed identified a development in awareness and knowledge about using diagrammatic representations in their mathematics teaching. Shulman (1986), and Ball, Thames and Phelps (2008) specifically in mathematics teaching, highlighted the use of external representations in the classroom being an important part of a teacher's specialised knowledge of the subject. Therefore, this project had a direct impact on this specialised knowledge for primary teachers. Looking at this issue of awareness and knowledge further, both the interviews with teachers and the observations of lessons identified a degree of progression in the sophistication of this knowledge of using diagrammatic representations:

Little or no use of diagrammatic representations

↓

Using a variety of diagrammatic representations

↓

Consideration of the progression for children between different representations

The qualitative findings of the project further suggest that in order to maximise the benefits for children, it is not a simple matter of using 'more' representations, but also to consider how the teachers and the children themselves can make the connections between different representations. This issue may begin to explain why in some studies (e.g. Sowell 1989) simply using diagrammatic representations in the teaching of mathematics does not necessarily impact on the learning of pupils. This consideration of how diagrammatic representations can be used is in agreement with the issues around the use of various representations highlighted in the review of the literature (Boulton-Lewis 1998; Cobb, Yackel and Wood 1992; Flevares and Perry 2001; Hall 1998). In terms of teacher knowledge, this progression also emphasises how teacher knowledge can be more integrated; taking Ball, Thames and Phelps' (2008) model for pedagogical content knowledge as an example, the possible connections between *knowledge of content and teaching (KCT)* in terms of the representations that can be used to explain a concept and *knowledge of content and students (KCS)* in terms of the potential difficulties faced by students is evident. Therefore, in exploring teachers' knowledge of the use of diagrammatic or other external representations in the future, the progression in terms of this integration of knowledge may be focussed upon further in future studies (we have carried out some preliminary work in this area in Barmby and Milinkovic 2011).

In highlighting the impact that the project had on children and teachers in the participating schools, the study has already acknowledged the complex mechanisms by which this impact could have come about. A particular factor influencing this complexity was the structure of the professional development programme used in this project, and we can reflect on the results of the study in order to inform future projects involving professional development of teachers. The model of professional development was based on the training of mathematics co-ordinators, introducing them to the key issues emerging from the research on using diagrammatic representations in the mathematics classroom. Informed by the research literature on teacher professional development, particularly with a focus on mathematics, characteristics of the training programme positively influenced the project's impact. Firstly, the project was able to impact on teacher knowledge (Garet et al. 2001; Borko 2004; Hill and Ball 2004). Secondly, the project developed teachers' awareness of how diagrammatic representations support or hinder children's learning (Wilson and Berne 1999; Franke et al. 2001; Garet et al.2001; Borko 2004), as exemplified above in teachers' consideration of the progression involved in using these representations in the classroom. Thirdly, teachers acknowledged that the opportunity to discuss ideas and to share practice with each other (Garet et al. 2001; Wilson and Berne 1999) was a positive aspect of the training. The one aspect where the current training programme was less successful, in terms of meeting key characteristics for professional development programmes, was supporting teacher collaborations which were sustained over time (Garet et al. 2001; Franke et al. 2001). The interviews with co-ordinators highlighted that little or no discussion with one another took place outside the training days, and also highlighted the short-term nature of the project. In terms of encouraging collaborative working between the co-ordinators and the class teachers as well, the results of the project were rather mixed. In some cases, the context of the school and the class teachers (e.g. NQTs) positively supported collaborative working. In other cases however, the influence of external pressures, time and unforeseen circumstance impacted negatively on these collaborations. These issues echo the difficulties faced by teacher professional development programmes highlighted by Rhodes and Houghton-Hill (2000) and Back et al. (2009) in the UK context.

In addition to supporting previous findings on factors that support or hinder the professional development of teachers, the present study also identified some additional considerations that would support this professional development and the transfer of research ideas into the classroom. The model of professional development used for this project had the advantage of drawing on co-ordinators to influence practice in schools more broadly. However, drawing on the results of this study, clear improvements can be made to the design of the training programme. It was clearly identified by the co-ordinators and the class teachers concerned that achieving a greater balance of inviting both the co-ordinators with the remit to implement changes and also the teachers implementing these changes, would result in the professional development having a greater impact. Another characteristic which would improve the impact of the professional development was to consider the timing of professional development. Provision at a point in the year when the professional development can be more easily incorporated in the planning of the school would facilitate the transfer of ideas from the professional development. Related to this issue, providing clearer advanced information to schools about the focus of any professional development and how ideas from this input might be implemented in classrooms even before the professional development itself, would support schools in looking ahead and consider their future planning (and not be subsequently limited by the existing planning of the school as occurred in some schools in this project). These three suggestions of a greater balance of staff to attend, the strategic timing and more advanced information on the professional development to be provided are therefore additional characteristics we can consider, alongside the characteristics for successful professional development already identified from the existing literature.

The other contributory factor influencing the complex mechanisms of the impact of the project was the research design of the study. The nature of the research design, with a given cohort of schools implementing the research ideas on diagrammatic representations over one term, meant that exploring the impact of the training programme over a sustained period of time was not possible. For further work in the future looking at the impact of these ideas on teachers and children, a research design that supports more the reflection upon and the development of the use of diagrammatic representations could be used. Burkhardt and Schoenfeld (2003) advocate the use of design experiments where instructional interventions based on theory undergo on-going revisions as a result of data gathered leading to subsequent reflections. Such an approach would more specifically support the sustained nature of the intervention and the reflections and developing awareness and knowledge by teachers (Korthagen and Kessels 1999).

Conclusions

The aim of the reported study was to develop the use of diagrammatic representations in the classroom, and to develop children's mathematical understanding, by introducing primary teachers to the research literature on the issues concerned. The study has shown that although the mechanisms for developing classroom practice and children's understanding was complex, the professional development programme reported in this study had positive outcomes on both areas. In addition, the complex mechanisms involved have provided insights into how carefully designed professional development programmes can inform classroom practice.

These insights provided implications both for the classroom practice of teachers, and also for the design of professional development programmes for teachers. In terms of classroom practice, the qualitative data from teachers and lesson observations emphasised the progression involved in the use of diagrammatic representations, not just using ‘more’ representations, but also considering the progression for children in using different representations. The study has shown that the increased use of diagrammatic representations does impact on children’s understanding, but the nature of this progression must be emphasised to teachers as well. This finding provides one explanation for why previous research on the use of diagrammatic representations in the teaching of mathematics have found their impact on pupils’ learning to be somewhat mixed. In terms of the design of the professional development programme, the study has provided implications for future work. For example, the involvement of classroom teachers implementing the research ideas, in addition to mathematics co-ordinators, considering the strategic timing of any professional development, and providing clearer advanced information about the training, would further support the transfer of research ideas into classroom practice. These characteristics, alongside a research design that would further support the refection of practice by teachers involved, can further enhance the professional development provided to teachers, and increase the identified impact of diagrammatic representations on the teaching and learning of mathematics.

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Table 1. Design of the input for mathematics co-ordinators

Training day	Timing	Content
First	Start of Autumn term (September 2011)	General research ideas on using diagrammatic representations, focussing these ideas on multiplication with considerations for classroom practice.
Second	Start of Spring term (January 2012)	Reflecting on the results and co-ordinators' experiences from the first term, and subsequently focussing the research ideas on fractions, with considerations for classroom practice.
Third	Start of Spring term (April 2012)	Reflecting on the results of the project and co-ordinators' experiences from the project in general, and looking at the overall implications for classroom practice and for future research.

Table 2. Study design including the professional input and the quantitative and qualitative data collection

Activities in the first two terms	Schools involved
Pre-test (quantitative)	All (8 treatment and 2 control schools)
Professional development training	8 Treatment schools
Start of term interview with class teacher	Half of the treatment schools (4)
Implementation of ideas from training throughout the term	Half of the treatment schools (as above)
Mid-term interview with co-ordinator	Half of the treatment schools (as above)
Start of term interview with mathematics co-ordinator, observation of class teacher, interview with class teacher	Half of the treatment schools (as above)
Post-test (quantitative)	All (8 treatment and 2 control schools)

Table 3. Interview questions for mathematics co-ordinators

Visit	Questions
Middle of term (~ 6 weeks following training)	1. What were your views regarding the first training day back at the beginning of September/January? What were the positives and what could have been improved? 2. Have you been able to implement any of the ideas from the training in the Year 3/5 classroom? Could you give examples of what has been done? 3. Have there been any wider benefits from the training? 4. What barriers or difficulties have you faced in implementing the training ideas? How have you tried to overcome these?
End of term (~ 12 weeks following training)	1. Have you been able to have discussions with the Year3/5 teacher about the project? If so, what have you discussed? 2. What, if any, changes to teaching has being part of the project resulted in? 3. Do you think the project has impacted on children's understanding of multiplication/fractions? If so, why do you think this is? 4. Do you think the project has impacted on your teaching and knowledge about the teaching of multiplication/fractions? 5. What difficulties have you faced in implementing any changes as a result of the project? 6. What improvements could we make to the project to ensure that it has more of an impact in the classroom?

Table 4. Interview questions for class teachers

Visit	Questions
Start of term (immediately after training)	1. Could you say a little bit about your teaching background – years of experience and responsibilities? 2. What are your views on teaching mathematics, for example you confidence in the subject, and your approach to the subject? 3. This research project is focussing on developing the use of visual representations or models in the classroom. To what extent do you use visual models in maths lessons? 4. Is the use of visual models promoted within the school? 5. How do you use visual models when teaching multiplication/fractions in Year 3/5? 6. What are the benefits do you think of using visual models? 7. What are the drawbacks of using visual models? 8. Is there anything else you would like to add before we finish?
End of term (~ 12 weeks following training)	1. Have you been able to have discussions with the maths co-ordinator about the project? If so, what have you discussed? 2. What, if any, changes to your teaching has being part of the project resulted in? 3. Do you think the project has impacted on children's understanding of multiplication/fractions? If so, why do you think this is? 4. Do you think the project has impacted on your teaching and

knowledge about the teaching of multiplication/fractions?
5. What difficulties have you faced in implementing any changes as a result of the project?
6. What improvements could we make to the project to ensure that it has more of an impact in the classroom?

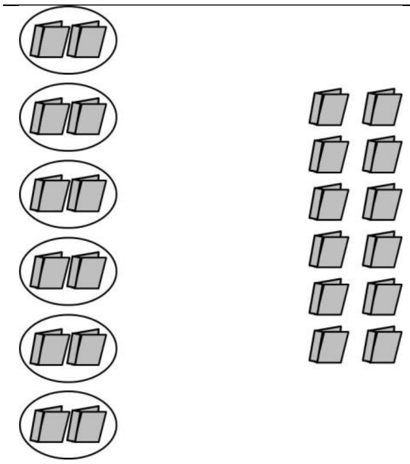

Figure 1. Equal groups (left) and array (right) representations for multiplication (6 times 2 books)

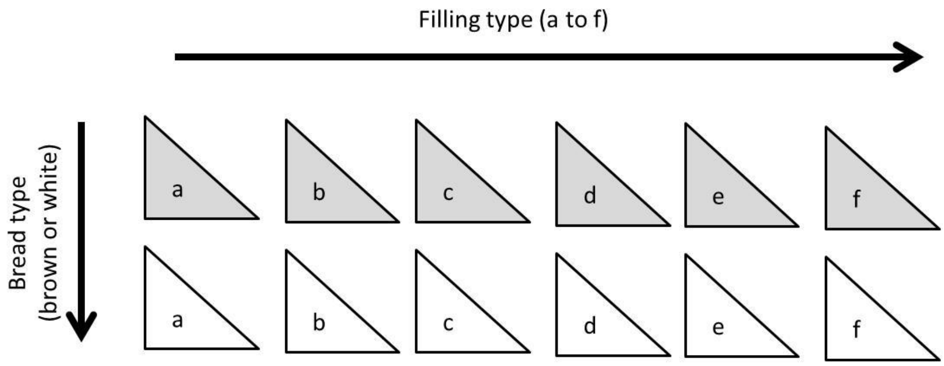

Figure 2. Cartesian product represented by an array (6 times 2 sandwiches)

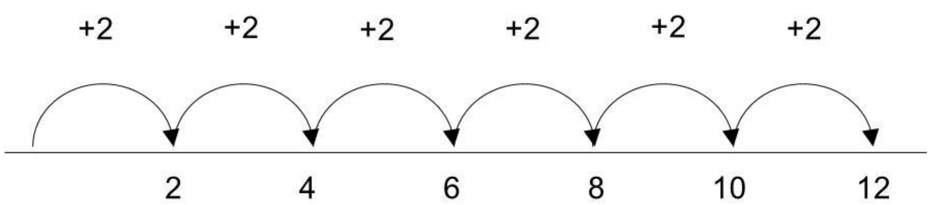

Figure 3. Number line showing 6 × 2

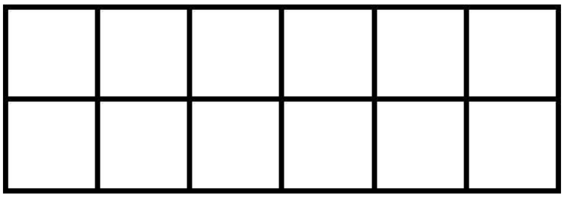

Figure 4. Area representation showing 6 × 2

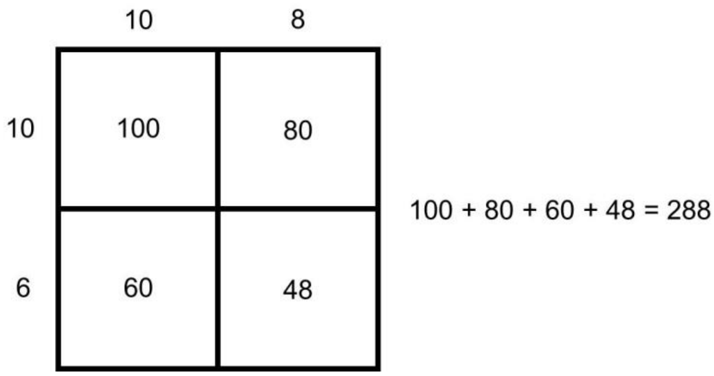

Figure 5. Grid method for 18 × 16

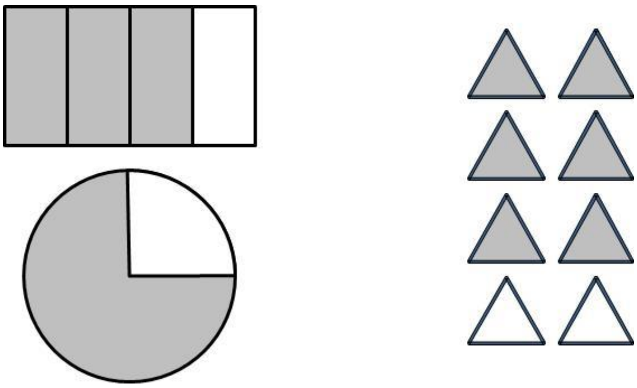

Figure 6.Part whole representations of fractions - continuous (left) and discrete (right)

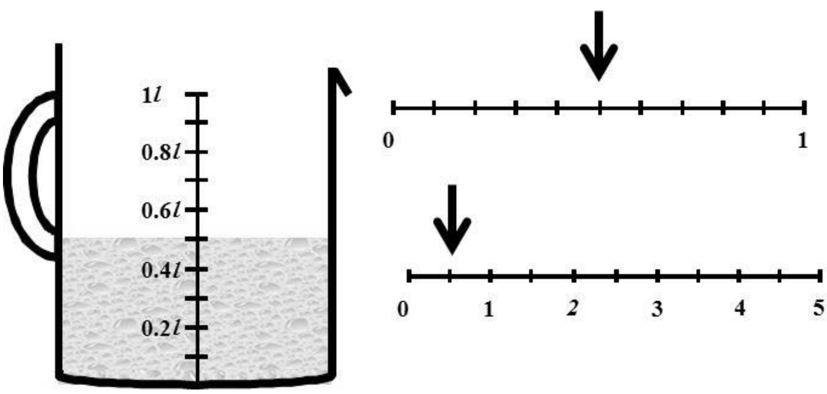

Figure 7.Measure representations of fractions – all showing ½

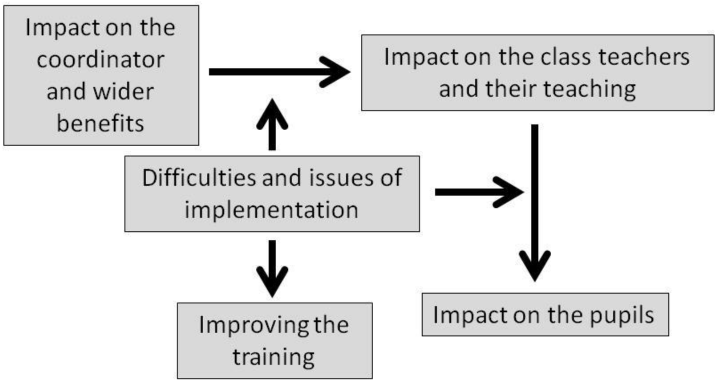

Figure 8. Categories emerging from the qualitative analysis of the interviews